Some characterisations of concurrency points in a triangle

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In geometry, we sometimes prove that a certain configuration Q has property P. If it turns out that Q is the *only* configuration that has property P, then we say that P is a characterisation of Q. For example, the centre of a circle is a point in the plane which is equidistant from every point on the circumference of the circle. In fact, the centre is the only point in the plane which is equidistant from every point on the circumference of a given circle. Thus we say that the centre of a circle is characterised by this property.

In this paper we find characterisations of some concurrency points in a triangle.

Theorem 1

In a triangle ABC, let AX, BY and CZ be cevians concurrent at an internal point M. If

$$\frac{AM}{MX} = \frac{BM}{MY} = \frac{CM}{MZ} = k.$$

then M is the centroid of the triangle and $k = 2$.

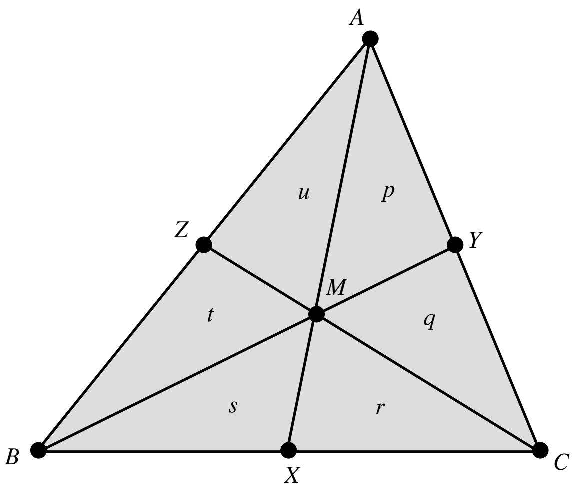

FIGURE 1

Proof

It is easy to see that the ratio of the areas of two triangles with the same perpendicular height is the same as the ratio of the lengths of their bases.

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Thus, in Figure 1, denoting the area of each of the six sub-triangles by the letters p, q, r, s, t and u, we have

$$k = \frac{AM}{MX} = \frac{p+q}{r} = \frac{t+u}{s};$$

$$k = \frac{BM}{MY} = \frac{r+s}{q} = \frac{t+u}{p};$$

and
$$k = \frac{CM}{MZ} = \frac{p+q}{u} = \frac{r+s}{t}.$$

Thus

$$\frac{p+q}{r} = k = \frac{p+q}{u},$$

so $r = u$. Similarly, $s = p$ and $t = q$. So $u + t + s = r + p + q$ and Area (ABX) = Area (AXC). Thus AX is a median of ΔABC. Similarly, BY and CZ are medians, so M is the centroid and $k = 2$.

Lemma

In a triangle ABC, let X be a point on BC such that $\frac{BX}{AB} = \frac{CX}{AC}$. Then AX is the bisector of $\angle BAC$.

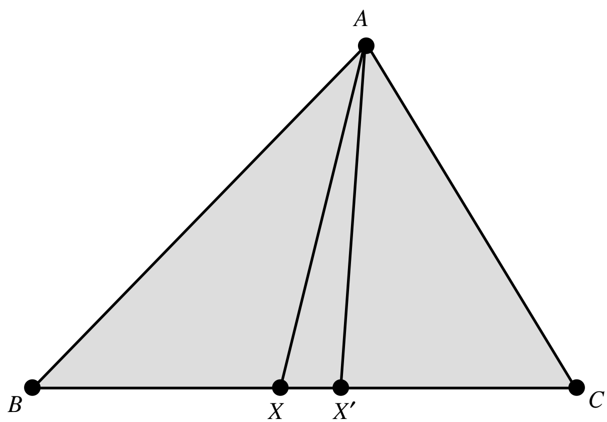

FIGURE 2

Proof

If the bisector of $\angle BAC$ meets BC at X, then

$$\frac{BX'}{AB} = \frac{CX'}{AC}.$$

Thus $BX' : CX' = AB : AC = BX : CX$ whence $X' = X$.

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Theorem 2

Let AX, BY and CZ be cevians of a triangle ABC meeting at an internal point M. If $\frac{BX}{CX} = k\frac{AB}{AC}$; $\frac{CY}{AY} = k\frac{BC}{AB}$ and $\frac{AZ}{BZ} = k\frac{AC}{BC}$, then M is the incentre of the triangle and $k = 1$.

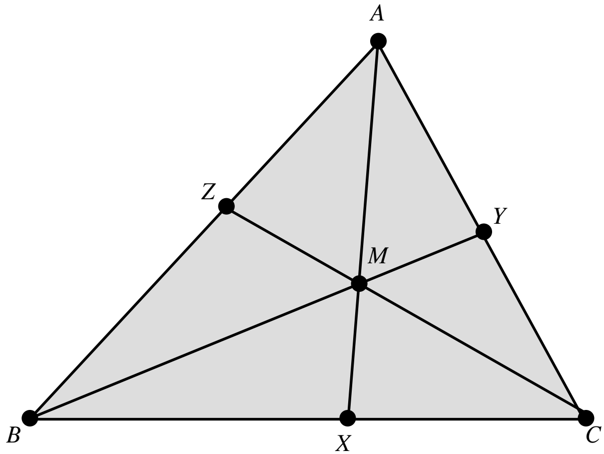

FIGURE 3

Proof

We have

$$\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = 1$$

by Ceva's theorem since AX, BY and CZ are concurrent. Thus

$$1 = \left(k\frac{AB}{AC}\right)\left(k\frac{BC}{AB}\right)\left(k\frac{AC}{BC}\right) = k^3.$$

So $k = 1$ and by the Lemma, AX, BY and CZ are the angle bisectors and M is the incentre of the triangle.

For the next case, we need another lemma.

Lemma

In a triangle ABC let X, Y and Z be internal points of the sides BC, CA and AB respectively such that the perpendiculars to the sides erected at X, Y and Z are concurrent at a point P. Then $AZ^2 + BX^2 + CY^2 = BZ^2 + CX^2 + AY^2$.

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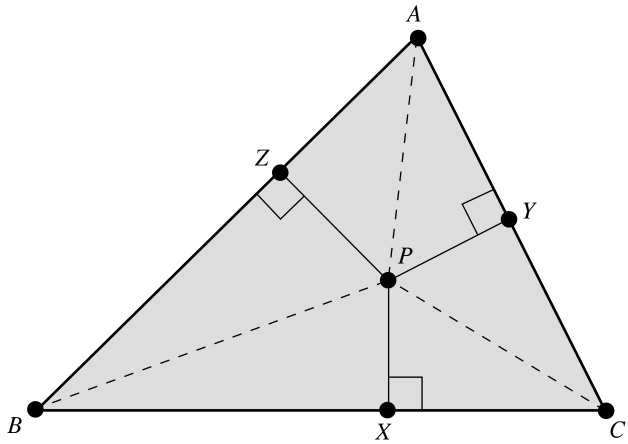

FIGURE 4

Proof

By Pythagoras,

$$AZ^2 = AP^2 - PZ^2;\ CX^2 = PC^2 - PX^2;$$

$$BX^2 = BP^2 - PX^2;\ AY^2 = AP^2 - PY^2;$$

$$CY^2 = PC^2 - PY^2;\ BZ^2 = BP^2 - PZ^2.$$

Adding gives the result.

It is clear that this result also holds if P is outside the triangle or on the perimeter, as the reader can see by drawing appropriate diagrams.

Theorem 3

In a triangle ABC let X, Y and Z be internal points of the sides BC, CA and AB respectively such that the perpendiculars to the sides erected at X, Y and Z are concurrent at a point P. If

$$\frac{AZ}{BZ} = \frac{BX}{CX} = \frac{CY}{AY} = k.$$

then $k = 1$ and P is the circumcentre of the triangle.

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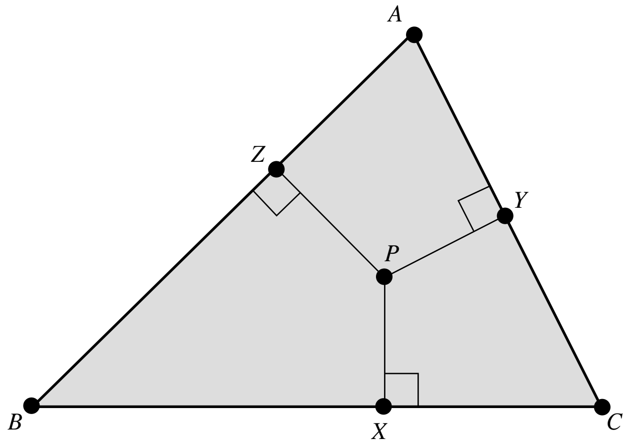

FIGURE 5

Proof

Denoting the segments AZ, BZ, BX, CX, CY, AY by z, r, x, p, y and q, respectively, for convenience, we have $x = kp$, $y = kq$ and $z = kr$. Then, by the Lemma,

$$z^2 + x^2 + y^2 = r^2 + p^2 + q^2,$$

so

$$k^2\left(r^2 + p^2 + q^2\right) = r^2 + p^2 + q^2.$$

Hence $k = 1, p = x, q = y$ and $r = z$, forcing P to be the circumcentre of the triangle.

Recall that the Gergonne point of a triangle is the (internal) point of concurrency obtained by joining the vertices to the opposite points of contact of the inscribed circle.

Theorem 4

In a triangle ABC suppose that cevians AX, BY and CZ are concurrent at an internal point P. If

$$\frac{AZ}{AY} = \frac{BX}{BZ} = \frac{CY}{CX} = k,$$

then $k = 1$ and P is the Gergonne point of the triangle.

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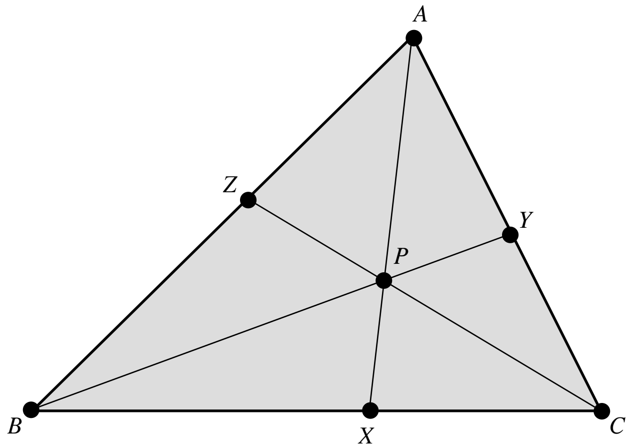

FIGURE 6

Proof

We have $AZ = kAY$; $BX = kBZ$; $CY = kCX$. By Ceva's theorem,

$$AY.BZ.CX = AZ.BX.CY = (kAY)(kBZ)(kCZ) = k^3(AY.BZ.CX)$$

so $k^3 = 1$ forcing $k = 1$. Thus $AX = AY$, $BX = BZ$ and $CX = CY$. Let $AZ = x$, $BX = y$ and $CX = z$. Then $x + y + z = s$, where $s = \frac{1}{2}(a + b + c)$ whence $x = s - a$, $y = s - b$ and $z = s - c$, and these are the tangent lengths to the incircle from the vertices. Hence P is the Gergonne point of the triangle.

We conclude by considering the orthocentre.

Theorem 5

In a triangle ABC suppose the cevians AX, BY and CZ are concurrent at a point P. If

$$AP^2 + BC^2 = BP^2 + CA^2 = CP^2 + AB^2$$

then P is the orthocentre of the triangle.

Proof

We use vectors. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{p}$ denote the position vectors of A, B, C and P with respect to an arbitrary origin O. Then the given condition reduces to

$$|\mathbf{p} - \mathbf{a}|^2 + |\mathbf{c} - \mathbf{b}|^2 = |\mathbf{p} - \mathbf{b}|^2 + |\mathbf{a} - \mathbf{c}|^2 = |\mathbf{p} - \mathbf{c}|^2 + |\mathbf{b} - \mathbf{a}|^2$$

whence

$$|\mathbf{p} - \mathbf{a}|^2 - |\mathbf{p} - \mathbf{b}|^2 = |\mathbf{a} - \mathbf{c}|^2 - |\mathbf{c} - \mathbf{b}|^2,$$

that is,

$$(2\mathbf{p} - \mathbf{a} - \mathbf{b}).(\mathbf{b} - \mathbf{a}) = (\mathbf{a} + \mathbf{b} - 2\mathbf{c}).(\mathbf{a} - \mathbf{b}).$$

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So,

$$(\mathbf{p} - \mathbf{c}).(\mathbf{b} - \mathbf{a}) = 0$$

implying that CP is perpendicular to AB. In a similar manner we can show that BP is perpendicular to CA and AP is perpendicular to BC. Thus P is the orthocentre of the triangle.

The vector proof removes the obligation of considering three cases where the orthocentre is inside, outside or on the perimeter of the triangle. Doubtless such characterisations exist for other triangle centres, which the reader might like to investigate.

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The answers to the *Nemo* page from November 2024 on equations were:

1.	GK Chesterton	Songs of education: the higher mathematics	
2.	Edith Nesbit	The Book of Dragons	Island of the Three Whirlpools
3.	George Macdonald	Robert Falconer	Chapter 18
4.	Jack London	John Barleycorn	Chapter 22
5.	W. M. Thackeray	The Book of Snobs	On Some Country Snobs
6.	James Joyce	Portrait of the Artist as a Young Man	Chapter 3

This month, it is time to focus on fractions and decimals. Quotations are to be identified by reference to author and work. Solutions are invited to the Editor by 23rd May 2025.

1. They did know it evidently: I saw quite well that they all, in a moment's calculation, estimated me at about the same fractional value. The fact seemed to me curious and pregnant; I would not disguise from myself what it indicated, yet managed to keep up my spirits pretty well under its pressure.

2. But the limit of desired knowledge was unattainable, nor could I ever foretell the approximate point after which I might find myself satiated, because of course the denominator of every fraction of knowledge was potentially as infinite as the number of intervals between the fractions themselves.

Continued on page 74.