Investigating and Supporting Teachers' Knowledge of and Responses to Students' Mathematical Thinking

Synopsis

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1. AN OVERVIEW

The thesis is an attempt to characterise teachers' knowledge of students' mathematical thinking, as it gets manifested in their practice. The current research in mathematics teacher education focuses on (a) the assessment of teacher knowledge through the use of standard instruments, and (b) supporting teachers through tasks that deepen their professional knowledge of the subject matter. Some researchers have argued that such a discourse does not capture the dynamicity of teachers' knowledge manifested in the classroom. The thesis is an attempt to respond to such a critique by presenting a way of systematically investigating teaching practice, in order to capture the dynamic aspects of teacher knowledge manifested in the act of teaching.

The thesis reports an ethnographic case study of the practice of four experienced elementary school mathematics teachers. Data was collected through observations, interviews, and formal and informal interactions with the participating teachers for two consecutive academic sessions. Evidences from teachers' classroom practices suggest that: (a) the knowledge of the teacher is not uniquely possessed by the individual but is a joint province of teachers and students in a classroom, and (b) the tools used to investigate the dynamic aspects of teacher's knowledge need to be reimagined, for instance, students' responses might help in unpacking some aspects of such knowledge.

The analysis revealed that teachers became more responsive to students' *anticipated* and *actual* ways of (mathematical) thinking from the first to the second year of the study. As teachers became more responsive to students' ideas, they experienced mathematical challenges in handling classroom situations. The thesis presents the *knowledge demands* underlying the teaching of a specific topic, decimal numbers, at Grades 5 and 6, respectively. These knowledge demands, arising from *contingent* classroom situations, were analysed to unpack the aspects of topic-specific knowledge required for teaching mathematics. Teachers were supported in handling these knowledge demands through *in-situ* support in the classroom, and *ex-situ* support through teacher-researcher meetings.

Through the nature of support, demanded by and offered to the teachers, the study witnessed the evolution of a community of learning involving teachers and researchers. The centrality of practice, both in investigating teachers' knowledge and in developing a process of supporting them, have implications for mathematics teacher education, research on mathematics teachers, and for bridging the gap between research and practice in education.

2. BACKGROUND

Chapter 1 of the thesis sets the background for the study. Section 2.1 locates the research study in the larger debate on making research meaningful to the practice of education. Section 2.2 discusses the relevance of the study in the Indian context.

2.1 Teacher-Researcher Partnerships: New Vision for Advancement of the Field

Cai, Morris, Hohensee, Hwang, Robinson and Hiebert (2017) suggest the creation of professional knowledge base for improving the practice of teaching through partnerships between teachers and researchers. Imagining, building and sustaining different kinds of partnerships between teachers and researchers requires a re-imagination of research methodologies in order to break the 'isolation' that teachers and researchers endemically suffer from.

The isolation between the research and the teaching community is a historical challenge in the Indian context. An attempt to bring the two communities together was made during deliberations on the National Curriculum Framework 2005 (henceforth NCF 2005), an intensive national exercise involving exchange of ideas between different stakeholders in education. Further, the National Curriculum Framework for Teacher Education (NCFTE, 2009) suggested a reformulation of goals of the existing pre- and in-service teacher education programmes in order to support teachers in implementing NCF 2005 in their practice.

Several challenges have been reported in implementing these reforms. Kumar (2008) asserted that the much-needed teacher education reforms, which were expected to follow from the reforms in school education, did not get operationalised. Although NCF 2005 acknowledged the teacher as a major change agent in helping students become independent learners, it did not engage with the constraints of a historically disempowered teacher who is accustomed to accepting the curriculum document and the textbooks as a given (Batra, 2005). Thus, the teachers became *objects of reforms,* with an expectation of implementing the proposed reforms in the classroom.

Research studies on teachers' classrooms (Takker, 2011) reported that teachers held a confused understanding of the propositions of the reformed curriculum. The teachers struggled to use reformed pedagogies due to their lack of experience with such pedagogies through their schooling or teacher education (Rampal & Subramanian, 2012). The changes noted in teachers' practices were minimal, sometimes a hybrid of the old and the new practices. Although workshops were organized to provide in-service training on the intended reforms, they did not engage with the realities of teachers' classrooms. Teachers have often raised concerns about the mismatch between the ideal classrooms imagined or assumed in the trainings or workshops, and their real classrooms.

2.2 Existing Research On and With Mathematics Teachers in India

In a review of literature on research in mathematics education in India, Banerjee (2012) found that studies on mathematics teachers are very few. A majority of research studies focus on identifying the causes of fear of mathematics among students, listing students' errors in specific problems, finding reasons for mathematics anxiety, and comparing different teaching methods. In the recent years, research has shifted its focus to hypothesising learning trajectories in specific topics.

There are tensions in the research on teaching. First, an observation of teaching is perceived as an act of performance evaluation, since the school inspectors use observations to judge a teacher and make decisions about their career advancement. Second, participation in research on teaching is often seen as a task (among several others) that the teacher is expected to perform. The perceived image of the teachers as objects of research (refer Excerpt 1) is quite similar to the popular image of the teacher in the enactment of a curriculum framework (as noted by Batra, 2005).

Excerpt 1: Teachers' perspectives on their role in research

TP	Researchers come and go, they are not actually interested in our (teachers') problems.
TV	You can tell me what you want to see and tell people through your thesis. I will teach using that method. I want to help you.
TN	See the researchers can spend time to analyse everything, but we have to teach, teach in the class, fast, complete the syllabus. We don't have so much time to think. This is my job.
TJ	Researcher can report one incident or performance, like results of a test. In teaching, every day matters.
TR	See I have a lot of work, so you tell me what you need. I can give that to you and then both of us are done.
Legend used: T = Teacher, followed by the first letter of the pseudonym of the teacher.	

How can teachers contribute to the research on students' conceptions? In what ways can research on students' thinking or learning trajectories be meaningfully used by teachers? Some such questions can lead us to exploring the nature of teacher-researcher partnerships. In the Indian context, where connections between research, school education and teaching are a significant part of the contemporary discourse, it would be relevant to ask - what kind of an image or role of teachers can be imagined in the research on teacher education and development, and also, how can teachers' knowledge contribute to the changing research questions? In this context, the accountability of the researchers studying teaching or learning in classrooms also needs to be figured. This points to the need for thinking about the dialectical ways in which research and teaching can support the development of these two fields and impact practice.

In Kumar's (2018) research study, mathematics teachers' beliefs and knowledge were explored and supported through professional learning community comprising of teachers, teacher educator and researcher. Situatedness, challenge, and community were identified as central to the design of professional development to help teachers reflect and renegotiate their identities. Initiatives such as these are significant in the Indian context for two reasons. First, they call attention to the features of a professional development programme which situate it in the context of classroom practice. Second, they offer an exemplar for approaches which can connect research more closely with practice, rather than offering mere accounts of practice.

2.3 Some Emergent Questions

In the Indian context, teachers are unaware of the research conducted in classrooms or about students. Besides, there is little in research practice that encourages researchers to make explicit connections with practice. Further, working through the bureaucracy of the school system, it often becomes difficult to conduct research which can be done in collaboration with teachers. Some questions which help in thinking about ways in which teachers can engage with the research meaningfully include: What kind of research would teachers find useful? What could be the role of teachers in a research which focuses on students' or teachers' views, perceptions, knowledge, etc.? How can teachers contribute to making research more grounded in practice? What are the ways in which teachers and researchers can work to improve students' learning?

The thesis reports an attempt to explore some of these questions through a participatory approach to understand and support teachers' knowledge of students' mathematical thinking and learning.

3. INVESTIGATING AND SUPPORTING TEACHERS' KNOWLEDGE

An investigation of teachers' knowledge has been an area of interest for a few decades now. However, the two strands of investigating and developing teachers' mathematical knowledge, have been researched separately. Chapter 2 of the thesis presents a review of literature on these strands to understand what all has been learnt, and the questions that still remain unanswered. Section 3.1 briefly discusses the frameworks used to understand mathematics teachers' knowledge. Teachers' knowledge of students' mathematical thinking, which is the focus of this study, is considered a subset of teachers' knowledge, discussed in Section 3.2. The literature on responsive listening to students, reviewed in Section 3.3, suggests a way of studying teachers' knowledge about students' thinking manifested in their practice. An important resource for developing the knowledge that teachers need to teach effectively is the research literature on specific topics, particularly the work on students' conceptions, reviewed in Section 3.4. Section 3.5 discusses the use of a framework to captures the connect between the assumptions of existing professional development initiatives, teacher knowledge and practice. To conclude, the questions raised from a reflection on the existing literature are discussed in Section 3.6.

3.1 Researching Mathematics Teachers' Knowledge

One of the ways in which teachers can be supported is by identifying and developing the knowledge needed to teach effectively. Shulman (1986) argued that teachers' knowledge is different from that of a mathematician. He classified teachers' content knowledge into subject matter knowledge (SMK), pedagogical content knowledge (PCK), and curriculum knowledge (CK), justifying how each of these knowledge types affect teachers' pedagogical judgments while teaching. For instance, PCK deals with ways of representing and formulating the subject to make it comprehensible for students.

Research studies following from Shulman's framework operationalised and refined the categories of teachers' knowledge, by designing and using instruments to measure it. One of the critiques of Shulman's work was that teacher knowledge was considered static (Petrou & Goulding, 2011). Fennema and Franke (1992) placed context at the centre of their framework on teacher knowledge. They argued that the dynamicity of teachers' knowledge can be captured by studying its interactions with teachers' beliefs. Research on students' thinking in specific problem types within arithmetic (Carpenter, Fennema, Peterson, & Carey, 1988) was followed with an impact analysis of structured knowledge about students' thinking on teachers' knowledge (of content, pedagogy, students' cognition) and beliefs in practice. A more detailed analysis of practice was offered by Ball, Hill and Bass (2005) who proposed the construct of Mathematics Knowledge for Teaching (MKT), which was classified into two broad categories, subject matter knowledge (SMK) and pedagogical content knowledge (PCK). The authors proposed studying 'tasks of teaching' to understand teachers' work. Hill, Blunk, Charalambos, Lewis, Phelps, Sleep and Ball (2008) designed a rubric to assess the Mathematical Quality of Instruction (MQI) which included an interaction of dimensions which characterize rigour and richness of mathematics in a lesson, and identified its relation with teachers' MKT. Ma (2010) used interview items, developed around tasks of teaching, to propose that a deep and thorough knowledge of the subject matter includes identifying key ideas in the teaching of specific topics and connecting these ideas with the structure of mathematics.

Further work on teacher knowledge continued to weaken the boundaries between SMK and PCK in Ball et al.'s (2005) framework by focusing on the interactions between the act of teaching and teachers' mathematical knowledge, leading to alternate frameworks. In their work on studying teacher knowledge in context, Fennema et al. (1992) stressed the need to look closely at the act of teaching. Rowland, Huckstep and Thwaites (2005) analysed pre-service elementary school teaching in British classrooms to propose the Knowledge Quartet (KQ) framework, which characterises teaching practices that invoke teacher's knowledge. The framework allows for integration of categories of teacher knowledge, for instance, connections between SMK and PCK, or between teacher knowledge and beliefs. Carrillo, Climent, Contreras and Muñoz-Catalán (2013) proposed a revised focus on the knowledge of the mathematics teacher (MTSK), which makes this knowledge different from the knowledge of other mathematics professionals and teachers of other disciplines. This focus, for instance, means ignoring the common content knowledge from Ball et al.'s (2005) framework, and broadening the scope of horizon content knowledge. Adler and Rhonda (2015) suggested focusing on the Mathematical Discourse in Instruction (MDI) from a socio-cultural orientation of learning in rural mathematics classrooms in South Africa.

The review of literature shows that the existing frameworks on teacher knowledge attempt to identify its components, especially those components that are missing from typical trajectories laid out by formal teacher preparation programmes (Ball, Thames & Phelps, 2008). Such frameworks have been criticized for at least two reasons. First, as Hodgen (2011) argues, "teacher knowledge is *embedded* in the practices of teaching and any attempt to describe this knowledge abstractly is likely to fail to capture its dynamic nature" (p.29). Second, the notion that the teacher acts as an individual in the process of teaching and learning, and hence teacher knowledge is uniquely the province of a teacher, needs to be problematized. Rowland and Ruthven (2011) suggest that there is a need to go beyond the individualistic assumptions about teacher knowledge and engage with the dynamic system in which teachers' work is located. The insight is that teachers' knowledge comes into play not when teachers are doing mathematics but when they are using it to teach mathematics (Fenemma & Franke, 1992; Hodgen, 2011).

3.2 Teachers' Knowledge of Students' Thinking

We noted several frameworks which have tried to conceptualise 'teacher knowledge' by categorising it into sub parts. One of the major arguments against the use of constructs such as specialised knowledge of mathematics teachers, or the measurement perspective on teachers' knowledge, comes from the difficulty in using them to understand teaching decisions made in the ebb and flow of teaching. Teaching decisions, such as choice of appropriate representations, pressing some learner meanings, responding to a student misconception, etc. are a part of teachers' routine and require rich knowledge base. Teachers' knowledge about students' mathematical ways of thinking informs their decision making in classroom.

Shulman (1986) proposed the construct of pedagogical content knowledge (PCK) which includes the knowledge of students' conceptions and preconceptions for the learning of specific topics. Ball, Thames and Phelps (2008) refined the construct of PCK, to knowledge of content and students (KCS), which includes the knowledge of common student errors, prediction about whether the students will find a task motivating and interesting, and the teacher's ability to hear and interpret students' emerging or incomplete thinking. Ball, Hill and Bass (2000) assert that teachers need insight and understanding of the content in order to identify students' errors and select appropriate representations for dealing with them. Such knowledge lies at the interface of a mathematical idea or procedure, and ways in which students think about it. Much of this understanding has developed from the research literature on students' thinking, learning trajectories, and cognition. Carrillo, Climent, Contreras and Muñoz-Catalán (2013) expand the notion of KCS to include the knowledge of ways in which learning theories can be utilised for teaching in classroom. This knowledge is not limited to *knowing* the theories or models of students' learning, but also how these can be used to orchestrate or plan learning experiences.

Llinares (2013) notes that a teachers' ability to identify the mathematical elements of the students' talk is a skill and can be defined as professional noticing. This skill of noticing, understanding and inferring from students' productions, allows the teachers to plan learning trajectories and make informed instructional decisions. Hodgen (2011) argues that mathematical knowledge for teaching needs to be distinguished from the mathematical knowledge required for practice. Rowland, Thwaites and Jared (2015) call it *knowledge in play*, suggesting that such knowledge gets activated while teaching in the classroom. They define 'contingency' to refer to unpredictable situations, arising from an unanticipated student remark, which are difficult to plan (Rowland & Zazkis, 2013). Knowing to act in the moment (Mason & Spence, 2000) is an important part of teaching. The literature described above leads us to conjecture that although these surprises are unanticipated, teachers can be prepared to handle such situations more effectively.

3.3 *Knowticing* and Responsive Listening

An awareness of students' conceptions might not be sufficient to support teachers' decision-making during teaching (Kazemi & Franke, 2004). Even (2008) proposes that the integration of knowledge about students' conceptions or ways of thinking and using this knowledge to inform practice, gives rise to a new object, called *knowtice*. Teachers' knowledge and noticing of students' thinking influences and gets influenced by how they listen to (and interpret) students' ideas. Since such listening depends on how knowledge is constructed in specific classrooms, an important part of teaching is listening and responding to unanticipated student ideas.

Doerr (2006) argues that expertise in teaching is not uniform, and cannot be achieved through the learning of a fixed set of constructs; rather it is knowledge that develops across varying dimensions and in varied contexts for particular purposes. Responsive teaching has two aspects – first, an aspect of listening to what students are saying and understanding what they are thinking (Empson & Jacobs, 2008), and second, responding pedagogically to students' thinking in order to facilitate learning. Potari and Jaworski (2002) stress both cognitive and affective sensitivity to students, as well as the skill in managing mathematical challenge and learning. Davis (1997) distinguished interpretive listening, where the teacher is attending to students' ideas from evaluative listening, where teacher's listening is filtered by prior expectations of how students ought to respond. A further category of hermeneutic listening points to the teacher's readiness to engage with students' ideas in changing them and teacher's own understanding. Empson and Jacobs (2008) draw a similar distinction between directive, observational and responsive listening. Doerr (2006) identifies three dimensions of teachers' knowledge of students' mathematical thinking – "(a) an understanding of the multiple ways in which students' thinking might develop, (b) ways of listening to that development, and (c) ways of responding with pedagogical strategies to support that development" (p. 256).

In contrast to the metaphor of a "map" of teacher knowledge described earlier, approaches on teachers' responsive listening emphasize the dynamic aspects of classroom interactions; they lay stress on the teachers' ability to anticipate paths that learners may take as they navigate the construction of new knowledge from what they have known previously. We argue that the literature on listening helps in illuminating how different components of teacher knowledge, such as Knowledge of content and students (KCS), knowledge of content and teaching (KCT), and specialized content knowledge (SCK) interact dynamically in the context of classroom interactions. Interpretive listening is strengthened by knowledge of how students interact with content, for example, by knowledge of common student errors and difficulties, which is a part of KCS. SCK deals with making the features of mathematical content visible to students through the choice and use of effective representations, justifications, etc. Teacher knowledge entailed in responding to students in pedagogically appropriate ways are a part of KCT, that is, knowledge of the affordances of representations, and of ways of deploying them. All these components are associated with how the teacher makes decisions during classroom interactions. The knowledge of students' conceptions has been found central to teachers' knowledge of and responses to students' ideas in the classroom. The nature of student difficulties or their struggles with the use of representations is topic-specific. Students find some (sub-) topics easy or difficult to learn. An awareness of the topic-specific knowledge on students' conceptions can be a useful resource for teachers.

3.4 Characterising Topic-Specific Knowledge Required for Teaching

Descriptions of topic-specific knowledge needed by teachers draw not only on studies of teaching practice, but also on prior research on student errors and difficulties. The results from such research, along with suggestions on how to design instruction to deal with student difficulties, imply a set of demands on teachers' knowledge. The topic selected by the participants was decimal numbers.

In an early study, Resnick, Nesher, Leonard, Magone, Omanson and Peled (1989) found that, when comparing decimal numbers of varying lengths and digits, students tended to judge the decimal number with more digits after the decimal point as larger. In a later work, Steinle and Stacey (2004) found that while some students believed that the longer decimal number was larger (similar to Resnick et al., 1989), other students believed that the shorter decimal was larger. Students provided different reasons for their choice for similar behaviour. Such errors are often accompanied by errors in decimal number operations. The analysis of students' errors when working with decimal numbers reveals their roots in students' prior knowledge of whole numbers and the difficulties faced in making the transition to rational numbers (Resnick et al., 1989). Behr and Post (1992) argue that decimals are an important extension of both the base ten place value system and of rational numbers, and can therefore be interpreted using either or both of these perspectives.

Specific suggestions to deal with students' difficulties include presenting numbers in fraction, natural number, and decimal forms to show the invariance among these representations (Vamvakoussi & Vosniadou, 2007). Desmet, Gregoire and Mussolin (2010) suggest that students need to work with several examples of decimal fractions, where varying digit values and length helps in creating conflict between their understanding of whole numbers and rational numbers. Takker (2019) suggested using counter examples and examining different cases in order to challenge students' conceptions arising from the incorrect use of prior knowledge. Brousseau, Brousseau and Warfield (2007) have suggested a curriculum for rational numbers, where decimals are used to approximate the measurement of continuous quantities, differentiating them from natural numbers where discrete and imprecise measurement is made possible.

Research on student difficulties and the related literature on teaching decimal numbers suggests that students need to restructure their knowledge of conceptions, rules and symbols learnt for whole numbers (Irwin, 1996) and fractions. What are the demands placed on teachers' knowledge suggested by this research? Firstly, teachers need to be secure in their own understanding of the magnitude of decimal numbers. Research suggests that this may not often be the case as teachers face difficulties similar to students in judging the magnitude of decimal numbers and understanding their density (Muir & Livy, 2012; Widjaja, Stacey & Steinle, 2008). Tirosh and Graeber (1989) found that practicing teachers face difficulty in justifying the procedure for multiplying a decimal with ten (also noted by Chick, Baker, Pham & Cheng, 2006). This is an example of knowledge that is specialized for teaching, that is, SCK.

The research on student errors and difficulties shows that such errors have systematic misinterpretations underlying them, which the teacher needs to be aware of (as part of KCS). Some of this knowledge overlaps with KCT since student responses are related to teaching decisions. For instance, consider the rule of annexing a zero to make the length of two decimal numbers equal. Swan (1990, cited in Steinle, 2004) suggests that an emphasis on this rule, by the teachers, without reference to the place value might provide correct answers but does not support conceptual understanding. We may think of such knowledge as a part of both KCS and KCT (MKT framework).

How do teachers' understand, interpret and deal with such student difficulties? Jackson, Gibbons and Sharpe (2017) reported that teachers attribute students' difficulty to their personal traits, or deficits in their family or community and deal with it by lowering the cognitive demands of the task. Further, although teachers attributed students' difficulty in learning decimals to the lack of instructional opportunities, they did not respond to students in ways that would enable participation in rigorous mathematical activity. Teachers who participated in this study also held similar notions about students. Thus, an important question was – How to draw teachers' attention to the 'mathematical' aspects of student difficulties?

3.5 Connecting Knowledge and Practice to Support Teachers' Learning

A large body of literature has identified the need for developing "practice-based tasks" to enhance teachers' knowledge of mathematics teaching. What is common in different practice-based approaches to professional development (PD) is an invitation for teachers to participate in professional learning communities, and reflect on their teaching practice using the knowledge gained from the literature and field experience. Bannister (2018) suggests that teacher development through participation in such communities has the potential for "humanizing mathematics teaching and learning" by impacting classroom learning. Several PD programmes have used students' work as an artefact to situate teachers' learning in their practice. Ways in which students' work have been used in studies, are influenced by the theoretical stance on teacher knowledge and learning. To illuminate the difference between different approaches to PD of teachers, I borrow Cochran-Smith and Lytle's framework (1999) on the relation between knowledge, practice and learning (refer Table 1).

Table 1: Teacher Knowledge, Practice and Learning (Cochran-Smith & Lytle, 1999)

Knowledge-Practice relation	Source of knowledge generation	Teacher learning
Knowledge *for* practice	Theory or formal knowledge generated by university professors and researchers for use by teachers.	Knowledge generated by experts is passed on to teachers for use in classroom.
Knowledge *in* practice	Practical knowledge embedded in teachers' work and generated from reflection on teaching practice of expert teachers.	Teachers get opportunities to probe knowledge embedded in the work of expert teachers and deeper their knowledge through interactions in a community.

Table 1: Teacher Knowledge, Practice and Learning (Cochran-Smith & Lytle, 1999)

Knowledge-Practice relation	Source of knowledge generation	Teacher learning
Knowledge *of* practice	Treating classrooms as sites of enquiry and using the knowledge developed by others (in the field) to interrogate practice.	Teachers learn by generating knowledge in their local contexts of practice being a part of inquiry communities and theorise their work.

Cognitively Guided Instruction (CGI) is a PD programme wherein teachers were provided the research-based knowledge on students' strategies when solving arithmetic word problems through workshops, and teachers' instructional decisions in the classroom were studied (Carpenter, Fennema, Franke, Levi & Empson, 2000). The findings indicated that experienced teachers possessed some intuitive knowledge about students' thinking, but this knowledge often remained fragmented and was not utilised for decision making in the classroom. The changes in teachers' beliefs and practice through participation in these workshops (Kazemi & Franke, 2004; Empson & Jacobs, 2008) makes it an instance of knowledge *in* practice.

The lesson study approach to PD originated in Japan, and gained attention from the international community after the TIMSS video study (Stigler & Hiebert, 2009). A group of teachers interested in teaching a particular topic meet and plan a 'research lesson', which is taught by a teacher, observed by other teachers and then reflected upon. With participation in different lesson study experiences, teachers learn to connect classroom practice to the broad curriculum goals, experience and discover novel practices, explore conflicting ideas between reform suggestions and their implementation, and improve their knowledge base (Lewis, 2000; Doig & Groves, 2011). Fraivillig, Murphy and Fuson (1999) argue that documentation of instructional practices of classrooms where teachers provide students with an opportunity to "explore mathematical objects and to synthesise their own mathematical meanings" is productive in generating descriptions of effective mathematics teaching. The approach seems to be guided by the theoretical proposition that knowledge is generated in the field, making it as example of knowledge *in* practice.

The existing PD programmes, similar to approaches such as CGI or lesson study, seem to be guided by the framework of generating knowledge *in* practice by promoting teacher communities, or of creating a cadre of professionals (teacher educators, university professors, or researchers) to generate and provide knowledge *for* practice to teachers. Frameworks such as these are limited in their scope in bridging the divide between research and practice.

Challenging the existing research in mathematics education, Cai et al. (2017) argue that the research is yet to identify the "grain size that is compatible with teachers' classroom practice" in order to seriously address the divide. In order for research to impact mathematics classrooms, teachers' role needs to be redefined in terms of identifying research problems and learning opportunities in classrooms. Bannister (2018) critiques the extant literature on teacher learning communities by arguing that little is known about the nature and conditions of teachers' learning. He suggests that collaborations with teachers be situated in working days and tied to the aim of supporting students' learning. More recent work on organising professional learning experiences focuses on evidence-based decision making where teachers are encouraged to learn in and from practice. Approaches where the knowledge generated by teachers and researchers (or other professionals) in collaboration can be used for improving classroom practice need to be examined. Brodie (2011) asserts that in developing nations, the focus of professional learning communities can be to "support deliberate, collective learning, draw on local data and the knowledge base" (p.157) available in the field. In this case, the knowledge that the researcher or facilitator brings to a professional learning space becomes as significant as that of the participating teachers. Developing knowledge *of* practice by reconfiguring the role of teachers and researchers to impact practice, is attempted through the study.

3.6 Reflection on Existing Frameworks of Teacher Knowledge and Learning

What do we learn from the existing frameworks of teacher knowledge and learning? Some questions that emerge from a reflection on the literature are: What is the role of practice in investigating knowledge? What kind of accounts of practice can be used to comment about knowledge? How is it useful to centre the professional development programme around the practice of teaching? Such questions have guided the work reported in this thesis.

An insight drawn from the study of contemporary frameworks on teacher knowledge is its situatedness in the context of practice. The knowledge used by the teacher gets triggered by the inter-animation of ideas from how students interact with the content, how the teacher makes sense of these different interactions and builds on them. The knowledge that gets discussed in the classroom is not individualistic. That is, teacher's knowing is fluid and co-constructed, and at any point calling a piece of knowledge as solely the prerogative of a teacher, is problematic. At the same time, the teacher's role in orchestrating such knowledge in some direction(s) cannot be undervalued. This complex balance of the knowledge that the teacher uses to handle teaching situations while attending to students' ideas, can be characterised as *knowledgeability* (Wenger-Trayner, Fenton O'Creevy, Hutchinson, Kubiak, & Wenger-Trayner, 2014), or *knowing* which shows continuity and fluidity in teachers' knowledge. Therefore, any abstract description of classroom might fail to capture the specific descriptions of practice which unpack the nuanced and dynamic engagement of the teacher in the classroom activity (refer Lampert 2001 for thick descriptions of teacher's work). Re-defining knowledge in this manner has implications for how the construct of teacher knowledge is investigated (that is, tools used to capture teacher knowledge) and ways in which it is supported.

Brodie (2011) suggests that one of the ways to capture the complex and dynamic nature of teachers' knowledge is to create textured descriptions of the difficulties faced by teachers when implementing the reformed curriculum. Further, a detailed study of teachers' existing practices can be used to identify aspects which can be leveraged to design support for learning (Cobb & Jackson, 2015). Taking these two arguments, the thesis presents an approach of engaging with the work scenarios of teachers to develop an understanding of the challenges faced by them *in situ* and design appropriate support structures. This approach, which takes the realities of teachers' work into cognizance and engages deeply with the practice of teaching, has the potential for the formation of learning communities involving teachers and researchers.

4. THE STUDY

The study describes how an engagement with the teaching practice can be used to understand and support teachers' knowledge of students' mathematical thinking.

4.1 Research Questions

The research study aimed to investigate teachers' knowledge of students' mathematical thinking as it gets manifested in their practice. Further, attempts were made to develop teachers' knowledge of students' thinking through the design and development of *practice-based tasks.* The following research questions were addressed through the study.

1. How does teachers' knowledge about students' thinking manifest in their practice and what are the ways in which it influences teachers' responses to students? How can responsive teaching be identified and characterised?
2. What are the nature of demands placed on teachers' knowledge during teaching? How can such demands be studied and structured?
3. How can teachers' knowledge of students' thinking be supported? What do teachers learn from the support provided by the researcher and how does it manifest in their practice?

Since teacher knowledge is dynamic and fluid, any claims about the strict presence or absence of a piece of such knowledge is avoided. Students' thinking is defined as mathematical ways in which students process an idea: it could be their ways of problem solving, making sense of representations, forming explanations, facing conceptual difficulties, their common conceptions, etc. Practice refers to the act of, as well as reflection on, teaching, that is, what teachers do in their classroom and ways in which they think or reason about it.

4.2 Contributions of the Study

The research study makes methodological and analytical contributions, some of which have been mentioned below.

1. The study presents a systematic account of recording and analysing teaching practice which helps in understanding how the knowledge of a teacher comes into play in the act of teaching. Thus, it offers a way for studying knowledge from the standpoint of practice. Such detailed descriptions of practice have the potential to shed light on the complex aspects of teaching, such as, the interactive nature of talk, affordances of representations, and so on.
2. Apart from an exploratory purpose, the practice-centred approach of the study extends to supporting teachers' knowledge in the contexts of their practice. The study offers an exemplar of a professional development initiative which examines and challenges teaching while building a support system for teachers. It explicates ways in which an abstraction of knowledge underlying teaching practice and a critical reflection on such knowledge, can be reliably used to characterise aspects of mathematical knowledge required for teaching.
3. The study demonstrates how research literature on students' conceptions can be drawn upon to design contextual tasks for reflection on teaching and learning.
4. The study exemplifies how teachers and researchers can collaborate to challenge the existing practices and create a shared knowledge base to support students' learning.
5. The study demonstrates sensitive and reflexive use of research methodologies to build trust and responsibility towards reconfiguring the role of classroom observations from their inspectorial purpose to that of analysis and critical reflection for enabling teacher learning.

5. PILOT STUDIES

Chapter 3 of the thesis reports the two pilot studies which provided insights into the process of investigating teachers' knowledge through a study of their practice, and on the nature of knowledge needed to support teachers. The first pilot study, summarised in Section 5.1, was a case analysis of an experienced school mathematics teacher, with the aim of understanding how the teacher's knowledge gets manifested in practice. Supporting classroom-based tasks were designed to unpack the mathematical aspects of teachers' knowledge about students' ways of thinking in a specific topic. For practicing teachers, routine interactions with students, along with research literature on students' thinking in specific topics, are an important source of knowledge about students' ways of thinking and responding. Section 5.2 reports the second pilot study, where both these sources of knowledge were integrated to design and test a teaching module. The module was designed based on the knowledge of students' thinking gathered from the research literature and was modified through interactions with students. The researcher taught as well as observed a co-researcher's teaching of the module, conducted student interviews to understand their ways of thinking, and integrated these in modifying the lesson plans.

5.1 Examining Teacher's Knowledge of Proportions[1]

Despite the extensive work done in the field of developing teachers' knowledge, there are difficulties in identifying its nature and extent (Ball & Bass, 2000). We know that teachers make conjectures about students' learning, listen and respond to them in the classroom routinely and share intellectual and affective moments with them (Lampert, 2001). All such decisions indicate teacher knowledge, and it is a study of knowledge from this standpoint that remains largely unexplored. Ball, Hill and Bass (2005) question whether this is due to the nature of methods that we use or the nature of (teacher) knowledge that remains tacit and unarticulated.

In particular, I focus on the teachers' knowledge about their students' mathematical ways of thinking. An investigation of teachers' views about students' subject-related capabilities has led to findings which can be broadly classified into productive and un-productive framing (Jackson, Gibbons & Sharpe, 2017). In productive framing, students' difficulties are attributed to instructional or schooling opportunities, while attributing them to inherent traits of students or deficits in family or community, constitute unproductive framing. Research studies (Windschitl, Thomson & Braten, 2011; Coburn, 2006) have reported that a majority of (pre- and in-service) teachers attribute students' performance to the personal attributes leading to unproductive framing. How such views about students shape teachers' knowledge and instruction, needs investigation.

The first pilot study aimed to investigate the (a) nature of teacher's knowledge about students' mathematical thinking and learning, (b) relation between teacher's knowledge and her responses to students' mathematical thinking, and (c) teaching practices which reflect manifestations of knowledge about students' thinking. The teaching of a mathematics teacher (TJ) in two Grade 7 classrooms constitutes a case. Audio-video records and researcher's observations of classroom teaching, task-based interviews and an anticipation task was used to collect data about teaching.

It was found that TJ closely followed the textbook while teaching proportions. Her interviews revealed that she expected students to use the algorithms and did not appreciate students' strategies of solving problems. She mentioned that the goal of mathematics learning was to learn algorithms. When asked to discuss students' difficulties in mathematics, she attributed them to students' lack of

[1] Takker, S. & Subramaniam, K. (2012). Teacher's knowledge of and responses to students' thinking. In Proceedings of *12th International Congress on Mathematics Education*. pp.4906–4915. Seoul: South Korea.

attention or their backgrounds. Her anticipation of and responses to students' ideas were often in terms of correct or incorrect. When requested to anticipate students' responses to a set of proportion problems designed using the research literature, she under-estimated students' capabilities. For instance, TJ anticipated that students would make calculation errors in solving the direct proportion problems and would not attempt the inverse proportion since it had not been taught. A detailed analysis of TJ's expectations from students, the students' actual responses, and TJ's reflection on the students' responses can be found in the thesis. A summary of the findings from this case study are:

1. Teacher's knowledge about students' mathematical ways of thinking is about individual students and often based on attributes such as (lack of) students' attentiveness and listening.
2. Teacher's response to the students while teaching involved judging students' oral and written responses as correct or incorrect, without probing the thinking underlying these responses.
3. When directed to think about students' mathematics, the teacher underestimated the students' capability, which in turn determined her choice of tasks.
4. While the teacher has a broad understanding of students' mistakes, the mathematical aspects of students' thinking such as prior knowledge, use of explanations, strategies to solve problems were unnoticed. Often the pedagogical strategy of repeating the procedure was used by the teacher if students faced difficulty in understanding.
5. The teacher's belief about students' capability (particularly, what they can do) was challenged, when she discovered that her anticipation did not match with the actual students' responses. A reflection on actual students' responses by using some of the students' ways to solve a problem and discussions around it, helped the teacher in appreciating students' mathematical thinking.

5.2 Using Knowledge about Students to Teach Early Algebra

The second pilot study focused on how can the literature on students' conceptions in specific topic be used to inform instruction. The literature in algebra education suggests that the transition from numbers to letter symbols requires well-designed instruction (Banerjee, 2008). Other identified challenges in the learning of algebra include understanding of equality, making generalisations, operating with letters, and flexibly dealing with procepts.

The study attempted to explore students' algebraic reasoning when exposed to *early algebraic* ideas through contexts like number sentences, pattern generalisation, proof and justification, etc. Tasks were designed on the premise that students make sense of new experiences based on their intuitive knowledge and therefore opportunities for them to explicate their thinking is significant to the plan of teaching. These tasks were developed based on the research literature, and modified based on interactions with students. The strategies used by students to solve these tasks, and the justification or explanation given to support their responses, were recorded.

The findings of the study[2] suggested that the tasks on number sentences helped in students' movement from purely computational strategies to relational reasoning, and later to generalised thinking as justifications. Along with the role that classroom culture and students' prior knowledge played in the development of this trajectory, we also identified how intermediate resources supported the trajectory towards generalised thinking. However, the movement from computational to generalised thinking in a flexible mode entails a significant role of the teacher, including awareness of the ways in which students' think. This helps in identifying the appropriate prompts and in planning for the unexpected student responses. Clearly, the research literature on students' thinking and a reflection on their responses can be used as resources for planning such a learning trajectory.

5.3 Learning from Pilot Studies

The findings from the pilot studies helped in forming a few hypotheses about investigating and supporting teacher knowledge in practice. First, experienced teachers have an intuition about students' ways of thinking and responding, but these ways are largely classified as correct and incorrect and not explored for their nuances. Second, teachers attribute students' difficulties to non-mathematical aspects, and engagement with the mathematics underlying students' thinking needs some pressing and direction from the researcher. Third, the teacher might underestimate students' mathematical ideas and contributions. The 'anticipation, testing and reflection' task has the potential to challenge teachers' knowledge and beliefs about students' capabilities. Fourth, knowledge of research literature supports a teacher in making decisions about what to teach, adopting a more open-ended approach to problem solving, and handling unanticipated moments. And last, making sense of students' responses in-the-moment is complex and is guided by several considerations such as goals of teaching, setting a culture, careful listening and guiding, etc. Handling students' responses is challenging, but teaching can be informed by the research literature on students' conceptions and actual students' responses.

6. RESEARCH DESIGN

The literature on teacher knowledge and the status of research in the Indian and international context opened up several methodological questions. The primary question was to think of a methodology

[2] Takker, S., Kanhere, A., Naik, S. & Subramaniam, K. (2013). From Relational Reasoning to Generalisation through tasks on number sentences. In Nagarjuna, G., Jamakhandi, A. & Sam, E. (eds.), *Proceedings of epiSTEME 5: International Conference to Review Research in Science, Technology and Mathematics Education*. pp. 336–342. HBCSE, India: Cinnamonteal.

which will make the dynamic aspects of teachers' knowledge visible. Ball and Bass (2005) suggest that such as understanding of the teacher's knowledge can be developed by systematically studying the 'work of teaching'. An overview of the study and its methodology form Chapter 4 of the thesis.

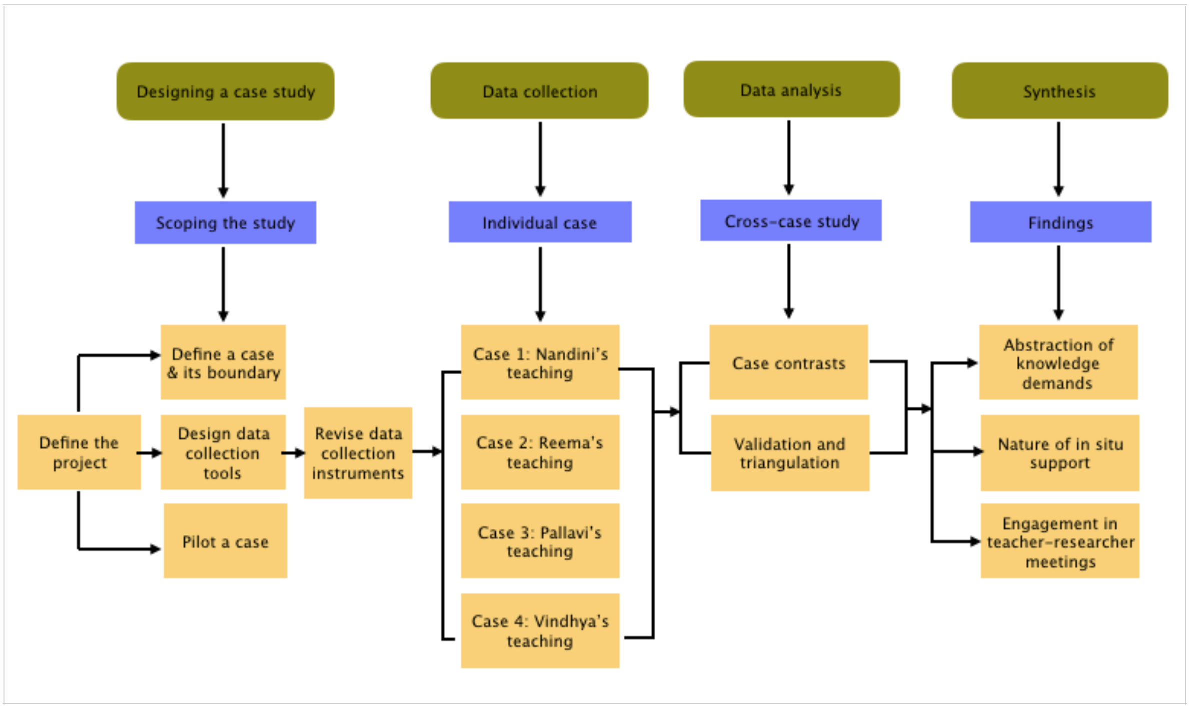
6.1 Case Study Methodology

A case study methodology was used to closely examine the work of teaching through a study of teaching practice. As Stake (1978, p.7) asserts, a case study features "descriptions that are complex, holistic, and involving a myriad of not highly isolated variables; data that are likely to be gathered at least partly by personalistic observation; and a writing style that is informal, perhaps narrative, possibly with verbatim quotation, illustration, and even allusion and metaphor". In this study, each teacher's teaching constituted a case, bounded by the interactions within and about teaching. Verbatims and actions of the teachers and students, the impressions of the researcher, descriptions of contexts and situations, etc. were used to enrich descriptions of practice with contextual details.

Each case was studied for its particularity and complexity by examining the teacher's individual and combined activity in situations such as teaching in a classroom, discussions with colleagues or parents, interactions with students, and so on. An ethnographic approach to observing teachers in different settings within the school, and an 'interpretivist orientation' (Harrison, Birks, Franklin & Mills, 2017) to understand their perspectives on their own practices, helped in grounding the study in the context of practice. An interest in understanding teachers' perspectives by being closer to their natural settings (Cresswell, 2013) helped in building a rapport, developing a vocabulary to discuss teaching, and creating a space for discussion on problems arising in the teaching.

Yin (2014) argues that precision or accurate reporting and a rigour in the process are central to case studies. In this study, four teachers were followed for two academic sessions and data was collected through various modes. In order to develop an understanding of a case in its real settings, case study methodology opens itself to a variety of data collection methods (Merriam, 2009; Harrison, Birks, Franklin & Mills, 2017). Teaching was understood by observing teachers' practices, seeking teachers' opinion about these practices, interacting with the students to understand their views about teaching, probing teachers' knowledge about students and their mathematical thinking, determining the considerations that guided their decision making, and supporting thinking aloud about particular events that arose while teaching. Triangulation of teaching practices was done by observing the repeated use of a practice at different occasions, discussions about these practices, and creating a simulated recall of these practices during reflection.

Figure 1: The study methodology



In order to engage teachers in discussions about their teaching, the researcher's role became participatory in the course of the study. A summary of how the case study methodology was used is summarised in Figure 1.

6.2 Participants and Settings

Four experienced elementary school mathematics teachers, and the students in the eight classrooms taught by these teachers over two years, participated in the main study. The researcher's interest was in studying the practice of mathematics teachers closely so the availability of teachers in the same school for a few years was necessary. This called for the use of a combination of purposive and

Table 2: Professional details of participants

Pseudonym	Educational Qualifications	Languages Known	Teaching Experience	Grades Taught	Subjects Taught
Pallavi	Bachelor of Science (B.Sc.), Bachelor of Education, ADSSCA Tech. (B.Ed.)	English, Tamil, Hindi	22 years	I to V (Age 6–10 years)	Mathematics, Environment Sciences
Reema	Bachelor of Science (B.Sc.), Bachelor of Education (B.Ed.)	Marathi, English, Hindi	20 years	I to V (Age 6–10 years)	Mathematics, Environment Sciences
Nandini	Master of Science (M.Sc. Physics), Master in Education (M.Ed. Systems Management), Bachelor of Education (B.Ed.)	English, Hindi, Malyalam	10 years 8 months	VI to X (Age 11–15 years)	Mathematics, Physics
Vindhya	Masters of Science (M.Sc. Mathematics), Bachelor of Education (B.Ed.)	English, Hindi, Telugu	25 years	VI to X (Age 11–15 years)	Mathematics, Physics

convenience sampling to select the participants. The four participating teachers taught in a school in Mumbai, which catered to students from mixed socio-economic and diverse linguistic backgrounds. The details of the school infrastructure and working conditions of the participants can be found in the thesis. The professional details of the participating teachers can be found in Table 2.

6.3 Phases of the Study

The research study was carried out over two consecutive academic sessions: 2011–12 and 2012–13, in three phases (refer Table 3). The aim of Phase 1 was to explore teachers' knowledge of students' mathematical thinking through a close study of teaching practice. The researcher interacted with the teachers before the lesson to understand their plan, and after the lesson to reflect on student responses or teaching decisions. After some understanding of teachers' practice, a long interview and an anticipation task was planned to engage with teachers' perspective on classroom observations and elicit their knowledge about students' mathematical capabilities. Phase 2, which happened in the second year of the study, included teacher-researcher meetings organised in the school, with the aim of creating a space for discussion on mathematical knowledge required for teaching. These discussions centred around the teaching of decimal fractions. The third phase was similar to the first phase, but had an additional component of the researcher supporting each teacher individually in the classroom. The data collection instruments can be found in the Appendix to the thesis.

Table 3:Phases of the research study

Phases of the Study	**1**	**2**	**3**
Objectives	Explore teacher's knowledge of students' mathematical thinking manifested in their practice.	Enhance teachers' knowledge of students' mathematical thinking through *ex-situ* support.	Examining and supporting teachers in using knowledge of students' mathematical thinking in their practice.
Duration	8 months	6 months	6 months*
Site of data collection	Classroom Staff room Other school premises	Mathematics laboratory in the school Researcher's institute	Classroom Staff room Other school premises
Modes of data collection	Classroom Observations, Pre- and Post-lesson interviews, Anticipation-reflection task, Long interviews.	Planning; observations, summaries, and reflections of teacher-researcher meetings.	Classroom Observations, Pre- and Post-lesson interviews, Anticipation-reflection task.
Tools for data collection	Audio records, Researcher notes, Documents, Video records (of some lessons).	Audio and video records, Researcher notes, Written summaries.	Audio and video records, Researcher notes, Documents.
Participant's role	Teach in their class. Discuss plan and reflections.	Participate in the meetings and later organise these meetings.	Teach in their class. Discuss plan and reflections.
Researcher's role	Non-participant observer.	Teacher educator.	Participant observer.
* Phases 2 and 3 happened concurrently during the second year of the study.			

6.4 Data Reduction and Analysis

The first level of data reduction was done by separating decimal lessons from the classroom observations of other topics. A coding scheme was developed through grounded ways of looking at the data of one teacher's teaching. The open coding of lesson transcripts included, "breaking down, examining, comparing, conceptualising and categorising" (Strauss & Corbin, 1990). The broad categories that emerged from coding were questions, explanations, and responses of the teacher and students. The coding scheme (refer Table 4 for sample codes) was validated and then used to analyse the decimal lessons taught by the other participating teachers.

Table 4: Sample codes from the coding scheme

Teacher Code	Example	Student code	Example
TQ-why	Why is $\frac{86}{10} = 8.6$?	SQ-how	How to find an equivalent decimal of 7.8?
TQ-textbook	Convert the following fractions to decimals.	SQ-why	Why is 36.0 equal to 36.00?
TE-procedure	First you count the number of zeroes in the denominator. Then start from the right and count that many digits. Then put the point.	SE-observe	Teacher, why is there no oneths?
TE-justify	$\frac{86}{10}$ Here 86 is made up of 8 tens and 6 ones. 80 divided by 10 is 8 and 6 divided by 10 is the same as 6 times 0.1.	SE-argue	No, teacher. When we multiply with 10, it becomes 10 times, not lesser.
TR-correct	Wrong, $\frac{86}{10}$ is not 86.10. It is 8.6.	SE-error	17.5 cm (an incorrect one word response)
TR-publicthink	Dev is saying that 2 is the common factor of 86 and 10. Do you agree or disagree? Why?	SE-justify	2 cm is equal to 0.02 m as 1 cm is a hundredth part of a metre, and 2 cm means two 1 cm parts of 100.
Legends: TQ - Teacher's question, TE - Teacher's explanation, TR - Teacher's response, SQ - Student's question, SE - Students' explanation			

A comparison of codes from the first to the second year revealed changes in teachers' practice. While the coding helped in capturing the broad changes in practice, it did not explicate the conceptual challenges faced by the teachers while exploring a different pedagogy. These conceptual challenges are defined as *knowledge demands* posed on the teachers in the course of classroom teaching. After breaking down each lesson into episodes, episodes which dealt with the same sub-topic in the two years of teaching by the same teacher, were paired for microscopic analysis. These "paired episodes" were used to abstract the knowledge demands posed on the teachers.

The data set from Phase 2, that is the teacher-researcher meetings, was transcribed using video and audio records, along with the summaries written by teachers and reflection notes of the researcher. This data was classified into episodes based on the nature of discussion around a particular idea. These episodes were categorised based on the nature of discussion. Themes were identified which acted as broad categories for classifying these episodes. The themes help in characterising the topic-specific knowledge required for teaching decimal numbers.

In Phase 3, the classroom data was analysed in the same way, as in Phase 1. Instances of *in-situ* support offered to the individual teachers were studied separately and analysed descriptively for the nature of support and the teacher learning from such support.

The data from student and teacher interviews, responses to the anticipation-reflection task, students' responses to the worksheet questions and informal interactions with the teachers, have been used to corroborate the observations, wherever necessary.

6.5 Dilemmas in Studying Teaching

Studying the complex nature of teaching, given the status of research and teaching (discussed in Section 2.2), is challenging. The researcher adhered to the ethical considerations, such as, anonymity of the participants, permissions for data use, seeking informed consent, etc. during the course of data collection, and while analysing data and presenting findings. Some of the dilemmas encountered in the course of this research on teaching are summarised below.

1. Since teachers' work is often inspected through classroom observations by senior officials and these judgments affect teachers' career trajectories, there was a reluctance to video record the lessons in the beginning of the study. While the permissions from the larger governing body and locally from the school authorities were granted, the video recordings were not insisted. As the intent of the research became clearer through the nature of interactions about teaching, participating teachers asked for video recordings of the lessons. Thus, the initial data from the first-year observations is audio recorded, while the rest of the data is video recorded. The sensitive use of data collection methods by respecting the concerns of the participants helped in creating a negotiated space and the emergence of the significance of videography for teachers' own use apart from using it for research purposes.

2. The effect of the researcher's presence on the natural settings was a concern as the teachers repeatedly requested comments (read: judgments) on the lessons observed. While their request implied an inherent hierarchy in the roles of a teacher and a researcher, after a few months this space was used to discuss teaching by drawing teachers' attention to the mathematical aspects.

3. While sustaining teachers' attention to the mathematical aspects of teaching was difficult, as teachers had other legitimate concerns that they were routinely struggling with, discussions on these issues were not discouraged. "Researchers need to reflect attitudes of compassion, respect, gratitude and common sense without being too effusive. Subjects clearly have a right to expect

that the researchers with whom they are interacting have some concern for the welfare of participants" (Cohen, Manion & Morrison, 2013, pp. 59–60). The reciprocity of the researcher in responding to the teachers' mathematical and non-mathematical concerns played a role in gaining teachers' trust in sharing their actual struggles, ideas and opinions.

4. In the plan of the research study, the researcher's role was that of a non-participant observer. This role varied from being 'observer as participant' and 'participant as observer' (Cohen, Manion & Morrison, 2013) depending on the situation. Teachers also took different roles apart from teaching, for instance, by observing a lesson, initiating conversations to reflect on teaching, bringing artefacts to the meetings for discussions, and so on. The reflexive roles of the researcher and teachers created opportunities for "boundary crossing" (Wenger-Trayner et al., 2014) which helped in developing a shared concern towards students' learning and in experimenting with alternate pedagogies while teaching particular mathematical ideas.

5. The field work required seeking permissions, building a rapport with the participants and the related others, convincing the need for and creating a space for discussion, working with the teachers' busy schedules and non-teaching responsibilities, navigating between different bureaucratic and other conflicting interests, organising meetings, etc. and amidst these creating a discourse on the reflection on mathematics in the classrooms. While it is recognised that qualitative research is time-consuming, as the researcher is expected to wait for the phenomena to occur in the natural settings, some structural or systemic issues need attention in reducing the time spent on several activities during fieldwork in schools. A step in this direction could be partnerships between schools and research institutions (elaborated later).

6. The different goals of a teacher and a researcher might conflict in such research. Teachers are burdened with the routine responsibilities such that discussions about teaching are a small subset of their work. A researcher's primary focus is however on magnifying and analysing every part of teacher's work. The conflicts in the role of teachers and researchers arise from a lack of space in the system and a vision of the roles that teaching and research play in the process of knowledge generation. The focus of the research on teaching needs to acknowledge this structural limitation and challenges arising from such conflicting goals.

7. It is almost a recurring question for conducting case studies about whether and to what extent they can be generalised. Questions such as - how is the uniqueness of this case relevant for other contexts need to be foregrounded. While no two teachers teach in the same manner, creating a discourse around teaching, the methods used, the contextual tools developed and the nature of discussions have a wider applicability in understanding diversity.

7. FINDINGS OF THE STUDY

The findings section is organised around the research questions on investigating teachers' knowledge through a study of practice, and supporting teachers' knowledge about students' thinking. Section 7.1 presents an analysis of the teaching observed in the two consecutive years of the study. It outlines the broad changes in the teaching practice from the first to the second year of the study, followed by a detailed analysis of the "paired episodes" from the teaching in two years. The paired episodes are selected based on the difference in the teachers' response to the contingent moments arising during the course of teaching. The analysis is used to abstract the knowledge demands posed on teachers as they attempt to teach responsively. An analysis of the nature of support, *ex-situ* in the form of teacher-researcher meetings and *in-situ* by collaborating with teachers in their classroom, provided to the teachers, and the teacher learning from it is reported in Section 7.2.

7.1 Topic Specific Knowledge Demands in Responsive Teaching

The data from classroom observations of the two years revealed changes in teachers' responsiveness to students' mathematical thinking. Two of the case studies, elaborated in Chapter 5 of the thesis, discuss the transition from more traditional to student-centred or responsive teaching. The coding of decimal lessons revealed the nature of change in the teaching in two years. For detailed analysis of how teachers dealt with students' mathematical ideas, a second-level analysis of paired episodes from teaching across the two years was done. Through this analysis, I argue that particular knowledge demands become more visible when teachers are in transition. Such demands are especially significant in moments when teachers deal with the ''contingencies'' that arise in the classroom (Rowland et al., 2005). Contingent moments may arise from an unanticipated student question or observation, or sometimes through a connection that the teacher makes between the mathematical ideas in play. The contingencies place demands on teachers' knowledge, as a teacher needs to evaluate whether and how these moments can be converted into learning opportunities (Rowland & Zazkis, 2013). Thus, the focus is on the dynamic nature of knowledge demands that arise in the course of teachers' listening and responding to students in contingent classroom moments, while teaching decimal numbers.

The two case studies discussed are of the teaching of decimal numbers by Nandini[3] and Reema[4]. The paired episodes for these two cases were analysed for the demands they place on teachers'

[3] Takker, S., & Subramaniam, K. (2019). Knowledge demands in teaching decimal numbers. *Journal of Mathematics Teacher Education, 22*(3), pp. 257–280.

[4] Takker, S. (2018). Knowledge demands placed on a mathematics teacher learning to teach responsively. In S. Ladage & S. Narvekar (Eds.), *Proceedings of epiSTEME7: Seventh International Conference to Review Research on Science, Technology, and Mathematics Education*, pp. 323–331, India: Cinnamon Teal.

knowledge in terms of handling the unanticipated moments. Their cases were selected as their classroom teaching offered maximum variation from the first to the second year. In the synopsis, one paired lesson from each teacher's classroom is presented. Knowledge demands abstracted from the analysis of all the 6 pairs of lessons can be found in the thesis.

Table 5: Frequency of codes in Nandini's and Reema's (paired) decimal lessons

		Nandini		Reema	
	Code	Y1DL1	Y2DL2	Y1DL1	Y2DL2
1	T Question - textbook	4	0	4	0
2	T Question - elicit	0	5	1	24
3	T Question - what	19	19	18	115
4	T Question - how	2	3	0	13
5	T Question - why	1	1	0	5
6	T Explain - tell	15	13	19	6
7	T Explain - procedure	8 + 1*	4	3	4
8	T Explain - justify	2	7	0	23
9	T Response - evaluate	17	7	1	6
10	T Response - restate	9	18	0	43
11	T Response - expand	0	7	0	20
12	T Response - argue	0	1	0	1
13	T Response - public think	0	3	0	23
14	S Explain - one word	42	26	34	196
15	S Explain - incorrect one word	15	0	4	5
16	S Explain - procedure	0	3	0	9
17	S Explain - justify	0	5	0	7
18	S Explain - new observation	1	12	4	7
19	S Explain - completes T explanation	1*	8	3	0
20	S Explain - adds to another S explanation	1	7	0	15
21	S Explain - argue	1	1	0	9
22	S Explain - evaluate	0	1	0	14
Legends		T - Teacher, S - Student, Y - Year, DL - Decimal Lesson, * indicates incorrectness			

The coded data from the cases showed an increase in the use of justifications by teachers (Row 8, Table 5) and students (Row 17); a decrease in rule-based explanations provided by the teachers

(Row 6); a considerable increase in the how and why questions asked by the teachers (Rows 4, 5); a change in teachers' responses to students' questions or utterances, from passing a judgment (Row 9) to re-voicing the mathematical idea in students' utterance (Rows 10, 11), using counter examples to argue (Row 12), and posing students' ideas for public thinking (Row 13). In addition, a new practice that emerged in students' talk was an evaluation of each other's statements with reasons (Row 22). The teachers encouraged students to articulate their observations about mathematical objects being discussed, and probed them for justifications.

While the coding scheme helped in capturing some broad changes in teaching practice, it was noticed that teaching responsively posed demands on teachers' knowledge. In the following sections, I present an analysis of one paired episode, each from Nandini and Reema's teaching, to abstract the knowledge demands posed on the teachers, as they respond to contingent classroom situations.

7.1.1 Introduction to Decimals by Nandini

In the first lesson on decimals in Year 1 (Y1DL1), Nandini asked students to guess the length of a duster. She listened to the students' estimates of 12, 15, less than 10 cm, etc. and measured the length using a ruler to be 17.5 cm. She introduced a decimal number as the number where a (decimal) point is used and explained that "point five is a part of full 1 cm" and the decimals are used "when there is no full 1 cm". She then asked the students to find the cost of half, quarter and half of a quarter litre of milk given the cost of one litre as 21 rupees. She drew a 5×2 rectangular grid to show the fractions $\frac{5}{10}$ and $\frac{2}{10}$, respectively. Students responded with "same as half" and "point five" for $\frac{5}{10}$, and with "two by ten", "point two" and "ten point two" for $\frac{2}{10}$. Nandini did not respond to these student utterances and shifted the discussion to the place value of digits in the whole number 256 as an introduction to "how to write a decimal number". She explained that the decimal place values are written like whole number place values "but with different words" (example "tens" and "tenths"). A student then asked, "what is oneths?" to which another student responded that it does not exist. The student's question was not taken up by Nandini for discussion and remained unanswered. Next, Nandini discussed place values in a decimal number 0.256. She explained that the places are counted from the "left to right side", and named the first place to the right of decimal point as tenths. She named the place value of each digit of a decimal number and told the rule for converting a fraction with denominator ten to a decimal number. She did not explicitly connect the rule to the just concluded discussion about place values in a decimal number. In the remaining part of the lesson, Nandini asked the students to draw a 10×10 grid. The grid was used to show fractions with denominator hundred. In the post-lesson discussion, Nandini explained that she always begins a topic by using it in a relevant context since the "need for learning a concept… creates curiosity".

In Y2, as in Y1, Nandini began teaching decimal numbers using a length measurement context. In the first lesson, she asked students to measure any five objects from their surroundings and write the measure precisely. Students expressed measures as ''half, half of half, point five, more than point five''. Nandini diagnosed that students understood that a decimal number is made up of a whole number part and a fraction part. In Lesson 2 (Y2DL2), she said that the decimal point ''separates the whole number side and the fraction side''. She wrote 7.39 on the board and asked students to identify the place value of each digit starting from the left. Students identified the place value of 3 as ''three by ten'' and ''tenths''. At this juncture, a student asked ''*Ma'am, oneths kyun nahi hota?''* [Teacher, why are there no oneths?]. Another student contested this saying that oneths exist. Nandini listened to these two students and revoiced the question to the whole class. She said, ''What she is asking, when you write a [whole] number we start with ones place. Ones place, tens place, hundreds place, thousands place, but here just after the decimal [point] we started with tenths place, one by ten. Why there is no oneths place after the decimal (point)?'' Students provided different explanations while agreeing or disagreeing that oneths exist. In the process, Nandini sought clarifications, asked questions, and offered counter-arguments. The explanation constructed by two students is reproduced below (see Excerpt 2).

Excerpt 2: Explanation for oneths - I (Y2DL2)

Line No.	Speaker	Utterance
49	GSt2	*Ma'am samjha. Iske andar jab karenge na to one ka part one hi rahega.* (Teacher, I understand. In this when we do [partitioning], a part of one will remain one.)
50	BSt1	No Ma'am. There is no oneths place in the decimal part.
56-57	BSt2	Ma'am, because one is a whole number and tenths means starting with ones, this whole number [one] has ten parts. And tenths here means three tenths, as three is in the tenths position [in 7.39]. So 3 parts of one whole.
58	GSt2	No. Three [times] one-tenth of a whole.
60-61	BSt2	Ma'am, ma'am, one there is a whole ma'am and then there is a tenths place because ones there is one whole and one part, and tenths means one whole has ten parts in decimal.

The students' justification used the relation between consecutive place values to state that one-tenth of ones is tenths and that partitioning a whole into one part will leave the whole intact. Two other students provided a slightly different argument (see Excerpt 3).

Excerpt 3: Explanation for oneths - II (Y2DL2)

Line No.	Speaker	Utterance
64	GSt3	Ma'am, first we write tenths as one by ten, but we cannot write oneths as one by one.

Line No.	Speaker	Utterance
65	T	Why not?
66	GSt3	Ma'am, because it is a whole.
67	BSt3	It is a whole number.
68-69	T	Whole number, okay okay. What she is saying is what you have studied in primary class.

The student (GSt3 in Excerpt 3) extended the meaning of tenths as one by ten to infer that oneths means "one by one", which is a whole number and not a fraction. By convention, its position is to the left side of the decimal point. After listening to different students' explanations, Nandini offered an argument based on contradiction. She began by assuming that oneths exists as a distinct place value and then rejected it. Following this discussion, students extended the place value names for the fractional part, by using "ten-thousandths, lakhths, ten-lakhths, and so on". The class discussed place values of different decimal numbers and placed them in a place value table. In the post-lesson discussion, Nandini mentioned that students "were making connections and extending the place values", justifying her decision to change her lesson plan to focus on it thoroughly.

In the first and the second year, Nandini drew on the analogy between the place values of whole number and decimal numbers to introduce the decimal numeral notation. The responsive nature of Nandini's teaching is evident in Y2, as in the moment where she appreciated the significance of a student's question, and then evoked a discussion that led from the student's question to a conclusion that is adequate in answering the question. To successfully manage this discussion, Nandini identified and supported threads that moved the discussion towards a conclusion, while continuously evaluating if the conclusion is within the reach of the students, recognizing the conclusion as it emerges, and bringing the discussion to a closure. From the teacher's moves we infer that *en route* to the conclusion, she listened to what the students were saying, continuously evaluating whether their statements (i) were accurate, (ii) based on knowledge shared by other students, or (iii) based on definitions that have been accepted, and (iv) led towards the desired conclusion and closure.

Such careful noticing and listening by the teacher are supported by rich knowledge and anticipation of students' thinking. In making an evaluation of the students' responses and in formulating her own responses, the teacher is called to draw upon topic-specific knowledge that is thick in terms of key ideas, as well as connections to other topics that the students already know. Some such ideas allowed students to progress further than the teacher expected. For instance, students extended certain definitions, a response appreciated by the teacher in her remarks to the researcher. The teacher's awareness of these definitions supports students in using them in an emerging argument. Thus, the knowledge demand concerns knowing potential ways of using this piece of knowledge in supporting an argument or an explanation.

The pieces of knowledge manifested in the discussion, such as particular definitions, are specific to the situation. Hence, particular enactments cover only a portion of the knowledge map related to the specific concept. Reflection on the episode might uncover further portions of this knowledge map, and other directions that the classroom discussion might have taken. To illustrate this, let us examine why students are led to ask the question about oneths. The teacher's references to the left and right side of the decimal point introduced an implicit metaphorical mirror located at the decimal point. While this is of some use in identifying the place names on either side of the decimal point, it suggests the presence of oneths. The correspondence between place names in the fraction part and in the whole number part is partly a matter of convention, but the fact that no distinct place exists for "oneths" is due to the underlying relation between consecutive place values. The base ten structure binds the place values in a relation of (positive and negative) powers of ten around the basic unit or "ones". Expressing the place value in terms of powers of ten shows that the unit's place is the point of "symmetry" $\{..., 10^3, 10^2, 10^1, 10^0, 10^{-1}, 10^{-2}, 10^{-3},\ldots\}$. This manner of clarifying the multiplicative relation between place values relocates the mirror at the "ones" place. Thus, the teacher might have chosen to lead the discussion towards recognising that the point of symmetry and the location of the mirror is the "ones" place and not the decimal point. A sense of the landing place of the discussion, that is, the statement or explanation that would bring the discussion to an adequate closure, and the knowledge entailed in managing a discussion towards such closure, is an important component of the knowledge demands made on the teacher. There are potentially different mathematical ideas that can be utilised to provide a justification for the non-existence of oneths, including, relocating the mirror to the units place as a reference point, a weakening of the mirror metaphor by focusing on place values, and relation with the fraction notation. We believe that Nandini implicitly recognised the mirror metaphor. Reflection on the episode could make this metaphor explicit, leading to deeper understanding of the affordances and limitations of the metaphor, including the mathematical understanding that the proper location of the "mirror" is not the decimal point. In this sense, although the knowledge that emerges in a particular episode is partial, it contains the possibility of elaboration to acquire deeper knowledge and understanding, which in turn, can lead the teacher to be better prepared to anticipate and listen to the students' utterances.

7.1.2 Introduction to Decimals by Reema

In the first year, Reema introduced decimals using the measurement context (Y1DL1) by asking students to look at their 15 centimetre (cm) ruler to guess and measure the length of the given objects. She pointed to the centimetre markings on the ruler “1 cm, 2 cm, and so on” and then referred to the first division between 1 cm and 2 cm as “point one” and extended this to showing “point 5, point 8 and point 9”. She asked the students to draw an ant of length less than 1 cm, a glass of length 11 cm with water up to 5 cm, and a pencil of length 7 cm. Although students used different ways of measuring and drawing the length, Reema told the students how to do the task. For instance, a girl student used the length drawn by her earlier to make a length of 7 cm, that is, she drew 1 cm length (of the ant), then doubled this length to get 2 cm, doubled it twice to get 8 cm and then took away 1 cm from this length. She used the 1 cm marking created on her finger using the length of an ant. In other words, she used a marking on her finger as a measure to draw the length of 7 cm. After this task, Reema asked students to cut a thread of length 10 cm and use it as a perimeter to draw a circle. Students worked on their drawings and completed the table on estimate and actual measures of the remaining objects for the remaining lesson. In the next lesson (Y1DL2), Reema introduced “tenths and hundredths” using the money context. She decided to omit a discussion on the length of the frog task (elaborated later), suggested by the textbook, as she found it “complicated for students to understand” as they just been introduced to decimals.

Like Y1, in Y2, Reema asked the students to measure the length of a few objects (she carried an envelope, comb, tin and marker to the class). The choice of the objects seemed deliberate, as two of them measured in whole numbers and the other two had measures in centimetres and half millimetres. Reema used these measures to introduce “half or point 5”. The whole class discussion on estimated lengths was tabulated on the board and students were invited to measure the actual length of the objects. In the next lesson (Y2DL2), Reema introduced the metre strip as an extension of the ruler and used it to show the relation between metre, centimetre, and millimetre. She posed a series of questions on the metre strip (refer Excerpt 4).

Excerpt 4: Metre strip task variations (Y2DL2)

Line No.	Utterance
68	Guess the length of this (strip).
119	It is divided into how many parts?
131	How much is half of this strip?
142	Five such parts (10 cm) would measure how much?
163	If from all the equal parts, I take one part, only one part, so what is the fraction?
176	What is the length of eight such parts?
181	In the whole strip, how many two two parts are there?

After creating familiarity with the metre strip, Reema introduced the length of the frog context. She began by asking students to guess the length of the frogs that they have seen and then consider the length of the shortest and longest frogs as 0.9 cm and 30.5 cm respectively. She asked students to find “how many frogs of length 1 cm can sit on the strip”. While the students were modelling, an iterative addition of 1 cm using their hand, Reema showed this action through jumps on the metre strip, drawn on the board. She restated and recorded students’ responses, such as, “100 frogs of 1 cm each”, on the board. Then, she changed the length of the frog to 2 cm and asked the same question. Students immediately responded “50” giving reasons using half. She asked them about the longest frog, which is 30.5 cm long. A student located 30.5 cm on the strip, when Reema mentioned that it is half centimetre or 5 mm more than 30 cm. Some students gave an incorrect response “71 cm, 75 cm” for the length covered by two frogs, but when probed to justify their responses revised their response. Students added the length 30.5 for the third frog and concluded that 91.5 centimetres are covered. Using the strip they counted from 91.5 to 100 and concluded the remaining length as 8.5 cm. Reema wrote the subtraction sentence “100 cm − 91.5 cm” on the board and verified the answer to be 8.5 cm. With support from the other students, Reema revoiced the whole discussion on the number of frogs of length 30.5 cm, decimal addition (adding 91.5 and 8.5), and subtraction (going backwards from 100 to 91.5). Followed by this discussion, Reema posed the question of “how many frogs of length 0.9 cm can sit on this (pointing to the metre) strip”. Initial student guesses were – “definitely 100, 100 divided by 9”, corrected by another student as “100 divided by point 9”, another student said “99 frogs”. Reema asked students to spend some time in solving this problem. A few students volunteered to explain their methods to the class. Reema recorded these responses on the board and invited students to think aloud about them. The methods listed on the board included – (a) adding 0.9 repeatedly, (b) subtracting 0.9 repeatedly from 100, (c) multiplying 0.9 with 10 twice, and (d) dividing 100 by 0.9. After asking students to repeat each others’ methods and evaluate them, Reema asked “how much is 0.9 cm less than 1 cm?”. The students stated 1 mm. Reema pointed to the metre strip, pasted on the board, and marked 0.9 mm starting from 0. After leaving a gap of 1 mm, she pointed to 1 cm and the students said “0.9 again”. Students extended the pattern and added 0.9 for every whole number, starting from 0. Together, the class kept a record of the remaining length from each 1 cm length (reproduced as Table 6).

Table 6: Reproduction of blackboard work on 0.9 cm frog (Y2DL2)

Number of frogs	1 frog	2 frogs	3 frogs	5 frogs	10 frogs	100 frogs
Remaining length	−1 mm	−2 mm	−3 mm	−5 mm	−10 mm	−100 mm

Students figured how many frogs sat in remaining 100 mm or 10 cm and concluded with 111 frogs. Reema reflected on the use of the frog context in a few teacher-researcher meetings. She mentioned that students were interested in the problem and identified that "ten times point 1 makes the space for 1 more frog to fit on the 1 metre strip". She also discussed how the use of a metre strip supported the "movement from tenths to hundredths". She noticed that students flexibly used the number line to represent decimals of different lengths, unlike the first year.

In Y1 and Y2, Reema used the measurement context, albeit differently, to introduce the decimal numbers. She began by asking students to measure a set of objects, which were more carefully planned (aligned with the purpose of introducing parts of a whole) in Y2. Reema's omission and selection of the length of the frog context is an informed decision and is an indication of her anticipation of students' mathematical capabilities. Further, how students engage with the context, and its affordance to teach specific mathematical ideas, emerged as some of the knowledge demands.

In Y2, Reema does not underestimate student capabilities but focused on how the context can be used to bridge students' existing knowledge with new knowledge. There is a difference in the way Reema is listening to students in the first year, where she saw the use of different strategies used by the students, and then told them the method of measuring a given length. In Y2, Reema works with the students' explanations, notices the mathematics underlying them, poses questions, and supports students in solving the problem. Her responsiveness to students' ideas is evident from her richer anticipation of their capability, appreciation of students' responses, mapping their progress by unpacking the mathematical ideas underlying their explanations and providing necessary scaffolds to enable problem solving. For instance, Reema supported students' explanations by organising the discussion around the context in ways which helped students connect the relevant parts of prior knowledge and connect it with the topic. Through a series of such questions or an exercise (Watson & Mason, 2006), Reema modelled the use of the metre strip as a tool to justify a response. Students used this understanding to formulate an explanation for the problem on length of frog.

Reema's teaching showed the use of linear measurement to invoke students' prior knowledge, introduce new knowledge of decimal fractions, make links between the context and the linear representation, support flexible movement among measurement units, and offer explanations using the representation. Unpacking the affordance of the context included using it to build connections between students' knowledge and mathematical content, creating a challenging task, and supporting students to solve a problem. Each of these moves requires deeper knowledge of the content.

The knowledge demands that emerged from an analysis of all paired lessons in Reema's and Nandini's teaching are detailed in Chapter 5.

7.2 Supporting Teachers' Knowledge of Mathematics for Teaching

Although Chapter 5 of the thesis reports the case study of two teachers, an increasing responsiveness towards students' thinking was found among all participating teachers. As teachers became more responsive to students' ideas, they needed different kinds of support. Section 7.2.1 presents an analysis of the support offered through teacher-researcher meetings (Chapter 6 of the thesis). This *ex-situ* support aimed to enhance teachers' knowledge of decimals by inviting them to participate in a community, to discuss their struggles, share and reflect on their pedagogic practices. Section 7.2.2 discusses how the challenges faced by the teachers in their classrooms led to seeking specific *in-situ* support from the researcher, and what the teachers learnt from it (Chapter 7 of the thesis).

7.2.1 Ex-situ Support

Teacher-researcher meetings (TRMs) were organised to create a space to discuss the knowledge needed for teaching decimals. These weekly meetings happened in the school in the second year of the study. The tasks designed for TRMs relied on the students' and teachers' work collected from the first year of data collection, primarily classroom observations and students' written worksheets. Twenty TRMs were held during the period of July to December 2013, each meeting continued for 62 minutes, on an average. All the four teachers and a few researchers participated in the meeting. For analysis, a transcript of each meeting was prepared and classified into episodes based on the topic of discussion. An example of how episodes were identified in a meeting is shown in Table 8.

Table 8: Episodes in TRM2

TRM 2	**Episodes**
Relation between whole numbers and decimal numbers: Focus on student errors	E2.1 - Recap of previous day's meeting
	E2.2 - From division to fractions and decimals
	E2.3 - Division of whole numbers and rational numbers
	E2.4 - How to introduce decimal numbers to students?
	E2.5 - Connecting fractions and decimals
	E2.6 - Defining a decimal number
	E2.7 - Worksheet on students' thinking in comparing decimal numbers E2.7.1 - Ways in which students order decimal numbers E2.7.2 - Mathematics underlying student responses
Legends used	TRM - Teacher-researcher meeting, E2 - Episodes in second TRM.

Themes were identified based on the focus of discussion within each episode. For instance, episodes on analysis of students' errors were classified under the theme of identifying students' errors and its sources. The analysis of TRMs is organised around these themes, summarised in Table 9. The rationale for each theme comes from the relevant literature and the data collected from the first year.

Within each task, the design and nature of engagement were analysed. In the interest of space here, I will discuss one task from each of the three themes.

Table 9: Categorisation of tasks used in TRM

Theme	Tasks
Developing an awareness of what students can think and do.	Task 1: Identifying students' errors and their sources
	Task 2: Anticipating and understanding students' responses
	Task 3: Modelling teaching decisions based on understanding of students
Building on the topic-specific knowledge of decimals.	Task 4: Connection between decimals, whole numbers and fractions
	Task 5: Decimal and non-decimal contexts
	Task 6: Use of linear and area model
Coherence in teaching mathematics	Task 7: Identifying key ideas in the teaching of decimals.
	Task 8: Designing and sequencing decimal problems
	Task 9: Connections in the curriculum

Task 1: Identifying students' errors and their sources

In TRM1 teachers were asked to list different kinds of errors that students make when learning decimals. One of the common errors which emerged was 3.17 > 3.5 and 0.30 > 0.3. Teachers were encouraged to think about the reasons that might lead to a particular kind of response.

Nandini proposed that students might focus on the digits of the number and ignore the length of the decimal number. Pallavi added that students tend to make more errors where the numeral part (refers to the whole number part) of the two decimal numbers to be compared is the same, but the other part (fraction part) is different. Reema predicted that, when comparing, students might treat 3.17 as a three-digit number and 3.5 as a two-digit number. The researcher connected the different explanations given by the teachers by mentioning the digits-based approach, where the students overgeneralise that 'more the number of digits, greater is the number'. This thinking explains why students might think that 0.30 > 0.3 and 3.17 > 3.5. In such thinking, either the decimal point is ignored to see the length of 3.17 as a three-digit number or only the length of the fraction part of the number .17 is considered. Through several instances, it was concluded that whole number thinking can influence the learning of decimal numbers. This was followed by Pallavi's remark on how to deal with such an error (refer Excerpt 6).

Excerpt 6: Pallavi's explanation for error (TRM1)

Pallavi	But whenever this type of problem comes no, what I do is, I just ask them to first compare the numerals. So, if the numeral is same, I just ask them to cut it so they are left only with the decimal part.

Other teachers were asked to comment on this approach. Vindhya mentioned using place value as an explanation. Pallavi explained "adding the zeros" to make the length of decimal numbers same. Vindhya proposed converting decimal numbers to fractions for comparison. This particular meeting concluded with the need to identify students' thinking underlying their responses, the mathematical sources of students' responses, the ways in which whole number thinking influences decimal learning, and different ways of dealing with such responses. All of these threads were expanded in the later meetings for further discussion.

Task 5: Decimal and non-decimal contexts

The question posed in TRM6 was thinking about where all do we see the use of a decimal point? The following situations came up – composition and cost of a drug, billing of items, overs in a cricket match, length (or height, distance, depth) measurement, currency transactions, temperature conversion, and measurement of weight and capacity. When probed to think of other out of school examples of the use of decimal numbers, teachers suggested weather forecast in the newspaper, measure of rainfall, marks and percentage, measures while tailoring, distance travelled in an auto-rickshaw, etc. They were asked to consider whether a measure written in feet and inches can be separated using a decimal point? After a brief discussion on the meaning of a decimal, the group re-examined all the contexts listed earlier. The list of contexts was now classified as: decimal point used as a separator with no relation between units (exercise numbers), using a point to separate parts of the same unit but the relation between units is not base ten (date, overs in a cricket match, feet and inches) and the decimal contexts. This lead to a discussion on the relation between the units in a decimal context, for instance, where is paisa in the representation Rupees 2.50. This discussion was extended to distinguishing cardinal, ordinal and nominal numbers; importance of positionally in the Hindu-Arabic numeration system; and for comparing measures in the later meetings.

Task 8: Designing and sequencing decimal problems

In TRM3, extending the discussion on the influence of whole number thinking on the decimal learning, teachers were asked to design some decimal problems that would help diagnose specific students' misconceptions. The problem types proposed by the teachers are summarised in Table 10.

Table 10: Problem types on comparison of decimals

Problem Type	Proposed by	Description of the problem	Example
I	Vindhya	The same whole number part but varied decimal part, but dealing with only tenths and later extending it to hundredths.	3.0, 3.9, 3.10
II	Reema	Take the same set of digits and change the position of decimal point.	3642, 364.2, 36.42, 3.642, .3642

Problem Type	Proposed by	Description of the problem	Example
III	Pallavi	Take different three digit numbers. Compare them with decimal point at different positions.	(a) 465, 599, 436 (b) 46.5, 59.9, 43.6 (c) 4.65, 5.99, 4.36 (d) 0.465, 0.599, 0.436

Teachers began considering the idea of variations within problem types. All the teachers used this discussion while they were teaching. A few other problems were created, for instance, use the given 0.25 part of a figure to draw the whole figure, and comparison of measures 0.02 cm and 0.02 m. During this discussion, teachers anticipated students' explanations to the problems, and also which explanations would need to be emphasised.

While the initial discussions in TRMs were led by the researchers and centred around students' artefacts, gradually teachers initiated these discussions using artefacts from their classroom.

7.2.2 In-situ Support[5]

In order to understand the process of change and the nature of support that teachers need to engage with the changed classroom practice, one of the cases was analysed. Pallavi's case was selected since she resisted the pedagogical practices such as listening to students' ideas or teaching why a method works, and held strong beliefs such as that the use of multiple methods confuse students, which led to unchanged classroom practice for some part of the second year. The analysis revealed that the *in-situ* support offered to Pallavi challenged her beliefs, knowledge and existing practices. The knowledge gained from such an experience led Pallavi to engage students' thinking in productive ways while teaching. Since the reported episodes are from the lessons on division, relevant literature on the topic was reviewed and ways in which the topic was dealt in the old and the new NCERT textbooks was contrasted. Significant differences were found in the approaches of the two texts. Unlike the old textbook, the new textbook offered a variety of methods and contexts for division.

Pallavi's interpretation of dealing with different strategies as proposed in the new NCERT textbook was to 'teach all the methods'. She did not seem to associate the choice of 'method' with the problem context. Observations over several lessons show that she explicitly taught students each of these methods, and then gave practice problems to use the same method repeatedly. She did not allow for students to use their own strategies or discuss why some strategies are more efficient than the others. When probed about the teaching of justification of an algorithm in class, she expressed that students were not developmentally capable of understanding the reason for why a method works and therefore her decision to avoid conceptual explanations while teaching. By specifying the use of

[5] Takker, S., & Subramaniam, K. (2018). Teacher Knowledge and Learning *In-situ:* A Case Study of the Long Division Algorithm. *Australian Journal of Teacher Education, 43*(3), 1–20.

a particular method when solving division problems and by breaking down a problem procedurally, Pallavi lowered the cognitive demand of the task (also noted by Jackson, Gibbons & Sharpe, 2017). In the second year, while reading the textbook, Pallavi noticed the chunking or the partial quotients method. In this method, convenient multipliers are chosen and the multiple is subtracted from the dividend. In other words, in a quotitive interpretation where the divisor is interpreted as the fixed size of a group or share, one has to reach the maximum number of groups/shares of divisor that can be taken away from the dividend. (Alternatively, in a partitive interpretation where the divisor indicates the fixed number of equal groups, one needs to arrive at the maximal size of a group.) The number of groups may be decided by the ease of arriving at multiples using doubling, multiplication with ten and its multiples, etc. Pallavi expressed her struggle in using the partial quotients method to solve division problems. She mentioned that it lacked procedural clarity and logic, found in the long division algorithm. Pallavi's comfort with the long division algorithm came from her confidence in using the method, following the steps sequentially, and its efficiency. Before her lesson, the teacher and the researcher solved a few problems using the partial quotients. While Pallavi grasped the method and used it to solve a few problems, she expressed her discomfort in teaching the method and proposed that the researcher teach a lesson. The researcher planned the teaching of this lesson and began teaching by posing the problem of dividing Rupees 75 among 3 children equally. The money context and the sharing meaning of division intuitively invoked different ways of solving this problem. The students solved the problem using chunking with convenient numbers and a combination of different sets of numbers. Students were invited to share their ways of problem solving, and observe similarities and differences between their methods recorded on the board. The question was then modified to, what if the number of children was 5. Without finding the exact share, students used proportional thinking to justify that the share of each child will reduce. While students were calculating to find the exact share, Pallavi began co-teaching the lesson. She posed the problem of dividing Rupees 127 among 5 friends equally and noticed the nature of students' discussions around the size of chunks. She found this method better as it kept a "track of place values". She invited the students to share their strategies and after listening to them, commented about the choice of multiples and the number of steps, link between the partial quotients and the algorithm, and the conceptual consistency in using this method. She discussed student strategies and challenges of dealing with such mathematical problem situations with the researcher, during and after this lesson. In the lessons that followed, Pallavi explicitly dealt with the relation between using partial quotients and the long division algorithm. While teaching, she figured that the place value structure is implicit in the division algorithm. The contrast between taking a digit based approach and the number as a whole was triggered by a student's explanation. It was during teaching that

Pallavi noticed and explicated that the underlying structure of the division algorithm is in finding the greatest partial quotient or with the highest place value.

It is important to note that working with a few examples using a different method along with the researcher, while planning the lesson, was not sufficient for Pallavi to develop an understanding of the method or to convince her to teach it. Pallavi's initiative of articulating her struggle with the method, and seeking support from the researcher by co-teaching in the classroom, marked an important shift allowing for a re-examination of existing beliefs and practices. In this interaction, Pallavi requested for a specific kind of support, which was unanticipated by the researcher. In the analysis, such an interaction is characterised as a 'contingent moment'.

Rowland and Zazkis (2013) use contingency to refer to the classroom events which are difficult for the teacher to anticipate or plan. I extend the notion of contingency to refer to those situations, in the teacher-researcher collaboration, which were unanticipated by the researcher, but demanded an actionable response in-the-moment. In this case, supporting Pallavi by co-teaching a lesson was a contingent moment. Pallavi's discussions with the researcher changed considerably, after this particular interaction. She began eliciting students' intuitive responses before introducing a topic, read and discussed the textbook content, and discussed students' responses with the researcher. We noted that identifying and responding to contingent situations in the teacher-researcher collaboration supported teachers' learning. The analytical construct of contingent moments, theorised from Pallavi's case study, was then used as a lens to identify and analyse such moments for other teachers.

Process of converting contingent moments into learning opportunities

What do we note about the process of identifying and responding to contingent moments from an analysis of all the four cases? First, although the analysis presents contingent moments as episodes of teacher-researcher interactions where teachers demanded individual support in the teaching of specific sub-topics, the emergence of such moments is part of a process. There were several such moments where the individual teachers sought support, but contingent moments marked a significant shift in the teacher-researcher relationship. The process of recognising and seeking support in such moments is continuous, it requires shared understanding about the learners, teaching and the goals.

Seeking support in such moments also involves building of the trust (through an engagement with the practice of teaching). The shared interest of the teacher and the researcher in enabling students' learning, and researchers offering support to the teachers in the process, is a marked shift from the commonplace understanding of inspecting or judging a teacher's practice. Such trust also demands that in the way that teachers are expected to be responsive to individual students' needs, the researchers who intend to support teachers must be prepared to act responsively, to offer the support that is useful for teachers. Such exemplar moments are analysed also to understand the nature of support that individual teachers need in the sites of their practice. How does such support enable teacher learning? Some aspects of the process can be discerned from the analysis of four cases:

1. Each teacher identified the challenge that she faced in a specific context of practice and created a space for dialogue with the researcher about it. The recognition of the challenge and acknowledging the need for support is important to trigger a discussion around it.
2. In the discussions, an alternate way of handling the mathematical idea was explored. While the pedagogical part of the discussion was often tentative, there was an openness on part of the teachers to try alternate ways. Further, the knowledge ensured in the changed pedagogy was thoroughly discussed in each case by the teacher and researcher.
3. An important aspect of each experimentation was students' response to a planned trajectory when it was enacted in the classroom. As the tasks planned in each case were tried out, the teachers noticed and uncovered the mathematical aspects of students' talk and discovered connections between different mathematical ideas (such as methods, representations, explanations).
4. Each such experience of planning and testing in the classroom was followed by a reflection on the students' responses, to identify the affordances of such teaching pedagogy (and the knowledge it entails) in terms of knowledge that gets constructed in the classroom. After explicating this knowledge, teachers made conscious decisions about organising further teaching.

The process of collaboration was deliberate, non-prescriptive, and negotiated with respect to the nature of support that teachers needed. The analysis suggests that noticing and identifying such challenges as well as preparing to teach requires rich knowledge base. Evidently, experienced teachers also struggle with the conceptual understanding of mathematical procedures, explanations and representations. An examination of the mathematical knowledge *in-situ,* grounded in the complex work of teaching, helps in identifying the knowledge demands posed on the teachers when teaching in a reform context. Preparing teachers with the topic-specific knowledge helps in noticing significant ideas and moments, and in experimenting with alternate practices. To conclude, identifying and responding to such contingent moments in teacher-researcher collaboration has the potential of creating a space inside the classroom for enabling teacher learning.

8. CONCLUSIONS AND IMPLICATIONS

The study aimed to investigate and support mathematics teachers' knowledge of students' mathematical thinking. Teaching of four experienced school mathematics teachers for two consecutive years was analysed. This section is organised around the research questions that were raised at the beginning and a few others that emerged during the course of the study.

The classroom observations revealed that teachers became more responsive to students' ideas while teaching in the first to the second year of the study. An analysis of classroom data helped in identifying aspects of responsive teaching. Section 8.1 characterises responsive teaching using the research literature and the changes observed in teaching during the study. It was evident that responsive teaching is challenging and poses special knowledge demands on teachers. Teachers' decisions in handling classroom situations were informed by their deeper knowledge of the subject (an insight also reported in the literature on teacher knowledge). Thus, the questions asked were – how to identify the knowledge demands placed on teachers while teaching responsively and what are the knowledge demands posed on the teachers while teaching decimals. Section 8.2 discusses the process of abstracting these knowledge demands from an analysis of practice, and summarises the observed and anticipated knowledge demands in the teaching of decimals. A part of the study was aimed at supporting teachers to handle the knowledge demands while teaching in the classroom. So the question was, how teachers were supported as they developed responsiveness to students' mathematical ideas. Section 8.3 is a reflection on developing responsive teaching through the *in*-situ and *ex-situ* support offered to the teachers. Section 8.4 discusses the evolution of a community of teachers and researchers through the study. The implications drawn from the study, suggestions for future work and the limitations are discussed in Sections 8.5, 8.6 and 8.7 respectively.

8.1 Characterising Responsive Teaching

The preliminary observations of classroom teaching and interviews from the first year of the study revealed that teachers attributed students' mathematical abilities to personal traits and background. It was found that the teachers who participated in the study were aware of students' backgrounds. This made teachers aware of the nature of support that students received at home. Some of the teachers' decisions, such as the need for students to become more fluent with the algorithms and rules, were based on the need to help students to excel. This knowledge about students' background interacted with an emphasis on algorithms, a goal of mathematics teaching that the teachers have experienced through their schooling. Further, teachers held the belief that conceptual understanding is "higher level" and can be gained only when students acquire some cognitive maturity. While the discussions with the researcher helped teachers in identifying some common student errors, teachers found it difficult to locate the sources of these errors and deal with them in ways that supported students' thinking. The analysis of classroom teaching indicated that teachers either ignored students' questions (unless seeking clarifications) or judged students' responses as correct or incorrect. When such questions were brought to teachers' notice in the post-lesson interactions, they mentioned the lack of time and the rush to complete the syllabus as factors affecting such choices.

Contrasting the data on teaching from the two years, it became evident that such choices or teaching decisions depend on the knowledge that the teachers bring to use while teaching. The decision to ignore students' utterances could emerge from the difficulty in knowing how to handle them in the class. The data suggests that teachers, despite their experience in teaching mathematics, struggle to (a) anticipate ways in which students make sense of the content being taught, (b) notice the mathematics underlying students' utterances, (c) decide whether erroneous strategies can be stated aloud considering how it might affect other students' understanding, (d) connect students' ways of thinking with the content planned for a lesson, and (e) estimate students' capabilities of engaging with the new content based on their prior knowledge. An anticipation of these challenges is also connected to beliefs about students' ability, in general, and the role of teaching formal mathematics. A general under-estimation of students' abilities was common among teachers.

In the second year, it was noted that teachers were listening to their students, interpreting the mathematics underlying their responses – sometimes along with the students or the researcher, and brainstorming ways of handling these responses in class. In the classroom practices, it was noted that teachers re-voiced students' responses, asked probing questions, gave students an opportunity to explain their responses (correct or incorrect), and either used or adapted students' ways of problem solving. A wide variety of teachers' responses to students (such as probing, seeking further information, connecting it with other ideas, challenging inferences made, etc.) indicates that teachers anticipated the mathematical potential underlying the initial students' response. A stronger teacher anticipation was corroborated with the task on anticipation and reflection on students' responses to a worksheet designed by the researcher. Clearly, teachers' *interpretive listening* (Davis, 1997) expanded their knowledge of the content and informed their decisions in the classrooms. Careful noticing of students' responses and making sense of them with the researcher and later other teachers also served a *hermeneutic* function.

The literature on responsive teaching suggests that teachers need to be able to listen to students' thinking, and plan and execute ways in which it can be developed. The act of listening and responding includes acknowledging and appreciating students' ways of making sense of the content, *knowticing* the mathematics underlying students' utterances, and using this knowledge to make informed teaching decisions. *Knowticing* requires that teachers listen to and interpret students' (incomplete, partial, incorrect, alternate), make connections between students' ideas and the key ideas in the teaching of a topic, and provide support that helps students learn. As revealed by the data from the teacher interviews, this process is complex, as teachers think about ways of interpreting and supporting student's ideas, while managing the participation and engagement of the other students in this interaction. Responsive teaching includes a complex mix of aspects such as challenging students in ways that help them think and articulate their ideas, at the same time help them reach a conclusion, and managing these discussions with individuals and groups in classroom.

During the course of the study, the pedagogical practices which indicated teachers' mathematical responsiveness towards students included – different ways of talking about students' strategies by identifying their source; connecting students' explanations to teachers' explanations; identifying with the struggles made by students and making attempts to link students' explanations with the key ideas of the topic; reading the textbook more closely and becoming more agentic in the selection of content, resources and contexts to be used for teaching; and using worksheets not just to assess students' understanding or offering more practice, but also for diagnosing and probing students' thinking. The responsive teaching poses knowledge demands on teachers.

8.2 Knowledge Demands in Teaching

This section addresses the questions – how can we abstract knowledge from a study of practice? and whether the knowledge abstracted in this manner is substantively different from the way knowledge is characterised by the existing frameworks on teacher knowledge.

8.2.1 Abstracting Knowledge Demands From A Study of Practice

The contemporary literature on teacher knowledge points to the need for creating descriptions of knowledge situated in the practice of teaching. The inclination to move towards practice-based descriptions has the potential to respond to the critiques such as lack of attention to the dynamic aspects of teaching or understanding knowledge through a study of work of teaching. The thesis offers some methodological insights into studying knowledge from an investigation of practice. While there was an awareness of the literature on the knowledge that the teachers need to teach effectively, the specifics of such knowledge were discerned from the study of practice. The knowledge required for teaching decimals was identified through an analysis of the knowledge demands placed on the teacher in the contingent classroom situations. As discussed in Section 7.1.1, these knowledge demands are closely tied to the decisions made by the teacher. In other words, different teaching decisions have the potential to uncover other parts of the teacher's knowledge. Therefore, apart from the discussion of the knowledge demands that were revealed through an analysis of classroom teaching, a reflection on the possible trajectories that a teacher might have taken has the potential to uncover the map of teacher knowledge. This map, unlike the abstract descriptions of knowledge, is topic-specific and is closely tied to the work of teaching, particularly teachers' in-the-moment decisions. Since the descriptions of practice are expansive, teacher's responses to contingent classroom situations, served as an appropriate grain-size (Cai et al., 2017) for an analysis of knowledge demands. An analysis of the "paired episodes" in the teaching of teachers in transition, offered a suitable context for the knowledge demands to become visible. To conclude, an abstraction of anticipated and actual knowledge demands from an analysis of paired episodes in contingent classroom situations is an important methodological tool to study knowledge manifested in teachers' practice.

8.2.2 Knowledge Demands in Teaching Decimals

A comparison of "paired episodes" helped in uncovering the topic-specific knowledge demands posed on the teachers. These knowledge demands can be suitably organised to map the mathematical knowledge needed for teaching of decimals. A summary of knowledge demands arising in the teaching of decimals include:

1. Affordance and limits of the analogy drawn between whole numbers and decimals, while teaching different sub-topics for instance, place value, operations, etc.
2. Using fraction representation along with the place value as a justification for procedures of working with decimals. For instance, using fraction representation when comparing decimals.
3. Consistent use of representations for different sub-topics within decimals. Identifying the affordance of the linear and area representation, and their suitability for the context of use.
4. Appropriateness of a context in teaching specific sub-topics within decimals. Examining different decimal and non-decimal contexts, and identifying their potential.
5. Identifying key ideas in the teaching of decimals and connecting them to form conceptual explanations. For example, connecting place value, positionality, base ten, distributivity, etc.

8.3 Developing Mathematical Responsiveness

It is evident that *knowticing* the mathematics underlying students' response requires knowledge. This knowledge is used dynamically. Teachers can be made aware of students' ways of thinking using research literature or through references to experienced teachers' practice. However, it is important to acknowledge that parts of such knowledge and connections get unpacked through actual interactions between the teacher, students and the content while teaching in the classroom. Therefore, teachers need both kinds of support, first from the knowledge that exists in the form of topic-specific research literature or as experiences of teachers, and second in the struggles arising from the challenges encountered in routine teaching.

How can teachers' knowledge of students' thinking be supported? The literature on topic-specific knowledge, and examining resources such as textbooks, are sources through which such knowledge can be developed. The other sources of this knowledge are the artefacts collected from the teachers' teaching. While the knowledge from the literature can prepare teachers in handling contingent situations, a reflection on different ways of identifying and dealing with such situations in the classroom feeds back into the corpus of knowledge required for teaching. The analysis of *in-situ* and *ex-situ* support offered to the teachers during the course of this study reveals that an interweaving of the knowledge from the literature with the actual practice of teaching developed teachers' knowledge. While the literature on topic-specific aspects was used to organise and plan teacher-researcher meetings, interactions with teachers inside and outside of the classroom encouraged sharing and reflection on the knowledge underlying teaching practice. For instance, the research literature on students' conceptions in decimals was used to (a) design problems given to students in the form of a worksheet used for the anticipation task, (b) organise the actual students' responses into categories or modes of thinking, and (c) examine similarities and differences in the way students responded to such problems.

Ways in which the artefacts from literature and actual practice were interweaved to support teachers' knowledge include (a) enhancing topic-specific knowledge, (b) developing knowledge of students' conceptions and capabilities, (c) using students' work as a spring board to discuss key ideas in the teaching of specific topics, (d) examining alternative pedagogies with their affordances and constraints, and (e) using classroom a site of conscious practice and reflection. These have been elaborated in the thesis.

8.4 Organic Evolution of a Community of Teachers and Researchers

An intensive engagement with the teaching practice led to the evolution of a community of teachers and researchers, with a shared interest in enabling students' learning. I will illustrate this using the concepts of "engagement, identification and boundary crossing" proposed by Wenger-Trayner et al. (2014) for the development of a community. While the teachers and researchers were aligned with their roles of teaching and data collection, respectively; an engagement with the others' role to reflect on the process of teaching, helped in identifying shared interests. The sense of community developed through the researcher's engagement with the teachers' struggles in practice and the teachers' interest in the reflections on practice and research literature, all these evolved during the course of the study. The flexibility in the role of the researcher also supported boundary crossing in events such as researcher as a co-teacher or co-planner, teacher as an observer, and so on. Teachers and the researcher identified more closely with each other in tasks such as designing a worksheet, planning a lesson, analysing a teaching episode, etc. The identification was weaker in tasks such as reading of research papers, marking students' papers, etc. However, respectful identification and dis-identification in different tasks helped in developing a shared accountability to the act of teaching. It is important to mention that an interest in and accountability for students' learning acted as major source of identification between the teachers and researchers. While the teachers' learning from this research study has been discussed, the researcher's learning from participation in this community is important. As a researcher, my *knowticing* of students' ideas and teachers' thinking has evolved. Further, the researcher's (as a teacher educator) mathematical sensitivity and responsiveness to the needs of individual teachers, a continuous process of learning, translated into action by supporting teachers in contingent moments.

8.5 Implications

1. The analytical constructs proposed by the study, such as abstraction of "knowledge demands" from an analysis of "paired episodes" in "contingent classroom situations" have implications for further research on mathematics teacher knowledge in two ways. The constructs offer a way of characterising topic-specific knowledge required for teaching from a study of practice. Further, the process of analysing knowledge demands offers methodological insights for capturing the dynamic aspects of teachers' knowledge by studying "paired episodes" of teachers in transition.

2. Discussions on specific episodes of teaching to identify the actual and anticipated knowledge demands entailed in different teaching decisions (or trajectories) is an important way of orchestrating learning in pre-and in-service teacher education programmes. Tasks on specific cases of students' work and teachers' activity in the classroom need to be used to develop teachers' knowledge of the subject matter (by focusing on the connections between its specific parts, such as SCK and KCS).

3. The study offers a way of designing and using classroom-based tasks and integrating them with the research literature to develop professional knowledge required for teaching mathematics. The methodological and analytical insights can be used to design professional development initiatives for other disciplines.

4. Insights from the study were translated into the curricular design for an innovative pre-service teacher education programme. The programme is designed to form communities of learning among student teachers, experienced teachers, researchers and teachers. Courses on school and classroom observations, and experiences such as following experienced teachers over long periods of learning, discussions on struggles of teaching, reading and using research literature on different topics to teach students, are woven into the design of this programme.

8.6 Suggestions for further work

First, the analytical constructs (such as contingent moments) proposed by the study can be used to identify knowledge demands underlying the teaching of topics other than decimals. A systematic analysis of knowledge demands can be used to create a map or landscape of mathematics teacher knowledge in specific topics. Such a map of teacher knowledge draws from the dynamic aspects of teaching and has the potential to imagine 'mathematical knowledge for teaching' in ways which are closely tied to the work of teaching. Second, the study proposes a way in which teacher-researcher partnerships can be used to understand and support students' learning. Other kinds of partnerships, for instance, through designing collaborative research projects with teachers, researchers as co-teachers, etc. can be explored in-depth. Third, different ways in which research literature can be linked with the actual practice of teaching to support responsive teaching, can be explored. These include making connections between the research literature and the artefacts from teachers' own practice, help in disseminating research findings in ways in which teachers find it useful and relatable. To conclude, an imagination of research projects and teacher professional development initiatives with the aim of forging stronger connections between research and teaching, are a potential way forward.

8.7 Limitations

The affective aspects of teachers' sensitivity and ways in which they interacted with the changing cognitive sensitivity of the teacher during the course of the study has not been addressed in the analysis. The participating teachers worked with different cohort of students in each year although that helped in creating a comprehensive map of teachers' knowledge. While broad changes in the teaching of topics other than decimals were observed, they have not been studied closely. Focused interactions among teachers and those with the researcher were limited due to systemic constraints such as strong framing of teachers' time and work in the school.

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