Synopsis

Developing a Learning Sequence for Transiting from Arithmetic to Elementary Algebra

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1.0 *Introduction*

Algebra as a domain in mathematics occupies a special position as a major analytical tool leading to higher mathematics and many other branches of science. It provides the symbols and techniques to represent and solve problems, and to reason, justify and prove within mathematics and other areas where mathematics serves as a tool. Many students fail to succeed in algebra and are therefore unable to enroll in advanced mathematics which is a gateway to many prestigious professions as well as academic careers. Thus, it is important to understand the conceptual changes which the students experience while moving to the middle school, especially due to the introduction of algebra, and identify ways to address the problems which arise in the course of its introduction.

In contrast to arithmetic, algebra poses a challenge to most students due to the new symbols it proposes and new ways of acting on those symbols. The notations and the conventions are both problematic and are not easily learnt by students. Further, it takes the students away from operations on numbers to computing with abstract symbols. It is no longer possible to process the symbols in an expression as a strict sequence of binary operations, ending in a numerical answer (Booth, 1988). The symbols need to be reinterpreted in new ways before they can be worked upon. The presence of the letter symbol complicates the situation as students do not understand the meaning of the letter as a number and either ignore it or consider it to have some fixed and arbitrary value or construe its meanings based on common appearances of the letter in many situations outside the domain of mathematics (Kuchemann, 1981; MacGregor and Stacey, 1997).

There are also differences in approaching problems in arithmetic and algebra. While in the arithmetic approach students can work from the known conditions and find intermediate numerical solutions to arrive at the solution to the problem, it is essential in the algebraic approach to use expressions to represent the problem situation using a letter for the unknown (Bednarz and Janvier, 1996; Stacey and MacGregor, 1999). Thus, in the context of arithmetic, students do not appreciate the purpose of recording operation sequences or representing problem situations as well as do not abstract the properties and rules of transformation which can be consistently applied while manipulating expressions (Booth, 1988). They only implement procedures for finding the numerical solution to a problem (posed using symbols or embedded in word problems) which may depend on the context or the numbers involved, and thus do not engage in general solution methods applicable over a range of problems (Ursini et al., 2001). The methods of teaching and learning generally used force the students to rigidly follow algorithms without any space for reflecting on them and for exploring properties and relations between numbers and operations. This is unhelpful to students in understanding the equivalence of different procedures, or their generalizability, making it difficult to shift to algebra. Students' poor skills in representing problem situations and weak understanding of transformation of expressions do not allow the students to move to the step of deducing or inferring about the situation, which is the crux of algebra (Booth, 1989).

In India, teaching of algebra generally follows arithmetic in the curriculum, which also would be the case with many other countries. Research over the last few decades has shown the complexities involved in the transition from arithmetic to algebra as described above and the interference in the learning of algebra from arithmetic. Some studies have cautioned against emphasizing the arithmetic-algebra connection as it leads to many misconceptions and is fraught with pedagogical hurdles (e.g. Lee and Wheeler, 1989). Others, in contrast, have pointed out the promises offered by focusing on the connection (e.g. Linchevski and Livneh, 1999; Carpenter et al., 2003). Although many research studies have explored the arithmetic-algebra connection and have identified the cause of many of the troubles in the teaching and learning of algebra, as will be briefly discussed below, there does not appear to exist a well elaborated model of teaching and learning symbolic algebra in the beginning grades which can help students build the connection between the two domains and handle the problems identified in the literature. This study aimed to develop a teaching approach which could bridge the gap between arithmetic and algebra and create meaning for symbols through two broad sets of activities: working with syntactic transformations and working with contexts that lend purpose to algebra. In the process, the study engaged in analyzing students' responses to the various tasks, and identifying the nature of the support that is required to make the transition. This fed back into the development of the teaching module, thereby evolving and clarifying the approach that facilitates students in making the transition.

1.1 The arithmetic algebra connection

Students' earlier experience in primary school arithmetic is largely one of computing single binary operations and the first exposure to multiple operations is in the context of evaluating arithmetic expressions which encode a sequence of binary operations. This requires following conventions in the form of order of operations so that a unique value is arrived at for each expression, even in the absence of brackets. Such tasks form the first connection between arithmetic and algebra where algebra encodes general rules and properties of operating on these arithmetic expressions, like the commutative, associative and distributive properties, which govern the nature of transformations that are possible on the expressions. Algebra provides the letter symbols to mathematically represent these properties in general terms and it is these properties which determine the rules of transformations for algebraic expressions, and which keep them equivalent. The conventions for operating on the expressions are so designed that they encode the structure of the expressions. Very often students fail to see this connection between arithmetic and algebra and thus are unable to make the required transition to algebra.

1.2 Hurdles in the transition to algebra

Students are habituated through arithmetic to obtain a ‘closed’ answer or a single number as the result, which leads them to misunderstand notations like 3+*x* and 3*x* as being equivalent. Expressions such as 3+*x* have multiple meanings in algebra (Wagner and Parker, 1999) and it is necessary to treat them as both processes and products/ objects or as flexible ‘procepts’ (e.g. Sfard, 1991; Tall et al., 2000). For example, 3+*x* can both be understood as a process of adding any number to 3 or the result of this process, namely, the sum of three and any number or three more than any number. Also, the ‘+’ and the ‘–’ signs can be thought of as operations of adding on or taking away (the most common meaning developed in arithmetic), as signs attached to a number used for representing change (increase or decrease) or as encoding a relation of more or less. Similarly, the ‘=’ sign is to be treated as a sign denoting equality or equivalence rather than as an instruction to compute, which is a meaning familiar from the arithmetic context. Whereas in arithmetic, an expression has a fixed meaning and denotation (value), in algebra it is important to separate the denotation of the expression from its meaning which describes the relation embedded in it; because the algebraic expression can be interpreted in various ways depending on the context. One must possess the ability to pay attention to these aspects flexibly, emphasizing one over the other depending on the context, which is a central point in developing algebraic awareness (Mason, 1996; Arzarello et al., 2001).

Many difficulties which students face while manipulating algebraic expressions can be understood by focusing on their understanding of arithmetic expressions and computations in arithmetic. Researchers (e.g. Chaiklin and Lesgold, 1984; Kieran, 1989, 1992; Linchevski and Livneh, 1999) have pointed out that the roots of the problem lie in students’ lack of awareness of the structure of arithmetic expressions. This does not allow the students to understand the properties of operations which can be consistently used in arithmetic contexts and which can be subsequently generalized to deal with symbolic algebra. Students often fail to judge the equality/ inequality of expressions like 345-237+489 with 489+345-237 or 237-345+489 without computation (Chaiklin and Lesgold, 1984) and are inconsistent while evaluating arithmetic expressions. They sometime solve an expression 50-10+10+10 as 50-30 and at other time would solve the expression 27-5+3 correctly as 22+3 (Linchevski and Livneh, 1999). Further they do not see a way of computing the expression 217+175-217+175+67 other than solving step-by-step from left to right and would even be tempted to cancel the '175's (ibid). These are errors due to misperception of structure of the expression and over generalization of rules of order of operations and the same are transferred while working on symbolic algebra. The rules of transformation are for the first time formally defined in algebra but do not make sense to them due to lack of a referent for the letter and validation of the rules, making the students feel that the rules of symbol manipulation are arbitrary. Thus, the reason for arbitrariness or meaninglessness which the students experience during their exposure to algebra is not due to applying or emphasizing rules of transformation, but due to the lack of emphasis on structure of expressions, making appropriate links with properties of operations and explanations for the rules, like distributivity, associativity (Kirshner, 2001). This cannot be simply solved by practicing manipulation of algebraic expressions but through specialized activities focusing on articulating and justifying the usage of rules in the classroom (ibid).

1.3 Approaches to teaching of algebra

Researchers' concern with students' poor understanding of properties of operations and structure of expressions and their resulting failure to deal with algebraic symbolism and its meaning and purpose led to various reconceptualizations of algebra. Many efforts have been made through research studies to convey to the students the essence of algebra and make sense of the symbols and operations on them. This includes introducing algebra through meaningful contexts like pattern generalization, using concrete materials, embedding algebra in problem solving situations, using technology supported approaches like spreadsheets, LOGO, CAS. The various approaches to algebra emphasize different aspects of algebra and may have certain limitations. Some of these approaches focus on creating meaning for the symbols, especially the letter, and the purpose of algebra, leaving the syntactic transformations to be handled by software. However, it has been realized that some basic understanding of symbols and syntax is required to make sense of the rich problem solving contexts or even judge if technology assisted solutions are correct or to use technology profitably in solving problems (Kieran, 2004). Further, it has been argued that when students work with syntactic transformations, they create meaning for the symbols by using them and acting on them. Therefore, separating the contexts in which meaning of the symbols are created from the syntactic aspects of algebraic symbols is not very helpful and both the competencies are required, which is the emphasis in this study.

Studies have also introduced algebra through the route of generalized arithmetic, which focuses on the structural aspects of the number system (Wagner and Kieran, 1989) and encodes the general rules of operations in arithmetic (Kaput, 1995). The “early algebra” studies by Kaput (1998), Carraher et al. (2000, 2001, 2003), Brizuela et al. (2000) and Carpenter, Franke and Levi (2003) are also efforts in the same direction, demonstrating in the process young children’s capabilities to understand symbols, to create them, to work with them and explain their reasoning and solution process. The generalized arithmetic approach is not limited to generalizing regularities in operations and patterns which is a major focus in the “early algebra” studies. It also encompasses a deeper understanding of the structure of expressions which is another line of work used in the studies with students in middle school. These studies try to enhance students' understanding of symbolic expressions and syntax of algebra, which is also the approach adopted by the present study. These researchers have attempted to exploit the arithmetic-algebraic connection, by focusing on the similarities in the two domains in different ways: (i) correct parsing followed by order of operations and exploration of properties of operations (e.g. Thompson and Thompson, 1987), (ii) procedural/ computational similarity (e.g. Liebenberg et al., 1998, 1999a, 1999b; Malara and Iaderosa, 1999; Livneh and Linchevski, 2003) or (iii) representational/ notational similarity (e.g. Booth, 1984; Malara and Iaderosa, 1999). Except for the study by Thompson and Thompson (1987) which actually trained students to perceive the structure of expressions and appreciate the constraint of certain transformations but in a limited situation, the other studies focused largely on computational features and their generalizations to make the transition to algebra. This always did not lead to the desired effect on the students and they still failed to see the equivalences in the transformation rules in arithmetic and algebra and continued to work on algebraic expressions similar to computational arithmetic without abstracting properties and constraints of operations. The present research study builds on these insights from the literature and proposes a way to deal with the arithmetic-algebra connection and tackle the errors due to faulty perception of structure of expressions which have been found to be hurdles in understanding symbolic algebra.

2.0 *Defining the research study*

The research study being reported here is a design experiment on grade 6 students from two schools in Mumbai. It tried to systematically investigate the arithmetic-algebra connection and explored the introduction of algebra as generalized arithmetic by enhancing and connecting students' prior knowledge of arithmetic to algebra and exploiting the structure of arithmetic expressions to learn algebra. In the process it aimed to identify precisely the arithmetic concepts and tasks which would help in making the transition to algebra. Its objectives were to strengthen both *procedural knowledge*, that is, the calculus of algebra – knowledge of rules, conventions and procedures for working on expressions, and *structure sense* – ability to think of an expression as having a value, to identify the components of an expression (surface structure) and to see the relationships of the components in an expression among themselves and with the value of the whole expression (systemic structure) (Kieran, 1989; Hoch and Dreyfus, 2004).

The teaching-learning sequence was not restricted to generalizing properties of operations from arithmetic by emphasizing the structure of the expressions. It was complemented by using tasks which took a more comprehensive view towards generalization – exploring and finding relations among numbers/ quantities, sequence of operations and shapes in patterns, representing and generalizing them and justifying and proving some of the patterns. These tasks provided opportunities to translate the informal processes or arithmetic structures into formal arithmetic or algebraic sentences, which is essential for an algebraic way of thinking. Thus, students learnt the syntax and rules of transforming expressions, with numbers serving as referents for the letter; together with the use of expressions as tools for generalizing, proving and justifying in problem situations. For a complete sense of algebra, one would need to build an understanding of both the syntactic (based on structure of expressions/ equations and rules which define the nature of possible transformations) and the semantic (based on meaning of the letter/ expression/ equation as derived from symbolic statements and problem situations) aspects of algebra. For example, one not only needs to understand the constraints on the possible transformations of the expression 12+3×5-18 but also appreciate the change in value when the expression is slightly changed, say, 3+12×5-18 whose explanation will require a semantic understanding. This kind of knowledge would also help while representing a situation using arithmetic or algebraic expression (e.g. distinguishing a representation $x+3\times2-5-x+4-x$ from $(x+3)\times2-5-x+4-x$).

Students were first engaged in reasoning based on syntactic transformations of expressions like evaluating expressions, identifying correctness of an evaluation procedure for an expression and justifying it, comparing and identifying equality of expressions (e.g. 23-14×34+65 and 23-14×65+34) and its implications for evaluating/ simplifying expressions. These tasks did not require students to generate the symbolic expressions but only to reason about equivalence or non-equivalence of symbolic expressions in various computational and non-computational situations based on rules of syntactic transformations. Hence these tasks are included in the category of reasoning *about* expressions. The purpose of engaging students in activities which required reflection on rules of transformations was to begin the separation of the meaning of the expression from the value of the expression in the context of arithmetic itself, where this is not essential but lays the ground for further algebra learning (see Arzarello et al., 2001). Disparate looking expressions could have the same value with different information/ relation contained in them and similar looking expressions could have different values. Moreover, the familiar arithmetic symbol system was used in the teaching approach as a 'template'[1] for the development of the new algebraic symbolism. It enabled the numbers to be gradually replaced by letters, initially understanding algebraic expressions as only computational processes (inventive-semiotic stage of Goldin and Kaput, 1996); before interpreting them based on the structure of arithmetic expressions (period of structural development of Goldin and Kaput, 1996). It is only after this that algebraic expressions and symbols can be considered independently as objects with certain properties which can

[1] The word 'template' is derived from Sfard's (2000) distinction between 'template-driven' phase and 'object-mediated' phase in the development of new signifiers/ symbols.

represent other entities and can be acted upon (autonomous stage of Goldin and Kaput, 1996 and object mediated phase of Sfard, 2000).

The tasks based on syntactic transformations exploiting the structure of the expressions could only help students to move from the ‘inventive-semiotic’ phase to the phase of ‘structural development’ but not to the ‘autonomous’ or ‘object mediated’ stage. To lead the students to this stage, they were engaged in a set of tasks which developed a culture of generalization, justification and proving, where algebra was treated as a tool for representing general relationships and concluding through manipulations on them. These tasks required the knowledge of rules, conventions and procedures for working on them and have been categorized as reasoning *with* expressions. However, it is important to note that the transition to the ‘autonomous’ or ‘object-mediated’ stage through reasoning *with* expressions is not the only way, this being considered most appropriate for this study. In fact, reasoning *about* expressions can itself lead to this advanced stage (e.g. complex operations on algebraic expressions, thinking of expressions as functions and exploring changes and transformations in functions).

The study also intended to observe and characterize the changes in students’ understanding of algebra in the context of the teaching sequence which was to develop as a result of repeated attempts to make it more coherent. The study did not aim to compare the efficacy of the instructional approach being discussed with other approaches. It aimed instead, at an internal understanding of its effectiveness by exploring the changes in students’ understanding and thinking processes as they developed new concepts and tools through interaction with the instructional sequence, and the possibilities it gave rise to in terms of student responses and the use of various concepts and procedures in different tasks.

3.0 *Designing the study*

The study was a design experiment (Cobb et al., 2003; Shavelson et al., 2003), the teaching-learning sequence evolving over five trials between 2003 and 2005. Design experiments are carried out in educational settings based on prior research and theory seeking "to trace the evolution of learning in complex, messy classrooms and schools, test and build theories of teaching and learning, and produce instructional tools that survive the challenge of everyday experience" (Shavelson et al., 2003, pp. 25). The first two trials (PST-I and PST-II) were preliminary and more exploratory in nature and the last three trials formed the main study (MST-I, MST-II, MST-III) which aimed at making the teaching learning sequence coherent. The teaching learning sequence co-evolved with the developing understanding of the research team about the phenomena under study as well as with the growing understanding of the students as evidenced from their performance and reasoning on various tasks.

3.1 Research questions

The study aimed to address the following research questions:

- What kind of arithmetic understanding would help in learning symbolic algebra?
    - How should the teaching of arithmetic expressions be restructured to prepare for a transformational capability with algebraic expressions?
    - How effective is such a teaching learning sequence in understanding beginning syntactic algebra?
    - Which tasks of the ones identified are more effective in making the shift possible from arithmetic to symbolic algebra?

- Does understanding the syntax and symbols of algebra support students in understanding the purpose of algebra and in the application of algebra for generalizing and justifying?

- What meanings do students attach to letters, expressions and syntactic rules of transformations in this learning approach?

- How do procedural understanding and structure sense of expressions mutually support one another?

3.2 Sample

The study was conducted in the research institute with grade 6 students (11-12 year olds) during the vacation period of the school in Summer (April-May) before the beginning of the school year and mid year (October-November). Each trail lasted for 11-15 days, with each session of approximately one and a half hour. Grade 6 is the first level when algebra is introduced to the students. The students came from nearby English and vernacular (Marathi) medium schools which catered to students from low and middle socio-economic backgrounds. Five schools were involved in the study at various stages of the programme but only two schools participated throughout the study. The choice of the schools was based on convenience; the first reason being their proximity to the centre and the second, due to a need for long term collaboration and support from the school to carry out the study. Students from these schools volunteered to attend the programme by filling in an application form distributed in the schools before the vacations. The final group of students attending the programme was randomly selected from the applicants. In the last two trials (MST-II and MST-III) the same students who attended MST-I were invited to attend the programme. 31 students (15 English medium and 16 Marathi medium students) who participated in all the trials of the main study were chosen for the final data analysis. The students had just appeared for their grade 5 year end exams when they came to attend MST-I and they completed grade 6 during MST-III. The pre-test performance at the beginning of the first main trial showed that the Marathi medium students were better than the English medium students in their knowledge of arithmetic. The English medium students did not undergo any algebra teaching in their school but the Marathi medium students were exposed to preliminary symbolic algebra in the school. The teaching was carried out with multiple groups (two to three) in each trial to see how the different groups responded to the same teaching sequence. The English group had between 20-30 students and the Marathi group had between 30-40 students in each trial.

3.3 Data collection and analysis

The data was collected through pre and post tests, interviews with a subset of students after the second and the third trial of the main study (MST-II and MST-III) which were video recorded and later transcribed, video recording of the classroom proceedings, students' daily work and teacher's log of the daily classroom processes. The post tests were long, containing approximately 25 questions and took around 2 hours to complete. The interviews (14 students after MST-II and 17 after MST-III) were held 8 weeks after the end of MST-II and 4 months after the end of MST-III. The tasks used in the interview were similar to the post tests and were restricted to only arithmetic expressions after MST-II, whereas it included both arithmetic and algebraic expressions and context activities after MST-III.

The data was analyzed both quantitatively and qualitatively with a focus on the nature of responses, the type and number of errors and the students' reasoning as inferred from their responses to tasks or from their explanations given in the interview. The analysis was carried out to ascertain the extent of students' understanding of concepts, rules and procedures in different tasks:

- Understanding of procedures – Evaluation/ simplification of arithmetic and algebraic expressions

Students were asked to evaluate arithmetic expressions which were of two kinds: simple expressions like 3+4×5 and 13-6+4 or complex expressions like -28+49+8+20-49 or 7×18-6×11+4×18 finding easy ways of evaluation. These exercises laid down the rules for operating on expressions and understanding the constraints on operations and dealt with the application of certain procedures on expressions, both arithmetic and algebraic to lead to numerical answers or simpler expressions.

- Rules for transforming expressions with brackets

Brackets were an important concept and were understood both as a precedence operation as well as connected to equality of expressions using bracket opening rules, for example, 23-(9+5)=23-9-5. This flexible understanding of brackets is important for algebra as the first meaning as precedence operation is used for purposes of representation and the second meaning associated with equality is needed to simplify algebraic expressions.

- Understanding of structure – tasks based on '=' sign (comparing simple expressions and filling the blank), identifying expressions equal to a given expression from a list without computation, generating equal expressions

A task that was used to develop the understanding of '=' sign was filling in the blank by computation so that the expressions are equal, e.g. 23+5=__-2. Many of these tasks deemphasized computations and instead focused students' attention on the structure of expressions, identifying relations among expressions and within an expression (e.g. 234+345=233__) in the process using students' intuitive understanding of operations and simple transformations like increasing and decreasing number/ terms, changing numbers and signs. Another set of tasks dealt with identifying and generating expressions equal to a given expression.

For example, given the expression 23+34×15+42, which of these are equal: 34+23×15+42 or 15×34+42+23. Later these tasks also were extended to include algebraic expressions.

- Context based tasks – letter number line, calendar patterns, think-of-a-number game, pattern generalization

The tasks used in this section dealt with observing and expressing generalities using algebraic expressions and subsequently with justifying/ proving them which required manipulating the expressions. The letter-number line was a generalized representation of the number line with the use of a letter, which was further used to carry out two tasks: journey on the letter-number line and distance between two points on the letter-number line. Calendar patterns required the students to represent the simple patterns between the numbers in a calendar using the letter and then explore various patterns in the arrangement of the numbers and justify them. Think-of-a-number game required the students to follow a set of instructions on a number and explain and justify the pattern in the answer with respect to the starting number. Pattern generalization involved the students in representing a general rule for the growing pattern in a sequence of shapes.

Through an analysis of these tasks, students' understanding of '=' sign, order of operations, transforming expressions, meaning of letter and expression, their ideas about representing a situation using the letter and manipulating the expression to arrive at a conclusion were explored. The effort was to examine students' use of the concepts and rules that they had learnt during the program and the extent to which their learning facilitated performance on various tasks. The analysis gave a sense of the nature of the concepts required to make the transition from arithmetic to algebra and allowed one to gauge the effectiveness of the teaching approach in enabling students to make the transition from arithmetic to algebra and in understanding the purpose of algebra.

4.0 *The teaching learning sequence*

The study intended to explore the arithmetic-algebra connection building on the structure sense of expressions but the exact nature of the tasks, procedures and concepts, which would enable the transition from arithmetic to algebra, were to evolve through the teaching. The teaching sequence evolved over five trials between 2003 and 2005 with multiple groups of students. In the paragraphs below, is described the gist of the understanding arrived through the engagement with the process.

4.1 The framework

The following general principles guided the development of the instructional approach.

- Using students' understanding and intuitions/ anticipations in the context of arithmetic to guide their learning of algebra
- Developing students' understanding of algebra by using and extending their experiences with symbols in arithmetic in specific ways
- Reasoning as a basis for learning

Students' knowledge of arithmetic was used as a foundation on which algebraic formalisms could be built. In this study, students' understanding of syntactic rules and conventions was developed and consolidated using their anticipations with respect to operations on numbers, thus tackling the pedagogical problem of teaching the syntax of algebra. By the end of primary school, students have had sufficient experience with numbers and basic operations, and are likely to have attained a level of familiarity and concreteness, which can be fruitfully employed to learn formal symbols and actions on them. Some of their expectations/ anticipations are correct (like, addition of two numbers can be done in any order) and some are wrong (like, subtraction of two numbers can be done in any order) which need to be brought to their notice and which they may be unable to correct by themselves. It was important for this teaching-learning approach to be aware of students' expectations and identify the situations which invoke these expectations, so that they can be gainfully employed to understand the meaning of operations, properties of operations and constrains on transformations. It is in this context that students were engaged first in comparing simple expressions (e.g. 234+436 and 235+437 or 428-129 and 429-128) which required them to make explicit their expectations regarding the operations of '+' and '–'; and then identifying equality of expressions, like 34+13×25+49 with 13+34×25+49 or 25×13+49+34 without computation. Discussions about possibilities and constraints of transformations (that is, about commutativity, distributivity and associativity) are critical in these situations.

The approach not only attributed meaning to the symbols by working on various tasks but also used them in communicating understanding. New ways of interpreting the familiar symbols were created in the context of arithmetic expressions that could subsequently be transferred to algebraic expressions. The students were made to focus away from computations and instead asked to attend to the information or description of relation that is contained in the expressions (e.g. 4+3 is not just 7 but also a relation 'three more than 4' or 'sum of 4 and 3'). Further, the numbers were attached with the signs preceding it to denote a signed number (like -2, +3), which could also represent a change (increase and decrease) in a state. This enabled students to move from an interpretation of expression as encoding a sequence of binary operations to focusing on the units in the expression as contributing to the value of the expression by increasing or decreasing it by certain amounts. This proved to be a very important concept while judging equality of expressions from a list to a given expression as stated in the previous paragraph. Students' expectations and understanding of symbols were tied together by engaging the students to discuss and reason *about* and *with* them.

The connection between arithmetic and algebra was established by building the content which had the following characteristics:

- Exploiting structure sense of expressions
- Use of structural concepts (Terms and '=')
- Explicating connections between arithmetic and algebra

The arithmetic algebra divide was bridged by exploiting the structure inherent in arithmetic expressions to connect arithmetic with algebra using the familiar symbols, thereby giving the letter a referent of number, and also by explicitly giving visual and conceptual support to the students to perceive the structure of an expression correctly. The visual cues allow the perception of the surface structure which is important to analyze the components/ units of the expression or equation. Understanding of systemic structure is required to act on the interpretation of the surface structure. In particular, understanding the '=' sign, equality of expressions and properties of operations are important aspects of structure sense. The reason for emphasizing the structure of expressions in the teaching approach was to link procedures with a sense of structure, so that instead of being two separate skills one following the other, they complement each other to form an integrated knowledge structure. Knowledge of structure of expressions provides scope for flexibly exploring procedures and strategies for computing expressions rather than applying the conventional rules for evaluation, which are rigid. This is an important characteristic of the approach taken in the study and which distinguishes it from earlier efforts (e.g. Livneh and Linchevski, 2003; Liebenberg et al., 1999a) of using arithmetic for teaching algebra.

The above was made possible by providing the students with a set of concepts, namely ‘term’ (e.g. terms in 12-5×3 are +12 and -5×3) and ‘equality’, which allowed them to correctly identify the units of the expressions and further understand the contribution of each part of the expression to the value of the whole expression. This approach is described as the ‘terms approach’. The terms could be simple term (e.g. +12) or complex term (e.g. product term: -5×3 or bracket term: -(4+6)) Also, these concepts helped in reformulating the rules for order of operations and bracket opening in structural terms, thus integrating the procedures more closely with structure of the expression. The precedence rules of evaluating expressions were replaced by the structural counterpart of flexibly combining terms. One could combine only simple terms and the product term had to be converted to a simple term before combining with the simple term: 4+5×2 = $\boxed{+4}\ \boxed{+5\times2}$ = $\boxed{+4}\ \boxed{+10}$ = $\boxed{+14}$. Else, two product terms could be combined if they had a common factor using the distributive property. It can be easily appreciated from the above that the value of the expression 5×2+4 will be the same as 4+5×2 but the value of the expression 5+4×2 will be different. Reordering the terms kept the value of the expressions invariant and thus terms could be combined in any order. In this way, the familiar processes of addition, subtraction and multiplication were converted into ‘objects’ (operations on signed numbers), not necessarily requiring computation at each step and could be combined flexibly by attending to the relationships between the terms in an expression. Thus, students were moved from ‘computing with numbers’ to ‘computing with expressions’ using properties of operations. Bracket was another important concept which was given a dual treatment: as precedence operation and a dynamic use connected with bracket opening rules and equality of expressions.

The terms approach not only created meaning for the operations but also afforded a more direct approach to tackling the structural errors (like, computing sequentially from left to right in the presence of a multiplication sign as in the above example 'LR' or detaching the negative sign, 24-6+4=24-10) and other inconsistencies in evaluating expressions which have been widely cited in the literature (Chaiklin and Lesgold, 1984; Linchevski and Herscovics, 1996; Linchevski and Livneh, 1999; Kieran, 1989). Further, this paved the way for learning manipulation of algebraic expressions which requires the flexibility in perceiving the information and interpreting the relationships embedded in an expression, so as to be able to operate on them. Essentially, the manipulation of algebraic expression follows the same rules of transformation as in arithmetic. All these reconceptualizations with respect to arithmetic allowed students to reason *about* expressions by engaging in discussions with respect to syntactic transformations and ideas of equality and invariance of value, without computation. Thus, on the one hand arithmetic operations were being reified, and on the other, understanding of algebraic manipulation was being developed on this understanding of arithmetic.

Contexts for algebra: The students were later introduced to the use of expressions in the contexts of generalizing, explaining and justifying (reasoning *with* expressions). The main ideas that students needed to grasp in this part are (i) the importance of representing situations for general cases, (ii) knowing that justification/ proof needs a general argument/ explanation (verbal or symbolic) not specific to particular cases, (iii) appreciating the purpose of transforming an expression, (iv) transforming the representation using valid rules and (v) interpreting the result. Students, in this study, were first engaged in simple representation tasks similar to the CSMS (Kuchemann, 1981) test items so that they could learn that representations could be made when all quantities were not given, with the letter/s denoting one or more of the unknown quantities in the situation. Continuing with the spirit of a generalized arithmetic approach that was adopted, students worked on tasks, like the letter-number line, think-of-a-number game, calendar patterns and generalization of growing patterns among shapes.

4.2 The development process

As the study evolved, some of the initial assumptions were modified to enhance the effectiveness of the sequence. The thesis discusses these modifications and the rationale for them. For example, the teaching-learning approach began with the assumption that to teach algebra one only needs to worry about building the structure sense for expressions. The first trial itself (PST-I) led to the modification of this assumption and more efforts were directed at consolidating the procedures of evaluating and transforming expressions and bracket opening rules in the second trial. In the first trial 'term' and 'equality' were found to be two concepts which had the potential to connect arithmetic and algebra. The second trial (PST-II) involved a two group experimental design to explore the extent of effect of arithmetic knowledge (procedure and structure) on algebra learning. The concepts of term and equality were used in this trial only for structure tasks, again separating procedure and structure of expressions resulting in a separation of arithmetic and algebra and a limited understanding of algebra. Discussion of the two pilot trials and a preliminary discussion of the results of these two trials can be found in Subramaniam and Banerjee (2004). In an effort to make the arithmetic algebra connection stronger in the third trial (MST-I), the concept of terms was used for both procedure and structure tasks. Terms were given visual salience by putting them in the boxes (e.g. the terms of 19 – 7 + 4 are $\boxed{+19}$ $\boxed{-7}$ $\boxed{+4}$). The rules for manipulating expressions in arithmetic and algebra were formulated differently, and on hindsight, these rules were not flexible enough and did not exploit the potential of the concept of terms fully. In the context of arithmetic expressions, 'terms' were used only to analyze the expressions before deciding the rule to be applied to evaluate it. In the context of algebraic expressions, 'terms' were used to identify like terms before adding or subtracting them by imagining them to be sum or difference of 'singletons' ($3\times x+4\times x = \underline{x+x+x}+\underline{x+x+x+x}$, Linchevski and Herscovics, 1996). The complementarity of procedure and structure could not be established and the connection between arithmetic and algebra did not get abstracted by the students. Simultaneously, contexts (like letter-number line, area and perimeter) were created to give meaning to the letters, using it to represent general relations so that students accept the non-closure of algebraic expressions. Students' poor knowledge of transforming algebraic expressions was a hindrance in using that knowledge in these tasks and they could not make sense of the use and purpose of algebra in the contexts.

The fourth trial (MST-II) was devoted to making the teaching-learning sequence coherent and radicalizing the structural treatment by making terms and equality as the key concepts which bound the whole sequence. The rules were made flexible and uniform across the domains and were structurally reformulated. Integer operations were also subsumed in the 'terms approach'. This was the first time that the precedence rules were completely done away with and was replaced by the idea of combining terms (which is nothing but adding integers) which has been briefly described in the previous section. New tasks like evaluating expressions using easy ways which required students to flexibly combine terms to minimize the steps for computing (e.g. $-28+49+8+20-49$ or $7\times18-6\times11+4\times18$), and generating equal expressions for a given expression (e.g. $25-3\times5+18$) were created which utilized the complementary nature of procedure and structure sense. The connections between procedure and structure sense and between algebra and arithmetic were established more securely. Efforts were also made to enable students to make sense of these algebraic symbols in contexts (like letter-number line and calendar patterns) and use them as a tool for solving problems. The last fifth trial (MST-III) was used for consolidating the teaching-learning process. It emphasized verbalization and articulation of various procedures and rules of evaluating/ simplifying expressions, rules of opening brackets and use of the concepts and rules learnt up till now in different tasks requiring reasoning. Students were also encouraged to explain and articulate their understanding of patterns in numbers and figures and generalize them verbally in the contexts created to embed algebra before moving to symbolic representations. This led to the opening of another dimension of the arithmetic algebra connection and needs further work.

5. Findings and conclusion

The study evolved from the indications made in various studies about the importance of building structure sense for arithmetic and algebraic expressions and the need to move away from computations to be able to connect the two domains. The analysis of the data revealed that the *radicalized* structural treatment of arithmetic (as is seen by the end of MST-II) with a deeper understanding of expressions and constraints and possibilities of transforming them enabled the transition to algebra by allowing flexibility in computing expressions. Identifying relationships between and within expressions and finding conditions for keeping the value of an expression invariant were the key ideas here. The effects of this approach are further elaborated below.

5.1 Procedural tasks

The students improved their overall performance in the procedural and structural tasks and understanding of rules. The students gained in flexibility while evaluating simple expressions (e.g. 3+4×5 or 13-5+7) and the more complex expressions (e.g. -28+49+8+20-49) finding easy ways of computing them, indicating their appreciation of the structure of the expressions and the ability to take advantage of it. There was a reduction in structural errors (due to faulty parsing, like ‘LR’ and detachment) but they did resurface in more complex situations, suggesting the lack of automaticity among students in the simpler contexts. Integer operation was another weak point resulting in low performance of the students in some items. The interviews and classroom discussions indicate that the students could avoid structural errors in the simple situations and were aware of uniqueness of the value of the expression even though one could use multiple ways of evaluating them. These achievements of the students were significant in the light of the results reported in the literature (cf. Liebenberg et al., 1999a; Malara at al., 1999). The flexibility in manipulating arithmetic expressions together with correct perception of structure of expressions paved the way for the manipulation of algebraic expressions.

By the last trial, most students were comfortable with simplifying algebraic expressions (e.g. $3 \times x+4+4 \times x-5$), applying the same rules as in arithmetic. Interviews with the students with respect to algebraic expressions after MST-III revealed their awareness of equivalence of all the steps in the process of simplification. For example, the expressions $3 \times x+4+4 \times x-5$ and $7 \times x-1$ are equivalent and so are the steps in between. Although most students were able to evaluate algebraic expressions for a given value of the letter even when they made sign and calculation errors; a few students, however, failed to substitute the letter by a number till the last trial. The students interviewed did not show any such difficulty.

The appreciation of the similarity between manipulating arithmetic and algebraic expressions was a difficult task and developed only in subsequent trials when attempts were made to focus away from computation in the context of arithmetic. Consistency in perceiving the structure of expressions and understanding the properties of operations that can be used in the context of arithmetic is an important step to move to algebra. The coherence in the teaching-learning sequence which was developed by MST-II (discussed in section 4.2) could be a factor influencing the change as is seen by the end of the last trial. The students successfully generalized their understanding of rules of simplification from the context of evaluation of arithmetic expressions to simplification of algebraic expressions, displaying the connection between the two domains in their understanding.

5.2 Rules of transformation of expressions with brackets

In the course of the program, more students learnt to use bracket opening rules to evaluate expressions but for some students, this was accompanied by a lack of appreciation of the meaning of the bracket as enclosing parts which have to be given precedence in operation. Both the written test and interviews revealed that the two notions of bracket were absorbed by some of the students as procedures and not as 'procepts' which did not allow them to anticipate the effect of removing and putting the brackets. They failed to simultaneously understand that the bracketed (sub-)expression could be substituted by either a number or another equal expression, which is an indication of an evolved 'proceptual' understanding, which is useful for generating representations for problem solving. Students made more errors when the bracket was preceded by a negative sign rather than multiplication sign. Additional suggestions about ways of dealing with the brackets which emerged as the structural approach evolved are described in the thesis.

5.3 Structural tasks

These tasks revealed students' deeper understanding of expressions. Students' responses revealed a fair degree of understanding of constraints and possibilities of transformations, properties of operations and anticipation of the result of those operations. They understood that terms can be rearranged to keep the value same or they can be changed in ways that the net result does not change, rearranging the signs or numbers changes the value, a positive term increases the value of the expression and a negative term decreases it. Research literature quoted earlier, both exploratory and classroom interventions, indicate the difficulty students in general have in understanding these ideas.

Students’ understanding of the ‘=’ sign, that it signifies the equality in value of the expressions on both sides of the ‘=’ sign, which is an important structural notion connecting arithmetic and algebra was elaborated through many tasks. Although students at times made errors in equalizing expressions by filling the blank (e.g. 23+4=__-3), they could judge both arithmetic and algebraic expressions for their equality/ inequality with respect to a given expression without computation and also generate expressions equal to a given one, focusing on the relationships between the terms and the transformations that were applied to it. In particular, classroom discussions of how a given expression could be transformed while keeping its value invariant led to significant revelations about students’ understanding. Further, these tasks served as better diagnostic and learning tools with respect to the understanding of equality than the more traditional task of filling in the blank.

Interviews also revealed students’ ability to identify equal expressions from a list of complex expressions and to compare them with the original given expression identifying the greater/ smaller expression in a pair. This was accomplished through a meaningful, rather than a mechanical, short-cut procedure, use of the concept of ‘terms’. Comparison of such complex expressions was unfamiliar to them and their flexible use of terms in the task was an important finding in the interview[2]. They performed well in the written test in both arithmetic and algebraic expressions, although there was a decrease in their performance in arithmetic expressions with product terms, where a few of them consistently failed to use the correct parsing/ unitization to identify the equal expression in the post test. A few students also faced difficulty in judging equality of expressions when it involved brackets, a problem which was noticed in the evaluation tasks as well. However, the interviews and the classroom discussions showed that they had strategies in place to deal with these tasks and to rectify their errors and they were clear

[2] Students had been exposed to tasks which involved comparing simple two termed expressions like 68-29 and 67-28, results of which are discussed in Naik, Banerjee and Subramaniam (2005).

about equality in value as an essential criterion for two expressions to be equal. The students further pointed out that two equivalent algebraic expressions (e.g. $3\times x+4+4\times x-5$ and $-5+4\times x+4+3\times x$) will have equal value for all numerical values of the letter. Two ways of justifying it were seen: by replacing the letter by the number in both the expressions to arrive at two arithmetic expressions which they knew would have equal values or directly inferring that particular cases would hold true since the general case is true.

5.4 Context tasks

Although the tasks discussed earlier had created in the students a predisposition for symbolic representations and thinking with an expression, fewer students could use these resources adequately for the tasks of reasoning *with* expressions or use this to appreciate the ‘purpose of algebra’. The issue is not simply one of transferring the abilities from the syntactic world to the context situations where algebra is to be used as a tool or of giving meaning to the letter by embedding them in contexts. Two elements that play an important part in these tasks are (i) the culture of generalizing, proving and verifying, with which the students had very little experience and (ii) students’ belief about the effectiveness of using algebra in these tasks.

In the initial trials, students either did not understand the goal of the task and therefore randomly manipulated the representation they had created, or knew the goal, wrote the correct answer in the end but could not manipulate the expression correctly to arrive at that answer. In the last trial, however with a change in the approach to deal with this issue which encouraged verbalization, some students engaged in algebraic thinking and used narrative arguments, often displaying a quasi-variable approach (Fujii, 2003), to convince others about the generality of a result or to draw conclusions. One must note however, that this did not necessarily require algebraic representation. A few also successfully used algebraic representations, could anticipate the goal and accordingly manipulate it to prove the result. Still, a few continued to repeatedly verify the conjecture/ proposition for specific instances, not realizing the limitation of the approach. This pattern of responses led to the understanding that students' abilities to manipulate algebraic expressions and their knowledge of transformation rules is put into use only after they understand the purpose of the task, the need for algebraic representation and can anticipate the goal. Otherwise, the manipulation of algebraic expressions in the contexts is random or the use of algebra is completely ignored. Possessing the syntactic knowledge of algebraic expressions predisposes students to think in terms of expressions within the contexts but does not guarantee success. Thus, besides the 'push' from arithmetic which lays the ground for initial understanding of algebraic symbols and expressions and reasoning about expressions (phase of structural development), one needs the 'pull' from a culture of generalization and the need for general justifications, not restricted to specific instances, to move to the autonomous stage.

5.5 Meaning of the letter and the expression

The emphasis in the teaching approach was on seeing an expression in flexible ways: as a statement expressing relationship and a value. One of the major hurdles in making sense of algebraic symbolism is understanding the meaning of the letter and the duality of the various symbols (see Wagner et al., 1999). From the analysis of the tasks in this study, it was found that, excepting a few, most students seemed to understand the meaning of the letter as a number and the dual meaning of the expression as something to be evaluated as well as expressing a relationship. Students could verbalize the meaning of simple expressions like 5+4 or x-3 (four more than five or three less than x) as well as see a statement like x-3+5=x+2 (in the context of a task on the letter-number line) as expressing a relation between x-3 and x+2 (x-3 is five less than x+2) and the fact that subtracting three and adding five to x leads to x+2. Instances of perceiving expressions in this dual manner were also seen in the tasks described above, especially in the structure tasks.

5.6 Procedure-structure connection

Students' responses in the tasks on reasoning *about* expressions in the context of syntactic transformations revealed the inter-linkages between procedure and structure of expressions. Their scores in procedural tasks and structural tasks are highly positively correlated. There is some indication to the fact that one needs a minimum competence in procedures to internalize and abstract those properties for perceiving structure and answer questions consistently related to them. A preliminary analysis of the data over three trials (PST-II, MST-I and MST-II) had revealed that the structural understanding of expressions developed as a result of consistent application of the rules and procedures over many situations sharing the structural features and that structure oriented approach to teaching helped in strengthening both procedural and structural understanding (Banerjee and Subramaniam, 2005). But, it is the qualitative data analysis, as discussed in the previous sections, which show the complementary use of these two senses and which allows students to work efficiently in both, predominantly procedural and predominantly structural tasks.

6. Conclusions

The study pointed out the purpose, strengths and the limitations of the various tasks used at different points of the study. It thereby elaborated on the specific supports, in the form of vocabulary, concepts, rules and procedures required for making the transition from arithmetic to algebra, without which it is difficult for students to see the connection between arithmetic and algebra. Further, a teaching guideline is proposed on the basis of this study for making a smoother transition from arithmetic to algebra.

The approach which was adopted and evolved during the study has the potential to substantially bridge the gap between arithmetic and algebra. The specific features of the approach which facilitate this connection are:

(i) building on students' understanding of arithmetic operations and intuitions

(ii) moving away from computation and emphasizing structure of the expressions

(iii) fostering an understanding of expressions in terms of information it contains, relationship embedded in it and the value it stands for

(iv) identifying concepts of terms and equality, which are structural and can help in consistently understanding rules of transformation of expressions

(v) reformulating the procedures of manipulating expressions in structural terms and using the same rules, terminology, notations and conventions in solving tasks in arithmetic and algebra

(vi) deepening the understanding of structure of expressions by focusing on invariance of value of expressions, thereby elaborating the understanding of equality and equivalence of expressions

(vii) choosing tasks so that procedures get connected with structure sense

(viii) explicit attention to the number as a referent for the letter

(ix) emphasizing the process-product duality or flexible 'proceptual' understanding through tasks

(x) developing the ability to communicate and reason with symbols

These are important aspects of the arithmetic-algebra transition and have been points of concern in many of the exploratory studies quoted in the introduction of this synopsis and elaborated in the thesis.

The approach succeeded in many ways in dealing with the syntactic and the semantic aspects of arithmetic and algebraic expressions. Although students' understanding of rules of transformations and operation sense was visible in the context of syntactic transformations and reasoning *about* expressions, it was not fully used while reasoning *with* expressions. Students could display algebraic thinking by the end of the last trial and convincingly explain their solutions but the transfer to the symbolic mode was not easy, even when they could understand the process of the representation and manipulation to draw conclusions. The unsatisfactory development of the teaching approach with regard to this aspect of algebra, largely guided by the assumption that knowledge of algebraic symbols and manipulation would directly lead to their use in contexts, was probably responsible for many of the effects seen in students' responses. Symbolic proofs/ justifications need to be preceded by developing understanding of the need for algebra and engaging students in verbalizing the process of solution, a point which was realized only in the last trial. It is hypothesized that reasoning *about* expressions may help in reasoning *with* expressions by enabling the students to think in terms of expressions.

The study tried to explore and show the potential of the approach in making the teaching and learning of the two domains, arithmetic and algebra, more coherent and connected. It was not designed to experimentally establish the efficacy of this approach with respect to the traditional or any other approach. One direction in which the study can be extended is to include problem solving by framing and solving equations with in the scope of the approach and also include rational numbers in the arithmetic expressions and as referents for the letter. Another challenge is to evolve the approach to incorporate non-linear algebraic expressions, multiple variables in expressions and operations on linear and non-linear expressions.

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