Developing a Learning Sequence for Transiting from Arithmetic to Elementary Algebra

A Thesis

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by

Rakhi Banerjee

Homi Bhabha Centre for Science Education
Tata Institute of Fundamental Research
Mumbai

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DECLARATION

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions. The work was done under the guidance of Professor K. Subramaniam, at the Tata Institute of Fundamental Research, Mumbai.

Candidate's name and signature

In my capacity as supervisor of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.

Guide's name and signature

Date:

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Table of contents

List of figures

List of tables

Chapter 1: Introduction

1.0 *Background: Mathematics and mathematics education*

Mathematics has an important place in the elementary school curriculum in India, as in all other parts of the world. It has a high status and value in the society, as an intellectual endeavour and as a subject which provides better opportunities for employment. It is seen as an "epitome of precision, manifested in the use of symbols in calculation and in formal proofs" (Lakoff and Nunez, 2000, p. xi). But, it is not in this symbolic rigour that the content of mathematics lies; it lies in human ideas (ibid.). Mathematics is mind dependent and not Platonic. Mathematics is shaped, developed and structured by the human mind. Mathematics began from the activities of counting, measuring and evolved into a discipline with its own concepts, problems and syntax, which could be used in many other disciplines (Jacquette, 2002). In the process, it became separated from the practical contexts and grew in its abstraction characterized by precision, generality and certainty. In the school curriculum as well, the subject was translated in a similar fashion, with emphasis on its abstract nature from the beginning. The alienation of most students from mathematics can be attributed to remoteness of the subject (its 'mind-independentness') and lack of a sense of meaning for the symbols, these being considered arbitrary marks on paper. This is also one of the reasons for the large number of students opting out of optional mathematics courses and in general, dropping out of school education due to failure in mathematics.

In an effort to get important and challenging mathematics accessible to students and strike a balance between the precision and rigour of mathematics and its "social construction", the new National Curriculum Framework (2005) of India aims to move away from the 'utilitarian' view of mathematics which equipped students with basic computational ability, to developing among students the ability to “think and reason mathematically, to pursue assumptions to their logical conclusion and to handle abstraction” (NCF, 2005, p. 42). By adopting this approach, the new curriculum framework intends to tackle the large number of dropouts from school due to failure in mathematics. The earlier reform movements in the United States of America, like the ‘New Math’ movement which introduced ideas like set theory and inequalities, followed by the ‘back to basics’ movement with focus only on basic skills had failed to achieve the desired outcome of getting students interested in mathematics, to develop mathematical thinking in them and retain them for advanced studies in mathematics (Resnick and Ford, 1981; Schoenfeld, 1987, 2002).

Subsequently, reform movements, including the one in India, have endeavoured to improve the quality of mathematics that is to be taught and learnt and include as many students in the teaching-learning process of mathematics as possible and teach them important and meaningful mathematics (NCF, 2005). The focus in this reform movement has been on multiple ways of approaching a problem, reasoning and verbalizing explanations, which are better indicators of procedural and conceptual understanding of students. Richard Skemp (1919-1995) pioneered the important distinction between ‘instrumental’ and ‘relational’ understanding underlying the difference between ‘knowing what’ and ‘knowing why’. He also pointed out the reason for students’ difficulty in mathematics in the nature of abstraction (reflective against empirical abstraction, as explained by Piaget) required for constructing concepts in mathematics. Hans Freudenthal (1905-1990) was another person who led the goal of making mathematics meaningful to students by starting a school of thought called Realistic Mathematics Education. This approach believed in beginning from a context that the students could relate to, and through the processes of ‘horizontal’ and ‘vertical’ mathematization, leading to context free exploration of properties and relationships within mathematics.

1.1 *The state of algebra education*

Within mathematics, algebra has occupied a very special place. The teaching and learning of algebra has been highly debated and contested in the last three decades with respect to its role and purpose as well as the aspects which need emphasis in algebra. Its abstract nature and demand for precision in symbolic recording has been a source of concern and failure for many students. This has led to the exclusion of large numbers of students since algebra is the gateway to success in many prestigious professions as well as academic careers. Algebra is a discipline which deals with the study of structures, and is a means for exploring patterns, relationships, and expressing them as generalities. It thus serves as a major analytical tool in many branches of mathematics and in other disciplines which use mathematics for reasoning, justifying and proving. It provides us with concepts, symbols, and techniques for working on symbols, using which one can first express a situation symbolically and deduce results with certainty, thus making it one of the most preferred symbol systems for purposes of reasoning. Much of the higher mathematics and sciences, takes algebraic expressions as the input, for example, for understanding and using functions, and in calculus.

1.1.1 The arithmetic algebra divide

Algebra is the domain which first takes students away from concrete situations and operations on numbers to abstract rules, properties and generalizations involving the use of letters, making it difficult for students to cope with this new domain. It poses the first major challenge of making sense of symbols and working on them, also because symbolic expressions are no longer amenable to a sequential processing leading to a numerical solution (Booth, 1988). Dettori et al. (2001) describe the break between arithmetic and algebra as consisting of a '*change in the nature of problem resolution*' (from performing step by step computations to defining relations and transforming them by means of formal manipulations) as well as a change in the '*nature of mathematical ob-*

jects' (not just numbers but variables, unknowns, parameters). Students have to move beyond looking at expressions as sequences of binary operations to working with the whole expression.

Notations in algebra are also responsible for many of the difficulties with algebra. 3+*a* cannot be added up to 3*a* as they are different kinds of terms, and denotes the sum of 3 and '*a*', while 3*a* stands for the product of 3 and '*a*' and does not denote place value notation. Together with these changes in notation, students have to deal with multiple uses of '+' and '–' sign: used as an operation sign for adding or taking away, sign of the number, denoting increase and decrease, and the relationship between two numbers. Similarly the '=' sign has to be considered as signifying the result of an operation or a set of operations and also the sign of equivalence between two expressions (Wagner and Parker, 1999). Over and above all these, one needs to construct the meaning of the letter, which also varies with the context in which it appears (unknown in an equation, generalized number in identities and variable in functions). Students have been found to have many misconceptions regarding the use of letter in mathematics, for example, that the letter is the short hand for an object, the letter which precedes another in the order of alphabets has a smaller value than the one which succeeds, different letters must always have different values, etc. (Kuchemann, 1981; Booth, 1984, MacGregor and Stacey, 1997; Stacey and MacGregor, 1997a).

Further, the difficulty for symbolic algebra arises from students' prior exposure to arithmetic computations. Students initially get familiar with binary operations and then preparation for algebra begins by introduction to evaluating a sequence of binary operations on numbers presented as a symbolic expression. Thus, procedures and conventions of operating on these sequences of numbers and operations are introduced so that one arrives at a unique value for the expression. But hardly ever are these procedures abstracted to form properties of operations, which are essential to understand the rules of transforming algebraic expressions. Attending to the properties of operations and structure of expressions is one of the essential prerequisites for learning symbolic algebra (Kieran, 1989a, 1992; Stacey and MacGregor, 2001, Linchevski and Livneh, 1999).

There are also differences in approaching problems in arithmetic and algebra. While in the arithmetic approach students can work from the known conditions and find intermediate numerical solutions to arrive at the solution to the problem, it is essential in the algebraic approach to use expressions to represent the problem situation using a letter for the unknown (Bednarz and Janvier, 1996; Stacey and MacGregor, 1999), distinct from solving the problem. Thus, in the context of arithmetic, students do not appreciate the purpose of recording operation sequences or representing problem situations. They also do not abstract the properties and rules of transformation which can be consistently applied while manipulating expressions (Booth, 1988). They only implement procedures for finding the numerical solution to a problem (posed using symbols or embedded in word problems) which may depend on the context or the numbers involved, and thus do not engage in general solution methods applicable over a range of problems (Ursini, 2001). The methods of teaching and learning generally used force the students to rigidly follow algorithms without any space for reflecting on them and for exploring properties and relations between numbers and operations. This is unhelpful to students in understanding the equivalence of different procedures, or their generalizability, making it difficult to shift to algebra. Students' poor skills in representing problem situations and weak understanding of transformation of expressions do not allow the students to move to the step of deducing or inferring about the situation, which is the crux of algebra (Booth, 1989a).

1.1.2 Reconceptualization of algebra: solutions and problems

Researchers' concern with students' understanding of properties of operations and structure of expressions and their resulting failure to deal with algebraic symbolism led to various reconceptualizations of algebra. Some of these approaches (to be discussed below) moved away from emphasizing a strong connection with arithmetic expressions, which was believed to involve pedagogic hurdles and to be a source of misconceptions and was considered difficult (Mason et al., 1985; Lee and Wheeler, 1989, Usiskin, 1988). However, studies have been done with students in the middle school level which have made efforts to explore and analyze the connection between arithmetic and algebra bringing a focus on the structure sense of the expressions in the two domains – arithmetic and algebra (Kieran, 1989a; Linchevski and Livneh, 1999, 2002; Livneh and Linchevski, 2003; Liebenberg et al., 1998; Liebenberg et al. 1999a), thereby showing the promise in the connection between the two domains. These will be discussed in detail in the next chapter.

Recent studies by Blanton and Kaput (2001), Carpenter and Franke (2001) add evidence to the power and accessibility of generalization, even with very young children at the beginning of primary school. In these studies, children were able to develop their symbolization, reasoning and analytical skills, through an emphasis on verbalizing explanations and negotiating meaning of the symbols. Also, studies by Carraher et al. (2000, 2001, 2003) and Brizuela et al. (2000) indicate that some children as young as in grades 3 and 4 can handle functions by treating each of the four basic operations as functions and use letters to both stand for an unknown and a variable. They can also solve linear equations building their own strategies for working on the unknown. There are many other studies which demonstrate young children's capabilities to handle simple algebraic situations (e.g. Carpenter et al. 2003; Ainley et al., 2003), not always using algebraic symbols but inventing ways (diagrams, non-conventional symbols) to explain their reasoning and the solution. These approaches capitalize on bringing forth the generality and pattern in the operations in arithmetic, which are within the reach of the young children and can provide a good ground on which the more complex algebraic understanding of both the syntax and the semantics of symbolic expressions, characteristic of middle school, can be built.

In this context, Schifter (1999) highlighted the importance of giving space and opportunity to students to explore alternative strategies to solve problems and explaining their understanding of the problem as well as the process of finding a solution. Both word problem contexts and calculation tasks can act as potential sources for learning about operations, properties and relationships, which can lead to the development of operation sense. The purpose of such tasks in the beginning years is to make the implicit understanding of students explicit, which with the use of appropriate symbols can be used in formal algebra. These studies represent a countertrend against the initial studies in the field of algebra education which deduced Piagetian stage-like features in acquiring the concept of letter/ variable (Kuchemann, 1981) and identified the lack of cognitive maturity as a reason for difficulty in dealing with algebraic symbols (Booth, 1984). So, the view that algebra is abstract and can be dealt only when the child has reached formal operation stage as identified by Piaget, has been challenged by several studies.

Researchers have also looked for alternative approaches to introducing algebra like pattern generalization (e.g. Rojano and Sutherland, 1991; van Reeuwijk and Wijers, 1997). It is a potentially rich way to introduce algebra and embeds algebraic symbolism in a meaningful context. Some studies, like the ones just mentioned, have shown the progress the students make and the understanding of algebra they display, while some others have pointed out the complexities involved in the task (Lee, 1996; Stacey and McGregor, 2001). The complexities arise chiefly because students fail to understand the requirements of the task; that they need to find a pattern which can be generalized, it needs to be denoted by a common rule in an algebraic form which can be inductively inferred from mechanisms rooted in counting. Also there are not sufficient empirical studies which show this approach to be better than others or the traditional approach (Stacey and MacGregor, 2001).

Numerous studies and experiments were carried out in the 1980s and 1990s on teaching students the meaning of the letter, the '=' sign, manipulating expressions and solving equations (e.g. Kieran, 1981; Booth, 1984; Chalough and Herscovics, 1988; Filloy and Rojano, 1989; Linchevski and Herscovics, 1996). In the process, many of them used concrete models to introduce the new symbols. The results are mixed with regard to efficacy in the teaching and learning of algebra. Some of the efforts, like trying to teach solving equations using the balance or area model, had limited success mainly due to the number of translations required between the abstract and the concrete world (Filloy and Rojano, 1989). Lins (2001), on the other hand, feels that these situations create a 'semantic field' which enables the students to develop symbols as well as attach meaning to them. A point which has been made against this approach is that it might be difficult for these students to move out of the concrete settings and generalize to other situations or even to treat the symbols on their own without considering the referent at each step (Balacheff, 2001). Also, the models and situations have a limited scope with respect to the concepts or procedures they can address and it is not possible to cover the domain of algebra using one such model/ situation.

Another approach to algebra is through the route of problem solving which mostly involves formulation of expressions and equations to mathematize real life or any other situation which is relevant/ meaningful, including situations within mathematics like justifying and proving. This is yet another fruitful activity in which students can be engaged but requires some basic knowledge of algebraic symbols to represent the situation – being aware that manipulating this representation would lead to a solution for the problem and correct manipulation followed by interpretation will complete the task. It is a matter of emphasis that one decides to teach the manipulation skills whenever required within the context or teach the basic skills before moving on to the more challenging situations as in the more traditional approach.

Recently the use of technology has opened many more avenues for approaching algebra, chiefly through the use of functions in the spreadsheet, and CAS environments and in other environments with multiple linked representations (Kaput, 1989). These have been especially useful in introducing algebra as functions, where different ways of representing functions can be highlighted as well as can be linked so that the changes in one form of representation is visible through changes in others. For example, changes made in the tables of values of a function are reflected in changes in the graphical representation and the symbolic expressions. These environments allow the students to focus on the more challenging aspect of problem solving – understanding the demands of the problem, correct representation and interpretation rather than manipulation of symbolic expressions. But for a successful use of these environments students need abilities to choose the correct keys to enter the data and process it further as per the requirements of the task, keep track of the solution process and judge the correctness of the solution. It is not clear that approaches using technologies such as CAS or multiple-linked systems can side-step this problem. Ball, Pierce and Stacey (2003) have used the concept of equivalent expressions to develop and monitor students' algebraic thinking in secondary grades equipped with CAS. These students need conceptual understanding and technical facility with algebra in order to see if the expression that they had entered or the one that CAS had simplified for them was the appropriate one and how it could be converted back to the standard form. Pierce and Stacey (2002, 2004) call it the 'algebraic insight', required to work with algebraic transformational activity. Algebraic insight is considered as being analogous to number sense in arithmetic and as a subset of 'symbol sense' as characterized by Arcavi (1994). Merely abandoning techniques of manipulating algebraic symbols does not lead to a better understanding of algebra as a basic minimum understanding of symbols is essential to be able to use these powerful systems (Kieran, 2004).

In spite of the possible advantages, this medium is not suitable for a country like India due to infrastructural limitations. Moreover, knowing the technique of solving the problem is equally important as solving the problem (Kieran, 2004) which need not necessarily be computer/ technological environment based as is contended by some researchers. She indicates research studies which highlight the recently developing consensus about the interrelationship between techniques and conceptual knowledge rather than opposition and that transformational activities can themselves form the ground for meaning making. One does not only understand the meaning of the symbols by attaching a referent to it from a context, but also by acting with and on the symbols. This action gives a sense of the operations and properties which can be associated with them. Moreover, symbols arise to express ideas concisely, but this act of expression itself using symbols broadens the idea, leading to new ideas and symbols (Arcavi, 1995). The objects in the algebraic world cannot exist without the processes on them; and thus the separation between algebra for problem solving and algebraic symbol manipulation is misplaced. Thus, it is once again essential to understand symbols flexibly: as processes and products which can be manipulated (Sfard, 1991; Dubinsky, 1991; Tall et al., 2000).

Finally, the choice of approaching algebra has to take into account students' ways of thinking, understanding and reasoning in the classroom as well as take into consideration the theoretical understanding of algebraic activity. Algebraic understanding according to Drouhard and Teppo (2004, p. 249) is "characterized by the way in which a student relates the sign and its meaning to a larger, connected set of relationships – that is, a way of representing, organizing, and acting mathematically within a particular syntactic structure". The fact that a student has made sense or meaning of a sign system/ symbols can be interpreted from the flexibility with which he/she can interpret it, use them in novel ways and explain procedures which go beyond the written steps.

1.2 *Rationale for the study*

The above description draws on nature of debates that have been prevalent in the area of algebra education. These debates, as we have seen, have largely dealt with clarifying issues related to the content of algebra, its aims and purpose and the possible ways of introducing it. Although many efforts have been made towards better teaching and learning of algebra in many other countries, there has not been much systematic research in mathematics education or even specific areas of mathematics in India. Like in any other country, students' performance in India is equally worrisome. In a recently conducted study in the top 200 schools in India by Educational Initiatives and Wipro (2006), it was found that only 23% of class 6 students could correctly solve an expression like10+30÷5-2 with 70% of the students writing the answer 6 for this expression. In an earlier study, 80% of grade 6 students in the sample drawn from two schools who had undergone instruction in algebra failed to add or subtract two algebraic expressions (Banerjee, 2000).

With the advent of the new National Curriculum Framework (NCF, 2005) and change in the curriculum together with the text books in the country, it is essential to undertake research in specific areas of mathematics and larger issues of approaches to teaching-learning of mathematics in the classroom, not only to fulfill the vision statement of the new curriculum but also to strengthen and broaden curricular choices. Also, as has been noted earlier, algebra being the gateway to much higher learning, failure in it leads to non-enrollment in many courses which use algebraic symbolism. Thus, it is important to understand the conceptual changes which the students experience while moving to the middle school, especially due to the introduction of algebra, and identify ways to address the problems which arise in the course of its introduction. In this thesis, an effort will be made to understand deeply the connection between arithmetic and algebra and propose a sequence for developing students' competence in algebra. This is important not only from the Indian point of view where there is an initiative to reform the curricula and include more students in the teaching-learning process, but also to throw light on the long standing debates in algebra education about issues of meaning, symbols, manipulation of symbols and purpose in algebra.

1.3 *Research questions*

The study aimed to address the following research questions:

- What kind of arithmetic understanding would help in learning symbolic algebra?
  - How should the teaching of arithmetic expressions be restructured to prepare for a transformational capability with algebraic expressions?
  - How effective is such a teaching learning sequence in understanding beginning syntactic algebra?
  - Which tasks of the ones identified are more effective in making the shift possible from arithmetic to symbolic algebra?
- Does understanding the syntax and symbols of algebra support students in understanding the purpose of algebra and in the application of algebra for generalizing and justifying?
- What meanings do students attach to letters, expressions and syntactic rules of transformations in this learning approach?
- How do procedural understanding and structure sense of expressions mutually support one another?

1.4 *Scope of the study*

The research being reported here was motivated by three issues: approach to algebra, major difficulties faced by students while learning algebra, and the need to systematically investigate the relation between arithmetic and algebra. The arithmetic-algebra sequence is followed in the traditional curriculum in India. From the viewpoint of acceptability to the teaching community, drastically changing the approach to algebra was thought not to be desirable. The focus was rather on an intervention in the classroom with the aim of making the transition from arithmetic to algebra smoother by exploiting the structure of arithmetic expressions. There is enough evidence in the literature to suggest that the link between arithmetic and algebra would be beneficial for the students to move to algebra. Some of these studies focused on notational/ representational similarity between the two domains and used that as the launching pad for algebra (e.g. Booth, 1984, Malara and Iaderosa, 1999). Some others used the computational properties of arithmetic expressions and generalized these properties of operations to algebra (e.g. Linchevski and Livneh, 1999; Liebenberg et al., 1998, 1999a, b; Malara and Iaderosa, 1999; Livneh and Linchevski, 2003). A few also used correct parsing followed by order of operations and exploration of properties of operations in order to make the transition to algebra (e.g. Thompson and Thompson, 1987).

Except for the study by Thompson and Thompson (1987) which actually trained students to perceive the structure of expressions and appreciate the constraint of certain transformations but in a limited situation, the other studies focused largely on computational features and their generalizations to make the transition to algebra. This always did not lead to the desired effect on the students and they still failed to see the equivalences in the transformation rules in arithmetic and algebra and continued to work on algebraic expressions similar to computational arithmetic without abstracting properties and constraints of operations. The present research study builds on these insights from the literature and proposes a way to deal with the arithmetic-algebra connection and tackle the errors due to faulty perception of structure of expressions, which have been found to be the main hurdle in understanding symbolic algebra.

The study capitalized on the intuitive understanding among students of numbers and operations developed during learning arithmetic and focused on the structural aspect of arithmetic expressions in order to develop a teaching-learning sequence for transiting to algebra. The objectives of the teaching-learning sequence were to strengthen both *procedural knowledge*, that is, the calculus of algebra – knowledge of rules, conventions and procedures for working on expressions, and *structure sense* – sense of the composition of the expression, how the components are related to the value of the expression and their relation among each other, for arithmetic and algebraic expressions. Meaning for symbols was created through two broad sets of activities: working with syntactic transformations and working with contexts that lend purpose to algebra. The study engaged in analyzing students' responses to the various tasks, and identifying the nature of the support (concepts, tasks) that is required to make the transition. This fed back into the development of the teaching module, thereby evolving and clarifying the approach that facilitates students in making the transition.

The research study being reported here is a design experiment and has been carried out with grade 6 students from two schools in Mumbai. Grade 6 is the level when students first learn algebra and there was no effort to lower the stage at which algebra can be introduced. Five trials were conducted with different groups of students between the years 2003-2005 to evolve the teaching learning sequence and much time was spent in developing understanding of syntactic transformations in the sequence. The first two trials were exploratory in nature and the last three form the main study. Thirty one students in two groups were followed as part of the main study for a year. The study began with the aim of exploiting the structure of arithmetic expressions for teaching algebra but the exact nature of arithmetic and the tasks that would enable students to perceive the structure and make the connection with algebra took time to develop. The role of procedures, rules, concepts in the whole sequence and in attempting various tasks had also to be clarified. The study did not aim to prove the efficacy of the instructional approach being discussed with respect to traditional or any other approach. It aimed instead, at an internal understanding of its effectiveness by exploring the changes in students' understanding and thinking processes as they developed new concepts and tools through interaction with the instructional sequence. The instructional sequence was repeatedly carried out with groups of students with the aim of evolving the sequence till some evidence was observed where students indeed saw the connection between arithmetic and algebra.

1.4.1 The content of algebra and the approach in the study

The approach that was developed keeping the aims in view, tried to deal with the semantic-syntactic problem which algebra brings with it. The approach adopted in the study could be characterized as 'generalized arithmetic' which treats algebra as encoding general rules of operations in arithmetic like commutativity, associativity, distributivity, thereby focusing on the structural aspect of the number system (Wagner and Kieran, 1989; Kaput, 1995). This was complemented using tasks which took a more comprehensive view towards generalization – exploring and finding relations among numbers/ quantities, sequence of operations and shapes in patterns and justifying and proving some of the patterns, where algebra was used as a tool. To put it in Kieran's (2004) words, the students were engaged first in *transformational activities* (focusing on the syntactic aspects like evaluating, simplifying, identifying equality and reasoning about these). The idea was to allow students to engage with understanding and articulating relationships, properties and processes and represent them in the context of arithmetic (which is largely computational in the traditional approach) and then simultaneously learn to use these properties in the context of algebra. The earlier studies (e.g. Kuchemann, 1981; Booth, 1984) showed lack of these ideas to be a major stumbling block for students for moving to algebra.

This was followed by *global/ meta-level activities* where algebra was used as a tool for problem solving, generalizing, justifying, proving and predicting (Kieran, 2004). These contexts are quite challenging and have demands more than knowing the meaning of the letter and its use in representation such as: (i) knowing the requirements of the task, (ii) deciding a plan of action based on this knowledge, (iii) choosing an appropriate representation, (iv) knowledge of transformation of expression, (v) goal directed manipulation, and (vi) interpretation of the result. It would have been difficult to work on these situations without any knowledge of manipulation of algebraic expressions in a paper and pencil situation. The interest was rather on exploring the connection the students make between the transformational tasks and the global/meta-level tasks, the students having undergone a specific set of instructions for the transformational tasks.

These two kinds of tasks are complementary and inculcate the understanding of algebraic symbols and their use in solving problems. It is only the algebraic way of solving problems which can ascertain a solution to be correct, complete and gives a deeper comprehension of the questions together with understanding of their solutions (Dettori et al., 2001). For a complete sense of algebra one would need to build an understanding of both the syntactic (based on structure of expressions/ equations and rules which define the nature of possible transformations) and the semantic (based on meaning of the letter/ expression/ equation) aspects of algebra. The syntactic aspect feeds into the more challenging problem solving part and while one tries to work with algebraic expressions and equations, one also simultaneously unravels many of the semantic features in the new ways of interpreting the same representation/ expression which vary in ‘intention’ but not in ‘denotation’ (see Arzarrello et al., 2001).

1.4.2 Overview of the thesis

In the context of this design experiment, data from various sources will be used to answer the research questions posed. In the next chapter (Chapter 2), review of literature will be presented to set the background for the study as well as to make an argument for the need for such a study by citing relevant research, both theoretical and empirical, which support (or oppose) the framework/ approach taken in the present study. The major issues identified in the teaching and learning of algebra will be discussed and some efforts which have been already carried out to handle those will be explained. Having set the stage for the present study, which also is an intervention to deal with the difficulties students face in algebra, the theoretical background for the teaching-learning approach will be presented in Chapter 3. There are two interconnected parts of the study: developing an instructional sequence for making the transition from arithmetic to algebra, and characterizing students’ change and development within the learning sequence. The designing of the instructional sequence followed some general principles like using students’ intuition and formalizing intuitions through the use of symbols. Further, emphasis was put on articulation and reasoning about mathematical situations using either language or symbols. The content for the sequence exploited the structure of the arithmetic expressions using two ‘bridge’ concepts: terms and equality. The connections were explicitly demonstrated by allowing for generalizations of procedures and rules from arithmetic context to algebra as well as explicitly reasoning with the syntactic transformation rules. Lastly, the teaching learning sequence included some activities to give a sense of meaning and purpose of algebra to the students. These contexts were of generalizing patterns and relationships and justifying and proving them.

Chapter 4 will deal with the methodology adopted for the study. The study is a design experiment and was carried over two years and five trials. The chapter will describe the implementation of the study, the sample, the tools used to collect data and a plan for analyzing the data. Chapter 5 will describe the evolution of the teaching learning sequence over the five trials. In the process, the nature of arithmetic knowledge that will be required to make the transition to symbolic algebra will be clarified as well as the nature of tasks that help in this transition. In the next three chapters (6, 7 and 8) analysis of the students' responses to the pre and the post tests to various questions will be taken up. This data will be substantiated, clarified and extended by the use of individual interviews with students, classroom discussions, daily logs of teachers and practice exercises of the students. The purpose of the analysis is to see the extent to which the students learnt the concepts, procedures and the rules taught in the study and how well they applied these ideas in situations of reasoning. The effect of students' understanding of symbols and syntactic transformations in contexts where they needed to apply these (purpose of algebra) will also be explored. The analysis of the responses, is expected to lead to an understanding of the relation between procedure and structure sense of expressions as well as interpret their understanding of letter, expressions and transformations on them. Conclusion drawn from this study will be presented in the last chapter (Chapter 9) where the research questions will be revisited and answers presented based on the analysis.

1.4.3 Limitations

The students who participated in the study came from two different schools which catered to low and middle income groups. One was a Marathi medium school and the other was an English medium school and both followed the state prescribed curriculum. But there were also differences in the groups of students with respect to their year end performance, their language capabilities, medium of instruction and the school setting but the differences were not systematically studied. Identifying the differences between the groups was not the aim of the study. Although factors, like the ones mentioned above, influencing learning in particular ways are important issues and must be explored, keeping in view the composition of the students in a classroom in a country like India with multiplicity of languages and socio-economic backgrounds, these were beyond the scope of the present study. Two different groups were taken to add insight about the different ways of responding to the same instructional sequence.

The study did not cover the whole of the algebra portion as it appears in the state prescribed curriculum. The focus was on the transition from arithmetic to algebra with a large amount of time spent in understanding arithmetic and concepts which can act as bridge between the two domains of mathematics. The discussion largely revolved around expressions, both arithmetic and algebraic, and various aspects of it like computation/ evaluation, equality, etc., which were considered to be the building blocks for all further learning. The algebraic expressions were all linear with a single variable and matched in structure the arithmetic expressions they were working with. More variables and complex syntax were avoided in this study, also because the students belonged to grade 6. To develop this framework into a full module for algebra learning, it would need to be elaborated to expressions with more than one variable and also expressions with higher degree. Expressions with the division operation in arithmetic, or rational coefficients or rational expressions were not dealt with as they involve more complicated notation requiring an understanding of fractions which was again beyond the scope of the study. Simple linear equations also need to be systematically dealt with and included in the teaching sequence.

The study was approached from the generalized arithmetic view to keep it close to the existing curricular sequence and its utility in the Indian context. Though it has its merits, as described earlier, other approaches, including the technology aided methods, were not explored. This in itself has its limitations. Some understanding of aspects of algebra, like the meaning of letter as a variable and its parameter use can also be understood in a function approach, especially with the use of spreadsheets, graphing calculators and linked representations. Further, the choice of the research design allowed an insider's view of the development process, making available rich data on students' thinking, reasoning and difficulties in understanding algebra while making the transition from arithmetic. But, the study design invovlign iterations of teaching did not allow the separation of the effect of refined teaching and more teaching. The study also did not seek to arrive at an independent assessment of the efficacy and efficiency of the teaching approach/method vis-à-vis any other method.

Chapter 2: Reviewing the arithmetic-algebra connection

2.0 *Introduction: Algebra as a domain of mathematics*

Algebra is an important domain of mathematics and the past three or four decades have seen a rise in the research studies in the area of teaching and learning of algebra. Researchers have elucidated the nature of algebra, the differences between arithmetic and algebra, the difficulties which students face and the reasons which make algebra learning difficult for students. Besides the exploratory studies which focused largely on students' understanding of algebra (spontaneous or after instruction), researchers have also conducted numerous teaching studies in an effort to make students understand the ideas of algebra and make the algebraic activity meaningful for them. Through this review of literature an effort will be made to situate the present study in the context of previous research, which has raised issues about the teaching and learning algebra, explored students' difficulties, hypothesized about causes of the difficulties in learning and has suggested possible measures aimed at overcoming these difficulties.

Researchers have suggested a variety of ways of approaching algebra, which entail certain possibilities and emphases on certain aspects. They have arisen as a result of efforts to find solutions to the many problems with the teaching and learning of algebra. The various approaches reflect differences in the conception of algebra leading to differences in the approach to instructional design. These approaches to algebra are briefly discussed below.

2.1 *Approaches to algebra*

Algebra as generalized arithmetic: Mason (1985), Usiskin (1988), Bell (1995) and Kaput (1998) have proposed this as one of the approaches to introduce algebra to students. Wagner (1989) suggested that algebra is generalized arithmetic in two senses: *generalized* arithmetic and generalized *arithmetic*. The first aspect focuses on the structural aspects of the number system - recognizing properties of operations like commutativity, associativity or distributivity and knowing when they can be applied. The second focuses on the numerical referent as the connection between arithmetic and algebra. For example, for statements like x+5=12 or $x+y=12$, the values of x and y can be found by subsequently replacing one or both of them by numbers. Kaput (1995) supplements this characterization, by describing generalized arithmetic as an approach that treats algebra as a 'language that encodes the general rules of arithmetic, particularly rules concerning the operations'. It involves generalizing in the context of arithmetic, often beginning in the system of integers, understanding their properties and operations and letting the mathematical structures play the core constraining role. Comparing 7+4 with 4+7 or 7-4 with 4-7, 3×(4+7) with 3×4+3×7 and 3×4+7 , 13-(7-5) and (13-7)-5 would be good examples which can lead to making explicit the general rules of order of operations. These rules, which also govern manipulation in algebra, have to do with when changing the order of numbers while operating makes a difference and when it does not. Mason (1985) cautioned that the success of generalized arithmetic lies in seeing algebra as an expression of generality (Wagner's first characterization mentioned above) and not just an extension of arithmetic on number symbols to arithmetic on letters. To develop awareness of generality, it is essential to see the particular in the general and the general in the particular (Mason, 1996). The expectation is to see equivalences in expressions based on their structure, that is, to know 3+5 is equal to 5+3 not only because both are 8 but also because the addition operation is commutative. All the above descriptions of generalized arithmetic focus on the structural properties of the number system and not simply replacing the letter with the number. Keeping with this description, one of the ways in which generalized arithmetic will be used in this thesis is 'algebra as encoding general properties of arithmetic operations'.

The key tasks in this approach are of translation and generalization with the letter as pattern generalizer (Usiskin, 1988). But the act of generalization is not restricted to the domain of arithmetic operations and requires engaging in an activity/ culture of generalization. The crucial aspect of learning to generalize is the "process of exploring a given situation for patterns and relationships, organizing the data systematically, recognizing the relations and expressing them verbally and symbolically, and seeking explanation and appropriate kinds of justification or proof according to level" (Bell, 1995, p. 50). In situations such as the above, it is meaningful to introduce the letter '*x*' without much cognitive hurdle as the purpose of using the letter is evident to the students. Moreover, contexts which lead to pattern generalization can be many and varied like, the arithmetic operations, pattern generalization based on calendar patterns, sequences of numbers, shapes, etc. (Bell, 1996; Arcavi, 1994). This is another way to look at generalized arithmetic which will play a role in this thesis.

Kaput broadened the discussion of generalized arithmetic and called it 'algebra for generalizing and formalizing patterns and constraints' which includes generalized arithmetic and generalized quantitative reasoning (Kaput, 1998). In contrast to generalized arithmetic, this second dimension of quantitative reasoning involves reasoning in contexts which are outside mathematics but can be mathematized. Thus, the generalized arithmetic approach allows for firstly, a gradual introduction to letter through generalization of arithmetic ('metacognitive teaching', Malara and Iaderosa, 1999), where the focus is on structure of operations and numbers (exploring both possibilities and constraints on operations) as well as using these while operating on variables and expressions. Secondly, it deals with broader pattern generalization activities where the letter is largely a generalized number. Many examples of studies dealing with this approach, tried with students in various grades, will be discussed later to show the understanding and power it can give to the students while working with numbers and engaging in generalizing activities. The extra emphasis being laid on this approach is mainly because the present study also takes such a view and is one of the main guiding principles of the study. Although many researchers have used this conceptualization of algebra in their exploratory studies of students' understanding as well as a teaching tool, some have also expressed their reservations about this approach (e.g. Mason, 1985; Lee and Wheeler, 1989; Linchevski and Livneh, 1999) which will also be discussed later.

Algebra as solving and forming equations (Usiskin, 1988; Bell, 1995) is another approach to introduce algebra where the variable is treated as an unknown. In contrast to an algebraic expression where the letter can take any value, the equation puts a constraint on the letter (Mason, 1985). Mental ways of resolving simple equations and ability to symbolically represent equation/s and manipulate them to solve for the variables/ unknowns are important in this approach (Bell, 1995, 1996).

'Syntactically guided manipulations' (Kaput, 1998; Mason, 1985) is a broader scheme to fit in all kinds of syntactic manipulations required to work with expressions or equations. Although in this case, the focus is on formal algebraic symbols and its syntax, Kaput (1998) believes that it is possible to act on these formalisms semantically. Mason (1985) detailed out the aspects which need careful attention while following such an approach:

- Discovering the possibility of manipulating expressions as a result of encountering different expressions for the same thing.
- Becoming aware that an expression is an entity in its own right.
- Realizing that an algebraic expression is something which has been built up and which can be 'unbuilt' or stripped down again.
- Deciding how to manipulate expressions and to what end.

It is clear from the above that the syntactic manipulation is guided by semantic aspects dealing with the essence and purpose of algebra: possibility of manipulating the expression to conclude a result and anticipating the goal in advance to decide a scheme of manipulation. This point will be revisited again.

Algebra as working with functions: This approach has the possibility to give insight into practical situations with the variables acting as arguments or parameters in tasks related to relations and graphing (Usiskin, 1988; Bell, 1995, 1996). The tasks require describing and writing a rule for the function, extrapolating, intrapolating, comparing graphs and studying the nature and properties of graphs of various kinds of functions. Kaput (1998) added that the ideas of correspondence and variation of quantities are very powerful and cut across and unify many different kinds of common mathematical experiences which can be introduced in elementary classrooms, like ideas about ratio and proportion.

Algebra as study of structures: For Usiskin (1988), this approach meant arbitrary marks on paper and the key task is to manipulate and justify. This apparently limited characterization is seen in a modified form in Kaput (1998) where he describes this strand as 'acts of generalizations and abstraction based on computations, where *the structure of the computation rather than its result* becomes the focus of attention, giving rise to abstract structures'. According to him, these structures have three purposes, (1) to enrich understanding of the systems that they are abstracted from, (2) to provide intrinsically useful structures for computations freed of the particulars that they once were tied to, and (3) to provide the base for yet higher levels of abstraction and formalization.

The fact that the above mentioned approaches to algebra are not mutually exclusive can be appreciated and as Mason (1985) says they are all 'routes to roots of algebra'. They should permeate the classroom and as the occasion arises the opportunities to bring in other notions of algebra should be utilized to the fullest extent. It will be difficult to conceptualize activities which will fall solely in one approach and not include aspects in other approaches, as that will make the whole effort contrived. One of the alternatives put forth by Kieran (2004), is to think of algebra not as ways of approaching it but as an activity which allows one to mix all the approaches and look at what can be done in and with algebra. Kieran characterized algebraic activity as consisting of generational activities (e.g. generating equations, expressing generalities from geometric patterns, expressions of the rules governing numerical relationships), transformational activities (e.g. collecting like terms, factoring, expanding, solving equations, simplifying expressions) and global/ meta-level activity (algebra used as a tool for problem solving, modeling, noticing structure, generalizing, justifying, proving and predicting). This last set of activities provides purpose to algebra and uses skills and abilities which are developed as part of the first two sets of activities. The traditional curriculum has emphasized 'transformational activities' at the cost of 'generational' and 'global/ meta-level' activities. It does not mean that transformational activities do not have any role or that they are meaningless. In fact in this thesis, the effort will be to show the meanings that can be generated within the transformational activities and ways, if any, in which they can be used in the global/ meta-level activities.

A lot of advancement has taken place with the introduction of technology, especially with the use of spreadsheets and CAS, in the teaching and learning of algebra. Introducing algebra using the path of functions and modeling is not difficult when sufficient support from technology is available. Some discussion on the use of technology was taken up in the last chapter and other detailed research review and examples of studies based on these approaches can be found in Nemirovsky (1996), Heid (1996), Kieran et al. (1996), Kieran and Yerushalmy (2004) and Thomas, Monaghan and Pierce (2004). Although these approaches are very fruitful and have given a new direction and meaning to the whole activity of algebra in the secondary and the higher secondary level, the primary setting for algebraic teaching and learning in the Indian context is the classroom without significant access to computers. Hence the study did not include these aspects of algebra teaching and learning. Instead the focus of the review in the sections below will be on studies which have used generalization approaches and a combination of semantic-syntactic approaches as a framework for their study. In the process, I will try to highlight the studies carried out either for the purposes of exploring students' understanding of algebra or using generalized arithmetic as a principle for teaching algebra.

2.2 *Exploring students' understanding of symbols in algebra*

2.2.1 Students' understanding of the letter

The earliest of the studies which had a major impact in the field of algebra education research were the CSMS and SESM studies. In the CSMS (Concepts in secondary mathematics and science) study, Kuchemann (1981) investigated the issue of structural complexity of questions and the meaning that students (13-15 year olds) associate with the letter. He found that students' understanding of the letter determines the complexity of the item and therefore facility with the item. One of the important outcomes of the study was the classification of students' interpretation and use of the letter symbol in various contexts. He identified six different ways in which children interpreted and used the letter, namely, letter evaluated, letter not used, letter used as an object, letter used as a specific unknown, letter used as a generalized number and letter used as a variable. The last three categories are higher levels of understanding of the letter than the first three which indicate no real understanding of algebra. He accordingly categorized students in various levels of understanding as below late concrete, late-concrete, early-formal and late-formal, which corresponded to the Piagetian developmental stages.

It is clear that if algebra is taken as recording the rules of arithmetic in a general fashion using letters, then the non-recognition of these rules and structures in arithmetic would definitely be a hindrance in the learning of algebra. Taking this as the basis for further work, Booth (1984) in the SESM study (Strategies and errors in secondary mathematics project) investigated the causes of the errors already described by the CSMS study through individual interviews of students and conducted small-scale teaching experiments based on the analysis leading to the development of teaching modules for trial with whole classes. In the study with 13-15 year olds, she found that the errors are due to students' poor understanding/ interpretation of the letter, alternative conceptions about the appropriate method, the non-appreciation of the need to symbolize and formalize the methods, relying on primitive methods and also due to confusion over the algebraic conventions and notations. The main task in the small-scale teaching experiment carried out with one group of six 13-year olds and two groups of five 14-year olds was to write instructions for a model computer so that it can solve the problem or a class of problems and write the print out or answer in each case (e.g. subtract 9 from 14, add 3 to any number, area of any rectangle). The teaching experiments revealed that the 'cognitive-readiness factor' could play an important role in the learning of algebra proficiently. It also showed that some of the problems in understanding the notion of letter and notations and conventions in algebra (like letter as a number or non-closure of algebraic expressions[1]) are caused due to poor teaching-learning material and can be remedied by "good teaching" whereas the errors relating to use and need for brackets proved to be more resilient. This study again highlighted the lack of awareness of structure of arithmetic expressions among students obstructing the possibility of generalizing concepts and methods in the context of algebra.

[1]In the context of algebra, it is essential to appreciate that all expressions will not lead to a closed answer. For example, $2x+3$ cannot be written as $5x$, that is, a single number, whereas in arithmetic an expression of the type $2\times5+3$ will lead to an answer denoted by a single number. Collis (1974) (as cited in Booth, 1988) called it as the 'Acceptance of Lack of Closure (ALC)'.

These studies emphasized the importance of building and appreciating structure sense in arithmetic. They also laid the ground for many future research studies which attempted to unravel students' understanding of letter and the reasons for the various difficulties students face in learning algebra. At least two more studies by Filloy and Rojano (1989) and Herscovics and Linchevski (1994) pointed towards cognitive factors as the reason for students' difficulties with algebra.

Filloy and Rojano (1989) termed students' inability to operate on or with the unknown when present on both sides of the '=' sign as a 'didactic cut'. They explained that the solution to the equation of the kind Ax+B=C can be arrived at by simply inverting the operations, which is an intuitive process, whereas to resolve the equation of the type Ax+B=Cx+D, students would need to go beyond the inversion and would need some elements of algebraic syntax. This requires the breaking of the barriers of the arithmetic domain and seeing the two sides of the '=' sign as expressions with a relation. An action which gives meaning to the sign of equality in such a situation is finding the number which makes the values on both sides equal. In the small teaching experiment they carried out with students (12-13 year olds) to teach linear equations, it was found that both concrete models (geometric area model) and concrete contexts (weighing balance) for understanding the meaning of equations are fraught with difficulties. Such models also do not help in learning the syntactic aspects of resolving the equation. They themselves observed that the use of such models and concrete situations were not very helpful, as the students first required translation by which abstract objects and operations are given meaning and later separation or detachment from the concrete model. They concluded that syntactic errors and the resolution of such complex problems cannot be spontaneously resolved by the students using their intuitive skills.

Herscovics and Linchevski (1994) and Linchevski and Herscovics (1996) indicated a 'cognitive gap' between arithmetic and algebra which could be characterized by students' inability to spontaneously operate on or with the unknown. Their results of interviews with seventh grade students did not confirm Filloy and Rojano's (1989) findings, in that most of the students were able to find a solution to an equation with a letter on both sides of the '=' sign but the solution methods were essentially those which were used for solving an equation with two occurrences of the letter on the same side, that is, by systematic substitution. Only in the case of equations with a single occurrence of the letter did the students succeed in using an inversion operation to find the solution and the substitution strategy was rarely used. This meant that the students were not able to operate on the letter. The researchers elaborated that it is not only essential to think of the letter as a number but also endow it with the operational properties of the number which render operations on the letter possible, again giving the generalized arithmetic flavor to algebra. The informal methods of solving such equations had an upper limit. In the individualized teaching experiment which followed, the researchers tried to bridge this gap between arithmetic and algebra by allowing the students to use their prior understanding of number operations. They were introduced to the notion of equivalent equations by making it possible for them to work spontaneously on and with the unknown by grouping and cancelling like terms and numerical terms and using inversion operations in equations like $23+n+18=44+16$, $n+n=178$, $3n+5n=136$, $17n+12n+36=210$ and $8n+11=5n+50$.

MacGregor and Stacey (1997) took the issue of finding the cause of students' misunderstandings about letter further and opined that lack of cognitive readiness can only explain their inability to work on certain algebraic problems but cannot explain misinterpretations of the letter. They gave evidence to support their argument through a large scale study conducted over several hundreds of students in grades 7 to 10 through the use of written tests and interviews in the age group of 11-15 year olds and traced the development of some of the students over two years. They found that many of the misinterpretations of letter are due to intuitive and pragmatic reasoning about an unfamiliar notation system, drawing on analogies with familiar symbol systems, interference from new learning and other subject areas that use similar symbols and from misleading teaching materials.

The explanations based on cognitive deficiencies/ factors suggest limitations intrinsic to the learner which create initial learning hurdles, whereas the last explanation suggests that it is due to general sense making phenomena and interference/ effects from various kinds of teaching and learning. Some more reasons have been put forth for the difficulties in learning the use and meaning of letters in algebra which are not based on cognitive factors, rather they are inherent in the nature of the symbols in algebra. Schoenfeld and Arcavi (1999) explained that the meaning of the letter is not determined by formal rules but is determined by the context of application of the letter, which makes it hard for the students to understand it. It encompasses both the usages: as a tool to express generality (for example, an even number is a multiple of two, so can be expressed as $2\times y$) and the dynamic aspect of continuous change in value (e.g. understanding an algebraic expression like $2\times a$-3 or a linear function $y = x$-5, where a and x can take any value in the domain of real numbers, and y will change depending on the value of x). Also a complete appreciation of the letter involves the knowledge that numerals label specific fixed elements of the set whereas the letter labels random, variable elements (Wagner and Parker, 1999). Students often show misconception regarding the use of the letter by thinking that it is one particular number.

2.2.2 Students' conception of '=' sign

Besides the letter symbol which, as most researchers agree, impedes students' development in the world of algebra, the '=' symbol has been found to be another crucial symbol connecting the passage from the arithmetic domain to the algebra domain. Kieran (1981, 1989) and Filloy and Rojano (1989) pointed out the importance of moving away from the arithmetical notion of '=' sign which signals following a sequence of instructions, to a notion which states a relation of equivalence between two sides of the '=' sign. Herscovics (1989), in his review of cognitive obstacles faced by students, explained that students often do not consider an equation as an expression of equivalence but as a description of relative size or the '=' sign as a symbol of association or comparison. For example, in the 'student-professor' problem (Clement et al., 1981 as cited in Herscovics, 1989), where there are six times as many students (S) as professors (P), students have been found to write the equation as 6S=P. However, their diagrammatic representation shows that they do understand the relative sizes of the two sets, six circles for the students' population compared to one for the professor. The 6 adjacent to S is like an adjective qualifying the S, instead of signifying the magnitude (ibid.). It is well known that the '=' symbol is a key to move from the procedural computational world of arithmetic to the structural understanding of 'equivalence' essential to begin learning algebra (Booth, 1988).

Kieran (1981) cites experiments carried out by various researchers to make students understand the meaning of '=' sign and describes her own effort in the area. In an experiment with six students (12-14 years), they were gradually moved away from the idea of answer on the right side to see arithmetic identities with multiple operations on both sides of '=' sign, which was a relational conception. This provided the foundation for later introduction of non-trivial algebraic equations. One of the numbers in the arithmetic identity was first hidden with a finger, then with a box and finally replaced by a letter. Occurrence of the letter twice in the equation could be easily understood in this scheme but this was not sufficient to understand the equation solving process which required an understanding of equivalent equations. The semantic understanding of '=' sign or equations, as two expressions having the same value, did not automatically lead to a syntactic understanding of the procedure of solving equations. In another review article (Kieran, 1989), she observed that students solving equations with multiple occurrence of the letter by trial-and-error, repeatedly substituting numbers in place of the letter (considered a semantic solution), were more aware of the structure of equations and relations between operations and made less errors (switching-addends: x+a=b considered equivalent to x=b+a and redistribution: x+a=b considered equivalent to x+a-c=b+c) than those who used the 'change-side-change-sign' method.

Many of the recent researches (e.g. Carpenter and Levi, 2000; Carpenter and Franke, 2001; Stephens, 2004a) focus on a flexible and relational understanding of the '=' sign from the very beginning (grades 1 and 2 onwards) because of its relevance and importance in the structural understanding of arithmetic and algebra. Nearly all of algebra learning exploits the idea of equivalence – first, equivalence of expressions: the fact that, despite changes in surface features, the application of valid transformations on an expression ensures the equality of all the intermediate expressions in a simplification process. And secondly, equivalence of equations, which keeps the solution of the equation the same despite changes in the form of the expressions on both sides of the '=' sign (Kieran, 1989).

2.2.3 Notational and conventional hurdles

The difficulties caused by the letter symbol and the '=' sign are not the only troubles in learning algebra. Algebra has been considered to be important due to its ability to express general rules and methods for manipulating the referent free symbols using well defined rules to lead to valid conclusions. Its notation system is precise, requiring correct recording of statements. Although in arithmetic one can get away with a wrong recording and a correct answer, but the consequences of this in algebra are critical (Booth, 1988). For example, in arithmetic one can afford to represent the area of a rectangle with dimensions (3+4) units and 2 units as 3+4×2 and calculate it as 7×2 but in the case of algebra representing the area of a rectangle with dimensions (a+2) units and 3 units as a+2×3 will be incorrect. On the other hand, its syntax is ambiguous leading to many difficulties for students in manipulating the expressions. Students have to be aware that the addition/ subtraction symbol can stand for both the process of computation and the answer to a question. For example, the expression x-4 indicates the performance of an operation, that of subtracting 4 from x, but at the same time the expression itself is the answer, when no value of x is provided. Similarly, the '=' sign can signal the answer to a problem as well as show equality/ equivalence between two expressions. Students must understand that the letter stands for a number, although some of their prior experience had exposed them to other uses of the letter, like shorthand for units of measurement (Booth, 1988). Some ambiguity is caused by following different convention in arithmetic and algebra. In arithmetic, concatenation means addition (number notation in decimal system or fraction notation) but in algebra concatenation implies multiplication. This is manifested in errors like evaluating $3a$ for a=2 as 32. Many of these problems arise due to excessive emphasis on finding a numerical answer to arithmetic problems which reinforces the operational rather than relational ideas and also leaves the students resistant to unclosed expressions as answers (Wagner and Kieran, 1989).

2.3 *Syntax and semantics of algebraic symbols*

It is clear from the discussion in the last section that symbols used in algebra contribute greatly to the difficulties students face in algebra. The studies indicated students' lack of structure sense, alternative/ wrong meanings for the letter attributed by the students and coping with correct but varied meanings of the letter in different contexts. Algebra encodes general rules and procedures of arithmetic, but has many distinctions with it like notations, conventions and shift in emphasis from the procedural in arithmetic to the relational in algebra.

2.3.1 The syntactic semantic divide

The discussions in the last section also indicate that having a sense of the algebraic symbols, like the '=' sign, letter/ variable is essential and would help students better understand algebraic expressions and equations, but this does not guarantee syntactic ability. Knowledge of the meaning of symbols is not enough as it does not automatically lead to the ability to work on them. True algebraic knowledge requires one to understand a situation, represent that using symbols and transform it till it matches the goal and interpret it back to the situation. So, embedding algebra in rich settings to lead to algebraic thinking and to create meaning for the symbols used in algebra cannot by itself be sufficient to use algebra as a tool for the problem solving. Also, learning to work on the symbols itself is a context which can create meaning for the symbols (Kieran, 2004). For example, identifying whether two expressions n^2+2n+1 and $(n+1)^2$ are equivalent or whether $23+45\times14-27$ is same as $14\times45+23-27$. Knowing the syntax and the properties of operations can guard the students against transformations which are not permissible, but tempting (Carry, Lewis and Bernard (1980) quoted in Bell and Malone (1993)). In the above example, a student who knows the syntax well would avoid adding 23 and 45 while evaluating the expression or would not consider $45+23\times14-27$ to be equal to the given expression. The reason for arbitrariness or meaninglessness which the students experience during their exposure to algebra is not solely due to working on rule based transformation tasks. It is also due to the lack of emphasis on structure of expressions, making appropriate links with properties of number systems and explanations for the rules, like distributivity and associativity (Kirshner, 2001). In the paragraphs below, research on students' syntactic awareness and ability and the connections of algebra with arithmetic will be discussed.

2.3.2 Algebra as a language

MacGregor and Price (1999) considered algebra as a language. They hypothesized and demonstrated a relation between students' algebra learning and meta-linguistic awareness of symbols and syntax in non-algebraic contexts. Similar to language proficiency, proficiency in algebraic symbols requires (i) knowing that numerals, letters and other mathematical signs can be treated as symbols detached from real-world referents, groups of symbols can be used as basic meaning-units like $x+2$, that is, having ability to reflect and analyze structural and functional features, (ii) recognition of well-formedness in algebraic expressions, e.g. $2x=10 \Rightarrow x=5$, but not $2x=10=5$, and making judgments about how syntactic structure controls both meaning and inference making, e.g. $a-b=x$ does not imply $b-a=x$ and (iii) mastering ambiguity: an expression can have more than one interpretation, depending on how structural relationships or referential terms are interpreted, e.g. knowing when brackets are required for ordering operations and awareness of potential for mistranslating relational statements to equations. MacGregor and Price argued that students, who are beginning algebra at the age of 11 or 12 years, possess symbol awareness as they know that words are arbitrary names, which can be represented as groups of symbols and they can use words in various ways (games, jokes etc.). What they lack is the awareness of syntax, which takes time to learn. While reading, context is sufficient to make sense of the situation. This is not so in the case of algebra where the contextual cues are not sufficient for interpreting algebraic notation as one has to attend to the order and arrangement of the symbols. Thus this aspect needs special attention in the teaching and learning of algebra.

2.3.3 Differences between arithmetic and algebra

For learning algebra, not only is it necessary to understand the language of algebra and know its differences with natural language, but also to shift attention from purely numerical solutions to methods and processes of representing and solving (Kieran, 1999). The prior experience of students in the field of arithmetic has been largely one of computation, where first they carried out the four operations on two numbers and later evaluated expressions with multiple operations following a sequential set of instructions, assuming that the written sequence of operations determines the order of computation (binary operation between each successive pair of numbers), leading them to a closed answer in the end. There is a radical change in the case of algebra where students cannot work sequentially; they have to look for like terms which can be combined and cannot expect to get a closed answer in the end (Booth, 1988). The students need to accept the fact that the unclosed expression of the form $2a+3b$ is a legitimate solution and stands for both the process of solution as well as the solution itself. Stacey and MacGregor (1994) noted that students conjoin to represent addition and sometimes even subtraction and multiplication because they want closure of answers. Davis (1975) while reporting the work of a seventh grade gifted student solving a linear equation ($\frac{3}{x} = \frac{6}{3x+1}$) elaborated the nature of cognitive demands such tasks pose (beyond knowing the mathematical pre-requisites) on students, like understanding the different meanings of '=' sign, choosing the one which fits the situation and understanding that the symbols stand for both the result as well as the process (in this case seeing x and $3x+1$ as both valid entities which can be multiplied and divided). The progress of the student is stalled due to his/ her inability to meet the above demands. The students do not see any problem in violating the structure while being creative in finding solutions and consider everything as 'rules of mathematics', without understanding the goal of the task.

This ability to view the unclosed expression as representing both the instruction to compute as well as the result of the computation has been called the process-product duality by Sfard (1991). Similarly, Tall et al. (2000) talks about flexible interpretation of the symbol as both process and object, viewing it as a 'procept'. Mason (1996) reiterated that 'algebraic awareness' consists of "necessary shifts of attention, which make it possible to be flexible in seeing written symbols: as expressions and as value; as object and as process" (p. 74). The students' inability to see the symbol as a 'procept' or both process and object leads to their non-acceptance of closure in solutions. Contrary to the understanding that the obstacles are in the mathematical systems of thinking and symbolizing, Stacey and MacGregor (1994) pointed out the interference effect from other notational systems which use algebra-like symbolism as well as students' own disabilities to distinguish between notations for repeated addition, multiplication and repeated multiplication, as the explanation for such behaviour. Other explanations include a loss of the numerical referent in the expression containing letters and non-appreciation of the relation of equivalence among the expressions on the left and right side of the '=' sign (Booth, 1988). In the case of such a loss of reference or unawareness, the expression does not stand for anything for the students which they can meaningfully manipulate using the rules of transformations they already know. The rules then tend to look arbitrary and meaningless.

Arzarello et al. (2001) argued that students' algebraic difficulties are due to their failure to "master the invariance of denotation with respect to the sense … as if there were a one-one correspondence between sense, denotation and formal expression, so that identifying all three, pupils remain with a trivial denotation: a symbolic expression denotes itself as a collection of signs" (p. 65). According to Arzarello et al., while an arithmetic expression has a fixed sense and denotation; an algebraic expression can be interpreted in ways that vary in sense. For example, an arithmetic expression 4+7 means the number which is seven more than 4 and stands for the number 11. But an algebraic expression $n(n+1)$ may mean the product of two consecutive numbers or the area of a rectangle with dimensions n and $n+1$. The denoted set in both cases is the set {0, 2, 6, 12, …} for $n \in \mathrm{N}$. Also, different ways of interpreting algebraic expressions resulting in a shift in sense can be achieved by one of the two following ways: first, transforming the expression according to a supposed denotation (e.g. to show that the product of two consecutive numbers can always be represented as the sum of the smaller number and the square of the smaller number, one would need to transform $n(n+1) = n^2+n$) and second, without any formal manipulation, inventing a new sense of the expression by looking at it with a supposed new denotation in a possibly new 'conceptual frame' (e.g. seeing $n(n+1)$ as the product of two consecutive numbers and seeing it as representing the area of a rectangle).

2.3.4 Students' understanding of structure of expressions

In addition to the difficulties caused due to differences in arithmetic and algebraic ways of working and notations and conventions, exploring students' understanding of the structure of arithmetic and algebraic expressions in particular and the connection between arithmetic and algebraic expressions in general, were two other main foci of research studies. Some researchers showed the limitations of students' understanding of structure of arithmetic expressions and their abilities to compute with expressions. Studies showed students' inability to judge equality of expressions without calculation (e.g. 685-492+947 and 947+492-685, Chaiklin and Lesgold, 1984), inconsistency in applying rules for computing/ simplifying (e.g. simplifying $4(6x\text{-}3y)\text{+}5\text{x}$ as either $4(6x\text{-}3y\text{+}5x)$ or as $4\times6x\text{-}3y\text{+}5x$, Kieran, 1989, 1992), sequential computation not capitalizing on the relations (e.g. being unable to find the answer of 17+59-59+18-18 without computation, Herscovics and Lincheveski, 1994), failing to use inverse relations (knowing $(3x+2)(5x-4)$ to be equal to $15x^2\text{-}2x\text{-}8$ but being unable to immediately identify the factors of the expression, Wagner and Parker, 1999). Similar to the context of number learning and place value, students while learning algebra need to "unitize" the polynomial expressions and treat them as single variables, as in factoring by grouping, for example, $\text{a}x+\text{b}x+\text{a}y+\text{b}y = (\text{a}+\text{b})x+(\text{a}+\text{b})y$, where (a+b) will need to be treated like a single variable to proceed further (Wagner and Parker, 1999). Hoch and Dreyfus (2004) also reported students' lack of structure sense among 92 grade 11 students while solving equations, where students solved the equation in the normal manner without attending to the relation between the terms (e.g. $1-\frac{1}{n+2}-(1-\frac{1}{n+2}) = \frac{1}{110}$ or $\frac{1}{4}-\frac{x}{x+1}-x = 5+(\frac{1}{4}-\frac{x}{x+1})$. The researchers found that the presence of brackets helped students in noticing the structure of the equation.

Putnam et al. (1987) examined 74 5th , 7th and 9th grade students' understanding of sign-change rules in addition and subtraction expressions with parentheses of the form a-(b+c), a-(b-c) in the context of numbers through interview. They found that students were generally successful in judging the equivalence of two story situations and justified them informally (e.g. sold the same number of things, together or at different times does not make any difference for a situation 18-(7+2) and 18-7-2) but they were more successful in situations of the form a-(b+c). Students' understanding of sign-change rules in the formal symbolic condition was weak even after instruction in algebra and they were less successful in a-(b+c) situation than in a-(b-c), contrary to their performance in the story situation. Students' responses did not use justifications based on structure but on computations, surface-level comparisons and often incorrect rules for operating on the symbols. The researchers found that situational referents in the form of stories for expressions with parentheses enabled students to justify their equivalence. This was an effort to improve students' symbolic understanding by using their intuitive understanding and strengthening referential meaning.

In a systematic effort to explore if lack of understanding of structure of arithmetic expressions really leads to errors in algebra, Linchevski and Livneh (1999) carried out a study on 6th and 7th graders and found students' difficulties in algebra in purely arithmetic contexts. They reported three errors in the case of arithmetic expressions: (i) detachment of a term from indicated operation (e.g. 23-6+7=23-13, due to a misinterpretation of the rule of order of operations as addition first or due to wrongly applying associative property), (ii) misunderstanding the order of operations (e.g. 5+6×10=11×10=110 or 24÷3×2=24÷6=4, move from left to right or multiplication before division), and (iii) jumping off with the posterior operation (e.g. in 217+175-

217+175+67 the 175s are cancelled because of the '–' sign following the first 175). Similar errors were observed by them in their earlier studies (Herscovics and Linchevski, 1994; Linchevski and Herscovics, 1996) in the context of algebra. Although the students generally showed consistency in their understanding and interpretation of the structure of the expressions, some of them were correct in one instance and incorrect in another. For example, in expressions with similar structure 27-5+3, 167-20+10+30 and 50-10+10+10, the rate of detachment error was different, least in the first and maximum in the last. This was attributed to some number combinations which are responsible for shifting the attention from structural to numerical properties in a way which changes the meaning of the expression leading to the assignment of a wrong value to the expression (Linchevski and Livneh, 2002). According to them, the difficulty arises due to a competition between structure and biasing number combinations together with many other factors.

Kirshner (2001) suggested that polynomial expressions have a complicated binary parse, and competence in algebraic skills is not so much about knowing rules but about coordinating pattern-based perceptual cues, a point that throws light on the effect of biasing number combinations. Kirshner and Awtry (2004) carried out an experimental design consisting of two treatments (2 lessons of 50 minutes and 1 review lesson of 30 minutes) on novices of grade 7 not exposed to algebraic symbol to assess the cognitive basis of algebraic symbol manipulation. One of the treatments involved an ordinary notational representation and the other involved a tree representation and both the groups learnt eight rules - 4 visually salient (e.g. $2(x-y)=2x-2y$) and 4 non-visually salient (e.g. $x^2-y^2=(x-y)(x+y)$). The tree notation was to act as a neutral medium to control the influence of position and spacing between symbols (visual salience) in standard notations. Students were presented with three items for each rule, two of which were recognition tasks (choose one from five options) and one rejection task (none of the options were correct). The analysis of the data revealed that for students taught in ordinary notation, recognizing visually salient rules was significantly easier than non-visually-salient rules. They also retained the visually salient rules more often than the non-visually-salient rules. The students trained through a tree-notation responded equally to the two sets of rules with a slight reversal in difficulty for visually salient rules, which is hard to explain. The rejection task showed similar results: constraining overgeneralization easier for non-visually-salient rules for standard notation but a slight difference for tree notation where students performed better in visually salient rules. The results indicate that "students engage with visual characteristics of the symbol system in their initial learning of algebra rules" (Kirshner et al., 2004, p. 242) and that this knowledge is not a result of declarative understanding of rules. In the light on the above, they argue that contextualizing algebraic rules in rich settings would not help as the source of the problem is not a result of de-contextualized, abstract learning but focusing on visual salience.

Fischbein and Barash (1993) hypothesized the existence of rules which served as models to explain the systematic errors in solving algebraic problems. For example, the model used while a student writes $(a+b)^2$ as a^2+b^2 is that of distributive property which acts as the prototype. Further, they argued that stronger models impose themselves over the weaker ones and eliminate them. Therefore in the above example, the incorrect formula a^2+b^2 is more intuitive than the correct one $a^2+2ab+b^2$. They provided evidence for their hypotheses in the case of exponents where students used a preliminary notion of distributive property (e.g. $\frac{4x^2-9}{2x-3} = 2x-3$ as $4x^2-9=(2x-3)^2$), as well as in the case of rational expressions where a more intuitive model of additive structures (which allows reduction by subtraction) was used incorrectly (e.g. $\frac{a+b}{a} = b$ or $1+b$). These students confused the ideas of 'terms' and 'factors' while simplifying algebraic expressions. Many of these students who had made the errors while simplifying algebraic expressions knew the correct rules, making the case stronger for the existence of conflicting models in the minds of the students. It is actually the lack of structure sense of expressions among students which leads them to over-generalize and use rules inappropriately in expressions which have different structures.

2.3.5 Exploring the arithmetic algebra connection

Some researchers focused on the more general issue of the connection between arithmetic and algebra and explored the extent of such a connection in the responses of students, who had undergone instruction in both the areas of arithmetic and algebra. Lee and Wheeler (1989) in their attempt to show the connection between arithmetic and algebra, found dissociation between grade 10 students' arithmetic and algebraic knowledge even when they could perform standard algebraic tasks correctly. These students could not coordinate their movement between the two worlds of arithmetic and algebra. They could not spontaneously use numbers to check their solutions in algebra. Instead of this step helping them to resolve the discrepancy in answers, it aggravated the dilemma, often choosing one (arithmetic or algebraic) over the other. When the expression was in the domain of arithmetic, they could not solve it by computation and went on to use the wrong algebraic identity to solve it. These students did not see algebra as generalized arithmetic and for them the rules worked differently in both the worlds.

Similarly, Cerulli and Mariotti (2001) present a case study of a student who could not think of the simplification of arithmetic and algebraic expressions as being the same process, one on numbers and the other on letters although she could handle both kinds of expressions well using appropriate rules of symbolic manipulation. She knew that two expressions, like $(a-b)(a+b)$ and a^2-b^2, are equivalent if their values are the same after replacing the letter with a number but considered the two expressions as independent and did not think that transforming one into another kept the expressions equivalent. The authors point out that although she was thinking of the expressions in the dual mode: of both a calculation procedure and as entities which could be compared, she did not relate the two expressions and the two kinds of computations: computing with numbers and with letters. This example demonstrates that students may acquire some structural and procedural conceptions of operations and algebraic expressions but may lack a comprehensive meaning of computing procedures, on letters and numbers. The authors add that the "key-point is that properties of the operations have to become rules of transformation, that is, instruments of computation, and in order to do so, they must assume a dual meaning (structural and operational): properties state the basic equivalence relations and function as instruments for symbolic manipulation" (p. 231).

Demby (1997) also suggested that the connection between arithmetic and algebraic procedures is complex and her study did not support the hypothesis of the analogy of procedures following from arithmetic to algebra. She asked students in grade 7 and the same students after a year in grade 8 to simplify algebraic expressions and she prepared a list of procedures used by the students to perform the task. Most of the procedures were spontaneous, although they had been taught the simplification procedure by the teacher using commutative and distributive properties and their geometric interpretation in the classroom. She identified seven different types of students' procedures: automatization (operations automatized, one simply knows the correct result), formulas (use formulas with variables), guessing-substituting (checking the answer by substituting with numbers), preparatory modification (changing the given surface structure into more elementary form), concretization (imagining some concrete model of the abstract operation), rules (when a rule is implicitly or explicitly stated while transforming and consistently used) and quasi-rules (the stated rule is used inconsistently). Some of the results that students obtained after transformations using these procedures were correct, though many were incorrect. The study brought forth the futility of the dichotomies between "formal" (routine algorithms/ school methods) and "intuitive" methods; between "deriving complex transformations from the basic properties" and "practicing algebraic rules in a quite mechanical way". A procedure like 'Concretization' is intuitive but of limited use and a procedure like 'Preparatory Modification' is mathematically sound but acts only as a preparatory step before using another procedure. Except for 'formulas' and 'rules' which were considered syntactic (based on form of the expression, solving $(8x\text{-}2x)^2$ on the basis of $(a\text{-}b)^2$), the others were considered semantic. The procedures termed semantic appealed to the meaning of the expression, e.g. changing $-2x^2+8-8x-4x^2$ to addition $-2x^2+8+(-8x)+(-4x^2)$ thereby affording more freedom while manipulating the expression or $3x$ and $6x$ are $9x$ similar to 3 apples and 6 apples are 9 apples. In contrast to the syntactic solutions which are based on rules of transformations, the semantic solutions try to find a referent for the symbols in some way. It was seen that the more successful students used semantic procedures more frequently than those who were less successful. Also, it was observed that a good command over algebraic transformation is more likely when the students use diverse types of procedures. Therefore an integration of semantic and syntactic aspects of school algebra with diverse types of methods was considered desirable.

2.3.6 Implications for teaching

The debate on the connection between arithmetic and algebra and of structure sense and procedures and its utility for teaching-learning is a long standing one. Nearly all of the studies quoted above were grappling with the issue of the importance of awareness of structure sense and its possible implication for the learning of algebra. Most of the researchers agree that algebra encodes the properties of arithmetic in a generalized manner and a grasp of the structure of arithmetic expressions would pave the way for learning algebraic symbol manipulation. But they are also cautious about the intended outcome of such a suggestion. Although Linchevski and Livneh (1999, 2002) expressed the view that an attempt to teach algebra as generalized arithmetic would help situate the structural rules of algebra in a meaningful context and give semantic validation to the procedures, they were skeptical about teaching arithmetic for algebraic purposes apprehending that the whole exercise would become meaningless and artificial. They suggested tasks based on equivalent structures of an expression, using transformations flexibly and creatively, and analyzing and unpacking familiar procedures as some of the ways to build the connection between arithmetic and algebra. A systematic investigation of the connection between the 'structure sense' in the contexts of arithmetic and algebra was recommended. Many others have also expressed doubts over using this connection of algebra with arithmetic for teaching as students do not find this connection easily accessible (Mason, 1985; Lee and Wheeler, 1989; Demby, 1997). Lee and Wheeler (1989) pointed out that the proposed building of arithmetic-algebra connection among students firstly requires a clear understanding of the connection between arithmetic and algebra. They argued that even if both the domains use the same operational signs, there are stark differences in the writing and manipulation of the expressions making it difficult for students to spot the connection. Wheeler (1996) pointed out the need to both facilitate the transition from arithmetic to algebra as well as tackle the intrinsic obstacles, such as those of notations, conventions and rules of symbol manipulation, which have been discussed earlier. Also since many of the errors discussed in the preceding paragraphs are known to be caused by over generalization of the rules from arithmetic, some researchers do not particularly like the idea of introducing algebra the arithmetic way and in fact argue for a 'rupture' between arithmetic and algebra rather than a transition between them (Lins, 2001). For them this pathway is littered with pedagogical, 'procedural, linguistic, conceptual and epistemological hurdles' (e.g. Lee and Wheeler, 1989).

Another apprehension is that the result of an approach which capitalizes on the arithmetic algebra link (following the generalized arithmetic approach) would be sophisticated 'symbolic arithmetic' where, even if difficult, it will be possible for the students to keep track of the referent for the letter while manipulating it (Balacheff, 2001). But the objective of algebra is to be able to manipulate expressions in a referent free manner and interpret the result of the manipulation back in the problem. As Balacheff (2001) puts it "the essential difference between symbolic arithmetic and algebra is a shift of emphasis from a pragmatic control to a theoretical control on the solution of the problems considered" (p. 256). The fact that the students can manipulate algebraic expressions does not mean that they have accepted or acquired the shift in the method from arithmetic. The algebraic statements are simply generalized procedures arrived at it by replacing the number with the letter (ibid.). It is not necessary that they derive confidence and control of the situation by using algebra and they understand that once a result has been shown using algebra, it is true for all possible cases, that it is correct and complete. This is a more serious problem that arises due to using arithmetic-algebra connection and can have stronger implications but I would suggest that these are stages in the development of understanding of the symbols in algebra and purpose of algebra. This point will be revisited again while discussing students' understanding of use of algebra in contexts.

In contrast to these views, Kirshner (2001) argued for an algebra curriculum which is structural in nature, building from undefined symbols and explicit rules, as against a referential approach to introducing algebra. He suggests that students should be exposed to the rules of algebraic manipulation rationally through specialized activities and be engaged in articulating and justifying their rule usage in the classroom. According to him, algebra learning is a matter of generating and consolidating subcognitive patterns rather than learning rules. He disagrees with the argument that the failure of traditional curriculum is due to its being rule based. This kind of understanding of patterns does not develop automatically by working with a symbol system and cannot be achieved by telling and practicing the rule. This is an important point which needs to be kept in mind while reading the first set of criticisms against the arithmetic-algebra connection and the thesis tries to find a way of communicating these patterns in expressions to the students. Some studies which build on students' arithmetic competence and enhance their understanding of structure of expressions to teach algebra will be discussed later.

2.4 *Theoretical models explaining difficulties with symbolic algebra*

Besides these empirical investigations of the connection between arithmetic and algebra and the causes for the difficulties with syntactic manipulation in algebra, theoretical models have also been suggested to account for students' difficulties in learning algebraic transformations. These models have been variously called as theory of reification (Sfard, 1991), encapsulation (Dubinsky, 1991) or flexible procepts (Gray and Tall, 1994). The focus of these theories is on the construction of abstract mental objects from processes, which makes manipulation on the higher order objects possible. These theories, which try to explain the stages involved in the formation of such objects among the learners are more 'domain general' and can explain and predict difficulties in different domains of mathematics. They have also been used to understand the causes of the difficulties in algebra.

A theory proposed by Sfard (1991) is called the theory of reification. According to this theory, abstract notions like number, expression or function can be conceived of in two ways: the operational (as processes) and the structural (as objects). She points out that there is a deep ontological gap between these two conceptions but they are complementary. Moreover, the journey from the operational to the structural is on a continuum gradually progressing from one to the other, and are not two discrete states, the end points being characterized by thinking of a set of symbols as instructions to carry out some action (operational) and thinking of a set of symbols as an entity in itself which can be manipulated using defined rules (structural). Instead of a dichotomy, this theory emphasizes a process-product duality, the process conception preceding the product. The movement from process to understanding the duality of process-product takes place in a series of stages. The first stage is called *interiorization*: 'getting acquainted with the processes which eventually give rise to a new concept'. These processes are operations on lower-order mathematical objects. A process is interiorized when it can be carried out through mental representations. The second stage is *condensation* which is a 'period of "squeezing" lengthy sequences of operations into more manageable units'. The processes now can be thought as a whole without going into all its details and steps. This is a long phase which lasts as long as the new entity is connected to processes. The final stage is that of *reification* when the notion can be treated as a full-fledged object. It involves an ontological shift which enables one to see something familiar in a new light. The first two phases are quantitative changes but the last one is a qualitative change where the process becomes an object or a static structure. Subsequently, this gets detached from the processes from which it has been constructed and gets attached to meanings which are associated with the new domain to which it belongs (e.g. seeing an algebraic expression $3(x+1)+5$ as a function mapping real numbers to themselves through linear transformation). Processes can now be performed on these newly formed objects. In this context, the role of names, symbols, graphs and other representations in condensation and reification is enormous.

This becomes a cycle where the newly constructed objects become the input for higher level objects/ concepts. She calls this a 'vicious circle' in that 'lower-level reification and the higher-level interiorization are prerequisites for each other'. This makes it also clear that the process of reification is inherently difficult – higher level interiorization is essential for reification and objects must also exist on which higher level processes can be carried out for interiorization to happen – and fewer people cross it. One of the implications of this is that the reification of the primary processes (underlying the given concept) is a precondition for the ability to deal with the secondary processes (applied to the given concept) and the latter is a precondition for the former.

Sfard's theory gives a framework for understanding the difficulty with algebra. In algebra, the symbols at each point have to be treated both operationally and structurally[2] (Sfard and Linchevski, 1994). The same representation encodes both the instruction of computation as well as the product of a computation (for example, x+4 is both the instruction to add 4 to any number as well as the result of a computation stating the relationship 'standing for a number which is four more than a given number'). This requires the reification of the process of adding 4 to any number leading to another number as an answer to stating the relation as a representation x+4. Further, to be able to consistently evaluate arithmetic expressions (primary process), one needs to know general rules which govern operations. These are algebraic in nature and also underlie manipulation of algebraic expressions (secondary process). It is the same rule of 'multiplication before addition' which allows 5+4×6 to be computed as 5+24 and disallows the simplification of 5+4x as 9x. Similarly, it is the same distributive law which allows solving 5×4+8×4 as 13×4 and 5x+8x=13x, but one has to withhold finding the intermediate results 5×4, 8×4, 5x and 8x and treat them as entities. This requires the students to move away from successively computing binary operations on two numbers to focus on the full expression with multiple operations and the properties which govern its computation/ simplification. Studies by Linchevski and Sfard (1991), Sfard and Linchevski (1994) describe the complexity in attaining a structural conception of notions like equations and inequalities and show that students are often led to a 'pseudo-structural' (mistaking the signifier with the signified) approach to dealing with symbols, which on the surface look structural but are in fact su-

[2] Considering algebra to be generalized arithmetic, Sfard and Linchevski (1994) correlate the operational phase with the rhetoric and the syncopated stages of the historical evolution of algebra and the structural phase with the symbolic stage of its historical evolution.

perficial. They have no connection with the underlying mathematical operations. A 15 year old student's solution to the inequality $x^2+x+1>0$ by treating it as an equation and concluding that it has no solution due to a negative discriminant, is a case in point. Such students do not realize that the processes required for solving an equation and inequality are different. The familiarity with the symbols in one context drives them to manipulate the symbols similarly, without appreciating the shift in meaning of the algebraic symbols in the context of inequality.

On similar lines, Gray and Tall (1994) and Tall et al. (2000) considered the duality (and not the distinction) between process and concept and the use of same symbols for denoting both, the process and the product or result of the process. They put forth the notion of '*procept*' to refer to 'the amalgam of concept and process represented by the same symbol' (Gray and Tall, 1994, pp. 121). According to them, an 'elementary procept' is the amalgam of three components, a *process* that produces a mathematical *object*, the mathematical *object* that is produced, and a *symbol* that represents either the process or the object. A *procept* is thus a collection of elementary procepts that have the same object (for example, the procept 6 includes the counting till 6 and representations like 3 + 3, 2 ×3 etc.). The flexibility in combining the conceptual and the procedural thinking to see the processes and the products/ objects in the same symbol is successful *proceptual thinking*. This kind of thinking is characterized by the ability to compress stages in symbol manipulation to the point where symbols are viewed as objects that can be decomposed and recomposed in flexible ways.

The trouble with students who are unable to work with algebraic symbols is precisely that they are unable to deal with the 'ambiguity' of the symbol as both a process and the product of the process. This leads to difficulties while combining these symbols (e.g. in simplification of algebraic expressions) as the students are fixated on the process conception of the symbols and consider an expression like $2+3\times y$ only as an instruction to compute which cannot be completed till the value of y is known. According to Tall (1992), it is essential to hold the operations in suspension in the case of algebraic expressions and consider them only as potential processes, whereas in arithmetic 'procepts' have an internal procedure and the value can be calculated (operational 'procepts') (Tall, 1992; Thomas and Tall, 2001). Through a computer driven approach ('cybernetic' approach) which included programming to understand equivalent expressions and practically carrying out computations (e.g. understanding equivalence of $2\times a+b$ and $b+2\times a$, for all values of a and b), using the same symbols, students were found to think about algebraic symbols proceptually (Tall, 1992). For example, one student rearranged $3x\text{-}5=2x+1$ as $3x=2x+6$, and concluded that the extra x should be equal to 6, without carrying out the whole process. This is the difference between a flexible 'proceptual' thinker, who could stop the manipulation when the solution was evident, and the procedural thinker who would go on till the end.

Mason (1996) while recommending the need for the awareness of the dual meaning of expressions: as entities or object, and as processes for algebraic thinking, described the development of abstraction as phases in a developing spiral – from experience in *manipulating* objects (physical, mental, or symbolic objects) to expressing this experience (*getting-a-sense-of*), to *articulating* the properties of such experiences as expressions of generality, and subsequently manipulating such expressions to search for further properties. 'The actual process of abstraction is considered to lie in the "delicate shift of attention" from seeing the expression as an expression of generality, to seeing it as a manipulable object or property' (Mason, 1989 as cited in English and Sharry, 1996, p. 137). For example, we learn that $2\times3+2\times5=2\times(3+5)$ and this can be generalized to $3a+5a = (3+5)\times a$. But this generalization is not sufficient to add $3a+5a$ with itself or multiply with itself. The learner must not only be able to manipulate these generalities but also remain aware of the calculations inherent in the expressions. English and Sharry (1996) proposed that algebraic abstraction involves analogical reasoning consisting of the articulation of expressions of generality from experiences with operations on lower level algebraic constructs, that is, operation on numbers and extracting relational commonalities between algebraic examples. These generalities can be subsequently manipulated as full-fledged mathematical objects. This theory highlights the need for exploring algebraic processes to construct abstract generalized models. Also it helps one understand that relational properties are the key to draw these abstractions.

Dubinsky (1991) based his theory of encapsulation on Piaget's theory of reflective abstraction. This theory is similar to Sfard's theory in many ways and has four stages given by the acronym APOS. The first two processes mentioned in Sfard's theory are similar to the first two of Dubinsky. An individual deals with mathematical problem situations by creating mental actions, processes and objects and organizes them into schemas. The object is formed by either encapsulation of the processes or the schemas. Dubinsky and MacDonald (2001, pp. 276) summarize their theory as follows:

> An *action* is a transformation of objects perceived by the individual as essentially external and as requiring, either explicitly or from memory, step-by-step instructions on how to perform the operation. … When an action is repeated and the individual reflects upon it, he or she can make an internal mental construction called a *process* which the individual can think of as performing the same kind of action, but no longer with the need of external stimuli. An individual can think of performing a process without actually doing it, and therefore can think about reversing it and composing it with other processes. … An *object* is constructed from a process when the individual becomes aware of the process as a totality and realizes that transformations can act on it. … Finally, a *schema* for a certain mathematical concept is an individual's collection of actions, processes, objects, and other schemas which are linked by some general principles to form a framework in the individual's mind that may be brought to bear upon a problem situation involving that concept.

Although Dubinsky (1991) used his theory to explain the difficulties of students in advanced mathematics, it can be used similarly to explain the problems at lower levels. As students beginning middle school bring their arithmetic habits along with them, it interferes with the algebra learning. The students' inability to move from actually carrying out each step (*action*) as in arithmetic, to see the steps as composing a *process* which can be mentally reversed, composed etc. leaves algebra almost out of reach for most students. Students fail to understand/ manipulate $3+2a$ or $3a+2a$ as they consider $3a$ and $2a$ as only a set of actions (like $3\times5+2\times5$ as a set of instruction – multiply 3 by 5, 2 by 5 and add them) and not as 'processes', which does not allow them to work with these symbols till the value of '*a*' is given. Once these are considered as 'processes', then they can be mentally imagined and combined with other processes. Finally, students can encapsulate these processes and carry out complex transformations on them.

Tzur and Simon (2004) and Simon et al. (2004) building on Piaget's theory of reflective abstraction have proposed a two stage theory of learning mathematical concepts – participatory and anticipatory, and a mechanism for explaining mathematical conceptual learning – reflection on activity-effect relationship (Tzur and Simon, 2004; Simon et al., 2004). Acknowledging the fact that learners can understand a new mathematical concept as long as they can assimilate it into their existing schemes/ structure, one needs to explain how learners make progress. The authors explain that learners' goal-directed activity and its effect (as seen by the learner) serve as the basis for the formation of a new conception. Faced with a task, the learners set up a *goal* based on their current conceptions which requires them to call up their *available activities* to meet the goal. They subsequently attend to the *results* of their actions, distinguishing the positive from the negative ones. The goal directed adjustments based on the results are the *effects* of the activity. The learner's reflection (may not be conscious) on these *activity-effect relationships* (mental comparison of the records of experience) lead to the first step of development of new conceptions. A *conception* can itself be though of as 'the ability to anticipate the effect of one's activity without mentally or physically running that activity' (Simon et al., 2004). Using this, Tzur and Simon (2004, p. 296) explain the two stages of learning mathematics.

> At the *participatory* (first) stage, the learners have learned to anticipate the effects of an activity and may also be able to explain why the effects derive from the activity. However, this knowledge is only available to the learners *in the context of the* activity through which it was developed…. In contrast, at the *anticipatory* (second) stage, the learners' use of the new activity-effect relationship is no longer limited to those times when they are focused on the activity through which it was developed. That is, at the anticipatory stage a learner independently calls up and uses a newly formed activity-effect relationship appropriate to the situation at hand (stressing original in author's description).

The learning of symbolic algebra from arithmetic as well as the errors many of the students make can also be explained by this mechanism. The arithmetic knowledge of the students acts as the assimilable scheme through which they make sense of the new activity of algebra. The initial goals of the students are determined and directed by this knowledge and therefore the students tend to display errors which are considered as being a result of interference of arithmetic ways of thinking. To be able to succeed in algebra, students through the feedback on their manipulation of symbols would have to engage with the reasons for their errors and distinguish actions which are permissible from ones which are not permissible. This reflection connecting the causes of the errors with the actions on the symbols would lead to a general understanding of arithmetic operations, the first stage for algebra. From this 'participatory' stage' where the conceptions are based on arithmetic, students have to move to the 'anticipatory' stage where the actions can be performed on the algebraic world without the arithmetic basis and the process can start all over again.

Summary

The review of a section of the literature related to the nature and causes of difficulties in algebra clearly shows that there have been many efforts to understand the issue. Some of the explanations for the problems faced in algebra are restricted to the context of algebra, understanding the meaning of various symbols used in algebra and learning symbolic transformations (e.g. Kieran, 1989; Booth, 1988; Linchevski and Livneh, 1999; Herscovics and Linchevski, 1994) while some are more extensive in their scope (e.g. Sfard, 1991; Dubinsky, 1991; Gray and Tall, 1994; Simon et al., 2004; Tzur and Simon, 2004). Most of the studies discussed above considered algebra as encoding rules in arithmetic and highlighted the importance of awareness of structural properties of arithmetic in helping students to learn the rules of transformation of the algebraic symbol system. Some researchers agree with this argument assuming that many years of experience in arithmetic and competent performance in it must lead to the appreciation and abstraction of the structure of expressions, leading to the domain of algebra which follows arithmetic in the hierarchically arranged curriculum. However, this does not usually happen in traditional classrooms due to emphasis on correct arithmetic procedures and answers. This could be a reason for some researchers questioning the usefulness of making the connection with arithmetic, based on their exploratory studies with students, where they have found the connection to be lacking or rigid. They therefore emphasize other ways of introducing algebra (generating expressions, equations, functions, modeling etc.), not basing it on arithmetic and not waiting for arithmetic instruction to end in the primary grades or for students to move from the concrete to the formal operational stage. In the process, although meaningful contexts are created for the generation of algebraic symbols, the meaning which can be created by transforming those algebraic symbols is lost. Arguments by Linchevski and Livneh (1999) and Kirshner (2001) for developing a connection between arithmetic and algebra emphasizing the structure of expressions have already been discussed earlier. This discussion will be continued in the next sections with examples of studies which are more successful in building and utilizing this connection. The research literature on errors/ difficulties in algebra and causes of the errors is vast and there is enough data to show that students do not make much sense of the traditional symbol manipulation which is devoid of any context. This led to the beginning of a series of research studies which focused on situating algebraic activity in contexts, numerical or otherwise, and giving meaning to the actions on the symbols.

2.5 *Algebraic reasoning and thinking*

2.5.1 Differentiating algebraic from arithmetic ways of thinking

To shift the emphasis from symbol manipulation and exploring students' understanding of symbols, one of the things which needed to be urgently addressed was the issue of 'algebraic thinking', which could lead to a fresh characterization of the activity of 'doing algebra'. Problem solving was one domain in which the distinction between 'arithmetic ways of thinking' and 'algebraic ways of thinking' was clearly visible. Bednarz and Janvier (1996) and Stacey and MacGregor (1999, 2000) distinguished arithmetic thinking from algebraic thinking in the context of problem solving. The researchers argued that questions/ problems which could be solved by operating with numbers to calculate numbers belonged to the category of arithmetic thinking. The characteristic feature of such a solution is that unknown quantities are successively calculated from the known quantities and serve as input for the next step.[3] A problem which necessarily requires algebraic thinking is solved by identifying the unknown, representing it by the letter, and describing the whole situation using the letter. It requires operating on and with the letter, as if the letter is known. The symbol is then used in a chain of deductive reasoning in statements of equivalence till the problem is resolved. For example, an arithmetic solution to the problem (Stacey and MacGregor, 1999) 'To rent a car from Ti-

[3] We have already seen the difficulties which students have in operating on and with the unknown: 'cognitive gap' (Herscovics and Linchevski, 1994), 'didactic cut' (Filloy and Rojano, 1989). Theoretical explanations for the discontinuity between arithmetic and algebra point out the distinctions between 'procedural and structural' (Kieran, 1989), 'process and object' (Sfard, 1991), 'procept' (Gray and Tall, 1994).

ger costs \$100 per day and 20 cents per km. How far can I drive, if the most I can afford to pay is \$240?' is the following:

\$240-\$100=\$140 (money to spend on kilometer charge).

Cost per km is \$0.20.

Number of kilometers that can be driven = money available ÷ cost per km = \$140 ÷ 0.20 = 700.

So I can drive up to 700 km.

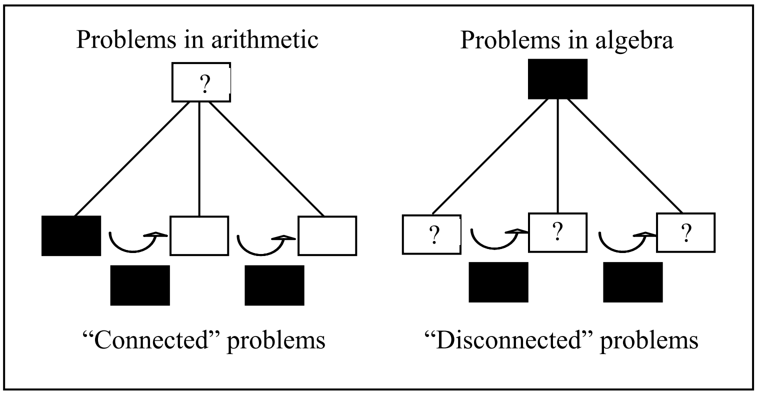


Figure 2.1: Bednarz's and Janvier's distinction between arithmetic and algebraic problems (1996, p. 124)

The above solution is arrived at by using a method called 'unwinding'. It is an intuitive method in which one begins with the last number and works backward in a step-by-step manner using the information (numerical data) given in the problem. On the other hand, the algebraic solution would require one to assume the number of kilometers that can be traveled to be x and set up an equation with the unknown number ($100+0.20x=240$) and manipulate it as if the x is known, as, for example in the operation of subtracting 100 from both sides. In this way, the letter is attributed with the operational properties of the number. The problems in arithmetic are "connected" as we work by connecting two given data and use arithmetic operations to arrive at the unknown in the end. Algebraic problems are "disconnected", it is not possible to establish a relation directly between the known data due to the presence of missing links (Bednarz and Janvier, 1996). The figure (Figure 2.1) illustrates the difference in the structure of the two kinds of problems. Attributing to the letter the status of something which can be worked on as if it is known, allows the algebra problem also to become connected. Therefore, students who use numeric trials as a strategy for solving problems are closer to algebraic reasoning than students who use other types of arithmetic reasoning, like unwinding.

2.5.2 Characterizing algebraic thinking

While the above description of algebraic thinking is restricted to problem solving in algebra and the use of symbols in a situation, some researchers also argue that it is possible for students to display algebraic thinking or reasoning in meaningful contexts without using letters or conventional symbols and not necessarily while working in the domain of algebra (e.g. Kieran, 2006). The letter symbol can be used as a tool if needed, but is not essential. Algebraic thinking can be displayed in activities such as analyzing relationships between quantities, noticing structure, studying change, generalizing, problem solving, modeling, justifying, proving, and predicting (Bell, 1995; Kieran, 2006). Some others (like Carraher et al., 2001) do not see introducing algebraic symbolism very early among small children (grades 2-3 onwards) as problematic. They emphasize notating relations among quantities and operations using symbols from the beginning, in situations like function notations for addition, multiplication, guess-my-rule games etc. (for example, $n \rightarrow n+3$, $n \rightarrow n+n$, or $2\times n$). If algebraic thinking is not restricted to the domain of algebra and the use of symbols, then it is possible to develop this type of thinking from the beginning which become the focus of ‘early algebra’. This did not obviously mean teaching traditional middle school algebra to smaller children but developing certain capabilities among children to think and reason using numbers and quantities relationally. Lins and Kaput (2004, pp 48) characterized algebraic thinking as:

> First, it involves acts of deliberate generalization and expression of generality. Second, it involves, usually as a separate endeavor, reasoning based on

> the *forms* of syntactically-structured generalizations, including syntactically and semantically guided actions.

Arithmetic serves as the basis for 'reasoning based on syntactically-structured generalizations'. It includes modeling relationships using numbers and operations, reasoning involving properties of operations and equality, and analytical thinking (Lins, 1992). In almost all characterizations of algebraic thinking, aspects of generality, generalization and expressing generality hold a central position, which is in contrast to arithmetic thinking where the focus is on specific situations, methods and values (Sutherland, in press). The point here is not whether one uses numbers or other symbols for purposes of reasoning, but whether one can think about general cases with the symbols (number or any other). Some researchers like Sutherland emphasize the need to manipulate and transform the generalized relationships. In this context, the use of algebraic symbols becomes essential as they allow for expressing the general as well as for manipulating general relations.

Davis (1985) described a step-by-step procedure for creating algebraic thinking among students: (i) valid experiences helping to build appropriate mental representations for the key concepts in algebra (ii) followed by discussions building on students' mental representations with emphasis on accurate explanations using simple language, and (iii) culminating in the development of an appropriate 'meta-language' for thinking about mathematical experiences using a non-misleading notation. A major problem with the students, in the studies reported in the last section, can be attributed to a drastic shift in approach, nature of thinking, symbols, conventions and notations for which they were not adequately prepared. There was indeed a huge 'cognitive gap' between arithmetic and algebra. The enunciation of algebraic thinking gave an indication of the amount of work that needs to be done prior to the introduction of symbolic algebra, which is typical in the middle and the secondary school. With this brief discussion on algebraic thinking, we move on to studies which exemplify the approach that encourages algebraic thinking from the early grades and understand the ways in which it has been implemented.

2.5.3 Early algebra and algebraic thinking

All the studies being reported here rely on students' intuitive understanding of numbers and operations and generalization abilities and encourage explication of these, thereby giving them formal understanding of the properties of operations. Some studies use only incidental and idiosyncratic symbols, just to expose students to the power of symbols and the extent to which it can help communicate among a group, with more time spent on argumentation and articulation of ideas. Some others develop the conventional symbols and notation in the classroom.

Algebrafying arithmetic

In exploratory studies conducted by Fujii (2003) and Fujii and Stephens (2001), they argued that the idea of quasi-variables is an important one which can bridge the gap between arithmetic and algebra. By quasi-variables, they mean a number sentence or a group of number sentences that indicates an underlying mathematical relationship which remains true whatever be the numbers used. This kind of reasoning relies on thinking algebraically, looking at structures and relations, without using letters and is different from reasoning which relies on calculating numerical values. They make the case that the missing number sentences (like 13 + 5 = ___ + 8) are not truly algebraic in the sense that these tasks can be completed by trial-and-error arithmetically, by calculating. But one does not deny their importance in understanding the meaning of '=' sign. Algebraic thinking, according to them, essentially engages students in patterns of generalized thinking. Although the idea of variable is not easily grasped by students with inherent difficulties in understanding the meaning of the letter (Fujii, 2003), generalizing patterns giving rise to numerical expressions in the many possible rich contexts of arithmetic or geometry are fruitful grounds to engage students to think in terms of quasi-variables. In a series of interviews with students in grades 2 and 3, using tasks which focused on generalizability and relational thinking like 'Peter's method' (solving 37-5 as 37+5-10) and filling the blank (746+__-262=747), they found that students in elementary grades (2-8) to possess and display abilities to think in terms of generalizations (like add the complement with respect to 10 of the number being subtracted and subtract 10) and look at relations between expressions and fill the blank without computation. Such relational thinking was associated with their ability to construct expressions with letter (arranging n-1, n+5, 7, 1 so that two expressions are equal) (Stephens 2004a, 2004b).

Saenz-Ludlow and Walgamuth (1998) describe a socio-constructivist teaching experiment with 14 third graders who participated in a year long project dealing with the '=' symbol and equality. The study showed the resilience of the procedural understanding of '=' sign and the unease in accepting the notion of quantitative sameness. It illustrated the potential of discussion, communication and explanation in the classroom and the effort required to construct the meanings of mathematical symbols (here '=' sign). In the process, the researchers bring forth an interesting discussion about the '=' sign. Cautioning against the thinking that the meaning of '=' sign is simple, they distinguish between the equalities in the expressions $a+b=b+a$ (nominal and quantitative sameness), $a+a=a+a$ (nominal and quantitative sameness but operating order not readily visible) and $a+a=(a+b)+(a-b)$ (nominal sameness broken but quantitative sameness preserved and operating order loses significance). The students need to understand '=' sign as quantitative sameness and simultaneously attend to operating command and the operating order of the addends. They conclude that "interpreting and symbolizing are different but complementary faces of the activity of constructing arithmetical meaning of equality through symbolic language (natural language and the mathematical symbols) or what could be called a *symbolic activity*" (pp. 185-186).

In teaching studies conducted by Carpenter and his colleagues (Carpenter and Levi, 2000; Carpenter and Franke, 2001; Carpenter, Franke and Levi, 2003), they showed that young children in the primary/ elementary grades are capable of learning and justifying generalizations about the underlying structure and properties of arithmetic, which can form the basis of algebra. The focus was on students' algebraic thinking, in particular students' abilities to generalize, articulate, represent and justify generalizations about the underlying structure and properties of arithmetic. The approach emphasized not only generalization but also representing mathematical ideas using symbols. They showed through whole class teaching studies that students in grades one through five can understand and articulate important properties of numbers and operations like adding a zero, subtracting the same number, commutative properties of addition and multiplication and develop a broader understanding of the '=' sign. In many of these instances, the students were engaged in 'true-false' and 'open number sentence' activities and were challenged to think of situations which remained invariant even though the numbers changed. Not only could they verbally justify such properties but with a little scaffolding in grades 4 and 6, they could use variables to represent the general statement (like $a-a=0$, $a+0=a$, $a+b-b=a$).

Schifter (1999) described students in primary grades (grades 1-3) exploring relations between numbers and properties of operations and unraveling for themselves deep properties like commutativity, associativity and distributivity. The contexts of the tasks for these students were word problems as well as calculations with numbers and operations. As the students engaged in the tasks, they were encouraged to come up with multiple ways of finding the solution and representing and justifying them. After having found the solution to a problem in different ways, the students were further asked to generalize their understanding to other pairs of numbers with regard to the same operation. Some of the students arrived at these properties by checking for numerical values of the two arithmetic sentences (viewing the operations as actions/ instructions on the numbers) and some others modeled the situation using blocks to justify the truth of the statement for all numbers (focusing on the relations between the numbers and the operations). It is on these numerical experiences that algebra is built and understanding based on this kind of algebraic reasoning can be easily converted into conventional notations and explicit knowledge.

Kaput and Blanton (2001) and Blanton and Kaput (2001) discuss some of the ways of algebrafying the elementary curriculum and present examples of classroom practice that actualize this goal. The activities involve generalizing arithmetic operations, their properties and reasoning about the more general relationships and their forms (e.g. properties of zero, commutativity, inverse relations, etc.), building generalizations about particular number properties and relationships (e.g. the sum of two odd numbers is even, finding regularities in 100 table, determining general properties based on place holder system etc.). This study shares many of its features with the studies described above by Carpenter et al. In examples from a grade 3 class, Kaput and Blanton (2001) show students' developing ideas about generalizations like 'adding two odd numbers always gives an even number', or counting the number of handshakes in a party systematically. The teachers in these classrooms try to algebrafy the particular number situations whenever possible. These generalization and formalization activities, according to Blanton and Kaput (2001), can build students' understanding of variable, function, and the '=' sign. The instruction utilized and developed students' symbol sense and introduced letters gradually through open number sentences or through generalizations of number properties, using students' abilities to conjecture and argue about the truth or falsity of the generalization.

Warren and Cooper (2001) report the development of an algebra curriculum for Australian schools for the grades P-7. They too used the arithmetic knowledge base to build understanding of operations, rules of operations, equivalence and the concept of variable. This served as a bridge between arithmetic and algebra making them aware of structure and connections, representation, exposure to multiple thinking/ reasoning styles, on which knowledge of algebra could be built using unknowns and patterns and relationships.

A functional approach to early algebra

In a slightly different approach towards early algebra, Carraher, Schliemann, Brizuela and their colleagues carried out research on students studying in grades 2 to 4 in Greater Boston to investigate their understanding of algebraic concepts, relations and notation. Their studies have been guided by the idea that arithmetic operations can be seen as functions, and that generalization is at the heart of algebraic reasoning (Carraher and Schliemann, 2002). According to them, one of the major reasons why students fail in representing or accepting algebraic notation is that notations are always used for computing and hardly for describing the relationships in the problem. Together with this are the limitations in the early mathematics instruction like restricted problem sets and focus on computation of particular set values rather than on relations among sets (Carraher et al. 2000). The emphasis on symbolic representations in their work is due to the belief that these open up new avenues of thinking and connecting and comparing with earlier learned ideas, as well as structure one's own mathematical thinking (Schliemann and Carraher, 2002).

Using students' initial understanding and intuitive ways of thinking and representing, students as young as 3rd graders were seen to use algebraic symbols and understand and represent additive and multiplicative relations as functions ($n \rightarrow n+3$ or $n \rightarrow 2\times n$) with minimal help from the teacher. The activities the students carried out include those with the letter number line, guess my rule games and activities focusing on functions. These students could fill out function tables and find the rule which could describe a function table. They learnt to see the equivalence of two such rules by checking if they got the same output for a given input. The researchers also emphasized proving the equivalence irrespective of the data set which gave rise to it, which was not so convincing to the students. It is important to note that understanding equivalence of algebraic expressions by manipulation without reference to the context is an important step in algebra. In their studies, students focused on generalizing quantitative relations from particular values and independent of any particular context. They allowed students to represent freely the problem situations using any of the means like language, pictures etc. but also ensured that the students moved towards conventional symbolic notation which they consider to be an integral part of algebra (Carraher et al., 2001; Brizuela et al., 2000, Brizuela and Schliemann, 2003). Third grade students could also understand graphs of linear functions and 4th grade students were able to solve algebraic problems using multiple representation systems such as tables, graphs, and written equations (Schliemann and Carraher, 2002; Schliemann et al., 2003; Brizuela and Schliemann, 2003). Contrary to the speculations of some researchers (Linchevski, 2001; Radford, 2001; Tall, 2001) regarding the ability of the students to operate on and with the letter in the letter-number line context, these later activities like guess-my-rule, functions, equations (Schliemann, 2003; Schliemann, 2002; Brizuela, 2003) showed that the students could learn to represent and think using the letter. Although these studies are promising, the overall situations were quite simple, mostly with a single appearance of the letter in an expression and a single operation sign where more intuitive methods can work and one still does not need to work on/ with the letter.

What the students achieved in the study described above is promising. Warren, Cooper and Lamb (2006) describe a similar teaching experiment with grade 4 students. The teaching sequence focused on the idea of functions where function machines were used to generalize and formalize arithmetic thinking. The relationship between input and output numbers in the function machine (arithmetic as ‘change’) was the point of attention, rather than on building understanding of operations on numbers so that the structure in the relationships is evident. The study also found that young children are capable of functional thinking in the sense that they could figure out the output number from the input number and sometimes even the reverse, and identify the change rule. However formal representation of these relations using arrow diagrams and equations had limited success.

Summary: The success stories

Compared to the earlier section of the discussion of the literature, which focused on students' failures and possible causes, this section highlighted what students can do given the opportunities and the means of carrying out meaningful tasks, giving some hope to the dismal picture of students' understanding about various aspects of algebra. The 'Early algebra' movement has given a direction towards improving the teaching and learning of algebra, preparing the children to make a transition to algebra. Lins and Kaput (2004) call these studies the 'happy stories' of algebra. Due to studies of the above kind, there is a growing awareness that children are capable of generalization and relational thinking. The students need to be given the opportunity in the classroom to articulate generalizations and explain their thinking, make conjectures and justify while working on any domain of mathematics so as to foster algebraic thinking from the beginning. They do not propose to teach the middle school algebra curriculum with all its symbolism and transformation rules but aim to expose the students to a kind of thinking which goes beyond the immediate and can reason about the probable. The studies reviewed here were those which largely relied on using arithmetic knowledge as the base[4].

Studies have also been carried out with students in the middle school (the normal stage for introducing algebra) aimed at improving students' understanding of symbolic algebra. Like many other researchers, the present study

[4] There would be many other fruitful approaches to induce algebraic thinking, using geometry and other branches of mathematics (Radford, 2001).

being reported here takes the view that symbols in algebra are a crucial part of algebraic thinking, enabling one to represent the situation and manipulate the representations to arrive at solutions which are not always intuitive. This is not to deny the role of verbalization and articulation in the process of generalization, but to highlight the tool which allows going beyond the surface features of a situation and the numbers involved to lead to patterns and general properties of numbers, operations, solution of a class of problems and further development of concepts/ ideas. In the section below, studies which have tried to develop some awareness of syntax of algebra and give meaning to syntactic transformations will be discussed.

2.6 *Developing understanding of symbolic algebraic expressions*

2.6.1 Modeling algebraic expressions

Chalough and Herscovics (1988) carried out a teaching experiment (three lessons) with six students in grades 6 and 7 (3 from each grade and belonging to weak, average and strong academic ability) trying to teach them algebraic expressions with meaning. They used situations which required representations by the students, like counting the number of dots in an array with only a row or column shown, finding the length of a line segment with the number of parts hidden and finding the area of rectangles with one of the dimensions or part of it as a letter and gradually increasing the complexity. Initially, they observed misconceptions with respect to notations like $3x$ is 32 if $x=2$ (concatenation) as well as weak conceptions of area. The students also had trouble understanding the idea of an unknown number of dots. The line segment problem had also to be readjusted from unknown number of parts to unknown length of each part. The students, even after the instruction, worked with another frame of reference (arithmetic), unless explicitly stated to work in the context of algebra. But by the end of the program, they succeeded in understanding the meaning of such algebraic expressions with even two variables. The teaching experiment did have some impact in making the students understand the meaning of the expressions, which was one of the most important issues in the research literature at that time. This experiment was restricted in its scope and operating on/ with the letter was not dealt with.

There have been many studies which tried to build a sense of symbols in algebra in modeling or other quantitative situations (real life or otherwise), like the one above, requiring representations of the relationships, which are largely generational activities (Kieran, 2004). These models although useful in giving meaning to the symbols have inherent limitations, in that the students remain tied to the concrete world of reference and often the rules of symbol manipulation are not generalized to work in contexts other than the one in which it was created. Moreover, it is cumbersome and sometimes impossible to use the same model for all concepts and situations as these need not be translatable to the world in which the symbols have acquired meaning. This requires one to use many different models simultaneously which itself can be very confusing. Also, it is important for students to learn to associate meanings with the symbol and to manipulate them in a referent free manner. This is an important ability while solving problems (not restricted to equation solving and inclusive of generalization and justifying tasks) where the contexts are first represented using symbols and then the symbols are transformed using rules, without considering the referent and finally interpreting the deductive chain of arguments back in terms of the situation. Balacheff (2001) argues that the validation of reasoning and the result in modeling situation is not internal to mathematics but to the situation and governed by constraints of the modeled world. In the above situation, it will be difficult to think about a quantity represented by '$-x$'(negative x) as lengths or areas cannot be negative.

2.6.2 Exploiting the arithmetic-algebra connection using its syntax

Researchers have also tried to induce meaning for the algebraic expressions or equations using number as the referent for the letter and build a sense of structure of expressions among students. Some of the literature has already been reviewed on students' understanding of algebraic symbols and their lack of structure sense. The studies which will now be described have exploited the arithmetic algebra connection by developing awareness for the structure of expressions, the letter automatically standing for the number.

Thompson and Thompson (1987) developed a special computer program called EXPRESSIONS which allowed students to manipulate expressions (arithmetic and algebraic) as well as constrain their actions on the expressions so that the structure of the expressions is not violated. The program was tried with eight students finishing grade seven over eight teaching sessions of 50 minutes each. The program presented an equation or an expression both in the sequential form and as an expression tree. This helped in highlighting the structure of the expressions. The students were taught order of operations, field properties as transformations of arithmetical expressions, identities and derivations. The program supported students' explorations of properties of operations by carrying out certain transformations as commanded by the students but not others which were wrong. Although, initially during the exploratory phase students committed errors by choosing the wrong buttons but once they internalized the structural constraints on the transformations, they found efficient solution strategies as well as made less errors. The analysis of the results suggest that mal-rules (perturbations of correct rules) are not naturally formed in environments which support explicit attention to structure of expressions and which impose constraints on students' actions, implying that attention to structure is important.

The problems designed for the instruction (e.g. exercises on order of operations involving expressions with and without brackets, changing the numeric expression 5×((4+3)+2) to (5×4)+((2+3)×5) or showing the identity $r\times(s/t)=(r\times s)/t$) required students to treat sub-expressions as a unit. Students needed to substitute sub-expressions in an expression for a letter in the canonical statement of a property or an identity. Letters were also introduced in expressions which required transformations. Through their experience of transforming numerical expressions, students appreciated that specific constituent elements are not important. The same transformation rules were used by the students in the context of expressions with letters without difficulty. The authors claimed that the students found the expression tree quite intuitive but also pointed out the fact that the study failed to test the students outside the computer environment and whether the students understood the identities as being applicable in other contexts as well. This study has been one of those which have explicitly tried to train students to perceive the structure of expressions. Despite the success of the approach in enabling the students to use the correct order of operations (which are hierarchical/ precedence rule based) and identities, it detaches the sign from the subsequent term which does not allow the students to see the effect of each component on the whole expression. The relationship of the components of the expression to the whole and between the components is not explicit. One would learn to transform expressions correctly without still getting a complete sense of the structure of the expressions.

APLUSIX (Chaachoua et al., 2004) is another environment for learning formal algebra using the structure of the algebraic expression where students carry out their own calculations as in paper and pencil situation (using functions like selection, cut, copy, paste, drag, drop) and learn through committing errors, unlike many other environments as the one described above where students use the commands like “commutativity” in the environment to transform the expression/ equation. The environment provides the feedback with respect to the denotation – equivalence of the transformed expression/ equation and a sense of the expression with respect to the goal of the task. The use of this environment has shown improvement in competences of students in grades 9, 10, 11 with a little help from the teacher and without any additional algebra instruction in the regular class.

Malara and Iaderosa (1999) report a research project with students in the age group 11-14 years which promotes learning algebra as a language through and for the study of problems, internal as well as external to mathematics. They studied if and to what extent an early introduction of letters in parallel with a constant work of reflection and control of the meanings of the symbols may limit or overcome well known obstacles and difficulties in algebra. This project also explored the connection between arithmetic and algebra and aspects of notation and convention, like parentheses, the '+' and the '–' sign, which cause difficulties in learning arithmetic or algebra. These need to be understood both as operators on numbers as well as signs for numbers. The researchers tried to deal with the confusion between the notations of addition and multiplication and, multiplication and exponents in the case of arithmetic by comparing them and bringing forth the similarities and differences between them. This developed understanding was simultaneously carried over to the domain of algebra. Students did not see the equivalence in the procedures and properties, which were known in the case of arithmetic, when a new or different symbolism with letters was used. The researchers observed that in the numerical expressions the students read the symbols as processes, working inside the bracket first, operating on the powers and adding but it was not possible in the case of algebraic expressions where they had to think about the form of the expression (e.g. $(2\times3)^2+2(2\times3)=6^2+12$ whereas in $(ab)^2+2ab=ab\times(ab+2)$). They recommended that algebra be introduced with a 'metacognitive teaching' of arithmetic with the algebraic aspects of arithmetic developed and explicated, which would allow students to use the correct algebraic code. Even though this project focused on generalizing properties and procedures of operations on numbers clarifying the notational difficulties, the results show the difficulty for students in perceiving the similarity and even engaging in the act of generalization in such a context. The recommendation put forth by the authors is therefore a vital lesson for any one trying to use arithmetic for teaching algebra and especially for the work being reported in this thesis.

In another study, Livneh and Linchevski (2003, 2007) tried to explore the connection between arithmetic and algebra among students of grade 7 in four schools over two years in Israel. Teaching modules were prepared purely in the numerical contexts to address the structural difficulties in arithmetic like, order of operations, detachment of the negative sign, '=' sign. These errors had earlier been identified in both arithmetic and algebraic context and therefore were considered to be impediments to achieving algebraic competence. The activities were designed to elicit cognitive conflicts and allow hypothesis testing in meaningful contexts. The contexts were chosen so that they were 'algebra compatible'; they reflected algebraic competence but in a numerical context (e.g. "Is 75-25+25=75-50" gets reflected in the algebraic situation "Is $16\text{-}4x+3x=16\text{-}7x$"). The intervention proved to be of help to students identified as 'students-at-risk' who made significant progress in the numerical and compatible algebra tasks. This suggested that teaching arithmetic for algebraic purposes could prevent some structural mistakes in beginning algebra and a carefully designed instruction in purely numerical context transfers its effect to algebraic one. The results also showed that although the at-risk students progressed in algebra-compatible tasks, the progress in other tasks (generalization, representation, word problems) was small. This instructional sequence drew on earlier studies which highlighted the need for building pre-concepts for algebra using pre-algebra activities (Linchevski, 1995). Number pattern generalizations, explicating the algebraic structure in numerical contexts (focusing on order of operations, brackets, detachment of an operation) and equation solving through numerical verification, equivalent equations through substitution, intuitive ways of solving, and forming equations were some of the activities proposed for pre-algebra.

Carrying over a similar line of research, Liebenberg et al. (1998) gave an overview of an approach which aimed to build students' understanding of structure of numerical expressions as a foundation for algebra. The structure of numerical expressions forms the input for algebraic expressions. They pointed out that while understanding the structure of expressions, students need to engage in both semantic and syntactic discussions and acquire a good command over order of operations before developing structure sense. In the study, they investigated the process of learning of structure of numerical expressions in grade 6 (Liebenberg, Linchevski, Sasman and Olvier, 1999). The students learnt the precedence of operation rules by analyzing and comparing the results for an expression using a scientific calculator and a non-scientific one and then constructing generalized rules. The study showed that the students found it hard to identify multiplication as a unit in an expression and wrongly generalized or over generalized the rule of precedence of the multiplication operation (e.g. $5\times3+2\times4\times6+7\times9 = (5\times3)+(2\times4)\times6+(7\times9) = 15+8+63 = 86$, $86\times6=516$; $5\times2\times6=5\times2+6\times2$). Students also had difficulty in generalizing the rules of operations from a structural perspective making it difficult to apply this knowledge to non-computational situations like judging equivalence of expressions or solving simple equations (e.g. $302+(79\times128)+29 = 302+(128\times79)+29$ because all the numbers are the same in the bracket as the given one or $5+5\times c=124\Rightarrow10\times c=124$).

In another effort they tried to teach the concepts of numerical and algebraic equivalence by focusing on both the procedural and the structural aspects of the expressions (Liebenberg, Sasman and Olivier, 1999). The aspect of numerical equivalence was tested in grade 6 and algebraic equivalence in grade 9. Most students justified the numerical equivalence using syntactic features (repeated the rules of finding the value of the expression) and not by using properties of the operations, but these students also had limited understanding of the rules of brackets except knowing that brackets are solved first. They gradually moved from this understanding of numerical equivalence to algebraic equivalence where two algebraic expressions were considered to be equivalent if they were equal for all values of the variable. The students were encouraged to focus on the properties of the operations to build equivalent algebraic expressions and to see that equivalent algebraic expressions can be substituted for one another. Equivalence of algebraic expressions was understood primarily numerically through substitution and not by transforming the expression. For example, students replaced the letter by numbers to see if $x+x$ is same as x^2 and since they are not equal for all values of x, they are not equivalent algebraic expressions. The same notion of equivalence was used to understand the use of letter in equations with a single variable (expressions which are equal for one value of the letter) and identities (expressions which are equal for all values of the letter). When tested in grade 9, although the students simplified the expression they could not confidently say whether the two expressions are equivalent. They did not accept the transformation process to be a valid way of ascertaining the equivalence of algebraic expressions and checked with a numerical value to be sure. Students' perceptions were also influenced by equation solving which they learnt to be expressions which are numerically equivalent for a specific value of the variable resulting in confusion between equivalent expressions and equations. The researchers realized that this result was a manifestation of the greater emphasis on procedures in the intervention. They therefore recommended focusing away from computational work to be one way to encourage students to adopt a more structural approach, which is the core of the idea in the transition from arithmetic to algebra.

Williams and Cooper (2001) describe two studies aimed at facilitating and assessing the teaching and learning of algebra, especially in terms of developing the meaning of operations, equals sign, and the variable. This study was also based on introducing algebra by capitalizing on the arithmetic algebra connection and the authors contended that learning complex algebra is facilitated by understanding similar structures in complex arithmetic. Unknowns, patterns and relationships were used to introduce the notion of variable. The first study was conducted with grade 8 students (twenty 40 min sessions over 4 weeks) and the emphasis was on reflection of their arithmetic experience and generalizations of those (differences between the four operations, procedures for simplification and equation solving). Simplification of algebraic expressions was introduced by translating the patterns in arithmetic into abstract schema, like multiplication as repeated addition, adding, subtracting, multiplying and dividing coefficients, multiplication as area, division as the inverse of multiplication and adding/ subtracting like things. Cups and counters (and at times 'apples') were used to model algebraic expressions, for example, $3x$ would be three cups and $3+x$ would be a cup and 3 counters. Students initially did not distinguish the modeling with cups and counters in the two cases above but slowly learnt it. Modeling something like $3(x+2)$ was harder which required the distributive property. After certain modifications of the teaching intervention, they reported that students appeared to understand the generalizations from arithmetic to algebra. Although students were reasonably successful with adding/ subtracting like things, they encountered problems due to lack of understanding of negative numbers.

The second study was also conducted with grade 8 students (ten 50 minute episodes). By the end of the unit, students' understanding of the equals' sign, expressions and equations improved and they were comfortable with non-closure of expressions. The results at the end of the study were similar to the results of the first study with difficulties persisting in the case of distributive property and negative numbers. Misinterpretation of notations was observed, especially mn as $m+n$ and not $m \times n$. The researchers proposed that a greater emphasis to be placed on operations, '=' sign, operation laws, expressions, equations and complex arithmetic and a gradual movement from operating arithmetically to operating algebraically. The modeling with concrete materials probably made the approach limiting as the students used it mechanically and remained in that world to attach meaning to the algebraic symbols. This could not be elaborated to more complex situations, especially the contexts of brackets and distributive property, where the students did not have much success. Negative numbers/ quantities also could not be treated in this situation. The similarity with arithmetic and properties of operations did not get sufficiently highlighted. Use of analogies like *apple*+*apple*+*apple* = 3 *apples* for $x+x+x = 3x$ is dangerous, as in spite of all efforts, there is more likelihood of misunderstanding the symbolism (letter as standing for object/ name rather than a number, e.g. $3a+2b=5ab$, 3 apples and 2 babanas are 5 apples and bananas). Also all these strategies work for very simple cases. The translation from one language (arithmetic) to another (algebra) was dealt with only at the surface level (replacing the number by the letter) and not fully engaging with the properties/ rules of transformations or equations or the meaning and denotation of the symbolic expressions.

2.6.3 Immediate lessons learned

There is no doubt that a strong understanding of properties of numbers and operations is an essential foundation for beginning algebra (Stacey and MacGregor, 1997). However, some of the studies, especially the ones by Liebenberg et al. and Malara and Iaderosa as described above, showed that students do not make the connection between arithmetic and algebra spontaneously, even when opportunities are created for doing so. The tendency to compute without reflection and using that to reason about situations when required without a focus on the properties of the operations is a common problem among students. Emphasizing correct procedures (as in Liebenberg et al., 1998, 1999a, 1999b) and/ or correct parsing (as in Thompson and Thompson, 1987) itself is not sufficient to make the connection between arithmetic and algebra, as very few students manage to abstract the structural features of expressions in the end. Therefore, Cerulli and Mariotti's (2001) suggestion (see section 2.3.5) of converting the properties of operations into rules of operation is very useful. Only when the focus is explicitly on rules of transformations and the discussion engages the students in explicating the possibilities and constraints of the transformation, is there some possibility of students making the requisite connection. In contrast to the referential approach where the meanings are drawn from the external domain, the 'structural approach' to introducing algebra, exemplified as above, builds meaning of the symbols internally, from the connections within the syntactically generated system, (Kirshner, 2001). The thesis builds on the lessons learned from this section of research and uses this as the framework of the teaching approach.

2.6.4 Other contexts and reasons for developing understanding of symbols

In another effort to give meaning to the symbols and manipulation on those as well as draw connections between students' natural language, informal notations, understanding of arithmetic symbols and procedures and formal algebra, Ainely, Wilson and Bills as part of the longitudinal 'purposeful algebraic activity' teaching project, have looked at the development of algebraic activity in pupils in the early years of secondary schooling (12 year olds) using a spreadsheet environment. The spreadsheet environment was used to guide the students into generational tasks which is meaningful within this environment and then moved away from it to engage in transformational tasks using transformations on 'non-letter-symbolic' representations. One set of data with three pairs (low, middle and high ability) of 12 year olds revealed no differences among the pairs with regard to transformational capability but differences with regards to generational abilities, where only the middle and high ability students were successful (Wilson et al., 2003). The low ability students however used the letters as in 'fruit-salad-algebra' and the middle ability students were not sure of the nature of the possible transformations on algebraic expressions. Another set of data collected through interviews with 12 pairs of 12 year olds (evenly distributed over achievement levels and gender) revealed their competence with transformation of not only simple expressions like $2a+3a$ but also the more complex expressions like $(a+b)-b$ (Bills et al., 2003). Students were seen to use substitution of the letter by a number or referred to the operation thereby activating the sense of the algebraic expression. The researchers view this to be an important activity for developing a sense of symbolic manipulation and stress the need for 'seeing the particular in the general' together with manipulating the general. The study showed the construction of meaning by the students as a result of the complex interaction in the spreadsheet environment through a back and forth movement between arithmetic and algebraic structures, natural language, informal notations, spreadsheet notations and formal algebraic notations.

This approach towards algebra contains a mixture of all aspects essential for algebraic activity. But the focus of the project was on generational and global/ meta-level activity with arithmetic playing a small role. Due to the nature of the spreadsheet environment, the letter/ cell number automatically, takes the number as the referent. It has an inbuilt potential to treat the formulas in the cells as functions and numbers in the cells as variables (referring to the cell and the column) or place holders (number container whose content can be changed). It also removes the students from solution of particular instances and leads to an awareness of solution for a family of problems, bridging the gap between arithmetic and algebra. Although this environment enhances one's understanding of the purpose of algebra and the representational nature of the symbols, it does not enable one to reflect on the general processes and properties of computation, which is an important component of sense making of the symbols and their transformations. Even when such tasks (as of identifying equivalences) are created, they will be tied to the results arrived at the end of the processes rather than the processes themselves. Shifting the attention between specific number and the general formula in each cell is definitely a challenge (Filloy et al., 2001; Dettori et al., 2001, Ainley, 2005).

Implications of the approach

The studies cited above emphasize syntax with all efforts made to give meaning to the acts of transformation on algebraic symbols. In the process, students do form some understanding about the meaning of the symbols themselves. Also, the studies display a range of ways of approaching the semantic-syntactic problem which algebra brings with it. Some of the approaches have been more fruitful than others and have more potential in addressing wider issues of algebra. It is somewhat clear through the research studies reported here, that only paying attention to the syntax by drawing on the similarity of syntax with arithmetic does not help much to alleviate the problem. Also students do not see the structural similarity between arithmetic and algebra spontaneously, even if they are posed with tasks/ questions which require the ability. There have not been many attempts to explicitly teach the structure of expressions to students which allows them to look at expressions flexibly as processes and as objects and combine both the syntactic and the semantic (reference for the letter) aspects of algebra. There are indications from the studies quoted in this section and the previous one that it is a promising approach and needs careful exploration. It is this task that was taken up in the present study.

2.7 *Contexts for algebra*

Besides the studies which have been discussed in the previous sections dealing with understanding the syntax and symbols, a lot of effort in the teaching and learning of algebra have gone in identifying situations which could lend meaning to the symbols and a context where these symbols could naturally arise, lending purpose to algebra. In fact, for many researchers, algebraic symbols derive their meaning from the contexts in which they are embedded. Syntactic transformations are considered meaningless (inferred from the poor performance of the students in tests) and introduction to algebra using this approach is considered to be the hardest. Many arguments in support and in contradiction to the above viewpoint have already been discussed in the chapter. It is also true that knowing syntactic manipulations does not necessarily reflect students' ability to think algebraically or to solve problems requiring algebra. Algebraic thinking and algebraic symbolization may develop asynchronously (Amarom, 2003). Thus, it becomes essential to analyze the role of contexts in introducing ideas of algebra as well as giving meaning and purpose to algebra. In this section, I will try to review some studies which deal with algebra in contexts, especially, contexts of generalization and justification/ proving, which are relevant in the context of this study. Most of the studies related to these issues have been exploratory in nature, trying to identify the problems/ issues which arise when students engage with these situations but some discuss the effect of these as instructional strategies in small teaching experiments. Equation solving is another popular approach which situates algebra but will not be considered in this review.

2.7.1 Pattern generalization as a context

Of the many possible situations, like modeling real life situations, pattern generalizing, problem solving, proving and justifying, generalization of geometric patterns has been found to be one way through which algebra can be introduced. This situation leads to algebraic symbolization quite easily as well as gives the letter a meaning (generalized number in this case). Mason (1985) considered expressing generality as one of the routes to the roots of algebra and opined that algebra provides a succinct way in which to express the observed generalities in patterns. He suggested a gradual progress for recording the general using symbols alone: first using words alone, then using words and symbols and finally using symbols alone.

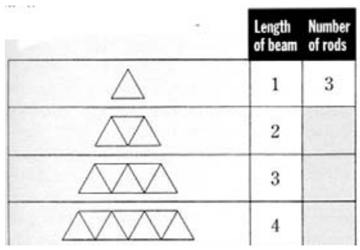


Figure 2.2: A sample problem from “Building Formulas”

van Reeuwijk and Wijers (1997) describe a unit of the ‘Mathematics in Context’ project whose main aim was to explore students’ construction of formulas on their own. Grade 7 students worked on the ‘Building formulas’ task where they had to generate a formula for a growing pattern. The students start by looking at the recursive relation between two successive positions by observing the change in successive figures. For example, while discussing the relationship of rods to beams (see Figure 2.2) they say “when the length goes up by 1, the number of beams goes up by 4”. Realizing the shortcomings of the recursion relation for prediction for larger positions they create direct formulas. Classroom observations revealed that students found many formulas which indicated the structures students saw in the situation. Students explored and explained the equivalence of formulas by reasoning from the context, applying or testing the formula in concrete situations (by substituting values in the rules) or by transforming one expression to the other using algebraic rules. In a test after the first half of the unit, students were found to give answers to various questions using recursive relations as well as direct formulas (using words, natural language and algebraic expressions) indicating their ability to describe relationships at various levels. The researchers observed that students freely used words, letter and a combination of both to explain the patterns.

Many others have found the pattern generalization task to be quite challenging. All of them reported faulty strategies (additive, recursive) rather than functional relations between the index/ figure number and the number of dots/

rods/ matchsticks to make the pattern to be the cause for the lack of success in the more complex patterns. English and Warren (1998) found that students could verbalize rules for simpler patterns (of the form $x+c$, ax) but not for the more complex patterns ($ax\pm c$). In the study by Lee (1996) a group of high school and another group of adults were involved in generalizing dot patterns in the form of rectangle (Figure 2.3a). Students were found to focus on borders (increasing number of dots in the border along the bottom and up the right side, 2, 4, 6 etc.) or boxes/ rectangles formed with dots whereas adults focused on number patterns (2, 6, 12, 20, …) leading to more success. The problem got aggravated due to poor knowledge of task requirement by most students.

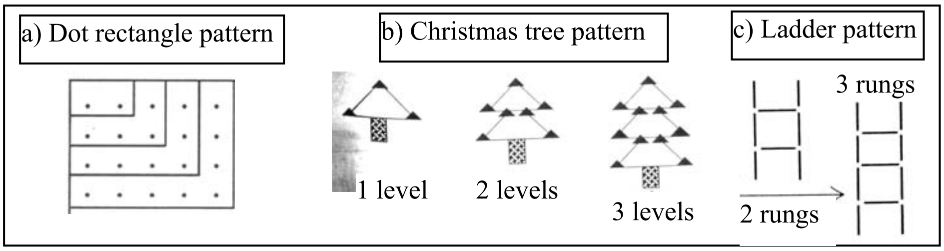


Figure 2.3: Patterns used in some studies

Stacey's (1989) study revealed the numerous strategies, many of them not likely to lead to the correct solution, used by students (aged 9-13) for working on pattern generalization task (for example, Figure 2.3b and c). Students across the grades tended to use similar strategies and models for generalization: counting method (Christmas tree with 20 levels will have 79 lights as count up by 4), difference method (number of matchsticks in a ladder of 1000 rungs is 3000 as every rung equals 3 matches), whole-object method (number of matches required for a ladder with 5 rungs is 17, so for 20 rungs, 17×4=68) and linear method (ladder with 1000 rungs will have 1000 matches on both the sides, 1000 in the middle and 2 in the end). Inexperienced students were less consistent in the models they chose and focused on easy relations not carefully building from simpler cases compared to the group of students exposed to problem solving. Many did not even check the rule with the concrete data. Some students were able to observe patterns and were able to describe them but these could not be used for predicting higher values (like, 'in between *x* and *y* there is four', 'there is three numbers missing' in a pattern given by the rule $y=x+4$) (Stacey and MacGregor, 2001; Ainley et al., 2003). Some others could find values but not express it algebraically, like when students find the value of successive positions by 'counting' the increase in each figure. According to Ainley (2003), generalizing the context is not sufficient to express the relationship in symbolic notations, and generalizing/ verbalizing the calculations (e.g. 'double the number of tables add two' and writing algebraically T2+2) could be a "bridge" which could support pupils in constructing the meaning for the symbolic expression of the relationship. Students' responses to such tasks in this study will be taken up in Chapter 8 (section 8.5)

Almost all the researchers agree and emphasize the need for verbalizing the rules and discussions and moving away from recurrent relations to functional relations which are suitable for generalization and going beyond describing the situation to describing the generalized procedures for counting, thereby also connecting it with the figure. Stacey and MacGregor (2001) point out that "to learn algebra students need to be able to recognize and articulate the process of arithmetic and the structures of relationships between numbers. One essential requirement for using algebra is that students can put their informal arithmetic knowledge into a formal arithmetic structure" (p. 148).

Sasman et al. (1999) varied the presentation of data in generalization tasks: either pictures together with numerical data or only tables of values and reported its effects on 8th grade students in Cape Town. The numerical table of values were presented as either 'continuous' (the input values for which the corresponding function values had to be calculated were included) or 'non-continuous' (input values were not given but presented verbally by the interviewer). The pictorial representations were chosen to be 'transparent' (function rules embodied in the structure of the pictures) or 'non-transparent' (function rule cannot be easily seen in the structure of the pictures). The results were not very different from those discussed above. They did not find any difference in the responses of the students with respect to the changes in presentation and the students were reluctant to find functional relationship between the variables. They worked almost exclusively with the number pattern (output values corresponding to the input), did not connect the rules with the structure in the growing pattern and favoured recursion methods or found simple proportional/ multiplicative rules ($f(x)=nx$). Although their rules worked for smaller values, they had to devise ways to adjust it for larger numbers. Some students managed to extend their recursion methods to a manageable strategy in the context of tables of values of the form $f(n) = (n-k)\times d+f(k)$, d is the common difference between consecutive terms. Errors were also seen when the numbers involved were 'seductive' (when $n_1=k\times n_2$, then $f(n_1)=k\times f(n_2)$). This was changed to $f(n)=f(a)+f(b)+f(c)$, where $a+b+c=n$ when the numbers were no longer seductive. The students came up with many strategies but they lacked the awareness and the skill to show that the rules are only hypotheses and need validation against the database. Contrary to many of the earlier comments against recursive methods, Lannin et al. (2006) showed the usefulness of the recurrent relations which can be connected to explicit rules whenever the former are not efficient for predicting larger positions/ values.

2.7.2 Issues and concerns

Pedagogical concerns

The pattern generalization task, even though promising, is not simple and there are issues which need careful thought. Stacey and MacGregor (2001) questioned the change from the traditional approach centered on equation solving which emphasized the letter-as-specific-unknown to patterns emphasizing letter-as-variable. They argue that there is insufficient evidence to indicate that an approach centered on the latter approach is better than the former. Most of the studies described in the previous paragraphs show students' difficulty in generating algebraic rules using symbols and focusing on functional relationships. These studies also highlight the fact that many students do not spontaneously verify or justify their generalization. At the most, they check it by referring to specific cases (Healy et al., 2001). Healy et al. (2001) further add that the students who generate an algebraic expression may understand the meaning of the letter as generalized number but may not understand the meaning of the expression in the context of the pattern, ignoring the structural aspects of the situation and focusing only on procedures. Lee (1996) sums up by saying that the difficulties are at three levels: perceptual, verbal and symbolic. In spite of these difficulties, focusing on the students' verbal ability and encouraging them to verbalize their understanding of the pattern helps them to make reasonable progress in the task. Some of the studies quoted above (e.g. Reeuwijk and Wijers, 1997; Ainely et al., 2003) give hope that the generalization task is potentially useful and that it can lead to useful discussions in the classrooms both about syntax and semantics of algebra.

Lee (1996) described how initiation into the generalization activity was like initiating into a "culture" where some ways of doing things are more correct than others. Some ways of looking at patterns and some ways of choosing symbols are more fruitful/ useful than others for purposes of generalization, representation and further manipulation to arrive at conclusions. Students fail to understand the limitations of a particular strategy of generalization or check the validity of a generalization. The challenge therefore is to negotiate the requirements of the task and arrive at a shared understanding, without which it cannot be effectively used to convey the meaning of symbols in algebra. Also, in many of the studies dealing with pattern generalization, as described in the previous paragraphs, the syntax issues are kept in the background and the whole attention is paid to the semantics. The syntactical problems were dealt with as they arose in contexts due to the participants' own initiatives and needs (as in Lee, 1996). The pattern generalization task creates opportunities for students to understand the activity of algebra, the meaning of the letter and explore the syntax of algebra. My contention is that the use of symbols in the process of generalization not only creates a situation for giving meaning to the symbols but also the symbols enrich the task. Moreover, unless the symbols have a role to play in the task besides representing the general, students may restrict themselves to verbal rules or the letter may just replace the words. It is important to explicitly focus on issues of representation and correct syntax as well as questions of equivalence of the rules for the same pattern, which students on their own may not attend to.

Algebra and generalization

Besides the pedagogical issues related to pattern generalization task, researchers have also commented on the activity itself and its nature and processes vis-à-vis algebra. Radford (1996) raised the issue of validity in generalizing results because the logic of generalization may differ for each student. He adds that 'the logical base underlying generalization is that of justifying the conclusion' (p. 111). The proof process in this case moves from the empirical (seeing the pattern in the numbers or the figures) to the abstract through the use of symbols (the general rule describing the pattern). This requires one to identify the features which are to be retained in the generalization and is thus directed by an anticipation of the goal. But algebraic thinking is analytical in nature and in this sense differs from generalization and the processes and objectives of these do not completely match. Generalization may not need algebra, while for algebraic thinking it is not sufficient to see the general in the particular but also express the generality using symbols (Kieran, 1989b). Mason (1996) argues for liberating generalization from just an empirical act to accommodate 'powerful generalization' which allows one to master a single example with 'appropriate stressing and consequent ignoring of special features' (p. 77) and use this as a generic example.

2.7.3 Other contexts for algebra: Proving and justifying

Many other contexts which may support students' developing understanding of both syntax and semantics, like contexts of generalization of specific number patterns in the natural number sequence or number arrangements in the calendar have also been suggested (Bell, 1995, 1996; Arcavi, 1994, 1995). Although these activities involve generalization, they are different from the pattern generalization activity which was discussed in the previous section. These tasks require one to first write the general relationships between the numbers explicating the structure of the situation and then justify/ prove the pattern that exists between them. These are rich contexts and afford many paths for representation and solution and provide a good opportunity to discuss semantics together with syntactic rules of transformation. The letter automatically takes the referent of a number in these contexts as the whole exploration is situated in the context of number patterns and the manipulation is carried out specifically to establish the generality of the pattern (Bell, 1996). The knowledge of syntactic transformations is more like a tool which when used at the right moment also makes the students aware of the need of the tool. The justification process can be of two types (Arcavi, 1995): (i) actions and processes that are analogous to the processes and actions carried on a single example (e.g. generalizing the pattern 1/2 -1/3 = ?, 1/3 – 1/4 = ?, 1/4 - 1/5 =?, 1/n – 1/(n+1) = 1/(n(n+1)), (ii) generalization and justification in the problem situation are completely different from the actions performed on a single example (e.g. what kind of numbers do we get as a result of the difference between the third power of a whole number and the number itself: that it will always be a multiple of 6). The second form of justification is more difficult than the first form.

Algebra as a tool and symbol sense

In all the cases, whether it is the pattern generalization context or the justifying/ proving, problem solving context, students need some basic understanding of symbols, their meanings and some ability to understand and carry out manipulations on the symbols. These situations not only use algebra as a tool, which means that students should know some thing about algebra, but also direct them to think with the symbols and learn to use symbols meaningfully. Some students, although able to handle the algebraic techniques of manipulating and transforming expressions, do not see algebra as a tool for understanding, expressing, and communicating generalizations, for revealing structure and for establishing connections and formulating mathematical arguments. It is in this context that Arcavi (1994) elaborated the idea of 'symbol sense', which is important from the point of view of both paper-pencil tasks as well as technology supported tasks, analogous to the idea of number sense (having non-algorithmic feel for numbers and which is different from doing arithmetical operations). Symbol sense, according to him, grows and changes by feeding on and interacting with other "senses", like number sense, visual thinking, functional sense, and graphical sense. Therefore, symbolic manipulations should be introduced from the beginning and should be taught in rich contexts which provide opportunities to learn when and how to use these manipulations. The goal for any algebra curriculum would be to instill in students "symbol sense" which includes: understanding and aesthetic feel for power of symbols, abandoning symbols and changing symbolic representations whenever needed, ability to manipulate and "read" symbolic expressions, checking the symbol meanings with one's own intuitions or expected outcomes and being aware of the fact that the symbols can play different roles in different contexts.

Emphasizing the shift in algebra from manipulative skills to conceptual understanding and meaning making, Booth (1989b) pointed out the importance of not only attending to '*what* is being represented in terms of the underlying structure and relationships in problems (the semantic aspects of algebra), but also to *how* these are represented (the syntactic aspects)' (p. 244). The other requirements for problem solving in algebraic context are (i) choosing the right kind of symbols for representation, (ii) anticipating the transformation rules to reach the goal, (iii) checking the solution process against the goal and (iv) ability to work with a referent free representation (Boero, 2001).

Further, Arzarello et al. (2001) used the two processes of 'condensation' (allows one to see different meanings in the same expression, thus stressing semantic control and creativity, that is, flexible relations between sense and denotation) and 'evaporation' (dramatic loss of meaning of symbols when they cannot be expressed in natural language and their construction and generation cannot be conceived) to explain the difference in ability of students to solve problems. The former process of condensation allows students to anticipate the path to solution and therefore carefully choose the representation to incorporate the relationships in the problem and the transformations required to reach the solution. The latter process of evaporation proceeds randomly, does not provide anticipatory power leading to the lack of representational capability and rigidity. For example, some junior secondary to university level students were seen to represent sum of two consecutive numbers as: $x+y$ or $2h+1+2k+1$ or $2h+1+2k+1+2$. This happens even though some of these students can understand and express the relationships using natural language or arithmetic but fail to use the algebraic code as a mediator between the goals of the problem and the relationships expressed in the problem. The good problem solvers could see the path of the solution process, generate appropriate representation and anticipate the transformations required to reach the goal. Students who 'have condensed' often show flexibility in understanding the sense and denotation of algebraic expressions and display a complex use of ordinary language with algebra. On the other hand, students who 'display evaporation' show rigid understanding of symbols and algebra marked by stereotypes, use symbols superficially and is not accompanied by natural language. There is indeed more semantic control when one works with numbers and where the general properties can be identified within the context of numbers. But to switch over to algebra, one needs to get rid of the 'extra-mathematical and procedural tracks and must translate them into symbolic expressions, which are highly synthetic, ideographic and relational' (p. 78).

Use of symbols for proving

The previous section elaborates on the characteristics required to succeed in problem solving tasks in algebra. The nature of control and anticipatory power is what distinguishes a successful problem solver from an unsuccessful one. Proving and justifying is one area where control, anticipatory skills and flexibility in perceiving representations as well as in creating them is of utmost importance. The pattern generalizing activity has attracted a lot of research (and this does not require much control on algebra), but studies based on other rich contexts which call forth students' understanding of symbols and engages them in the algebraic activities of justifying and proving are rare to find. Due to the possibilities created by technology, more research on adding context to algebra can be found in the context of spreadsheet, CAS (e.g. Rojano and Sutherland, 1991; Healy et al., 2001; Ainley et al., 2005) where representations can be linked and explored. Word problem situations involving formulation and solving of equations and the recent early algebra studies (e.g. Kaput and Blanton, 2001) are some examples in the paper-pencil medium. Moreover, contexts focusing on ideas of proof and justification are difficult for students as they are unable to follow or produce deductive arguments as well as understand the meaning of proof. But if the focus of proof shifts from formal rigour to understanding and communication, it is more likely that students would display a reasonable understanding of proof and its need and purpose (Hanna and Jahanke, 1993). High school students have been found to engage in meaningless symbol manipulation while proving a proposition (like sum of two consecutive numbers is an odd number) and they also chose different letters for representing the numbers or considered 'x' to be even and 'x+1' to be odd (lee, 1996). Further, they were not convinced by the general argument and therefore substituted numbers in place of the letter to verify the conclusion.

A recent study by Healy and Hoyles (2000) explored conceptions about proof in algebra among high-attaining 14 and 15 year olds (2,459 students from 94 classes and 90 schools). The study while exploring the nature of the arguments which are considered as proof, found that students thought narrative or numerical instances as better and convincing arguments for proof of a proposition. But they believed that an algebraic argument (correct or incorrect) is required for convincing the teacher or getting marks. They were however, aware of the limitations of empirical examples for proving. The narrative arguments were found to be accompanied by deductive reasoning as well as empirical examples and were valued for their generality and explanatory power. The authors argue that for many students the empirical data convinces and the narrative arguments (words/ pictures) explain and algebra does not figure in either of these roles. Lastly, most students were aware that a proof is general enough and proves all specific cases but some believed that conjectures need to be proved for specific cases as well. The students held the following conceptions of proof: to establish truth of a statement (half), to explain (more than one-third), to discover and systematize new ideas and theories (1%). Some (more than one-fourth) had no idea of the meaning and function of proof. This study indicates the importance of words and pictures as a medium of communicating and explaining generalities. For any effective intervention in the classroom with respect to proving and explaining, one will have to begin with such techniques before moving on to formal symbols and their necessity.

The explanations given by Booth (1989b), Boero (2001), and Arzarello et al. (2001) in the previous section relate directly to the findings of the study by Healy and Hoyles (2000). The fact that many of the students preferred the incorrect but algebraic argument for a conjecture to please the teacher is because they understood algebra to be composed of arbitrary sets of symbols and their manipulation. Other students who used words and pictures for explaining the results, saw the general in the particular. But they still do not have enough control on algebra to be able to anticipate an appropriate representation followed by transforming it. Similarly, those students who choose numerical instances as proof for a conjecture either see the general in the particular and therefore the conjecture is obvious or they have no more tools at the present to work on the task meaningfully.

2.8 *Conclusion*

The research reviewed in the sections above gives an idea about the extent of research carried out in the field of symbolic algebra, this being not exhaustive. The review of the research was restricted by the approach to the studies, namely, generalized arithmetic including other generalizations from patterns of shapes and numbers. The early large scale studies like CSMS and SESM assumed algebra to be encoding the general rules and procedures of arithmetic. These studies highlighted students' difficulties with the letter symbol and also pointed out that while some of the difficulties can be remedied by appropriate teaching, some are more resilient. These were also the first in a series of studies to follow which highlighted the importance of building structure sense for expressions. Many researchers explicated the differences between arithmetic and algebra and explained the causes for the difficulties. Reasons like 'didactic cut', 'cognitive gap', intuitive reasoning, suggest inherent limitations among students in operating on and with the unknown. Other difficulties are more due to the differences in thinking patterns in arithmetic and algebra. The narrow understanding of '=' sign and students' computational habits of sequentially solving arithmetic expression are two major obstacles in learning formalisms in algebra. Researchers also highlighted the necessity to see expressions as both process and product in order to manipulate the expressions. Using concrete models and concrete contexts did not prove to be of much help in the process of making algebraic activity meaningful as these are limited in their scope of generalization in other contexts, especially abstract symbolic expressions. The power of algebra comes from referent free manipulations.

Researchers have been grappling with the issue of the connection of arithmetic and algebra and whether developing an awareness of structure of expressions could help bridge the gap between arithmetic and algebra. They have also explored the possibility of introducing algebra through the route of arithmetic. Although much of the earlier studies pointed out students' difficulties in learning algebra due to the interference from learning in arithmetic and lack of structure sense, later studies indicated the value of learning arithmetic for understanding algebra. These later studies explicitly focused on the structure of expressions and delved into exploring properties of numbers and operations. There are two issues involved in this debate which need to be separated – learning rules for computation and attending to the structure of expressions. One may feel that learning the rules well and competent performance in arithmetic would eventually and automatically lead to the abstraction of the structural properties of expressions. However, this is rarely the case as is evidenced from the researches which point out dissociation between students' understanding of rules and using them in non-computational tasks. Studies have shown that although some students become competent in manipulating expressions, they fail to abstract the properties or relationships of numbers and operations.

On the other hand, explicitly drawing students' attention towards structure of expressions through direct teaching seems to hold some promise in learning rules of evaluation as well as understanding the constraints on transformations but has not been systematically tested and tried out. Many of the studies reported here indicated the fruitfulness of such an approach, but in such studies students' work with expressions were strongly tied to the context of calculation and verification. This proved to be a hindrance for the students in making the desired connections between structure of expressions and procedures of manipulating expressions and arithmetic and algebra. In the investigation being reported here, the approach developed has radicalized the use of structure of expressions to enable the developing structure and procedure sense to play a complementary role to each other, rather than one following the other. This approach will be explained in the next chapter.

It is also well understood and appreciated that unless the syntactic transformations are embedded in some context, students do not develop the capacity to use algebra as a tool. Pattern generalization from shapes is one such activity which has been widely used in research. Although this is promising, it is not simple to implement. Many of the studies reviewed indicate the nature of complexity involved while students work on pattern generalizing activities. The recursive relationships which students easily attend to are an obstacle for focusing on the functional relationships. This is essential for generating an algebraic rule to be used later for prediction. But once students understand this, it can lead to many explorations about syntax and semantics of algebraic expressions. Similarly, exploring number patterns in calendars and various other places can be challenging exercises for students to understand the meaning of the letter and transformations on expressions as well as extend their understanding of generalization. These situations lead to interesting discussions about verification, explanation and proof besides representation which serve an important function in the algebra curriculum, namely, taking control of a problem situation using algebra and knowing its advantages over other methods of solving.

Chapter 3: Theoretical background for developing the teaching-learning sequence

3.0 *Background*

The discussion of previous research studies in the last chapter revealed the range of difficulties which students face while learning algebra. Also much effort, both empirical and theoretical, has been made to understand the reasons for those difficulties. Finally, a discussion was undertaken exploring the various routes to approach algebra together with illustrations from a set of studies which have made an attempt to improve the teaching-learning situation of algebra among students in the middle grades. The studies reviewed indicate that though some of the errors related to the meaning of the letter and accepting unclosed expressions, can be tackled with a carefully chosen teaching strategy, some others, like making sense of the manipulation of expressions, are more complex, resistant to teaching and not easily resolved.

The older research tradition focused mainly on the symbolic competence of students and made efforts to give meaning to symbolic manipulation through figurate representations such as areas of rectangles with dimensions as letters or dot arrays with some dots hidden and used problem solving (mainly equation solving) as the context of application. In the more recent research studies, the emphasis has shifted towards meaning making and understanding/ appreciating the purpose of algebra, for which various contexts have been created elucidating the use and meaning of letter, expressions, etc. This has happened sometimes at the cost of neglecting symbolic manipulations which have been relegated to the background. Symbolic manipulation is often thought to be mechanical and a function that can be taken over by sophisticated calculators and other technological support systems like computer algebra systems. The assumption behind this movement is that students should be exposed to challenging situations where the maximum attention should be paid for formulating the problem algebraically, identifying the correct representation and a method of solving the problem rather than spend time on mechanical manipulation. We saw however, from discussions in chapters 1 and 2, that understanding the structure of syntactic expressions is essential to learning successfully even in these technologically supported environments. Appreciation of properties of operations, awareness of rules and conventions and knowledge of manipulating expressions is essential while working on any problem. These are important skills and would be helpful in making sense of symbols in complex situations. At different points in solving the problem, one needs to decide the correctness of the procedure and the usefulness of path taken with respect to the problem, even if the transformation of the representation can happen in a technological world. Also it is difficult to build a sense of structure of expressions by being immersed in the problem situation while attending to numerous other requirements of the problem and students may get easily bogged down by the complexity of the situation. Moreover, manipulation of expression builds conceptual knowledge and structure sense. Some support for the above argument is also found in Kieran (2004) when she points out that the transformational process can itself be embedded with meaning and has epistemic quality. There is some grain of insight in the frequent use of rules and concepts for manipulating expressions, which need to be unraveled by the students. Hence the generational activities that focus on the meaning and purpose of algebra should be complemented with transformational tasks. As a solution, Bell (1995) advocates interspersing the two kinds of tasks, problem solving and syntactic transformations.

3.1 *Research study on the transition from arithmetic to algebra*

Learning algebra, on the basis of the above description, can be conceived of as requiring students to understand syntactic transformations (based on structure of expressions/ equations and rules which define the nature of possible transformations) as well as the semantic aspect (based on the meaning of the letter/ expression/ equation as derived from the symbolic statements or problem situation). Accordingly, this study was formulated to deal with developing among students two aspects with regard to algebra: ‘reasoning *about* expressions’ (dealing with syntactic and semantic aspects of symbolic expressions) and ‘reasoning *with* expressions’(going further and developing a culture of generalization, justification and proof). One of the main aims of the study was to generate a teaching-learning sequence for beginning algebra which strengthens both procedural knowledge, that is, the calculus of algebra – knowledge of rules, conventions and procedures for working on expressions, and structure sense – sense of the composition of the expression, how the components are related to the value of the expression and their relation among each other – for arithmetic and algebraic expressions. For this, students were engaged in reasoning based on syntactic transformations of expressions in computational and non-computational situations like evaluating/ simplifying expressions, comparing expressions, identifying equality and its implications for evaluating/ simplifying expressions. Further, this knowledge was used in contexts where algebra was treated as a tool for representing, expressing generalities, verifying and proving. The study also intended to observe and characterize the changes in students’ understanding of algebra in the context of the teaching sequence which was to be developed. It was expected that the students would learn to deal with both the syntactic as well as the semantic aspects of algebra by first transforming expressions and beginning to appreciate the possibilities and constraints on transformations and then using algebra in contexts which lent meaning and purpose to the use of algebra.

Thus, the study consisted of two interconnected goals: to prepare a teaching learning sequence and to characterize students’ learning while transiting from arithmetic to algebra. The designing of the teaching learning sequence can be thought of in two parts: overall principles guiding the construction and characteristics of the content designed for transiting from arithmetic to algebra. The overall principles guiding the teaching learning sequence were the following:

- Using students' understanding and intuitions/ anticipations in the context of arithmetic to guide their learning of algebra
- Developing students' understanding of algebra by using and extending their experiences with symbols in arithmetic in specific ways
- Reasoning as a basis for learning

The connection between arithmetic and algebra was established by building the content which had the following characteristics:

- Exploiting structure sense of expressions
- Use of structural concepts (Terms and '=')
- Explicating connections between arithmetic and algebra

Figure 3.1 summarizes and captures the framework guiding the teaching sequence in this study, which also evolved with the study.

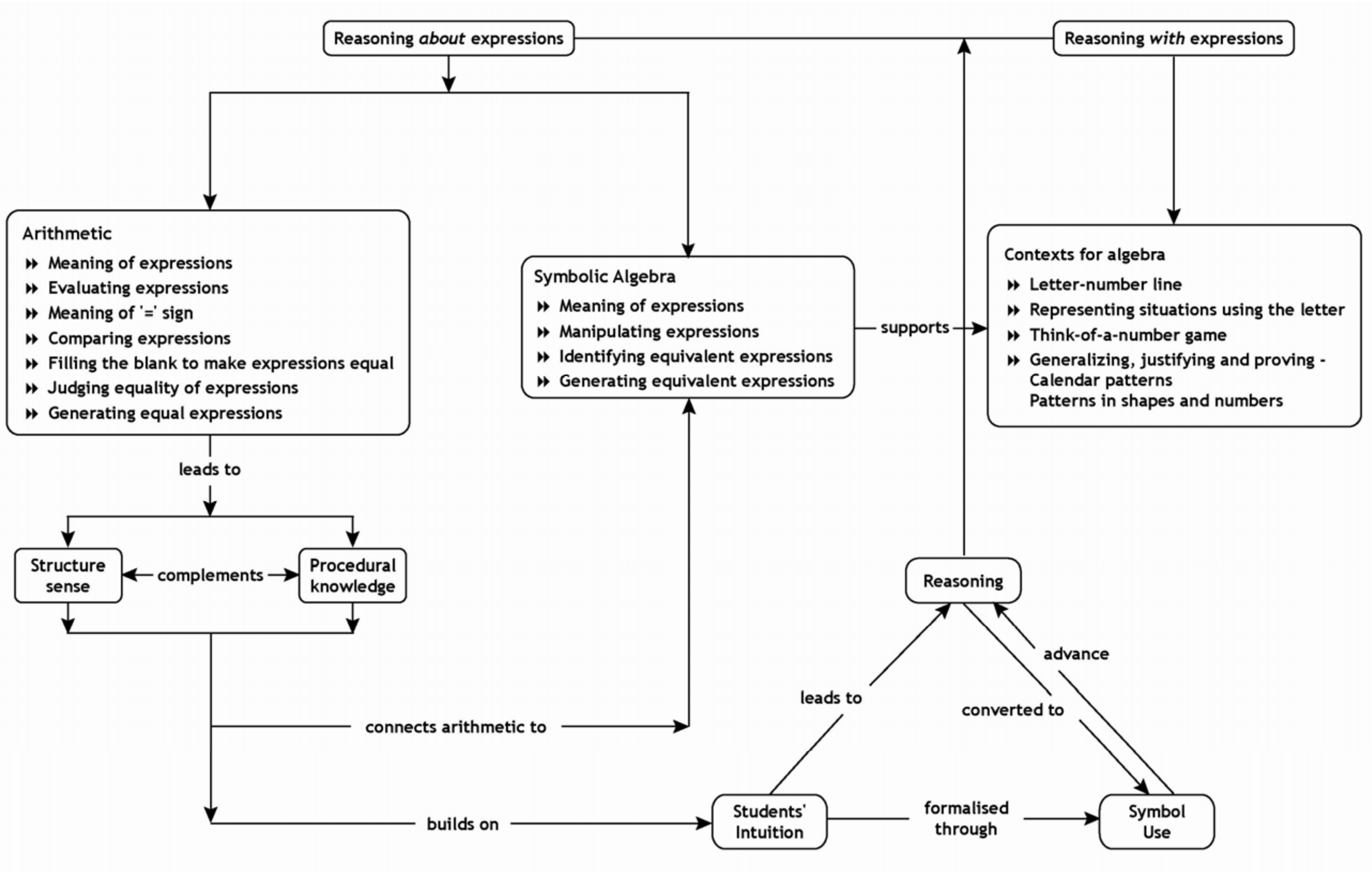


Figure 3.1: Framework guiding the teaching approach

3.2 *Developing the teaching-learning sequence*

3.2.1 Using students' understanding and intuitions/ anticipations in the context of arithmetic to guide their learning of algebra

As students begin to learn algebra, they encounter a representational system that is new and unfamiliar. Their understanding and use of the new signifier is at first guided by the old discursive habits and forms and is 'template-driven' (Sfard, 2000). As students progress, the new symbols acquire meaning leading to a stage of 'object mediated' use of symbols, allowing them to be used as representation for something else. This becomes possible due to the expectations and verifications derived from experiences in the old discourse and knowledge of rules and procedures of the old signifiers (ibid.). Whereas in the template driven phase, the use of symbols is rigid and there is no awareness of reasons for why things work; in the object mediated stage, the symbols are used flexibly. The same line of thought can be seen in Goldin and Kaput's (1996) description of the process of development of representational system drawing on a variety of ideas. These are:

- Inventive-semiotic stage: New characters are created or learned, and from the outset are used to symbolize aspects of a prior representational system. This prior system acts as a template for the development of the new system. The new characters/ symbols do not truly symbolize but are actually aspects of the prior system that they represent, which could be an obstacle for further learning/ progress.

- Period of structural development: The construction and development is driven by structural features of the earlier system. This process makes use of the symbolization that was established in the first stage and syntax for the new system is built. At this stage there is no scope for symbolic relationships other than those or contradictory to those which already exist as they are derived from the earlier domain.

- Autonomous stage: The new system separates from the earlier one. It can now form symbolic relationships with systems other than the one which had acted as the template. Transfer of meaning from old to new domain becomes possible.

The students' knowledge of arithmetic forms the prior system on the basis of which it becomes possible to absorb the new symbol system of algebra. This connection, may not, in fact, be made by students spontaneously, and they may assimilate the use of the letter in algebra to other templates, such as those provided from the domain of language or labeling in graphical systems. Hence this perspective of arithmetic, as a symbol system, leading to algebra needs to be adopted and explicitly incorporated in the teaching approach[5]. This approach has been called the generalized arithmetic approach. In this view, initially the letter replaces the number and the algebraic expression can be considered to be computational processes derived from an understanding of computing arithmetic expressions (e.g. considering $x+5$ or $2x+5$ as only a set of instructions to be followed given the value of the letter). The new symbols can be subsequently interpreted and used based on the structure of arithmetic expressions (e.g. rewriting $8-3x+4x$ as $8+x$ using the distributive property or not rewriting $5+3x$ as $8x$ due to structural constraints). It is only after this that they can be considered independently as objects with certain properties which are ready to be operated upon without the use of the template domain. The understanding than an algebraic expression has a value, that it can be combined in various ways with other algebraic expressions, that it is a function, that the letter can take any value in the domain over which the function is defined, etc. are features of an 'object' perception of the algebraic symbols. This would

[5] The possibility of other domains providing prior representational systems for the learning of algebra, such as geometry, cannot be denied (Radford, 1996).

lead to the establishment of a representation/ symbol system with its own semantics and syntax. The development of such a representation/ symbol system would depend of course on the exposure of the students to different number systems which can act as a referent for the letter and with each new exposure the meaning of the algebraic expression and knowledge of its structure will be elaborated.

In the present study, students' knowledge of arithmetic was used as a foundation on which algebraic formalisms could be built. Their understanding of syntactic rules and conventions was developed and consolidated using their anticipations regarding operations on numbers, thus tackling the pedagogical problem of teaching the syntax of algebra. By the end of primary school, students have had sufficient experience with numbers and basic operations. These are likely to have acquired properties similar to concrete objects, which can be fruitfully employed to learn formal symbols and actions on those. Their knowledge of arithmetic shapes their expectations in the situations they encounter later. It is important to be aware of and identify these expectations and the situations which could invoke these. Some of their expectations/ anticipations are correct and some are wrong which need to be brought to their notice and which they may be unable to correct by themselves. Students intuitively understand that the sum 34+29 would be less than 34+31 as adding a smaller number would lead to an answer which is less than the expression where a bigger number has been added. The counterpart of this, where a subtraction operation is used is less intuitive and many times their expectations and anticipations go wrong. Generalizing from the above situation, they may say that 34-17 is more than 34-16. It is these tendencies to generalize which contain the essence of algebra that need to be capitalized and rightly channeled.

Thus, in this approach, concepts, rules and tasks had to be framed as part of the teaching sequence which provided opportunities to work on their expectations, strengthening the right ones and correcting the incorrect ones. For example, students believe that changing the order of operations does not change the value of the expression. In fact, very young children can add two numbers by flexibly changing the order of the numbers, starting with the bigger numbers and counting up to the smaller number. Although this is true in the context of addition (5+8) and multiplication (5×8) operations where the commutative property holds true, it is not true for subtraction (8–5) and division (8÷5) where they have over generalized their expectations. This leads to errors in the context of evaluating expressions where more than one operation is involved (e.g. in 4+5×6 or 12–4+6) and where it is hard to ascertain the operation which is to be computed first. In such expressions, it is essential to identify the units involved so that the roles of the operations can be distinguished. In 4+5×6, the '×' operation scales up the '5' and not '4+5', and therefore '×6' cannot be treated as a unit scaling up 4+5. Similarly, in 12–4+6, the '–' sign is only attached to '4' and not to '4+6'. Errors of a similar kind are also seen while the students engage in identifying equal expressions from a list (e.g. given 34+13×25+49, which of the following are equal: 13+34×25+49 or 25×13+49+34. Linchevski and Livneh (1999) observed similar errors as discussed above in the context of algebra, which have their roots in arithmetic. Most of the studies dealing with structure sense (e.g. Chaiklin and Lesgold, 1984) demonstrate students' inability to form units correctly and their tendency to over generalize their expectations (discussion in Chapter 2, sections 2.3.4 and 2.3.5).

One of the reasons for this may be the fact that the procedures for evaluation are learnt in isolation, without giving any recognition to students' expectations and their understanding of the various operations. Also many times students do not grasp the generality of the rules and the fact that they should be applicable in all similar situations. Other factors like the presence of certain numbers (biasing number combinations induce errors as seen in Linchevski and Livneh, 1999) or inappropriate use of rules serving as models/ prototypes which are more intuitive (see Fischbein and Barash, 1993) provoke them to solve the expression without following the rule (see section 2.3.4). This could explain the inconsistencies seen in their evaluation procedures. There is no connection of the evaluation procedure with important concepts like the value of the expression, that the value is unique or that some transformations on an expression keep the value invariant and all the expressions involved in the process have equal value. There is no discussion about constraints and possibilities on transformations which make students think that the rules are indeed arbitrary. This notion gets carried over to algebra as well, leading to a lack of understanding of manipulation procedures in algebra. The present study tries to bridge the gap by naming the units (called 'term' in this study) of an expression in ways (to be described later) so that the conventions get subsumed in it. In this way, evaluation of expressions, contribution of each part to the value of the expression and effect of changes made to any part on the value can be addressed simultaneously. These are important skills and not only lead to a deeper understanding of arithmetic expressions but also enable the transition to symbolic algebra by considering it as encoding the structure of expressions and general properties of operations ('the phase of structural development', Goldin and Kaput, 1996).

3.2.2 Developing students' understanding of algebra by using and extending their experiences with symbols in arithmetic in specific ways

Symbols and symbolic representations were an integral component of the instructional approach. The aim being to connect arithmetic and algebra, it was essential to use symbols in ways which made evident the connection as well as elucidate the power of these symbols. Thus, the context which lent meaning to the symbols in algebra was drawn from within mathematics, namely the domain of arithmetic, rather than contexts outside mathematics. The limitations of referential systems borrowed from outside mathematics are well known and have been discussed at various places in Chapter 2 (see for example, critique of Chalough and Herscovics, 1988, section 2.6.1). To recapitulate the limitations briefly, it is often difficult for students to create meaning for symbols in a referential world and subsequently to work in a referent free environment which is essential for algebra (Balacheff, 2001, pp. 51). All students may not abstract the same properties and relationships from the situation as the instructor expects (Gravemeijer et al., 2000), the symbols not necessarily corresponding to single fixed entities in the referential world.

Further, the contexts/ activities which are created to bring in real-world situation to the classroom are often superficial with the complexities skillfully removed and suited to the level of the students (Ainley et al., 2004). If the situation taken is very complex, then it is difficult to monitor students' mathematical thinking or their learning and development in any area. They, therefore proposed 'purpose' and 'utility' as the two characteristics of tasks which could engage students in meaningful and challenging activities. 'Purpose' of the task should lead to a meaningful outcome for the learner, in terms of actual or virtual product or solution of an engaging problem and 'utility' should lead to a understanding of ways ('how' and 'why') in which an idea or technique is useful. Also, while solving problems, it is cumbersome to translate a new situation into the familiar referential world in which the symbols have been interpreted earlier and then solve it. It is not necessary that any given situation can be understood in terms of the referential world in which the meaning of the symbols were created. This requires the use of many models simultaneously or one after the other for the teaching and learning of algebra and its associated symbolism. Moreover, in many of the approaches the role of syntax of expressions and manipulating expressions in a referent free manner takes a back seat, with attention focused on only meaning making, which may limit algebraic understanding.

Not only is it the case that symbols need to be attributed with some meaning, by analogy with an existing system or by creating a system, but symbols themselves are important objects in mathematics which carry and convey information and meaning about other mathematical objects, relations or situations. In line with many other researchers (Cobb, 2000; Sfard, 2000; Bazzini et al., 2001), in this research study, it is believed that it is in the use of symbols that the meaning for the symbols is created. Meaning making and symbolization are intricately linked, with the use of symbols generating meaning in that situation and the development of meaning leading to the modification of symbols. Thus it is not possible to separate out the use of the symbol from its meaning, that is, the signifier from the signified; neither precedes the other (Sfard, 2000). According to Sfard (2000), mathematics cannot be perceptually mediated but only mediated with the help of 'symbolic substitutes' of objects under consideration. It follows from this that the meaning of the symbol would have to be developed through use, negotiation and discussion in the classroom.

Using the above as the background, the present teaching sequence exploited the use of symbols which students were familiar with from their experience in arithmetic. One of the main resources which could be capitalized in developing the sequence was students' intuitions in arithmetic. However, as discussed earlier, it was necessary to find ways to formalize these intuitions. This led to the exploration of new ways of using the same symbols to communicate and reason and also elaboration of the list by including the letter and operations on the letter. The first step in the process involved looking at symbols as also describing relations rather than only as instructions. So 4+3 no longer indicated only the operation that yields answer 7 but also as a relation standing for a number which is 'three more than four'. One can think moreover of other expressions standing for the same number, like 2+5, 10-3 etc. Similarly, x+3 would be understood as a relation. Number was the referent for the letter (new signifiers) and manipulation procedures in arithmetic were generalized to those in the context of algebra. 3+4×2 and 3+4×x have similar structure and therefore the rules for manipulating them should be similar.

Further, the numbers were attached with the signs preceding it to denote a signed number (like -2, +3), which could also represent a change (decrease or increase) in state or a relationship of greater/ less between two numbers/ quantities. While comparing the expressions 25+14 and 26+13, one can denote the total change in the later expression as +1-1=0 (26 being one more than 25 and 13 being one less than 14). The operation signs '+' and '–' were used as operations as well as signs of the numbers. These usages of symbols could be arrived at by translating from the natural language, which was accessible to them from their arithmetic experiences. Also, this short hand representation of the changes in expressions can be elaborated into complete mathematical sentences showing equivalence or otherwise (like for the above example it will be 26+13=(25+1)+(14-1)=25+1+14-1=25+14). The symbols helped in compressing the long verbal explanations for tasks like the above example into succinct expressions for easy communication to others as well as in progressing to higher levels of learning mathematics[6]. This is an additional feature of symbols, that it encapsulates explanations into short precise statements which can be manipulated, when achieving the same through the verbal mode would be quite complex and often impossible (Kaput, 1989).

In this study, symbols were considered important not only to describe situations (representational use) and find solutions to them, but also symbolic expressions and syntactic rules formed the focus of many activities. Students were engaged in discussions about order of operations, possibilities and constraints on transformations of expressions, identifying equality and equivalence of expressions. For example, one not only identified the constraints on the possible transformations of the expression 12+3×5-18 but also needed to appreciate the change in value when the expression is slightly changed, say, 3+12×5-18 whose explanation will require a semantic understanding of the

[6] In the history of evolution of algebra, from rhetoric to syncopated to symbolic, what differentiated the stages was firstly the presence or absence of letters, but more importantly, coming to know that one can operate on the letters similar to numbers (Puig and Rojano, 2004). This changed the nature of the problems as well as the solving process.

operations. This kind of knowledge would also help while representing a situation using arithmetic or algebraic expression (e.g. distinguishing a representation $x+3\times2-5-x+4-x$ from $(x+3)\times2-5-x+4-x$). It is also through these ways that one can attribute meaning to the signs/ symbols and develop algebraic understanding by relating them to a larger set of relationships – ways of representing, organizing and acting within a syntactic system (Drouhard and Teppo, 2004). Students' knowledge of symbols, syntax and syntactic rules/ transformations were further used to work on contexts like pattern generalization from growing patterns of shapes and properties in number arrangements in calendar and charts, proving and justifying them, which gives purpose to algebra. Working on such tasks leads to getting a sense of algebra, its purpose and role in structuring particular kinds of experiences and using symbols and transformations on symbols to make sense of these experiences (ibid.).

3.2.3 Reasoning as a basis for learning

Developing reasoning abilities has been considered to be an important aspect of teaching mathematics. Russell (1999, p. 1) argues that

> First, mathematical reasoning is essentially about the development, justification, and use of mathematical generalizations. …Second, mathematical reasoning leads to an interconnected web of mathematical knowledge within a mathematical domain. Third, the development of such a web of mathematical understandings is the foundation of what I call "mathematical memory", what we often refer to as mathematical "sense", which provides the basis for insight into mathematical problems. Fourth, an emphasis on mathematical reasoning in the classroom, as in the discipline of mathematics, necessarily incorporates the study of flawed or incorrect reasoning as an avenue towards deeper development of mathematical knowledge.

Algebra, being a field which builds on generalizations and relationships drawn from various other areas, is explicitly used for purposes of reasoning. As has been noted in the previous chapter, there has been a recent shift in focus from simply doing algebra to thinking and reasoning algebraically. This requires one to explore and articulate relationships and patterns among numbers and operations on them (like properties of operations: commutativity, asociativity, distributivity, patterns among consecutive numbers etc.), in situations and among quantities. To be able to reason algebraically does not necessarily mean working with formal symbols. From the written performance in standard algebra tasks of manipulating expressions one cannot discern whether or not a student has difficulties with the formal syntactic structure or whether he/ she can understand the generality of the processes. On the other hand, asking students to explain their solution methods or choice of strategies leads to richer responses from students. Greenes and Findell (1999) recommended that children should be provided with opportunities for algebraic reasoning from the very beginning which focus on the 'big ideas' like representation, balance, variable, proportionality, function and inductive and deductive reasoning which can lead to symbolic representation and have multiple solution paths (see section 2.5.3 in chapter 2 for more details on studies based on these ideas). Students can use their intuitions and their preferred ways of thinking to reason about mathematical situations as has been seen in many studies (e.g. Carpenter and Levi, 2000; Kaput and Blanton 2001). Not only does reasoning, especially through open discussion in the classroom, encourage students to participate in algebraic thinking, they also learn in the process powerful and meaningful mathematics which can be of use in future. Students negotiate meanings, share ideas among themselves and are exposed to various strategies and their explanation and justification.

In the teaching approach adopted in this study, certain tools comprising of concepts and rules were provided as aids to the reasoning process of the students through explicit teaching. The purpose was to use them repeatedly in situations which would lead to a stable understanding of expressions and would also support the intuitions and strengthen the concepts and rules themselves. The focus being on expressions, which is one of the core elements in algebra, activities revolved around various aspects of expressions, arithmetic as well as algebraic. Further, these tasks could be considered to be of two major kinds: reasoning *about* expressions and reasoning *with* expressions. Students reasoned *about* expressions when they evaluated expressions by choosing an appropriate strategy of computation. This required an understanding of the ways in which the value can be found, which was articulated through discussions regarding when changing the order does not make any difference to the value etc. It included engaging with expressions containing brackets (34-(12+7) or 3×(5+4) etc.) and understanding the ways in which brackets make a difference to the value of the expression. Reasoning about expressions was also involved when they compared two expressions, generated expressions equal to a given one or judged for equality of two expressions with and without computations. These activities provided the context where students explained their solutions to each of the tasks and discussed more strategies and reasoning styles which would involve generalizations about operations. The engagement of the students in such reflective activities with regard to syntactic based transformations was to begin the separation of the meaning from the value of the expression in the context of arithmetic itself. Although, this is not essential within arithmetic, it lays the ground for further algebra learning (see Arzarello et al., 2001). Disparate looking expressions could have the same value with different information/ relation contained in them and similar looking expressions could have different values. Thus, transformation of algebraic expressions using valid rules would keep the value same but change the meaning contained in it, which is an important step in learning algebraic manipulation and working on algebraic problems. Also, violating the constraints on operations would change the value of the expression.

Reasoning *with* expressions involved representing the situations using expressions and transforming them to derive inferences. These contexts required simple representation of relationships between quantities, pattern generalizations and justifying and proving certain patterns. It was expected that to explain their understanding of the situations and tasks, students would engage in articulating their explanations verbally, create representations which can be used to communicate in the classroom and use the concepts and rules which had been taught earlier to work on the representation. In this way the teaching sequence afforded to incorporate students' ideas and symbols as well as push it towards the direction of formal algebraic symbols.

3.3 *Bridging arithmetic and algebra*

3.3.1 Exploiting the 'structure sense' to connect arithmetic and algebra

As mentioned, the teaching approach exploited the structure inherent in arithmetic expressions to connect arithmetic with algebra using the familiar symbols, thereby giving the letter a referent of number. Further the approach sought to provide visual and conceptual support to the students to perceive the structure of an expression correctly. At this point it is important to clarify that 'structure' in the above sentence does not mean the broad algebraic structures but the particular structure of expressions. In the course of exploring the structure of expressions, elements of algebraic structure do not become transparent, although aspects are implicitly focused: commutativity, associativity, distributivity and at times the notions of inverse and identity. In the context of this study, following Kieran (1989a) and Hoch and Dreyfus (2004), *structure sense* means the ability to think of an expression as having a value, to identify the components of an expression (surface structure) and to see the relationships of the components in an expression among themselves and with the value of the whole expression (systemic structure). Surface structure is important to perceive the expression or equation and for analyzing the components in it. Knowledge of systemic structure, on the other hand, allows one to act on the interpretation of the surface structure. In particular, understanding the '=' sign, equality of expressions and properties of operations are important aspects of structure sense. It includes understanding possibilities and constraints on operations and overlaps with and is facilitated by certain aspects of 'operation sense' (Slavit, 1999). It has also been called relational understanding by some researchers (Fujii and Stephens, 2001). This is in contrast to *procedural knowledge* which only focuses on ability to correctly execute procedures and not test for the understanding of expressions or relationships within an expression or among expressions.

The reason for emphasizing the structure of expressions in the teaching approach was to link procedures with a sense of structure, so that instead of being two separate skills one following the other, they complement each other to form an integrated knowledge structure. The once emphasized dichotomy between procedural knowledge and conceptual knowledge (which in the case of algebra includes an understanding of structure) no longer appear to be valid as concepts feed into the formulation of procedures which in turn strengthen the concept. That is one reason why it has become important not to ignore procedures and instead integrate the two aspects, procedure, concepts and/ or structure. This would also allow the possibility of turning the familiar symbols and processes on them into objects (Sfard, 1991) which have their own properties and which can be further manipulated to lead to higher order mathematics. Moreover, knowledge of structure of expressions provides an explanation for the procedures and scope for flexibly exploring procedures and strategies for computing expressions, rather than applying the conventional rules for evaluation, which are rigid.

In traditional curricula, it is expected that students would abstract the structure of the expressions on their own by repeated use of standard procedures on expressions. But this does not occur with most students, as seen in the numerous studies quoted in the previous chapter. One reason could be that students are not exposed to situations where such a skill or knowledge is important, computing and arriving at answers being the goal of mathematics. Therefore, it is essential that tasks which can induce students to develop and apply structure sense must be a part of the instructional sequence. Since the emphasis on computation hampers perception of structure (as seen in sections 2.3.4 and 2.6.2) the tasks need to focus away from computations. As has been noted earlier, students by the end of primary school form sufficient understanding of arithmetic, even if they are not expressed in the standard computational tasks. Such understanding can be formalized and guided towards the development and use of new symbols in algebraic expressions.

The perception of structure seems to be determined by the length and complexity of expressions, the kind of operations involved in the expression and the feasibility of calculation as a way to arrive at the answer in the situation. An expression which is long involving more than one operation on numbers which are also large poses a challenge for computation. Only then does one attend to the structure of expressions. For example, an expression like 234 + 125×347 - 129 is more likely to attract students' attention to structure, if they are required to judge the equivalence of this expression with another expression like 125 + 234×347 – 129. In contrast, comparing an expression like 12 + 4×6 with 4 + 12×6 is less likely to elicit attention to structure since students can take recourse to calculation. This was observed in the initial trials. Appendix-I gives the list of tasks that were used in the trials, the tasks in the domain of 'reasoning *about* expressions' are relevant in the context of the above discussion on connecting procedure and structure of expressions and arithmetic and algebra.

3.2.2 Role of structural concepts: 'Term' and 'equality'

In order to understand the structure of an expression, learning to parse the expression correctly is an important skill which needs to be developed. In this study, students were provided with a set of concepts which allowed them to correctly identify the units of the expressions and further understand the contribution of each part of the expression to the value of the whole expression. Also, gradually these concepts helped in reformulating the rules for order of operations and bracket opening in structural terms, thus integrating the procedures more closely with structure. It was expected that the various tasks such as evaluating/ simplifying expressions, comparing, identifying equal expressions etc., would draw on these concepts and in turn reinforce them. These tasks sometimes involved the use of only surface structure and at other times included both surface and systemic structure, in the senses described earlier.

The two concepts of 'term' and 'equality' which helped in making the structure explicit were identified during the study. 'Term' in the traditional Indian school text books is used for no other purpose but to identify like and unlike terms for simplifying algebraic expressions. Its full potential lies in parsing expressions correctly: arithmetic or algebraic, thus also connecting arithmetic and algebra. Figure 3.2 (read from top to bottom) shows the use and the place of 'terms' in the present study. The two concepts of 'term' and 'equality', which have been called 'bridge concepts' (Subramaniam and Banerjee, 2004; Subramaniam, 2004) enable the students to explore properties of operations on numbers and possibilities and constraints on transformations as well as to carry out the various tasks which require manipulating expressions and judging their equality. The concept of term plays a crucial role in delineating the surface structure of an expression whereas the concept of equality is important for understanding the deeper systemic structure of expressions.

Terms are units of the expression demarcated by '+' or '–' signs, the '+' and the '–' signs attached to the number following it, which can be transposed without changing the value of the expression. For example, in the expression $12 + 3\times5$, $+12$ and $+3\times5$ are the two terms, with the former being called a simple term and the latter a product term. Terms are of two kinds: simple terms and complex terms. Complex terms include product term and bracket term (For example, $-(3+5)$). The factors of the product term can be numerical (like $+2\times3$) or variable (like $+2\times a$). $+a$ or $-a$ can also be rewritten as product term ($\pm1\times a$) with one variable factor and the other factor being 1.

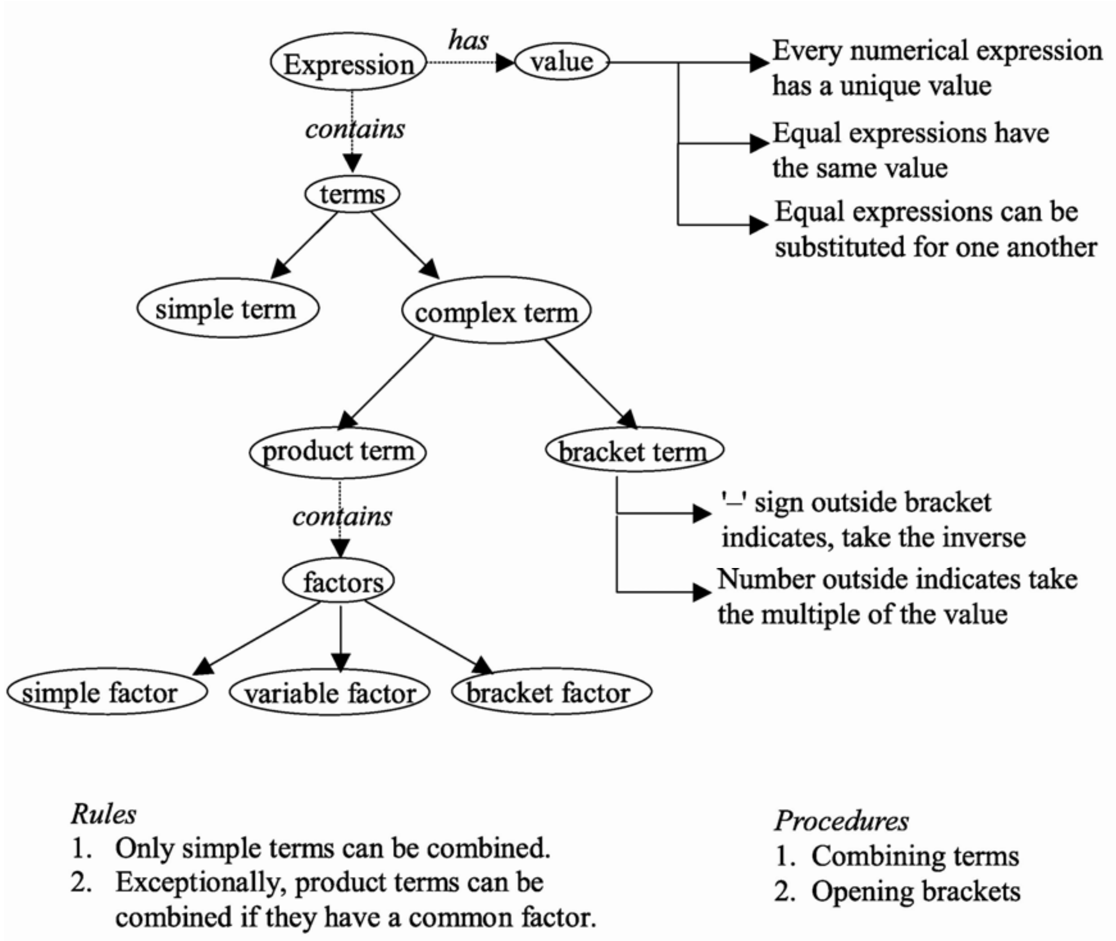


Figure 3.2: Role of terms in the teaching approach. (This is a modified version of the map found in Subramaniam, 2004)

The rules for evaluating expressions are reformulated structurally using the idea of combining terms, replacing the usual precedence rules. Combining terms is based on the idea of compensation, that is, equal and opposite terms cancel each other. Simple terms can be combined easily like the following.

12-5 = $\boxed{+12}$ $\boxed{-5}$ = $\boxed{+7}$

Product terms can be combined with a simple term by converting the product term into a simple term.

4+5×2 = $\boxed{+4}$ $\boxed{+5\times2}$ = $\boxed{+4}$ $\boxed{+10}$ = $\boxed{+14}$

Two product terms can be combined if they have a common factor using the distributing property.

$3\times2 + 7\times2 = \boxed{+3\times2}\ \boxed{+7\times2} = \boxed{+(3+7)\times2} = \boxed{+10\times2} = \boxed{+20}$

Further, the idea of combining terms, together with the intuitive idea that positive terms increase the value of the expression while negative terms decrease it, is a step towards making the processes of addition and subtraction into objects. Also, by attaching the numbers with the signs, new symbols have been created from the old ones but with meanings which can preserve and guide students' intuitions and expectations. As can be seen, this approach subsumes both, the operations on the signed numbers as well as allows for flexible evaluation of expressions by allowing terms to be combined in any order. This approach is called the 'Terms approach'.

Implicitly, by attaching '+' and the '–'signs to the terms, integers are always added while terms are being combined and there is no subtraction operation on integers. The newly formed structural rules for evaluating expressions are equivalent to the precedence rules which are commonly used. The various types of terms have been defined in such a way that there is no contradiction in the two ways of solving an expression. It simply replaces with the structural counterpart the procedural terminology of evaluation like 'do multiplication first' and 'move from left to right in the case of expressions with only + and – signs' or acronyms like BODMAS[7]. For example, in an expression 14-5+3, 5+3 cannot be computed first according to the conventional precedence rules, but many students make this error called detaching the '–' sign (Linchevski and Herscovics, 1996). In such cases the 'Terms approach' not only parses the expression in an unambiguous way (which otherwise is only possible through

[7] In fact, the acronym BODMAS (computing an expression in the order – starting with Bracket, Of, Division, Multiplication, Addition and lastly Subtraction) is misleading as it suggests that addition should be done before subtraction which may not always lead to correct answer.

the use of brackets and not very helpful in the context of algebraic expressions) but also provides the flexibility to compute in any order leading to the same value.

The formulation of these two concepts strengthened the syntactic aspect of teaching-learning sequence. They not only created meaning for the operations but also afforded a more direct approach to tackling the structural errors which have been widely cited in the literature (Chaiklin and Lesgold, 1984; Linchevski and Herscovics, 1996; Linchevski and Livneh, 1999; Kieran, 1989a). The approach provided the tools in the form of concepts, rules and symbols to 'reason *about* expressions' without computations. For example, by identifying the terms in the expression 27 – 34×12 + 17, one can see what components make up the expression as well as see how the components affect the value of the expression. The positive terms +27 and +17 would increase the value of the expression whereas the negative term which is quite large in magnitude -34×12 would decrease the value of the expression. It is also possible to observe that interchanging the positions of 27 and 34 would not only change the value of the expression but also that the resulting expression 34 – 27×12 + 17 would be greater than the original expression as the negative term is now smaller in magnitude (and hence decreases the value of the expression by a lesser amount) as well as one of the positive terms have increased. It is thus clear that such a transformation would not keep the value of the expression invariant. But other transformations like combining terms in any order, transposing the terms, extracting a common factor from two terms (simple or product) are possible and would keep the value of the expression same. It is in this manner that the approach attributes meaning to the operations and procedures as well as strengthens students' procedural and structural understanding.

Bracket was another symbol which was given a lot of attention. Its significance lies both in enclosing parts of the expression which get precedence in operation and can be replaced by a number on evaluation, as well as in affording an important transformation: removing brackets from the expression using bracket opening rules to get equal expressions. Students need to be comfortable with these two meanings of brackets to be able to appreciate the structural difference in two expressions like 34-(12+5) and 34-12+5, as well as to use brackets as a tool for representation purposes. In the context of symbolic algebra and transformation of algebraic expressions, the second meaning related to transforming to obtain equal expressions is more important, since it is not possible to always evaluate the expression inside the bracket. However, in the various contexts which lead to algebra, one needs to use brackets to enclose parts of the expression which in principle need to be evaluated prior to carrying out further operations. For example, the instruction 'add two to a number, and then multiply the result by three' cannot be written as $x+2\times3$ but must be written as $(x+2)\times3$. In this study, the bracket opening rules were reformulated using the concepts of term and equality in conjunction with the ideas of inverse and multiples. A leading '–' sign for a bracket term indicates that the inverse of the bracketed expression (or its value) is to be taken. With this extension, the whole of integer addition and subtraction is brought under the terms approach. For example, the inverse of 4+6 is –(4+6) which is equal to -4-6 and -10 is the inverse value of the expression 4+6. A number multiplying the bracketed expression indicates that a multiple must be taken. For example, 3×(4+8) is three times the sum 4+8 which is equal to 3×4+3×8 and 36 is three times 12 (the sum of 4+8). The same concepts and rules are used in the context of algebraic expressions by exploiting the structural similarity between the two kinds of expressions with the number being the referent for the letter.

Further extensions

Another extension is required to include the division operation in the teaching approach, although it was not dealt in the present study. Division by a number can be considered as multiplying by the multiplicative inverse of the number and thus can be represented as a fractional notation. Therefore, 35÷6 has one term $\boxed{+35\times1/6}$ or $\boxed{+35/6}$

7+15÷3 contains two terms $\boxed{+7}$ and $\boxed{+15\times1/3}$ or $\boxed{+7}$ and $\boxed{+15/3}$.

The challenge in this case is firstly to make the students understand the equivalence of fraction/ rational number notation with division/ sharing. Students would also need to appreciate the equivalence of the two notations: explicit product term notation (with unit fraction as one of the factors) and the fraction notation (can be called the 'fraction term'). In this way, again by emphasizing the structure of the expressions, tendencies to sequentially process information in the expression will be reduced as well as will help in converting the division into fraction notation. This is important keeping in mind that algebraic expressions will soon have rational numbers as coefficients and the letter will also take rational numbers as referents, for which meaning has to be created so that manipulation can be performed on them. The rules for combining the terms will be similar to the existing ones and will also need to be extended to deal with the complexities of the fraction notation. A fraction term can be combined with a simple term only if it can be converted into a simple term. Else, the value of the expression can be represented as a mixed fraction or an improper fraction. Two fraction terms with common denominator can be combined in a manner similar to combining two product terms with a common factor. Two fraction terms which cannot be converted into simple terms or do not have a common denominator, will have to be converted into equivalent fractions so that they have common denominator and then the two fraction terms can be combined. Further, the explicit parsing will enable students to distinguish expressions like x^2+2^2 from $(x+2)^2$ and eventually may lead them to use the correct rules in expressions like $\frac{x^2-9}{x-3}$ and not simplify it as $x-3$, either by considering $x^2-9 = (x-3)^2$ or considering $\frac{x^2-9}{x-3} = \frac{x^2}{x} - \frac{9}{3}$.

3.3.3 Explicating connections between arithmetic and algebra

Using the 'terms approach', on the one hand, arithmetic operations were being reified, with the ability to look at relations between the terms without necessarily computing them at each step and using properties of operations for combining them, and on the other, manipulation of algebraic expresions and the meaning of the symbols was being developed on this understanding of arithmetic. One of the important differences between arithmetic and algebra is that the processing of the information given in an algebraic expression cannot happen sequentially, unlike in arithmetic. The approach (the concepts together with reformulated rules) prepared the students to become flexible in perceiving the information, interpreting the relationships and further simplifying it using the same rules, properties and constraints on operations. Some preparatory steps need to be carried out while simplifying algebraic expressions before the rules can be applied, for example, separating the like terms from the unlike terms. In this context it is essential to know which transformations can be carried out without changing the denotation of the expression. For example, to simplify an expression $8 \times x + 15 - 3 \times x - 7$, the two terms $+8 \times x$ and $-3 \times x$ can be combined as they are the only two product terms in the expression with a common variable factor and this is possible because the denotation does not change by changing the order of terms. Also the terms $+8 \times x$ and $+15$ cannot be combined as one is a simple term and the other a product term with no common (variable) factor between them. In a paper-and-pencil situation, the validation of such knowledge can only be in the context of arithmetic where there is a possibility of evaluating the expressions and checking for equality. This is not possible in the context of algebra, where the rules of transformations are generalized from the context of arithmetic. In fact, the act of substituting a value for the letter to verify the correctness of the answer takes one back to arithmetic.

Further, manipulating algebraic expressions would lead to equivalent algebraic expressions (like, $2\times x+8=2\times(x+4)$) that may occasionally be more compact (like $2\times x+5\times x=7\times x$), but not to closed single number answers as in arithmetic. This involves a strong understanding of the concept of equivalence of algebraic expressions. Permissible transformations on an expression do not change the value of the expression, and each step in the simplification process keeps the algebraic expressions equivalent, which is an important idea in algebra. Many of the difficulties in manipulating expressions in algebra can be attributed to the lack of this important concept. Inconsistent and arbitrary use of rules for manipulation stems partly from the fact that the students do not know that each step leads to equivalent expression.

The equivalence of algebraic expressions is defined as functional equivalence; on substituting the same value of the variable, both expressions result in the same value. However, operationally, equivalent algebraic expressions are obtained by making valid transformations on the expressions. It is important therefore to ensure that transformations find a place even in working with arithmetic expressions, so as to allow students to integrate the concept of equivalence with valid transformations. This is an essential part of the terms approach, and the flexible approach to evaluating arithmetic expressions together with anticipating results of operations prepares students for this important concept.

It was decided not to use the compressed notation of algebra (concatenation) and keep the '×' notation as in arithmetic, thereby maintaining the similarity in notations and bridging the gap between arithmetic and algebra in the syntactic world. Nunes (1997) pointed out that compression of representations open new relationships which can be understood only by connecting them to the new operations, making it hard for students to understand them. Discussion and verbal explicitations of aspects of notations with regard to algebraic expressions and comparing and contrasting with arithmetic play an important role in making sense of the expressions and their transformations. Once again reasoning *about* expressions could be a key to understand these links.

3.4 *Creating contexts for algebra*

The tasks described above belong to the category of 'reasoning about expressions' and could take the students from the 'inventive-semiotic' stage to the 'template driven' (Sfard, 2000) phase of 'structural development' (Goldin and Kaput, 1996). To move the students to the 'autonomous' (ibid.) or 'object mediated' stage (Sfard, 2000), tasks belonging to the category of 'reasoning *with* expressions' were used (see Appendix-I). This is not to claim that 'reasoning *about* expressions' cannot take the students to this advanced stage but this path was not considered suitable for this study. For example, complex operations on algebraic expressions, thinking of algebraic expressions as functions and anticipating changes in function are good activities in the 'autonomous' stage.

The second part of the study consisted of exploring students' use of the letter and expressions in situations which lent meaning and purpose to algebra. The contexts chosen for 'reasoning *with* expressions' were of generalizing, proving and verifying. The important ideas to be grasped in this part are (i) the importance of representing situations for general cases, (ii) knowing that justification/ proof needs a general argument/ explanation (verbal or symbolic) not specific to particular cases, (iii) appreciating the purpose of transformation on a representation, (iv) transforming the representation using valid rules and (v) interpreting the result. Most students do not appreciate that situations can be represented in a general manner without instantiating it. For example, to represent Mohan is four years older than Shalini whose age is represented by k years, students often choose a value for 'k' (say 15 years) and then find the age of Mohan (in this case 19 years). Booth (1984, 1988) pointed out that this happens because students have very primitive means, symbols as well as procedures, of working on situations which does not allow them to represent general procedures. A good grasp of processes of arithmetic and articulation of structures of relationships between numbers is essential prerequisite for using algebra (Stacey and MacGregor, 2001). This allows one to translate the informal processes or arithmetic structures into formal arithmetic or algebraic sentences. However, Balacheff (2001) argues that students' familiarity and comfort with arithmetic could itself be an obstacle for expressing generalities as they may not feel the necessity to use algebra where the problem could have an arithmetic solution.

Even if students manage to represent the situation using letters, they may not know how to use the representation or to manipulate it in order to obtain a meaningful result. They have difficulties in both constructing a proof and in following it using deductive logic (Healy and Hoyles, 2000). Further, they are not convinced about the truth of the solution arrived by manipulating the algebraic expression (Liebenberg et al., 1999b, Cerulli and Mariotti, 2001). Instead they often substitute the letter by a number and check for the correctness. The purpose of algebra is not clear to most students and there are many hindrances in the path. Similar is the case with generalization. Students do not see the reason behind generalizing patterns and expressing it for purposes of prediction. They use methods which are not suitable for predicting the value of an unknown position (English and Warren, 1998; Stacey, 1989; Sasman et al., 1999). These issues have been discussed earlier in sections 2.7.1 and 2.7.3.

In this context, the tension between arithmetic and algebra emanates from the fact that arithmetic solutions begin and end in number manipulations without ever establishing the generality of the result. Algebra is the only way by which one can prove and justify general results and solutions. In this study, the effect of knowledge of syntactic transformations on representing situations and manipulating them in the given context to arrive at conclusions was explored, that is, to use Kieran's (2004) terminology, the role of transformational activities on global/ meta-level activities.

Students, in this study, were first engaged in simple representation tasks similar to the CSMS test items so that they could learn that representations could be made when all quantities were not given, with the letter/s denoting one or more of the unknown quantities in the situation. Continuing with the spirit of the study which follows a generalized arithmetic approach, students worked on tasks which required generalizations of patterns and relationships between numbers. These situations were further exploited to give way to 'reasoning *with* expressions' which expected them to conclude some general idea/ rule from their representation (e.g. Take three consecutive numbers. Show that the sum of the first and the last is twice the middle number). In such an example, the students can indeed work with numbers and see the truth of the statement but it cannot be proved for all cases unless algebra or some visual/ verbal explanation, which is general enough, is used.

The purpose behind the use of these tasks was also to explore possible ways of making the role of algebra accessible to the students. One way to circumvent students' inability to deductively work on representations was to engage them in articulating the problem and their explanation for it. In the process, symbols other than formal algebra could be used to communicate and convince oneself of the explanation. The use of arithmetic (number) representations was allowed in this study which students were familiar with and which they could relate to meaningfully as the problems dealt with number patterns arising from many contexts. Also the general proof was closer to the particular numerical instances. The belief was that such actions would enable the students to see the 'general in the particular' (Mason, 1996) and gradually move to the symbolic level by appreciating the goal of the task. At this point, they could use their knowledge of syntactic transformations. As with any other aspect of mathematics learning, it is important to consider the process of arriving at the proof, and not only emphasize proof as a product (Healy and Hoyles, 2000; Heinze, 2004). The same is true for generalization; one must look at the process of arriving at the generalization rather than just the general rule in the form of an algebraic expression.

In the following two chapters, the methodology of the study and the evolution of the teaching approach will be described. In further chapters, analysis of the data will be carried out in the light of this framework in order to explore the extent to which the students displayed the requisite understanding across the three trials of the main study.

Chapter 4: Description of the design experiment

4.0 *Methodology*

Identifying a design for conducting research which would adequately help in answering the research questions is crucial for carrying out any research. The design chosen is not only the one suited to answer the questions being raised in the research study, but also reflects the assumptions underlying the research. It captures the development in understanding of the field under study and the theoretical and methodological advancements made in the areas of learning and teaching in education. Over the years, methods of conducting research have changed from largely quantitative experimental designs to qualitative (descriptive studies, case studies etc.) and mixed method designs. Researchers are no longer content with laboratory experiments with strictly controlled variables and they have moved into complex settings such as school systems, classrooms, workplaces where it is no longer possible to answer the questions using traditional designs (Kelly and Lesh, 2000a). They point out that "the needs of learning and teaching, and the descriptive, analytic, and communication needs of the community of researchers should help bring forth and test a diversity of research methods. These methods may include those borrowed from other traditions (e.g. anthropology) as well as those emerging from within the practices of mathematics and science education research (e.g. teaching experiments, design experiments, action research)" (ibid., pp. 35-36).

One can find two reasons for changes in the research design. First, there is a growing interest in answering questions which are more complex, like means and mechanisms of learning in the context of tasks or learning environments in the classroom within a socio-cultural context. Recent studies not only provide empirical results about certain processes under investigation but also enhance theoretical understanding of the area concerned (Collins et al., 2004;

Shavelson et al., 2003; Lobato, 2003). Second is the view that the researcher is a participant in the whole process and not an outsider, co-constructing knowledge with the students and the teacher in a dynamic environment. The purpose of the research is to understand and describe the complexity of the system, develop models of thinking about learning and teaching and analyze the change in the participants' understanding of subject matter and their meaning making process. The data is collected iteratively over cycles observing complex behaviour, and also accepting theory ladenness of both observation and method in the research (Kelly and Lesh, 2000a).

There are various ways to investigate the complex system of teaching-learning. Teaching experiments are a class of research methodologies which encompass varied research foci: development of students, development of teachers, development of ideas in groups, teams, individuals, classroom instructional environments, and instructional activities including use of software (Kelly and Lesh, 2000b). What binds them together is the fact that all these "focus on development within conceptually rich environments that are designed to optimize the chances that relevant developments will occur in forms that can be observed" (ibid., p. 192). The time range of such experiments is flexible (from few hours to months) and so is the setting (interview sessions to whole class learning environments).

4.1 *The design experiment*

The focus of this study was to develop among students an understanding of symbolic algebra, using their knowledge of arithmetic as a foundation. Design experiment (e.g. Cobb et al., 2003) was found to be the most suitable for this particular research as there did not exist a fully formed teaching-learning module which could be tested against a control group. Such a module was to evolve over time. A few characterizations/ definitions of "design experiment" are:

> Design experiments are extended (iterative), interventionist (innovative and design based) and theory oriented enterprises whose "theories" do real work in practical educational contexts (Cobb et al., 2003, p. 13).

> Design experiments, based on prior research and theory and carried out in educational settings, seeks to trace the evolution of learning in complex, messy classrooms and schools, test and build theories of teaching and learning, and produce instructional tools that survive the challenge of everyday experience (Shavelson et al., 2003, p. 25).

As is clear from above, design experiments are conducted for the purpose of formative evaluation of research and it systematically tries to study learning processes in a context defined by the means of supporting them (Collins et al., 2004; Cobb at al., 2003). The design is put into practice and tested and revised based on experience to lead to the development of some local domain-specific theory; addressing theoretical questions and issues delineating why it works or understand the relationships between theory, artifact and practice (Collins et al., 2004; Brown, 1992; Cobb et al., 2003; DBRC, 2003). The theory intends to "identify and account for successive patterns of student thinking by relating these patterns to the means by which their development was supported and organized" (Cobb et al., 2003, pp. 11). Gravemeijer's (2001) methodology of *developmental research* embodies fundamentally the same ideas.

The present study started with a conjecture (due to its evolving nature it is not a hypothesis, Confrey and Lachance, 2000) of finding the extent to which arithmetic could be used for the purposes of teaching children algebra. The assumptions/ conjectures had to be progressively tested in order to bring about the necessary coherence in the teaching sequence. Previous literature in the field of algebra education had shown the difficulties students face while transiting to algebra if they did not know arithmetic well, especially if they lacked the understanding of the structure of expressions. These have been discussed in detail in Chapter 2. The difficulties, their reasons and the non-existence of a well elaborated model of teaching and learning of algebra using arithmetic led to the formulation of this research study to systematically explore the effect of arithmetic on teaching and learning of algebra. To build the sequence, it was required to first identify suitable teaching-learning materials and then to elaborate it by making the required connections between arithmetic and algebra. The elaboration and modification of the approach was carried out through a series of five teaching trials. Over each of the trials that was conducted, strengths and limitations of the concepts, ideas and tasks were identified leading to a suitable modification of the sequence in the next trial of teaching. This was continued till there was some connection seen between arithmetic and algebra in students' understanding as well as in the teacher-researcher's judgment.

The first two trials were more exploratory in nature and the last three trials formed the main study which aimed at making the teaching learning sequence coherent. The research aimed to study the developing understanding of students in a context where the teaching approach explicitly built bridges between arithmetic and algebra, by giving visual and conceptual support for the understanding of expressions. This was to be achieved by building strong structure and procedure sense of arithmetic and algebraic expressions. The study further intended to explore the connections between procedure and structure sense; and between arithmetic and algebra. The teaching learning sequence co-evolved with the developing understanding of the researchers about the phenomena under study as well as with the growing understanding of the students as evidenced from their performance and reasoning on various tasks. The study did not aim to measure the efficacy of the approach taken vis-à-vis the traditional method of teaching algebra or any other approach. Nor did it aim to show its efficiency in terms of time taken for teaching. In fact the teaching approach took some time to evolve and attain the required level of coherence and completeness due to the nature of the study. Evidence for understanding and use of the concepts introduced during the study in the students' performance in various tasks was looked for, with a focus on the links between understanding of arithmetic and algebra. In particular, the study hoped to identify concepts that facilitated students' understanding in both domains, namely 'bridge concepts' and loci of transfer. Another intended outcome was to identify principles for teaching and learning of algebra and to operationalize these principles through a detailed teaching learning sequence.

4.2 *Research questions*

- What kind of arithmetic understanding would help in learning symbolic algebra?
    - How should the teaching of arithmetic expressions be restructured to prepare for a transformational capability with algebraic expressions?
    - How effective is such a teaching learning sequence in understanding beginning syntactic algebra?
    - Which tasks of the ones identified are more effective in making the shift possible from arithmetic to symbolic algebra?
- Does understanding the syntax and symbols of algebra support students in understanding the purpose of algebra and in the application of algebra for generalizing and justifying?
- What meanings do students attach to letters, expressions and syntactic rules of transformations in this learning approach?
- How do procedural understanding and structure sense of expressions mutually support one another?

4.3 *Sample and location of the study*

The study was conducted in the research institute with grade 6 students (11-12 year olds). The students came from nearby English and Marathi medium schools which catered to students from low and middle socio-economic backgrounds. Four schools (E1, E2, E3 and M1) were involved in the study at various stages. Schools E1, E2 and E3 were English medium schools and M1 was a medium school. Out of these four schools, two schools (E1 and M1) were involved throughout the study. The choice of the schools was based on convenience, the first reason being their proximity to the centre and the second, due to a need for long term collaboration and support from the school to carry out the study. The students were called to the institute (Homi Bhabha Centre for Science Education) during the vacation periods of the school in Summer (April-May) and mid year (October-November) for 11-15 days with each session lasting approximately one and a half hour. The academic session of the schools start in the Summer (June). Application forms were distributed in some nearby schools before each of the vacation periods of the school. The students for the program were then randomly selected from the applications received from interested students of these schools.

The differences between the schools were not systematically studied as part of the research being reported here. The school E2 catered to people working in a prestigious government organization, the schools E3 and M1 were state government aided schools and E1 was a missionary school funded by the Church. The school E2 followed the national curriculum and the text book provided by the central government and the other schools followed the curriculum of the state and the text book prescribed by the state government. The students in all the English medium schools came from various parts of the country and therefore did not share a mother tongue, unlike the school M1. Although these students in the English medium school understood English, most of them were not fluent in the language to be able to explain their thoughts. But all of them understood and spoke Hindi, the national language of the country. In this study, they were allowed to discuss and respond in any of the two languages - English or Hindi. The students from the school M1 were taught in the same language (medium, Marathi) in the study as their medium of instruction in the school. They had good command over the language they spoke. Further, during the teaching intervention, the students from E1 were found to be less efficient in systematic writing than students in M1. Many of the students in M1 appeared for the scholarship tests conducted by the state for which they were given special training by the school.

Granting that there were differences among the groups of students, no effort would be made to explicate the reasons for difference in their performance during the study, as that is not the focus of this study. It is acknowledged that the exact nature and reason for the difference would be interesting to pursue in a separate study. The medium students (M1) were better to start with as seen from their pre-test performance and they continued to perform well in the study. The aim here was to trace the understanding and use of the concepts by the students during the various trials.

Students' participation and continuation in the study were both voluntary and low pressure. They had no pressure of either completing home work or preparing for the tests during this study compared to the school situation, which made it a fun filled situation. In the school they are forced to complete the home work. Much of the time is spent in routine mechanical computations and they have to commit to memory a whole lot of information for the tests. The time spent in the teaching sessions was considered to be sufficient for learning the concepts and skills introduced and they were not required to prepare for the tests. Although the students knew that they had to appear for a pre and a post test in the beginning and the end of each trial, this did not bother them. Small tests were also conducted from time to time during the trials about which the students were not intimated in advance. Teaching was conducted as a whole group activity and the tasks were mostly done individually by the students, although they were free to talk to their peers. After they had finished working on the task and were ready with their solutions, students typically had to explain their strategies and solutions in whole group discussions. A few activities, which were considered challenging, were carried out in pairs. This too was followed up by the whole class discussions.

In the school during grade 6, students learn integer operations, evaluation of arithmetic expressions, manipulation of algebraic expressions, evaluation of algebraic expression upon substitution and solving linear equations in single variable. Text books following both the state and the national curriculum treat these areas as distinct without making any explicit connection, presents them as formal bodies of knowledge, explaining the rules/ properties which govern the operations. The text books give examples of situations which require the use of integers or letters to represent events/ phenomena. Further, problem solving is restricted to situations leading to simple linear equations in one variable. Classrooms are largely teacher directed and focused on students mastering the rules of transformation and learning to solve the problems in the respective areas. Teachers follow the text books and try to explain the solved examples and the problems given in the exercises at the end of each unit. All students then appear for tests which are conducted periodically by the school and success and failure is determined by their scores in the test.

4.3.1 The trial cycles

In all, the study consisted of five trial cycles. Each trial cycle involved between 11-15 days of teaching, with approximately 1.5 hours of teaching per day. The first two trials were pilot trials and were more exploratory in nature (PST-I, PST-II) and the last three trials form the main study (MST-I, MST-II, MST-III).

Pilot study trials (PST)

The first trial of the pilot study (PST-I) was conducted in Summer (April) 2003, with students from only one school (E1). The trial was announced in the school before their summer vacations and students were invited to volunteer for the course. The students were finally selected randomly from the list of applicants. Eighteen students participated in this trial. The students had just finished their grade 5 exams and were awaiting their results. The students had not as yet had any instruction (especially in algebra) for grade 6 in school. The course lasted for 13 days with sessions of one hour each. I was the only teacher in this trial and the students were taught arithmetic.

The second pilot study trial (PST-I) was held in October-November 2003, with 82 students in three groups (25+23+34) from three schools (E1, E2, M1) participating in the trials. The trial was, as before, announced in the schools before their mid-year vacations and students randomly selected from the applications for the trial. A few students who had participated in the earlier trial from school E1 chose to participate again in this second trial. The students were in the middle of their grade 6 in the school and had recently had some instruction in arithmetic expression evaluation and integer operations before they came for the trial. The students from schools E1 and E2 were mixed and then divided into two equivalent groups (Group-I and II) based on their pre-test performance and given separate treatments. Group-I was taught both arithmetic and algebra whereas Group-II was taught only algebra with some instruction in an unrelated domain (geometry), to maintain the total interaction time among the two groups. The students from the school M1 formed the last group-III and were given the same treatment as to Group-I. Four researchers from the centre including me, were involved in the teaching and formed the research team. The units of arithmetic and algebra were taught by two different teachers in Group-I but by one teacher in Group-III. Algebra and geometry were also taught by two different teachers in Group-II. The course lasted 11 days with each session being one and a half hours long.

	Pilot study trials		Main study trials			Final analysis
	PST-I	PST-II	MST-I	MST-II	MST-III	
English	I: 18 (E1)	I: 25 (E1+E2)	A_1: 23 (E1+E3)	A_2: 28 (E1)	A_3: 21 (E1)	English: 15 (E1)
	-	II: 23 (E1+E2)	B_1: 29 (E1+E3)	C_2 (E1)*	-	
Marathi	-	III: 34 (M1)	C_1: 38 (M1)	B_2: 42 (M1)	B_3: 22 (M1)	Marathi: 16 (M1)
Total students	18	82	90	70	43	31
Sessions	13×1 hr	11×1.5 hrs	16×1.5 hrs	13×1.5 hrs	11×1.5 hrs	40×1.5 hrs

Table 4.1: Sample used for the study in the five trials (* indicates group not followed and reported in the study)

Notes: I-III, A_1-A_3, B_1-B_3, C_1-C_2 = Groups which participated in the study. E1, E2, E3, M1 = Schools which participated in the study

Table 4.1 gives an overview of the sample used in the five trials of the study: number of groups, number of students who participated in each trial, number of sessions, duration of each session and the sample for the final analysis.

Main study trials (MST)

The first trial of the main study (MST-I) was conducted in Summer (April) 2004, with 90 students from three schools (E1, E3, M1) participating in the study. This was a fresh batch of students who had just finished their grade 5 exams and were waiting for their results. The trial was announced in the schools and students volunteered to attend it as earlier. The students from schools E1 and E3 were combined and two equivalent groups: Group A_1 (23 students) and Group B_1 (29 students) were formed[8]. Students belonging to school M1 formed the Group C_1 (38 students). Groups A_1 and C_1 followed the

[8] The number of students in the equivalent groups is not the same as some students dropped out before or on the last day, when the post test was held.

same teaching-learning sequence whereas Group B_1 followed a slightly different approach but the same core ideas were taught. Groups A_1 and C_1 were taught both arithmetic and algebra whereas group B_1 was taught mostly arithmetic and a little algebra. Four teachers were involved in this phase as well, with three of them being researchers and collaborators in the project from the institute. One high school teacher was involved in this phase who taught arithmetic to Group A_1, algebra being taught by me. The other groups were taught by a single teacher. The course lasted for 16 days with a session of one and a half hour each day.

In continuation of the first trial, the second trial of the main study (MST-II) was organized in October-November, 2004. Two schools (E1 and M1) and two groups of students were involved in this phase. Only those students from these two schools who had earlier attended the trial MST-1 were invited again to attend this follow up trial. The sample was restricted to only two neighboring schools due to the cumbersomeness of the design with large number of students. Students from Groups A_1 and B_1 who attended MST-I were combined and a group of 28 students (Group A_2) made for MST-II. A few new students joined Group C_1 of MST-I and the new group was called Group B_2 in MST-II with 42 students. These groups then followed the same teaching-learning sequence. Students during this phase were in the middle of grade 6 and had received some instruction in school on evaluating arithmetic expressions and had been recently introduced to integers. A fresh batch of students was also called during this trial and 26 students attended the programme to form the Group C_2. They were taught both arithmetic and algebra using the same core ideas as the other groups but a slightly different sequence. Since this group was not followed, I would not report the results of this group but would point out some interesting ideas that emerged from this trial as part of the evolution of the approach and the teaching-learning sequence which will be suggested in the end. Three teachers (researchers and collaborators from the centre) were involved in this cycle with each of them taking care of a group and they taught both arithmetic and algebra to the respective groups. The course involved 13 teaching sessions of one and a half hour each.

The third and the final trial of the main study (MST-III) was held in Summer (April) 2005. The same students who had attended the first and the second trials (MST-I and II) were invited for this final phase of the course. 21 students of Group A_2 (henceforth A_3) and 23 students of Group B_2 (henceforth B_3) attended the trial. These students had just finished their grade 6 exams and had been participating in this study for a year. In the course of the instruction in school in Grade 6, students in group B_3 had been introduced to evaluation of algebraic expressions for a given value of the letter and manipulation of algebraic expressions including addition and subtraction of two algebraic expressions. These were carried out in the traditional way by collecting like and unlike terms separately and further extracting the common factor between the like terms. Two teachers (researchers and collaborators from the centre) were involved in this last trial taking care of a group each. The course lasted for 11 days with sessions of one and a half hour each day.

Sample used for analysis

Amidst this constant inflow and outflow of students, throughout the trials, it is important to analyze the performance of those students who were constant across the main study. At the end of the main study, two groups of students from two different schools (E1 and M1) had participated in all the trials, which lasted a year. These were 15 students from the English school E1 and 16 students from the school M1. The reason behind having two groups of students was not for the purpose of comparing them but to broaden the study. It was of interest to see how the two groups of students would respond to the teaching-learning sequence and how they progressed with time. In the thesis, the focus will be on the performance of only these students who had participated in all the trails, which will be tracked and discussed. This is essential so as to limit and make sense of the large amount of data that was generated due to the involvement of many students over various trials. Moreover, in the present design of the study, where the teaching learning sequence progressively evolved, discussion of the performance of those students who did not go through the entire study trials would be incomplete with respect to some aspect of the learning sequence. The contribution of the other students who participated in each of the trials will be considered as part of the overall development of the teaching-learning sequence, their responses providing valuable feedback in each of the trials.

4.4 *Data collection*

A part of the data was collected by administering pre and post tests to the students during each trial phase. The tests were modified in each trial as the teaching-learning sequence changed or evolved, although many basic concepts and the corresponding items remained the same or were similar. The concepts, skills and procedures tested were:

(i) evaluating and simplifying arithmetic and algebraic expressions,

(ii) filling in the blank to make two expressions equal,

(iii) comparing expressions with and without calculations,

(iv) judging equality of a given expression to other expressions,

(v) using the letter to make simple representations and

(vi) problem solving tasks embedded in context.

The tests consisted of items which had to be solved with the working shown, as well as items which had multiple options. The students were required to mark right or wrong against each of the options. The post test in each trial included most of the questions asked in the pre test together with the concepts and skills taught and learnt during the particular trial. The post tests had an average of 25 questions and usually students worked on them for two hours. (See Appendix – IIA, B, IIIA, B, IVA, B for pre and post tests of MST-I, II, III respectively). The tasks used in the study and the items in the test were fairly new for the students, except simplifying and evaluating arithmetic and algebraic expressions which some students had encountered in the school.

Interviews were also conducted with the students at the end of MST-II and MST-III to get a better picture of students' understanding and reasoning about numbers, operations, expressions and symbols and how they used the concepts and skills learnt during the trials to respond to the tasks. The students chosen for the interview performed at an average and above average level in the tests and were active (but not necessarily correct) in the classroom discussions. Interviews were not conducted with very low performing students. The written responses of many of these students in the post test of MST-II showed interesting errors and inconsistency in reasoning across tasks. Fourteen students (6 English medium and 8 medium) were interviewed after 8 weeks of the end of the second trial of the main study (MST-II). Although the overall performance of these students improved in the post test of MST-III, they were interviewed again after MST-III. These students and some additional students totaling to seventeen (8 English medium and 9 medium) were interviewed 4 months after the end of the final trial of the main study (MST-III). Three of the students interviewed had not appeared for one of the three post tests but had participated in all the trials.

The students interviewed after MST-II were posed tasks only in arithmetic whereas the students interviewed after MST-III were posed tasks both in arithmetic and in algebra. The tasks that were common in both the interviews were of evaluating expressions and identifying expressions equal to a given expression. In the second interview, students' responses to some other tasks related to their understanding of the use of letter, algebra and transformation of algebraic expressions were also explored. Contexts of think-of-a-number game and pattern generalization were used to elicit their understanding of the use of algebra and letter, which they were familiar with. The interview was largely structured with minor changes made at the time of the interview depending on the students' response (See Appendix – VA and B for the interview schedule used after MST-II and III respectively). The students were first asked to complete some tasks on a sheet of paper and then in a one-on-one conversation with the interviewer the student explained his/ her answer/ solution for the task and responded to further additional probes. In the case of tasks of evaluating arithmetic expressions, the additional probe consisted of an explanation for an alternative solution, its correctness (MST-II and III) or a request for another way of finding the solution and whether two ways of evaluating would give the same answer (MST-III). In the task of judging equality of an expression to a given list of expressions, the additional probe consisted of asking for the reasons for the students' judgment (equal or not equal) (MST-II and III) and if the expressions were judged unequal, to compare them for more/ less (MST-III). Students were also explicitly asked whether the value of the two expressions being compared will be the same if they are equal (MST-III). In the context based tasks embedding algebra, besides solving the task, students were additionally probed with respect to their understanding of algebraic expressions, transformations on them and their equivalence.

Data was also collected by recording students' daily work and coding their responses each day on a sheet of paper. Also the teacher-researchers maintained a log file of the daily proceedings in the classroom. All classes were video taped for later transcription and analysis of the required portions. The interviews were also video and audio taped and later transcribed and coded for analysis.

4.5 *Data analysis*

The data from different sources was analyzed with a focus on the nature of responses, the type and number of errors and the students' reasoning as inferred from their responses to written tests or explanations given in the interview. The analysis was carried out to ascertain the extent of students' understanding in different task domains of concepts, rules and procedures:

- Understanding of procedures: Evaluation/ simplification of arithmetic and algebraic expressions
- Rules for transforming expressions with brackets
- Understanding of structure – tasks based on '=' sign, identifying expressions equal to a given expression from a list without computation, generating equal expressions
- Context based tasks – letter number line, calendar patterns, think-of-a-number game, pattern generalization

The tasks which have been categorized as 'procedural' required the students to use the rules and procedures for operating on expressions taught during the study, in the context of arithmetic and algebraic expressions to lead to numerical answers or simpler expressions. Although the rules themselves were structurally formulated, they could be procedurally applied to evaluate/ simplify the expressions. The tasks categorized as 'structural' deemphasized computations and instead focused students' attention on the structure of expressions, and on identifying relations among expressions and within an expression. In the process, the tasks elicited students' intuitive understanding of operations and anticipations with respect to simple transformations like increasing and decreasing number/ terms, rearranging numbers, terms and signs.

The analysis of the different kinds of tasks sought to throw light on the following specific aspects of students' understanding: order of operations, transformation of expressions, understanding of '=' sign, equality/ equivalence of expressions, meaning of letter, ideas about representing a situation using the letter and manipulating the expression to arrive at a conclusion. The effort was to examine students' use of the concepts and rules that they had learnt during the trials in the various tasks and the extent to which their learning facilitated performance on various tasks. This is compared and contrasted with what is reported in the literature about students' difficulties with formal syntactic algebra, structure sense of expressions and their knowledge of rules of transformation and equivalence. The analysis of the data leads to an understanding of the extent to which the approach used for transiting from arithmetic using 'reasoning *about* expressions' based on syntactic transformations to algebra is effective. Further, it gives a sense of the nature of the concepts required to make the transition from arithmetic to algebra. For a complete understanding of algebra, it is essential for students to be able to use algebra in contexts. Therefore, data was analyzed to see if knowledge of syntactic transformations, which have been attributed some meaning in the context of transformations and invariance of value, would help in 'reasoning *with* expressions'. Since the study is a design experiment, I would try to capture the changes in students' responses as the teaching learning sequence evolved with respect to the above mentioned aspects.

The data from all the sources, that is, pre and post tests, students' daily worksheets, video recordings of classroom, teacher's log file and interview transcripts were used for the analysis. The performance of the students in the pre and the post tests in each of the trials was analyzed both quantitatively and qualitatively. The percentage of correct responses was calculated for each of the items as discussed earlier. Also the quantitative data was used to find if there was a significant difference in the performance between the pre and the post tests on similar items using t-tests. Correlation between procedure and structure tasks, and arithmetic and algebra tasks were found. Graphs were plotted to illustrate the data wherever suitable. The responses of the students in the tests on various items were coded to explicate the extent of facility as well as the difficulties students faced while working on the tasks. All interviews were transcribed and coded for the correctness of the solution to a task, the appropriateness of the explanation to an additional probe (alternative solution, explanation for a judgment, identifying more/ less of two expressions) and the changes made by the students during the interview on a particular task. Video recordings of classroom sessions were transcribed whenever required to clarify certain points. Data from the classroom and interview transcripts will be used to corroborate and explain the results obtained from the post test as well as to give a glimpse of the capabilities the students demonstrated at various points while working in groups. Actual interview transcripts will be used to illustrate the nature of explanations given by students to various tasks.

Chapter 5: Evolution of the teaching approach

5.0 *Introduction*

The teaching approach evolved over five trials between 2003 and 2005. The aim was to develop a sequence which builds bridges between arithmetic and algebra and strengthens students' sense of the structure of expressions. As has been discussed in Chapter 3, the development of the teaching sequence followed a set of principles, and the content had a set of characteristics which were to act as a guide in the teaching intervention. Of course, the teaching approach did not start with a full fledged list of principles and characteristics; rather they evolved as a result of the trials. The feedback received through students' classroom responses and performance in the tests, and the subsequent discussions between the teacher-researchers, led to decisions to continue, change and modify parts of the sequence, the tasks and approach to ideas, which were incorporated in the next trial. This was repeated till some coherence was observed in the sequence of tasks and the concepts and skills required for working on them, inferred from students' consistency in responding to various tasks.

The process may be termed a 'mathematics teaching cycle', as described by Simon (1995). The first teaching trial was based on an understanding of the major areas of concern in algebra and the difficulties students face in learning algebra. It was also motivated and guided by some of the existing intervention studies, which have been reported in Chapter 2. A teaching sequence was constructed which aimed at identifying instructional material as well as testing their efficacy, sequencing and identifying pre-requisite concepts or skills needed for developing structure sense. Tasks were chosen, adapted and modified from the existing literature for the trials. Students' intuitive as well as formal ideas about operations, symbols and procedures were given due importance in the classroom, allowing the students to articulate their reasoning, so as to be able to build on them. During the enactment of the teaching sequence in the classroom, the students were engaged in making sense of the tasks and the expectations from them (e.g. that they have to explain their solution, that they have to understand the explanations given by others) and the teacher was engaged in observing and making sense of the students' responses and actions. This led to changes not only in the subsequent trials but also small immediate changes, with regard to examples, and explanations in the same trial. Accordingly, the '*hypothetical learning trajectory*', that is, the teacher's prediction as to the path by which learning might proceed (ibid.) was suitably adjusted each time. In the following pages, discussion specific to the evolution of the teaching approach through the five trials will be taken up.

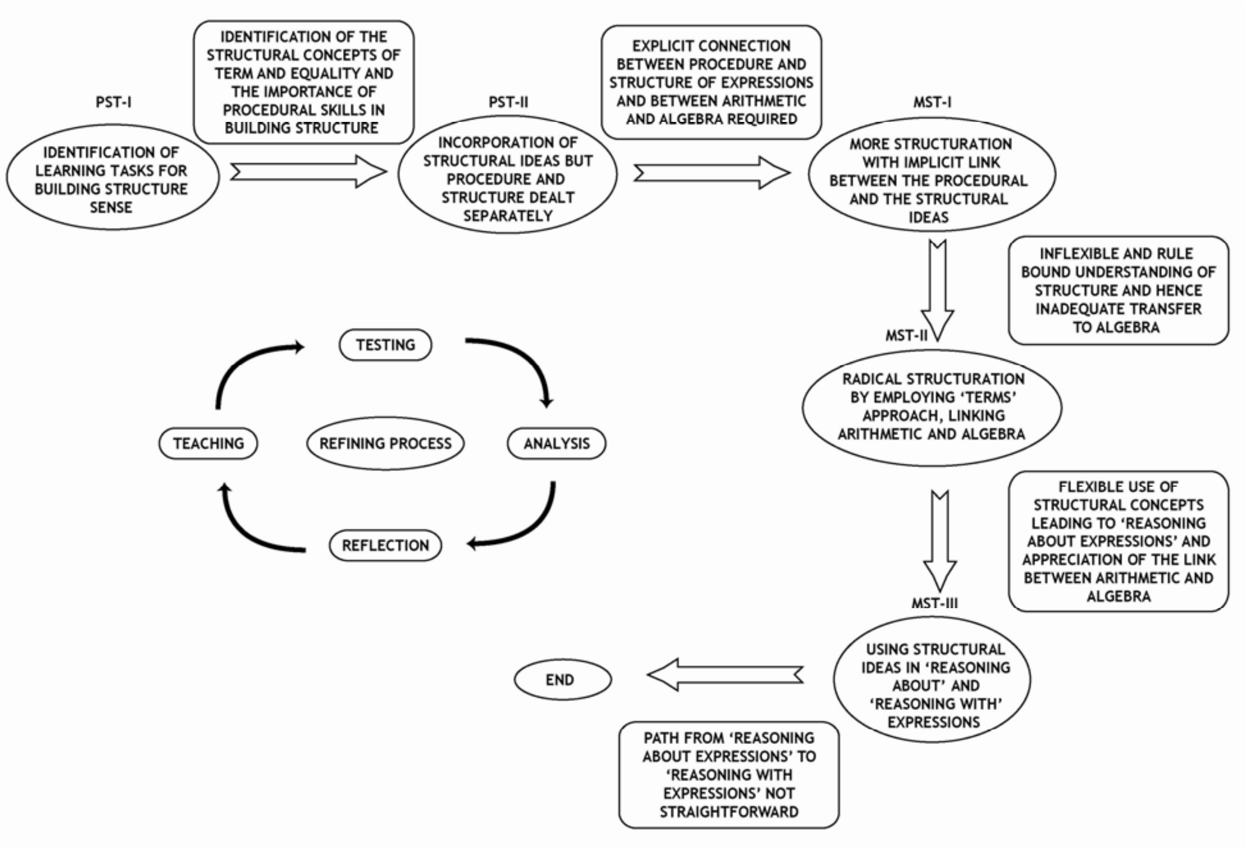


Figure 5.1: The process of evolution of the teaching sequence through the five trials

Figure 5.1 is a schematic depiction of the evolution of the teaching approach through the five trials. It highlights the focus of the trials (in the oval shaped figures) and the outcome of the trial (in the rectangular boxes).

5.1 *Pilot study trials*

5.1.1 Pilot study trial-I

The first trial cycle of the pilot study (PST-I) was aimed at exploring instructional material which can induce a sense of structure of arithmetic expressions among students. The trial began with an introduction to simple two and three term arithmetic expressions and verbalizing their meaning. The purpose was to inhibit students from computing expressions spontaneously as a result of many years of practice with arithmetic expressions. This was supposed to be the first step towards building an appreciation for structure of expressions. Students wrote expressions for a given number and verbalized its meaning. For example, they learnt that 5 + 8 stands for the number 13 and conveys the information that it is ‘8 more than 5’. They could use many ways to express the same information (e.g. sum of 5 and 8), which led to an exploration of different phrases like ‘more than’, ‘sum’, ‘difference between’, ‘less than’, ‘product of’, ‘times’ and ‘quotient’. Different expressions could stand for the same number, conveying different information about the number. They also learnt to write arithmetic expressions for a relation expressed verbally. The key idea was to understand that the expression stands for a number which is the value of the expression and contains some information about the number which is stated as a relationship between two or more numbers.

Rules of evaluating simple expressions, like 13 – 5 + 8 and 6 + 2 × 4, were explained to them in the traditional fashion by explicating the precedence rules. For an expression containing only ‘+’ and ‘–’, the computation had to be carried out from left to right and for expressions containing ‘×’ and ‘+’ or ‘–’ sign, multiplication was to be carried out first. These procedures of evaluation were further reinforced by differentiating the meaning of the expressions by verbally stating them. For example, 9 – 3 + 4 is four more than the difference between nine and three whereas 9 – (3 + 4) is difference between nine and the sum of three and four, suggesting the difference in the way the computation is be carried out. The rules of order of operation reflected and determined the meaning of the expressions. Of course, stating the meaning of all expressions was not always easy but sensitivity towards differences in meaning was created. The division operation was not used and the examples were carefully chosen so as not to lead to an answer involving negative number as these students had no acquaintance with integers. In the subsequent trials however, although students learnt the meaning of expressions, the procedures for evaluating expressions were not connected with their meaning.

Another major aim in the first trial of the pilot study was to elaborate the meaning of the '=' sign as a symbol denoting the structural relation between two expressions. Like in many of the studies discussed in Chapter 2, students in this study too had an operational understanding of '=', taking it to separate the question from the answer of the problem. Tasks were designed so that students understood the symbol as a relationship between two expressions on the two sides of the '=' sign. The students were required to compare expressions like 23 + 4 and 27 – 1 using the signs <, =, > or fill in the blanks so that the expressions on both sides of the '=' sign are equal (e.g. 25 + 8 = ___ + 12). The belief that when students are pushed to work on tasks without calculations, they can focus on the structure of expressions, led to formulation of tasks where students had to judge whether pairs of expressions, which were related like 27 + 32 and 29 + 30, were equal or unequal. While working on this task, students were comfortable in anticipating the answers as long as the expressions involved only positive terms but this anticipation broke down when it involved a negative term like in 56 – 6 and 57 – 9 or 56 + 58 + 1 and 57 + 59 – 1. They could not anticipate the change in the result because of the simultaneous transformation required on both the numbers in the presence of the '–' sign and inability to keep track of the transformations. They tended to use the same kind of reasoning as in the case of only '+' sign (27 + 32 and 29 + 30), leading to wrong comparison in the presence of '–' sign.

It was felt that there was a need for better understanding of the syntactic aspects of expressions, especially knowledge of brackets could give the students tools necessary to reason about such expressions. Hence bracket opening rules were introduced by comparing expressions involving brackets with other expressions (e.g. 12 – (6 + 4) with 12 – 6 – 4 and 12 – 6 + 4) and identifying expressions whose value did not change. The rules were arrived at by inductive generalizations based on such comparisons. Verbal meaning of the expressions was again used as a means to explain the reasonableness of the rules. In the above example, sum of 6 and 4 is to be subtracted from 12 and therefore if 6 has been subtracted from 12 then 4 more needs to be subtracted from it, to keep the value same. All bracket opening rules were based on such meaning and were named 'adding a sum part by part' (13 + (3 + 4)), 'adding a difference part by part' (13 + (5 – 2)), 'subtracting a sum part by part' (12 – (6 + 4)) and 'subtracting a difference part by part' (12 – (6 – 4)).

Subsequently, a different version of the earlier comparison task on judging equality/ inequality of pairs of expressions was introduced: finding the value of an expression given the value of a related expression (e.g. If 234 + 487 = 721, then 235 + 488 = ?) (adapted from van den Heuvel-Panhuizen and Gravemeijer, 1993). This task had a well defined goal of finding the value of an expression (with which they are more comfortable due to their experience in arithmetic) compared to the earlier one which only required one to compare two expressions without reference to their answers/ values. Students could readily give verbal justification for these questions, but the symbolic justification introduced by the teacher was not very easy for them to understand, nor would its relevance have been apparent to them. The presence of negative numbers continued to trouble them in these exercises. At this stage the idea of terms, as components of expressions, was introduced to them and they quickly verified and learnt that rearranging terms does not change the value of the expression. The students identified terms in expressions and wrote them by using commas as separators (e.g. in 12 + 4 – 3 the terms are +12, +4, –3). The idea was used by them to even generate expressions equal to a given expression by rearranging the terms. This concept opened the possibility of using it in all the other tasks the students had been working prior to this like comparing expressions, finding the value of the expression given the value of a related expression, which required some rearrangement of terms to justify the answer as well as help in avoiding the problem with the negative numbers.

Lessons learned and implications

The completion of this preliminary trial indicated the modifications required for evolving the teaching-learning sequence and the need for better understanding of developing and using the structure sense built in the context of arithmetic for purposes of learning algebra. The teacher-researcher's insistence on using standard procedures and symbolic expressions for communicating the reasons for tasks, like comparing arithmetic expressions without computation or finding the value of an expression given the value of a related expression, made it difficult for students to appreciate the role of the symbols. Thus it failed to tap the students' intuitive understanding about operations and their verbal explanations which could be fruitfully used to build structure sense and deepen the meaning of the various symbols. Besides this, it provided important feedback regarding the concepts and skills which would be required to build this sense. It was clear that the concepts of 'term' and 'equality' would play an important role in building not only structure sense but also combine with it a sense of operations required to make judgments about expressions (e.g. comparison of expressions, judging equality). Terms and equality could be subsequently used in the context of algebra to identify terms in an expression and equivalent algebraic expressions derived by rearranging terms. But the utility of 'terms' was thought at this stage to be restricted to only comparison based tasks and not in computational tasks. Its potential to make computations flexible requiring mental/ physical rearrangement of terms was not realized. The students were supposed to make the connection between the perception of structure learnt in the context of comparison based tasks and the computation procedures on their own. At this stage no algebra was taught and the role of arithmetic in learning algebra was not very clear, especially in view of the skepticism expressed by various researchers about using arithmetic for the purposes of algebra (e.g. Lee and Wheeler, 1989; Linchevski and Livneh, 1999).

5.1.2 Pilot study trial-II

A two group experimental design was formulated in the next trial (PST-II) to explore the extent of effect of arithmetic knowledge (procedure and structure) on algebra learning. On the basis of the pre-test performance, students from the English group were divided into two equivalent groups. One of the groups (Group I) was given instruction in both arithmetic and algebra and the other (Group II) was given instruction only in algebra with some experience in an unrelated topic – geometry. An additional group (Group III) was chosen and was given the same instruction as Group I. Overall, nearly the same amount of time was spent in each topic (arithmetic and algebra) for each of the groups, Group II getting some extra time for simplifying algebraic expressions as they had no instruction in arithmetic. The algebra part was taught by the same teacher for groups I and II and arithmetic was taught by a different teacher to Group I.

The instruction covered the following topics: (i) meaning of arithmetic and algebraic expressions, (ii) evaluation of arithmetic expressions, (iii) simplification of algebraic expressions, (iv) comparing expressions with and without computation, (v) filling the blank with and without computation to make two expressions equal, (vi) identifying equal expressions from a list of expressions and (vii) tasks requiring the use of letter in simple representations. As in the previous trial, students who received instruction in both arithmetic and algebra started by forming expressions and verbalizing the meaning of expressions. They also wrote expressions for verbal sentences, restricted to two term expressions. This was immediately followed by elaborating the meaning of the '=' sign from the 'do something' instruction to include a relation of equality between two sides of the '=' sign. The tasks used were those as earlier: comparing expressions using '=', '<', or '>' signs, or equalize by filling the blank. Students learnt to evaluate arithmetic expressions in the traditional manner following precedence rules and working with expressions whose answers were positive numbers.

The concept of 'term' was introduced next, first, to parse expressions and then, to use in the contexts of identifying equal expressions and comparing expressions. It was assumed that students would be able to connect procedures, with structure of the expression without any explicit connection in the teaching sequence. Students identified terms in expressions like $14 + 5 \times 6$ and $13 - 4 + 7$ as $+14$, $+5\times6$ (a simple term and a product term) and $+13$, -4, $+7$ (all simple terms) respectively. It was verified that rearranging terms does not change the value of the expressions and this was used to identify equal expressions from a list of expressions and to generate expressions equal to a given expression. For example, they identified an expression like $79 + 13 \times 65 + 91$ to be not equal to $13 + 79 \times 65 + 91$ and generated an expression like $243 + 357 - 129$ as an expression equal to $243 - 129 + 357$ (similar to tasks used in Chaiklin and Lesgold, 1984; Linchevski and Livneh, 1999). They further had to compare expressions of the type $25 + 18$ and $26 + 18$ or $35 - 18$ and $36 - 19$ using the signs '<', '=', '>' giving reasons for their choice of the sign. The students learnt to use the concept of term as a tool to deal with these tasks. Students gave verbal reasons for these tasks which were further modified and elaborated by the teacher. An extension of this task was to fill the blank with a term so that the expressions on both sides of the '=' sign are equal (for e.g. 26+36 = 25+35__). This extended their ideas about the '=' sign as well as required them to look at relationships between terms and expressions.

Bracket opening rules were taught by situating them in contexts where the use of brackets would be meaningful. The contexts were such that two ways of computing will lead to the same correct solution. For example, 'Team A has scored 23 runs and then batsman B1 scores 4 and batsman B2 scores 2 runs. How do we write the score of Team A?' This could be represented as 23 + (4 + 2) or 23 + 4 + 2. On computing both the expressions, they were found to be equal. The rule induced was: signs are not changed if there is '+' sign to the left of the bracket. A similar example for '–' sign to the left of the bracket was 'Team A has scored 59 runs. Then one of the umpires fines the team for breaking a rule and reduces the score by 4 runs. Later it was also found that the score board had been showing 2 runs more than the actual. Now what is the score of Team A?' This could be written as 59 – 4 – 2 as well as 59 – (4 + 2). Again both these expressions were found to be equal as their values were equal. The rule induced from such pairs of expressions was that if there is a '–' sign to the left of the bracket then the signs inside the bracket are changed. A situation for distributive property was also created and the same principle of representing and solving it in two ways was used. For example, 'if Anita buys 5 notebooks at Rs 4 each and then she buys 3 more at the same price, then how much money did she pay?' This can be written as (5 + 3) × 4 or 5 × 4 + 3 × 4 which would give the same answer. The students also counted the number of dots in arrays in two different ways and represented it. This was called the Distributive Property of multiplication over addition or subtraction. The students then practiced such tasks of bracket opening. It was hard for students to understand that it is only the '–'sign to the left and not to the right, which leads to a change in sign of the terms inside the bracket on removing it. But distributive property is applicable for '×' sign to the left and right of the bracket.

After the students had some exposure to evaluation of arithmetic expression and the basic activities with regard to '=' sign, students studied arithmetic and algebra simultaneously. They were introduced to algebra in a traditional manner through a variety of context such as the following:

- 'Guess-the-number' game: Students guessed the number which will replace the box in equations like $\square + 3 = 13$. The box was soon replaced by a letter and they had no trouble in associating the letter with the number to be found.

- Representing simple situations of area and perimeter using a letter: These were similar to the CSMS (Booth, 1984) test items. The letter could take many values in these situations and the meaning of simple algebraic expressions as describing a relationship and standing for a number could be conveyed through this task. Students were encouraged to verbalize the meaning of such expressions as generated above and also to write expressions for verbal sentences. For example, $x + 3$ is a number which is 'three more than x'.

- Simplification of algebraic expressions: Students used the idea of multiplication as repeated addition to understand addition and subtraction of monomials. They added such monomials by writing the products as sum of 'singletons'. For example, $2 \times c + 3 \times c = \underline{c + c} + \underline{c + c + c} = 5 \times c$. The take away model was used for subtraction of monomials, $6 \times d - 2 \times d$ was understood as 2 d's taken away from 6 d's, leaving 4 d's ($6 \times d - 2 \times d = d + d + d + d + \cancel{d + d} = 4 \times d$). Subsequently, simplification of multi-termed algebraic expressions was taught by identifying like and unlike terms and then using the rules of monomial addition and subtraction as discussed earlier. They were told the conventions of algebra, like +3 and $+4\times d$ cannot be combined to get $7\times d$ and verified through substitution. As in arithmetic, students here also identified expressions equivalent to a given algebraic expression. The '×' sign was replaced by '•'. Students also applied bracket opening rules learnt in the context of arithmetic while simplifying expressions.

- Contexts for algebra: Lastly students were given some context in which they could use algebra, namely the think-of-a-number game. They learnt to use the letter to represent the number thought by any person and write the corresponding algebraic expression. They had to simplify the expression to prove the result.

The control group (Group II) did not learn any arithmetic. They were introduced to algebra in a similar manner to the other groups, that is, through Guess-the-number game. Verbalizing the meaning of expressions, writing the product as repeated addition, identifying terms and like and unlike terms and simplifying expressions were done in the same way as in the other groups. The teaching approach was similar to that in the school textbooks and was rule based emphasizing manipulation of algebraic expressions without reference to procedures in arithmetic expressions. The fact that rearranging terms does not change the value of the expression (required to simplify algebraic expressions) or that $8 \times t - (6 \times t + 2) = 8 \times t - 6 \times t - 2$, were merely told without any explorations regarding the values of the expressions as a result of these transformations. A lot of time was spent in revising and practicing these rules for bracket opening and simplification of the expressions. Think-of-a-number game and a session on pattern generalizing was introduced to situate algebra in a meaningful context.

Lessons learned and implications

At the end of the second trial, students in groups I and III, who had received instruction in arithmetic and algebra, showed substantial improvement in the post test in the arithmetic tasks over the pre test as well as performed better than Group II (Subramaniam and Banerjee, 2004). In the structure based tasks also, the groups I and III performed better than Group II and were able to understand ideas of equality/ equivalence of expressions better but even these groups had limitations in their understanding of structure. In expressions with more complicated structure involving brackets, they could not anticipate the results of the operations mentally. The two groups (especially, Group I) were also able to apply their understanding of the structure of the expression using the concept of terms to identify equivalent algebraic expressions where Group II was again not as successful. However, Group II performed better than the other two groups in simplifying algebraic expressions where a lot of time had been spent in practice exercises.

Even though students in the arithmetic+algebra groups (Groups I and III) appreciated the similarity in the surface structure of arithmetic and algebraic expressions, it was hard for them to simplify the expressions. The 'algebra only' group (Group II) made more structural errors like 'conjoining' while simplifying expressions and could not understand the meaning of the expressions. As the vocabulary being used for the two kinds of tasks, evaluation/ simplification (procedural) and identifying equal expressions (structural) were different; students, in this trial, on their own could not make the required connections. Also, there was a difference in the procedure for manipulating arithmetic and algebraic expressions, the first one being based on the rules of order of operations and the second one based on collecting like terms and adding and subtracting the monomials. Besides the disparity which has been discussed above, the difference in the teacher teaching the two units could have also made some impact in physically separating the two domains of arithmetic and algebra.

It was evident that knowledge of arithmetic could be fruitfully used in the context of algebra but the link between arithmetic and algebra had to be much stronger than what was embodied in the approach of the 2nd trial. This was needed in order to exploit the properties developed in the context of arithmetic in algebra. Arithmetic helped develop the structure sense for expressions and meaning for the expressions. The concepts of 'term' and 'equality' were seen to be important to bridge the gap between procedures and the structure of the expressions. The second change required was in better and explicit connection between the procedures of arithmetic and algebra. Also a better understanding of negative numbers and operations on them was proving to be necessary. Further, the contexts created for learning bracket opening rules were found to be quite cumbersome and could not be used to remember the rules. The contexts were distracting, with too much information which did not allow the structure to become apparent. Nor was it feasible to translate each expression with brackets into the contexts created to understand them, so students largely relied on rules, over-generalized them and applied them wrongly.

A negative outcome of the instruction was seen in the students' inability to see simple expressions like $x + 2$ as both a relation '2 more than x' and an instruction for computation 'add 2 to x'. They could see it only as a relation due to the emphasis on this type of reasoning in the trial. This needed to be handled in future trials and could have been the consequence of separation of the procedural and structural skills, with emphasis on the latter. Over five Thursdays during the months February and March 2004, the same students from one of the English medium schools (E1) were called and an activity with the number line was tried. It was called the letter-number line (Carraher et al., 2001) and was constructed by generalizing the relations between the consecutive numbers on the number line and denoted the distance of that point from 'x'. The letter number line was thought to embody both: a process and a relation. The act of moving on the number line indicated the process of adding or subtracting a number to/ from ‘*x*’ or any other position on it (e.g. moving five steps to the right of *x*, one reached the position x+5 and moving three steps to the left of x-1, one reached the position x-1-3=x-4). Each point on it also signified a relation: x+1 is one more than *x* and x-2 is two less than *x*. Constructing the letter-number line and finding and representing the relations between the points on the letter-number line formed the crux of the activity. This activity was incorporated as a key element of the instructional sequence in the next trial.

5.2 *Main study trials*

5.2.1 Main study trial-I

The first trial of the main study (MST-I) began with a fresh group of students and with the feedback from the earlier trials. Groups A_1 and C_1 were given instruction in both arithmetic and algebra. For these two groups, the concepts of ‘term’ and ‘equality’ were made the central theme of the teaching-learning module. The focus in group B_1 was largely on arithmetic but used the same core concepts as the other two groups. An effort was being made in Group B_1 to develop an approach to teach operations on negative numbers, which had been identified to be critical for learning algebra. Again two separate teachers taught the units of arithmetic and algebra to group A_1. One of the main aims of the trial was to make the teaching learning sequence more coherent with the minimum number of rules and procedures to manipulate expressions in both arithmetic and algebra.

The first exercise of learning to write expressions for a number, expressing its meaning verbally and vice versa was carried out as was the practice in the earlier trials. The number line (with both positive and negative numbers) was introduced next to reinforce these multiple meanings of the expressions: as an instruction to compute a number and as a relation between two numbers to denote a third number. For example, the depiction 11 – 3 = 8 on the number line meant 'eight is three less than eleven', or 'eleven is three more than eight', 'three is eight less than eleven', 'three steps to the left of eleven is eight' and 'three steps to the right of eight is eleven'. Students compared the numbers on the number line, found the magnitude of the difference between two points and made the corresponding arithmetic expression as in the above example. Although they were introduced to negative numbers in this program, they had no knowledge of operating with these numbers. It was therefore necessary to carefully choose the expressions so that at each step one arrived at a positive answer.

In contrast to the earlier trials, the concept of term was introduced before students learnt to evaluate expressions. They learnt two kinds of terms: simple terms and product terms. The terms were put in boxes to increase their visual salience for the students.

For example, the terms of 19 – 7 + 4 were written as $\boxed{+19}$ $\boxed{-7}$ $\boxed{+4}$.

Although the rules of evaluating expressions included a step of identifying terms, they were largely precedence based. For an expression containing only simple terms, it was to be solved from the left to the right direction.

For example, 24 – 6 + 8 = $\boxed{+24}$ $\boxed{-6}$ $\boxed{+8}$ = 18 + 8 = 26.

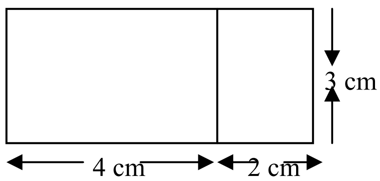
For expressions containing a product term, the product term was first 'simplified' to yield a simple term and then the expression with only simple terms was to be simplified as above.

For example, 15 + 3 × 4 = $\boxed{+15}$ $\boxed{+3\times4}$ = 15 + 12 = 27.

If the expression contained brackets, then terms were to be identified after the expression was free of brackets by first solving the bracketed sub-expression.

Students were exposed to the same tasks as discussed in the previous trial enhancing their understanding of equality and '=' sign. Some of these tasks were based on computation (comparing expressions using <, =, > and filling the blank to equalize two expressions) and some required them to attend to the structure of the expression and had to be completed without computation (comparing simple two termed expressions, identifying equal expressions from a list and generating equal expressions). As before, the students used the concept of terms to identify equal expressions in a list and generated equal expressions by rearranging terms. This was the only transformation used to generate equal expressions in this trial. The comparison of expressions like 34-17 and 35-16 or 28+47 and 29+46 was carried out using the concept of terms by the students, the judgments often supported by verbal reasoning. This was also the first time that the teacher made efforts to support students' verbal reasoning using symbols by denoting the change with the help of integers. For example, while comparing the expressions 34-17 and 35-16, one can note that both the terms in the second expression has increased and the change can be denoted as +1+1=+2. Other strategies like using the 'take away' model was also used in such situations. This was simple enough for the students to understand and use in their own reasoning and began the process of creating new symbols. The negative sign continued to trouble the students due to their poor anticipation with respect to the minus operation and very little understanding of negative numbers as well.

Students were taught bracket opening rules, which had been identified through the earlier trials as important for both procedural and structural understanding of expressions. Constant effort was being made to improve the teaching of brackets and get across the multiple meaning of brackets to the students. Compared to the earlier trials where these rules were situated in a context, students now learnt them in the context of equal expressions by comparing and matching the values of some expressions without bracket with a given bracketed expression. The rules were induced by repeatedly computing the value and checking when the values of two expressions are the same. For example, they compared the value of 21 – (4 + 5) with 21 – 4 + 5 and 21 – 4 – 5. After a series of such comparisons, it was concluded that for a negative sign to the left of the bracket, the signs of the terms inside the bracket changes and for a positive sign to the left of the bracket, the signs of the terms inside the bracket does not change. Distributive property of multiplication over addition and subtraction was derived in the context of area of rectangles. Students represented the area of the rectangular figures (see Figure 5.2) in two ways and since both the ways gave the same result and represented the same area, the expressions were equal. The model was not very effective because of students' poor understanding of the concept of area. They knew it only as a multiplication of two numbers without any understanding of the rationale behind it and the quantities which are to be multiplied.



Area = 4 × 3 + 2 × 3

Area = (4 + 2) × 3

(4 + 2) × 3 = 4 × 3 + 2 × 3

Figure 5.2: Area model for distributive property

To connect the procedures of evaluating/ simplifying arithmetic and algebraic expressions, students were asked to evaluate expressions like 83 – 5 + 28 + 5, 76 + 38 + 24 – 8, 18 × 6 + 12 × 8 + 18 × 2, where one could find easier ways of computing them by focusing on the relationship between the terms, than by using precedence operations. Non-sequential computation is an important stepping stone for simplification of algebraic expressions. This task required the use of procedures and rules and structure sense for expressions. The effectiveness of this task was restricted as the students did not possess the flexibility to work on this task. The students had to mentally/ physically rearrange the terms of the expressions in such a way that they could solve parts of the expression easily. This was counter intuitive, as they had learnt to evaluate expressions sequentially, where the role of terms was restricted to identifying the precedence rule to be applied. Students' lack of knowledge of integer operations was another factor restricting their performance in this task, reinforcing the experience of the earlier trials.

Students were also taught algebra simultaneously with the arithmetic instruction, trying to build on their recently acquired knowledge of procedures and structure of expressions. Algebra or the use of letters was once again introduced through the Guess-the-number game and open ended expressions like $x + y = 12$. The letter-number line (as discussed in section 5.1.2, PST-II, pp. 159-160) was soon constructed to reinforce the notion that the letter stands for a number and that it can take any value. Five 45-minute sessions were devoted to the discussion of the letter-number line and other tasks on it. Students at first responded by treating the letters as alphabets (already identified in the literature), by denoting the number to the right of 'x' as 'y'. The fact that the letter stood for a number and could be used for representing relationships between numbers was not apparent to them. This was seen even though they had been briefly exposed to the idea of letter standing for a number through working on simple guess-the-number game (e.g. x+6=19, x=?) and completing open sentences like if x+y=16, then x=?, y=? as mentioned above. After some discussion, it was accepted that x, x+1 (one more than x), x+2 (two more than x and one more than x+1), x-1 (one less than x), x-2 (two less than x and one less than x-1) could represent the numbers on the letter-number line. These expressions were the simplest examples of unclosed expressions and could be easily accepted by students.

Students carried out similar activities on the letter-number line as they did on the number line. They compared the expressions on the letter-number line and calculated the magnitude of the difference by counting the number of jumps between the two points. This was extended leading to symbolizations like $m + 4$___ $= m + 1$ or $m - 2$ ___$= m + 1$, or writing the complete representation for single step journeys as shown in Figure 5.3 (x-2+5=x+3). The students were encouraged to verbalize the meaning of these sentences as well, for example the first sentence means 'm+4 is 3 more than m+1'. This again reinforced the duality of expressions: something which could be operated upon and conveyed a relation.

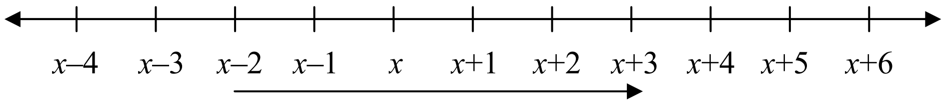


Figure 5.3: The letter-number line and simple journey

Simplification of algebraic expressions was carried out as earlier; writing monomials as sum of singletons ($3 \times a = a + a + a$) and by collecting like terms in multi-termed expressions, followed by either extracting the common factor or using the rules of adding/ subtracting monomials. In another effort to connect the manipulation procedures in arithmetic and algebra, which students failed to see spontaneously, the task of evaluating algebraic expressions for a given value of the letter was introduced. In order to reduce the conjoining error while transforming algebraic expressions, it was emphasized that since a product term cannot be added to a simple term before converting it into a simple term; therefore $3 + 4 \times d$ cannot be written as $7 \times d$. However, students did not seem to have difficulty in appreciating the similarity in surface structure of arithmetic and algebraic expressions and identified and generated equivalent expressions to algebraic expressions using the idea of rearranging terms with ease. These three tasks of evaluating algebraic expressions for a value of the letter, simplifying algebraic expressions and identifying equivalent expressions were considered to be complementary and were thought to be sufficient to understand the connection between arithmetic and algebra.

Students' knowledge of manipulating algebraic expressions was put to test with the activity of finding the distance/ difference between two points on the letter-number line by representing it as an algebraic expression and manipulating it to find the difference. Two 45-minute sessions were devoted to the think-of-a-number game, which allowed the students to use algebra as a tool for justifying/ proving. Both these tasks not only required them to represent situations using a letter but also manipulate the expressions to arrive at a conclusion. Another task which required only representation was of finding the area and perimeter of rectangles with one of the dimensions being a letter. The students found creating symbolic representations quite difficult in all the tasks, and were more successful in verbal statements describing the situations. They lacked not only a sense of the letter and its purpose but also the meaning of the operations on the letter. All of this was introduced to them too soon, without adequate preparation

Group B_1 received instruction mostly in arithmetic – integer operations and evaluation of simple arithmetic expressions. The idea was to develop an approach to introducing operations on negative numbers, which was being excluded in each trial till now and the need for this understanding was being felt strongly. Integer operations were introduced in this group using a context where negative numbers were represented as outstanding electricity bills and positive numbers were considered to be the income that a farmer earned or received. A lot of time was spent in developing this context and subsequently using this context to operate on integers. The model was found to be quite difficult for handling integer operations and was dropped from later trials. This group also worked on the number line and the letter-number line, learnt evaluation/ simplification of arithmetic and algebraic expressions and bracket opening rules in a similar manner as was done in the other two groups but spent less time on this part of the instructional sequence.

Lessons learned and implications

At the end of the first trial of the main study, although the instructional sequence had become somewhat better compared to the pilot trials, it had still many loose ends and had scope for further improvement and coherence. It was believed that the visual salience of terms and using it in both procedural and structural tasks would help students in making the required connections between arithmetic and algebra for the students. The arithmetic and the algebra parts of the teaching sequence used different vocabularies for evaluation and simplification which tended to confuse students and added to their uncertainty about how to approach the task. The concepts of term and equality were used in both the domains but to different extents. In the arithmetic part, these concepts had been used in all contexts but the use itself was more rule/ procedure bound rather than giving any real sense of structure which could be used flexibly in any situation. Therefore, the transfer of these to the algebra context was limited. For example, students did not spontaneously see the similarity in the structure of the expressions and why the same rules of transformation should be used while simplifying the expressions $3 + 4 \times d$ and $3 + 4 \times 2$. There were differences also in the ways they solved/ simplified the expressions. In the case of $3 \times 5 + 6 \times 5$ students first solved the product terms and then added the simple terms, but in the case of $3 \times b + 6 \times b$, they decomposed them as singletons and found the number of b's in the sum.

Students also did not see the notational similarity between 3+3 and $x+x$ spontaneously, on which much of the algebraic manipulation was based. Notational difficulties (for example, is $t+t+t+t+t$ equal to 5×5 or $5\times t$ or $5+t$?) continued to trouble students and these indicated serious misunderstandings about the letter. Students' responses to a few items of adding and subtracting monomials like $5\times b$-b and $8\times c+c$ point to the source of difficulty for the students. Their responses consisted of 5, 0 and $4\times b$ for the first one and $7\times c$, $8\times c$ and $9\times c$ for the latter one. They evidently failed to make sense of the 'singletons' 'b', 'c'. The teacher's emphasis on the right answers leading to rules of procedures and very few discussions focusing on the wrong answers hampered understanding and the progress of students. Students repeatedly committed the same errors. The similarity seemed to be forced by the teacher. The simplification of algebraic expressions still remained a mechanical procedure governed by seemingly arbitrary rules.

The tasks of evaluating algebraic expressions for given values of the letter and evaluating arithmetic expressions by easy ways were devised to bridge this gap, which also did not prove to be successful. Lack of proper understanding of the letter and rigidity in rules of evaluating arithmetic expressions together with rule bound understanding of structure of expressions were adjudged to be the reasons behind this. This is seen from the analysis of the students' responses to the various tasks and the classroom responses from the video recordings (more discussion in Chapters 6 and 7). The students failed to see the continuity and commonality between the tasks which had to be explicated by the teacher time and again. Procedural and structural understanding did not complement each other and remained two different aspects. Definitely, only the presence of some structural notions in the teaching-learning sequence was not sufficient for students to understand the connection between arithmetic and algebra and treat algebra as generalized arithmetic.

The students also could not use their knowledge of syntactic transformations on algebraic expressions in the context activities. They could not spontaneously identify the terms of the expressions generated in context and simplify the expressions using valid transformation rules and neither were they encouraged to do so. There was a big gap between syntactic transformations and making sense of them in contexts which required a much more sophisticated understanding of use of symbols as tools. It was unclear at this moment what could help bridge this gap. It was assumed that their knowledge of simplification processes could be used in the contexts without any support/ help. The answer to this problem could not be settled at this point as students' understanding of transformations on expressions and their sense of structure was also unstable. In the next trial, another effort toward removing some of these problems and designing an effective and coherent teaching-learning sequence was made.

5.2.2 Main study trial-II

In the second trial of the main study (MST-II), those students who had attended the first trial of the main study (MST-I) were invited and two groups were formed (A_2 - English and B_2 - Marathi) as explained in the previous chapter (see section 4.3.1, pp. 138-139). Two teachers took care of the different batches and taught both arithmetic and algebra. A fresh batch of students formed Group C_2, who were also taught arithmetic and algebra by a separate teacher but followed a slightly different sequence with the aim of exploring different approaches. Since the students in the groups A_2 and B_2 had previous exposure to the course, the trial began with a two day revision of the main ideas they had learnt in the first trial. A delayed post test, which was conducted four months after the end of the first trial MST-I (not being reported in the study) showed that there was almost no retention of the concepts, procedures or rules they had learnt in the first trial. The revision session was conducted just to ascertain the extent of retention/ forgetting. Most of the concepts, rules and procedures were not taught again but only posed as tasks in the classroom to which the students as a group responded. They had not completely forgotten the basic concepts that they learnt in the summer trial (MST-I), but they never got the opportunity to use them in the school. They did not remember any of the algebraic manipulation they had learnt as they had failed to develop an understanding of the rules of transformation in algebra in MST-I. A pre test was subsequently conducted after the two days of revision.

Some minor but important changes were made during the revision with respect to the procedures of evaluating expressions and bracket opening rules. The vocabulary of adding and subtracting terms in an expression was replaced by the vocabulary of combining terms. Since they were in the middle of grade 6, they had been introduced to integer operations in the school. Although they began combining terms with the use of rules of integer operations learnt in the school, this did not give them the required flexibility to work on the tasks. Soon they were introduced to the 'annihilation' model of adding integers through the use of positive and negative cards. Thus integer addition got subsumed in the 'terms approach'. The fact that two equal and opposite terms (terms with opposite signs) compensate each other was used as the guiding principle for combining terms. The key observation here is that positive terms increase the value of the arithmetic expression and negative terms decrease the value. The same number of negative and positive cards cancelled each other, more number of positive cards in the collection left behind positive cards after cancellation, more number of negative cards left behind negative cards after the cancellation. For example, to compute 12-5, it was thought of as consisting of two terms +12 and -5. When these terms are combined, five pairs of positive and negative cards get cancelled and seven positive cards are left behind resulting in the answer +7. Similarly, in the case of -12+5, five pairs would get cancelled as before, but seven negative cards would be left behind resulting in the answer -7. This change brought about many other changes due to the possibilities it gave rise to in the classroom. Now the students were free from the rigid left to right evaluation procedures and could combine the terms flexibly, the only constraint being the structural rules already described for identifying and combining terms.

The rules for evaluating arithmetic expressions were reformulated using this more structural vocabulary of combining terms. Simple terms could be combined directly and a product term could be simplified into a simple term which could then be combined with other simple terms. Further, two product terms could be combined if they have a common factor using the distributive property. Implicitly, the idea of combining terms also meant computing in any order, not adhering to a particular direction, but it took some more time for the students to reach this stage. This was facilitated by their newly acquired knowledge of integer operations, which was nothing but combining two simple terms. The purpose of writing the terms was not restricted to deciding the order of operations but to work with those units themselves. This modification was inspired by the difficulties students faced while working on the task of evaluating using easy ways in the earlier phase. It brought out the possibility of integrating the evaluation of simple expressions and more complex arithmetic expressions using easy ways by freeing the students from following precedence rules or any specific sequence of computation.

Thus students could now evaluate expressions like 28 – 12 + 32 + 12 by mentally cancelling -12 and +12, without the need to move in a fixed direction. They could also be initiated to manipulate expressions in a goal directed manner, thereby building anticipatory skills, an important aspect of algebra learning. Students were required to work with more complex expressions, for example, to show an expression like 19 × n – 8 – 5 × n + 1 to be equal to 7 × (2 × n – 1). The effort in this trial was to integrate the two domains of arithmetic and algebra and highlight the similarity in procedures in the two domains, with the structure of the expression being the only deciding factor in the choice of the procedure.

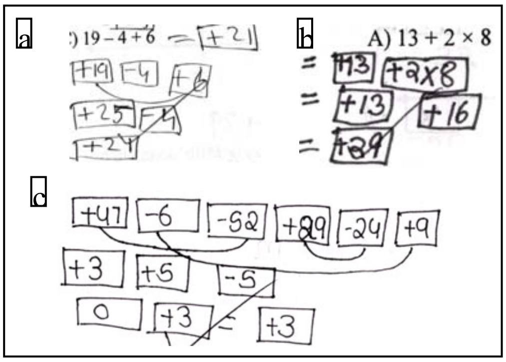


Figure 5.4: Students' solutions to arithmetic expressions during MST-I

The difference in their solutions to these types of expressions after MST-I and MST-II can be seen in Figures 5.4 and 5.5. It is not difficult to notice the flexibility in students' solution (students' responses and errors in these tasks will be discussed in the next chapter). The rule for combining the product terms could also be used for combining product terms with variable factor in the case of algebraic expressions, which is what most students preferred and chose to do later, when the task was introduced, than to break the product terms into sum of singletons and rewriting the result as a product, as was done in the earlier trials.

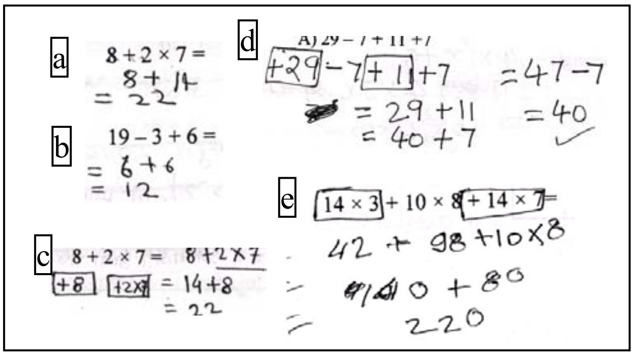


Figure 5.5: Students' solutions to arithmetic expressions during MST-II

The other change made during these two days of revision was with regard to the bracket opening rules. The concept of 'bracket term' was introduced at this stage to make explicit the role and the significance of the bracket and to reduce errors in evaluating expressions with brackets. For example, in the expression 12 – (5 + 2), +12 and –(5+2) are the terms, the latter one being named the 'bracket term' which were further classified as positive or negative bracket term. In this trial, the emphasis was on equality between the two expressions, one with bracket and the other without bracket. The students were told to focus on the relation between the terms inside the bracket and the terms after the bracket has been removed. They subsequently framed the rules for opening bracket when a positive or negative a sign was to the left of the bracket, that is, for a positive or a negative bracket term. It was hoped that highlighting the bracketed term would take care of ignoring the bracket as well as over generalization of the rules which the students were often seen to make earlier - not only changing the sign for a negative bracketed term but also a positive bracketed term, and associating the '+' and the '–' sign to the right of the bracket. Similarly, rules for distributive property of multiplication over addition and subtraction for expressions of the form $8 \times (9 - 4)$ were made. The students spontaneously saw expressions like the one above as one single term, a product term with a bracketed factor, and the term could be positive or negative. Equality of expressions, like $8 \times 9 - 4$ or $8 \times 9 - 8 \times 4$, were judged against the expression $8 \times (9 - 4)$ and the rule formulated. This was accompanied by a verbal explanation for the rule: 'eight times the difference between 9 and 4 is same as the difference between eight times nine and eight times four'. This is because eight times nine is greater than eight times the difference between nine and four, and to compensate for that effect one needs to subtract eight times four and not just four.

Some more ideas for introducing bracket opening rules were tried with a separate group of students (Group C_2). These were based on students' intuitions and pattern perceiving abilities. From the information that $8 - 4 + 3 = 7$, the students inferred that $-8 + 4 - 3 = -7$ or an expression which could stand for -7 would be $-8 + 4 - 3$. These expressions were understood as 'inverses' of each other. The negative sign outside a bracket was interpreted to mean 'take the inverse value of the expression in the bracket' or equivalently 'take the inverse expression'. So $-(8 - 4 + 3) = -8 + 4 - 3$. Taking the inverse of the expression meant changing the sign of each of the terms inside the bracket. A plus sign before the bracket meant 'take the value of the expression in the brackets' so that no part of the expression needed to be changed. Similarly, the students referring to the expression $8 - 4 + 3$ (=7) inferred that $16 - 8 + 6 = 14$ or an expression which would stand for 14 is $16 - 8 + 6$. This actually meant doubling the expression $8 - 4 + 3$, that is, $2 \times (8 - 4 + 3) = 16 - 8 + 6$, where each of the terms have been doubled.

One of the important activities added in this trial was to generate equal expressions for a given expression, the expressions involving only simple terms or simple and product terms (e.g. 23-35+49, 16-12×7+34), thereby exploring the possible transformations which would keep the expression equal. In the process, it was expected that they would learn the constraints and the possibilities on transformations. The idea was also to separate the meaning of the expression from the denotation, which is not necessary in the case of arithmetic but is a difficult idea to understand in the context of algebra, where one needs to interpret expressions flexibly (Arzarello et al., 2001). Disparate looking expressions could stand for the same value and similar looking expressions could have different values. In this task, as the surface structure of the expression was changed repeatedly, the value remained invariant. Students used various transformations, like re-ordering the terms, putting brackets, splitting a term into sum, difference and product, adding and subtracting the same number, increasing and decreasing two or more terms in such a way that they compensate each other. This was a good ground for revisiting and discussing the various rules of evaluating expressions and bracket opening, the need for brackets, as well as anticipating results without necessarily calculating it. They also identified whether expressions in a list were equal to a given expression where the transformations were not restricted to reordering terms but many other transformations, as were seen in the classroom. The intention was to push the students to look at expressions in various ways, other than those which they could think of, and attend to the transformations which keep the value invariant. The same task was carried out for algebraic expressions as well once they had some familiarity with manipulating algebraic expressions which is discussed below. This task too depended on students' competence and flexibility in performing integer operations (esp. addition) and mentally carrying out various operations on the terms (e.g. combining/ splitting) and their ability to handle brackets.

Continuing with the structure based tasks, as in the previous trials, students (i) compared expressions without calculations using the signs <, =, > and (ii) found the value of expressions given the value of a related expression. The small efforts which were made in the previous trial to introduce symbolism in students' justifications to these questions were further systematized. Students were gradually directed to use only symbols in order to substantiate their responses. In the process, the possibility of capitalizing on their intuitions and converting their verbal explanations into symbolic representations was recognized, which has larger implications for reasoning and arguing in mathematics classroom.

Simultaneously, students worked on the letter-number line (4 sessions of 45 minutes each). Attempts were made to consolidate students' understanding of the letter-number line which they had failed to grasp in the earlier trial. Their correct reproduction of the letter-number line was more a result of memorization than understanding, which could be seen in their inability to correctly find out numbers to the left or right of a given number (like whether the number to the left of 53 is -52 or 52). The relation between them was strengthened through a repeated replacement of the letter in the letter-number line by a number and completing the 'portion' of the number line (given the number 27 on the number line, fill three numbers to the left and right of it). The fragility in their understanding of the relations between numbers on the number line was revealed through this task but by the end of the trial they had successfully learnt the order relations in integers as well as the meaning of the letter-number line.

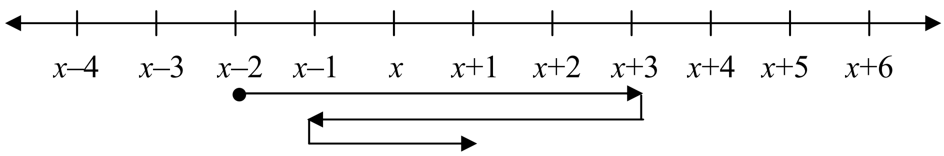


Figure 5.6: Journey on the letter-number line

Two activities were carried out on the letter-number line: (i) the journey on the letter-number line (Figure 5.6) and (ii) calculating the distance between two points on the letter-number line. The 'journey' task was an extension of a similar task in the last trial where students completed sentences like x-2___ = x+3. The multi-step journey task required the students to represent a depicted journey using an algebraic expression, which was a simple sequential representation. Further, they had to simplify the expression to verify if the end point of the journey corresponded to the one shown in the diagram. For example, if the expression representing the journey was $x - 2 + 5 - 4 + 2$, then they would verify by simplification that the end point for this journey would be $x + 1$. Students' responses to these tasks will be taken up for discussion in Chapter 8.

Sun		7	14	21	28
Mon	1	8	15	22	29
Tue	2	9	16	23	30
Wed	3	10	17	24	
Thurs	4	11	18	25	
Fri	5	12	19	26	
Sat	6	13	20	27	

	10	17
	11	

A	B	C
D	x	E
F	G	H

Figure 5.7: Calendar task

With the realization that a simple representation task (e.g. representing area of a rectangle with a dimension unknown) is not sufficient to necessitate the use of algebra, another activity was introduced where algebra could be used as a tool. This was the calendar patterns task (Bell, 1996) (3 sessions of 45 minutes each) where students had to represent the relation between the numbers in the rows and columns, find patterns in the arrangement of numbers in a calendar (e.g. in Figure 5.7, A+H=C+F=$2x$), represent them and justify that the pattern would hold for all similar arrangements of numbers. This was a very challenging task and quite difficult for students to succeed. Discussion of students' work on this task will also be taken in Chapter 8.

Lessons learned and implications

The arithmetic-algebra module finally seemed to have evolved to a level where the approach adopted ('Radical terms' approach) allowed the students to attach meaning to expressions containing numbers as well as letters. It also gave them the required flexibility and opportunity to use the concepts learnt during the trials in various situations and tasks thus making the unit coherent.

Many of the tasks, like evaluation using easy ways, generating equal expressions, required both procedure and structure sense and therefore these two aspects got complemented, rather than one following the other. By the end of this trial, there was continuity and coherence in the various tasks between symbolic manipulations in arithmetic and algebra as well as with regard to procedural and structural tasks. The approach also made it possible to turn the familiar operations like addition, subtraction into objects (positive term and negative term) which could themselves be manipulated (combined in any order). It was possible for them to suspend the operations at each step and look for relations between the terms and then decide to combine them. The concepts of 'term' and 'equality' gave students not only visual support but also a vocabulary for communicating their reasoning about expressions.

It was realized that integer operations which are indispensable for algebra can get absorbed into the terms approach. Some ways of relating the concepts and the rules were also found in the ideas of 'inverse' and 'multiple'. These ideas were tried with only a small group of students and still needs more systematic exploration for its effect on the evaluation/ simplification tasks. Each group was taught by one teacher, which would have made some difference in students' understanding. At least there was no physical separation of the domains – arithmetic and algebra, for students to internalize. What this module still lacked was continuity between symbolic algebra and using these manipulation skills in the situations/ contexts created for using algebra (reasoning with expressions). The students failed to spontaneously use in the contexts of representing and proving/ justifying, the manipulation skill that they had already acquired. The time spent on these activities was also minimal. There are various issues other than manipulating expressions when tasks require algebra as a tool for proving, justifying and generalizing. The next trial focused on these aspects of the teaching sequence.

5.2.3 Main study trial-III

In the third and final trial of the study (MST-III), only those students who had attended the earlier two trials were invited. As in the last trial, they formed the two groups, A_3 (English) and B_3 (Marathi) with a different teacher for each group. These students had by now completed grade six in the school. The focus of this trial was on verbalization and articulation of various procedures and rules of evaluating/ simplifying expressions, rules of opening brackets and use of the concepts and rules learnt till now in situations. Students were encouraged to point out the major mistakes which students might make while evaluating simple and the more complex arithmetic expressions or explain different ways of evaluating expressions, thus developing their meta-cognitive abilities. Quite a lot of time was spent in discussing with students the evaluation of arithmetic expressions involving brackets, which often required a greater analysis of the terms and coordinated use of more than one rule for computing the value. Efforts were made to build among the students flexibility in understanding brackets: as precedence operation and substituting the bracketed part by an equal expression, which could be used while evaluating the expressions.

The idea behind continuing the structure oriented tasks was the same. The emphasis in this trial was on verbalizing the general principles which would keep the value of the expression invariant. Students' ability to reverse operations mentally so as to see which transformations lead to the same value and which change the value could be observed on such occasions (students' responses will be discussed in Chapter 7). The tasks based on '=' sign were changed from the earlier trial and elaborated to use the structure of expressions while justifying the response. Examples of these tasks are: $37 + 52 = 39 + 51$ ____, and $327 + 239 = 329$ ____ (see Stephens, 2004a). Students were helped by the teachers to write full symbolic sentences as reasons for filling the blank by a term, gradually transforming students' verbal explanations into expressions which looked like the ones below (i). The purpose here was to gradually shift the students to complete symbolic representation from the shortened ones they were familiar with till now. This was not simple for them to write on their own and they made errors in the process, but they seemed to make sense of these expressions. For example, in 37 + 52 = 39 + 51 ____,

39 + 51 = (37 + 2) + (52 – 1) = (37 + 52) + (2 – 1) = (37 + 52) + (1) ……(i)

Students could read the relations in these kinds of sentences, like '39 + 51 is 1 more than 37 + 52' and therefore the blank should be filled by –1 or in case of 37 + 52 ___ = 39 + 51, the blank would be filled by +1 as '37 + 52 is 1 less than 39 + 51'. This was another occasion where the expressions were being treated as an entity, not simply as computational procedures. Although this elaborate reasoning with formal arithmetic expressions was accessible to the students, they often used a shortened version of the symbolic reasoning keeping track of the transformations on each of the terms (Figure 5.8).

$$\begin{array}{l} 37 + 52 = 39 + 51 \;\_\_\_ \\ \qquad\qquad\quad +2 - 1 \\ \qquad\qquad\quad = +1 \end{array}$$

Figure 5.8: A student's solution to the problem

The aim in this module was not just to make the connection between arithmetic transformations and syntactic algebra but also for them to appreciate the power of the symbols for purposes of reasoning. Although the above task deals with invariance of values and can be considered to be part of reasoning *about* expressions, this task also systematically attempted to use transformations on arithmetic expressions for reasoning to arrive at the conclusion, that is, reasoning *with* expressions. The arithmetic sentence as in (i) above needs to be flexibly interpreted as a relation between the two expressions so that one distinctly sees the term which when used to fill the blank would make the expressions equal. This is an important skill, especially in algebra, to shift one's attention to see different meanings/ sense in an expression as per the requirement of the task or the demands of conclusion to be drawn.

This time, the 'representation only' tasks were not used at all and instead two activities were used which enabled one to reason *with* expressions, using the rules and procedures learnt till now in the context of syntactic transformations. The students worked on two tasks of 'think-of-a-number' game (2 sessions of 45 minutes each) and pattern generalization from geometric shapes (4 sessions of 45 minutes each). The 'think-of-a-number game' required the students to justify the pattern in the answer with respect to the starting number arrived by following a sequence of instructions. For example, 'Think of a number. Subtract 2 from the result. Multiply it by 2. Add 8. Subtract the original number. Subtract 1. Subtract the original number. Add 2. What do you get? Explain why everyone gets the same number'. This was repeated in order to exploit the rich possibilities for explanation/ justification which it offers and had not been capitalized upon during MST-I. The teacher guided the students to form an algebraic expression as a representation for the situation, again converting their verbal arguments into symbols. Subsequently, they were encouraged to make similar problems for their peers in pairs, which strengthened their understanding of inverse operations as well as the use of verbal explanations to keep track of the transformations on the starting number and the need to switch to symbolic representations with increasing complexity of the problems.

The pattern generalization task was also carried out in pairs and required them to first continue the pattern for a couple of positions and then predict the value for distant numerical positions before making the generalized rule. Two extensions of the activity were carried out: to show the equivalence of the different rules for the same pattern, and to predict the rules for positions $n+1$, $n+2$ etc which laid the ground for substitution of 'n' by '$n+1$'. This was also another context where verbalizations of generalizations could lead to the formation of symbolic rules and further reason with them.

Both the tasks used in this trial were dependent on students' understanding of the letter, the expression and the use of algebra (esp. for proving, justifying and generalizing) besides knowing syntactic manipulation. These tasks gave rise to opportunities to discuss the need for brackets, the use of bracket opening rules, the correctness of the rules and symbolic conventions and the meaning of the letter. Allowing students to verbally explain their reasoning and using them to generate symbolic representations allowed students to make sense of the activity as well as better participation of the students in the tasks in the classroom compared to the previous trials. Students' responses and performance in these tasks will be discussed in Chapter 8.

Lessons learned and implications

The third trial brought an end to the series of trials. The concepts of 'term' and '=' were found to be useful in giving meaning to the operations and strengthening the structure sense even in the more complex situations, as were used in this trial. The students also comfortably used these ideas whenever required. Although the students confidently explained syntactic transformations in tasks of reasoning *about* expressions, they did not display the same ability in tasks of reasoning *with* expressions. The contexts chosen for conveying the meaning and purpose of algebra were quite challenging and required further deliberations about ideas of proof, generalization to understand the requirements of the task. Syntactic transformation of the symbolic representations was only one aspect of the task, which seemed to fall in place once the requirement of the task was clear to the students. There was indeed a dilemma here: if the situation considered was simple enough for students to comprehend, then they preferred to give verbal explanations and did not find the use of algebra necessary; and if the situation was complex, then it did not allow the students to proceed. This is an important constraint on the design of the teaching sequence.

5.2.3 *Conclusion*

After these five trials, a sufficient number of activities and tasks had been tried out which could make possible students' transition from arithmetic to algebra with some understanding. In the trials in the main study, some concepts or tasks where the performance or understanding was not very satisfactory were modified or elaborated and repeated. For other tasks, students did work on the tasks with some extensions and complexities added as the trials progressed.

The trials firstly, led to the formulation of a teaching learning sequence for beginning algebra based on developing a structural understanding for arithmetic and algebraic expressions and further using them in situations which focused on meaning and purpose of algebra. A teaching guideline based on the above experience is proposed in Appendix VI. Students in the process learnt to use symbols in various ways and for different reasons: reasoning *about* expressions while working on tasks based on syntactic transformations, and reasoning *with* expressions while working on contexts using algebra as a tool. Secondly, the trials were instrumental in developing a framework for the research study as well as in identifying the principles which would allow the transition from arithmetic to algebra. The methodology adopted made it possible to engage with the teaching learning process over a long period of time and also teacher-researchers' reflections on the actions led to subsequent modifications in the conjectures, choice of the tasks and understanding of students' responses and actions.

The discussion in this chapter did not include students' responses and their performance in the tasks, which will be taken up in the next three chapters. The focus here was on an analysis of the teaching sequence and progressive development in it. The data for the pilot studies will not be analyzed further. The first chapter discussing the analysis will deal with students' understanding and performance in various procedural tasks of evaluating/ simplifying arithmetic and algebraic expressions and their knowledge of rules of transforming expressions with bracket. The second chapter would deal with their understanding and performance in the predominantly structural tasks, that is, their understanding of equality. The third analysis chapter deals with students' performance in contexts where algebra is used as a tool.

Chapter 6: Analysis I: Understanding of rules and procedures in arithmetic and algebraic expressions

6.0 *A brief overview of the chapter*

This chapter deals with analyzing and describing students' understanding of rules and procedures in the context of arithmetic and algebra and the connections that they make between the two domains while transiting to symbolic algebra from arithmetic. As discussed earlier (see Chapter 2, esp. sections 2.3.3 and 2.3.4), many of the difficulties with symbolic algebra arise due to non-appreciation of and inconsistent use of the arithmetic rules and procedures. The procedures and rules for evaluating arithmetic expressions were reformulated in the teaching approach in more structural terms, rather than as precedence rules, to include the important ideas of unambiguous parsing of expressions and flexible ways of operating on the expression resulting in equal values (described in Chapter 3, section 3.2.2). Procedures for transforming algebraic expressions were generalized from arithmetic expressions by explicitly pointing out the similarity in the structure of the expressions and hence the rules to be applied for manipulating the expressions. The letter, in the process, took number as its referent. In this way students' intuitive understanding of arithmetic was taken into account and developed as a template on which a new symbolic system of algebra was built, to help the students move from the 'inventive-semiotic' stage of representation to the phase of 'structural development' (Goldin and Kaput, 1996).

In this chapter, students' performance will be analyzed to explore the extent of the use of the concepts, rules and procedures taught during the study by them while solving tasks which were predominantly procedural. In the process, students' achievements and their failures in making connection between the procedures in arithmetic and algebra during the three trials of the main study (MST-I, MST-II, MST-III) will be explored. This will also indicate partly the effectiveness of the teaching approach in building among students transformational capability for algebraic expressions. The effectiveness will be judged on the basis of the students' ability to simplify algebraic expressions, the strategies used to simplify them and their explanation of the simplification process of algebraic expressions together with their ability to move flexibly between the two domains as required. Other facets of this arithmetic-algebra connection will be explored in the next two chapters. The tasks which will be analyzed in this section are: (a) parsing of expressions (b) evaluation of arithmetic expressions – simple expressions, expressions with brackets and more complex expressions using easy ways, (c) understanding of rules of bracket removing, (d) simplification of algebraic expressions, and (e) evaluation of algebraic expressions for a given value of the letter. These tasks are categorized as procedural because (i) they require the students to apply rules and procedures that they have been exposed to on the expressions and (ii) the solution has a clear direction in which it proceeds, that is, towards a numerical answer or the simplest algebraic expression and in that sense procedural and algorithmic. Some of these tasks, however, especially the task of evaluation of arithmetic expressions using easy ways, reveal student responses that go beyond mere application of procedure and are based on structure sense. These will be discussed later in the chapter.

The quantitative analysis of student responses consists of the data from the 15 English medium and 16 Marathi medium students who attended all the trials (N=31), that is, of the pre and the post test scores before and after each trial. The qualitative data will be drawn from the written responses of students in the tests, interviews with a subset of the students, daily practice sheets and classroom discussion. The interviews tried to elicit students' responses to tasks similar to the ones in the post test. Students first wrote the responses to certain questions and then in a one-on-one interview explained their solution as well as answered the additional probes used by the interviewer.

The results of a delayed post test conducted after some months of MST-I showed that the retention of the concepts, procedures and the rules was quite poor among most students. The data from this test will not be analyzed here. It was therefore decided to conduct the pre test in MST-II after exactly two days of revision. The concepts, rules and procedures were not completely forgotten but had been unused for a long time. This together with interference from the school learning could have resulted in the low delayed post test scores. The students had recently been exposed to evaluation of expressions and some ideas about integers in the school, when the students came to attend MST-II. The concepts and the procedures taught, and the vocabulary used in the school was quite different from those adopted in this approach. To some extent, interference due to these different approaches was inevitable. This factor must be kept in the background while interpreting the results. The pre test for MST-III was conducted on the first day of the trial.

6.1 *Evaluation of expressions*

6.1.1 Parsing of expressions

As explained in Chapter 3 (section 3.2.2), in the teaching approach the parsing of an expression into terms was a basic step for implementing the procedures of evaluation and simplification as well as for analyzing the structure of expressions. Students learnt reasonably well to parse the expressions by identifying the terms of an expression early in the first trial. The retention of this concept among students was very high. Figure 6.1 shows the students' performance in identifying terms of expressions in the three trials in the three kinds of expressions: those containing only simple terms (e.g. 19-6+7), simple and product terms (e.g.

2+3×4), and algebraic expressions (e.g. 3×x-4+6×x+10) (See for example, Q.20 in Appendix IIB, Q.12 in Appendix IIIA). The performance of the students in this task is nearly perfect with occasional errors like splitting the product term or forgetting the sign of the term or not attempting an item, especially during MST-I. Competence in this task did not always lead to improved performance in tasks which depended on this ability. In the sections below, implications of this ability for other tasks will be explored.

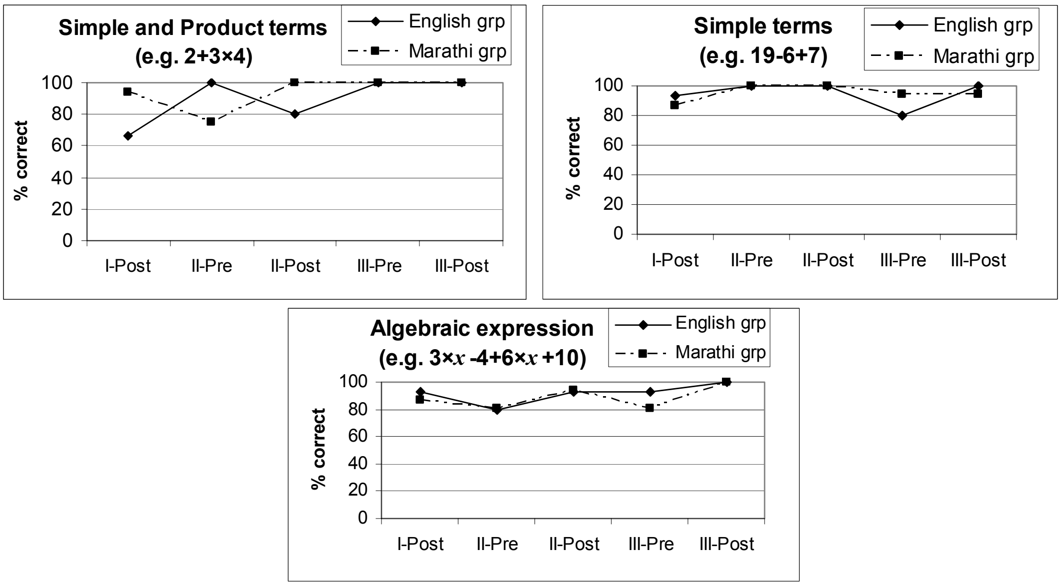


Figure 6.1: English and Marathi medium students' performance in the three trials in parsing expressions of three kinds ($n_{english}$=15, $n_{marathi}$=16)

Note. There was one item of each kind in each of the trials. There was no such item in the pre-test of MST-I. I-Post = Post-test (MST-I), II-Pre = Pre-test (MST-II), II-Post = Post-test (MST-II), III-Pre = Pre-test (MST–III), III-Post = Post-test (MST-III).

6.1.2 Evaluating simple arithmetic expressions

Simple arithmetic expressions contained either only simple terms (e.g. 19-3+6) or a simple term and a product term (e.g. 7+3×4) (For example, see Q.3, Appendices IIA, IIB, IIIA, IIIB). The students initially (MST-I) learnt to evaluate simple arithmetic expressions by analyzing the terms to decide the precedence rule. Due to students' unfamiliarity with negative numbers, expressions were so chosen that computation would lead a positive answer. This was a constraint on the nature of problems and discussions which could follow. Gradually (MST-II and III) they learnt to combine terms flexibly to evaluate expressions, without focusing on the precedence rules. Figure 6.2 shows the percentage of correct responses for evaluation of simple arithmetic expressions in the three trials of the main study.

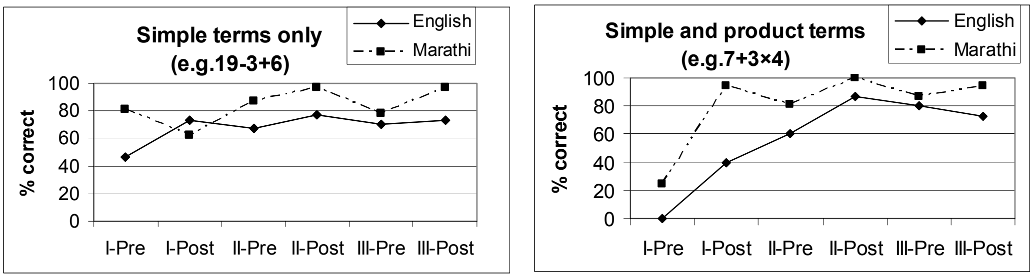


Figure 6.2: English and Marathi medium students' performance in evaluating simple expressions in the three trials ($n_{english}$=15, $n_{marathi}$=16)

Note. There was one item of the type 'simple and product terms' in each of the trials. There was one item of the type 'simple terms only' in the Pre-test and Post-test of of MST-I and Pre-test of MST-II. Thereafter, there were two such items. I-Pre = Pre-test (MST-I), I-Post = Post-test (MST-I), II-Pre = Pre-test (MST-II), II-Post = Post-test (MST-II), III-Pre = Pre-test (MST–III), III-Post = Post-test (MST-III).

Pre-test performance of the Marathi group is high for these simple expressions in the beginning of the first trial of the main study. Further, it reaches a high level of performance at the end of MST-I and this is maintained, except for post test of MST-I in the expression 19-3+6 where this performance was lower than in the pre-test. The errors in the post test of MST-I however were all calculation errors (discussed later). The English group gained steadily in proficiency. It must be noted that in this task and in others, the Marathi group was better than the English group even at the beginning of the study. Both the groups gained significantly (at .01 level in MST-I and at .05 level in MST-II in the t-test) in the items after the instruction. In MST-III, the English students' performance in the post test was slightly lower than in the pre-test in one item.

Sample item	Errors	MST-I		MST-II		MST-III	
		Pre	Post	Pre	Post	Pre	Post
7+3×4	LR	18	6	5	2	3	1
	CE	0	2	0	0	1	3
	Others	4	0	1	0	1	0
	Not done	5	2	1	0	0	0
19-3+6	Detachment	3	0	3	0	2	3
	CE	2	7	3	4	3	2
	Others	1	0	0	0	1	0
	Not done	5	3	1	0	0	0

Table 6.1: Number and type of incorrect responses in solving simple arithmetic expressions across the trials (N=31)

Note. LR = Left to right error, CE = Calculation error,Others = Errors which could not be classified as LR/ CE, Not done = number of instances of not attempting the item.

The analysis of error patterns also shows a trend of increased competence in these tasks (see Table 6.1). Expressions with only '+' and '–' signs (e.g. 19-3+6) do not create much conflict with students' intuitive ways of evaluation, that is, moving from left to right sequentially. Sometimes, due to the presence of certain numbers coupled with over generalization of the rules of operations (associativity and commutativity) or incorrect integer addition/ subtraction operation, students solve an expression like 19-3+6 as 19-9=10, detaching the negative sign from the succeeding number ('detachment' error). Even with the 'terms' approach where the terms could be combined flexibly, this error could not be dealt with, mainly due to students' poor knowledge of integer addition and subtraction operations. Integer operations were not extensively dealt in the study but many approaches were tried across the study. By the end of MST-II, it was realized that it could be subsumed in the terms approach and indeed the students' solutions to evaluating such expressions in the post test do not show any detachment error but resurfaces in MST-III.

The expressions with '×' (e.g. 7+3×4) on the other hand, creates a situation of conflict for the students as it can no longer be solved from left to right sequentially (as 10×4=40, 'LR' error) and the only correct way of solving it is to first simplify the product term. Despite instruction on the rule and analyzing the expression by identifying the terms of the expression, many students made the error in MST-I, which gradually reduced in the subsequent trials. The visual cues with regard to correct parsing of expressions, leading to forming the correct units in the expression ('terms'), were not internalized easily by the students. One reason for this could be that in MST-I, the concept of term was used to decide the precedence of operations – a procedural interpretation, rather than a more structural interpretation which allows flexibility in combining terms as in MST-II (see discussion in Chapter 5, section 5.2.2). The impact of such an approach on other tasks will be discussed in the sections below. However, both the structural errors do resurface in more complicated situations where students tend to find quick ways of finding solutions without paying enough attention to the structure of expressions.

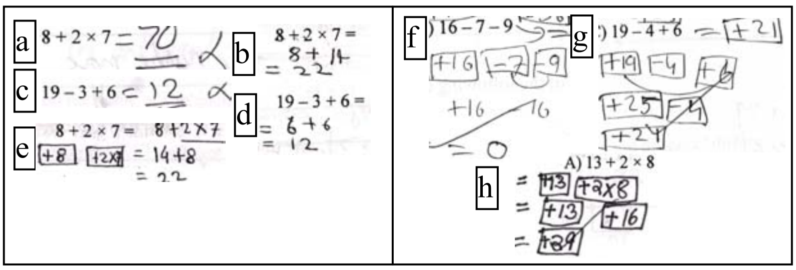


Figure 6.3: Examples of students' solutions of evaluation of simple arithmetic expressions in Post test: a-e: MST-I; f-h: MST-II

Figure 6.3 shows some solutions by students in the post test of the two trials. The solutions are representative of the students' ways of evaluating simple expressions. The solutions (a) to (e) are from MST-I and (f) to (h) are from MST-II (solutions in MST-III are similar to the later). As described in Chapter 5 (section 5.2.1), in MST-I students used the concept of terms only for identifying the conditions for applying precedence rules and therefore one does not see the solutions to be integrally linked with the concept of terms, as in MST-II. Students made structural errors ('LR') in MST-I which are well known and have been discussed in Chapter 2 and did not completely accept the idea of writing terms and using it to minimize the errors. They continued to solve sequentially and many did not identify the terms. Even in MST-II, there were a couple of instances of not identifying terms or making structural errors despite writing the terms. But largely, the shift in emphasis from procedural precedence rule based computation to learning to effectively use the structural analysis of the expression giving flexibility to evaluation of expressions could have made some difference in the acceptance of the 'terms approach'. Solutions like (f) and (g) in Figure 6.3 were never seen during MST-I.

Students' understanding of evaluation of simple expressions as revealed through the interview

The nature of changes that were taking place in the students' understanding with respect to evaluation of such simple expressions as has been discussed above can be clearly seen in their interview responses. Fourteen students (6 English and 8 Marathi) were interviewed after two months of completing MST-II and seventeen (8 English and 9 Marathi) after four months of completing MST-III and their ability to evaluate and explain the computation of simple arithmetic expressions was explored. The students were asked to evaluate two simple expressions in the interview: one consisting of a simple term and product term and the other consisting of only simple terms (similar to those in the test). The interview procedure as described in Chapter 4 (Section 4.4) was used which consisted of the student solving the expression and explaining it followed by an additional probe requiring the student to judge whether an alternative solution is correct or asking the student to find another way of solving the expression (see Q.A (1 and 2) and Interview schedule (Tasks 1 and 3) in Appendix VA, Arithemtic test Q.1 (A and B) and Interview schedule: arithmetic (Tasks 1 and 2) in Appendix VB).

Table 6.2 summarizes the students' responses to the evaluation of the above two kinds of expressions (that is, with only simple terms or with a product term) in the last two trials (MST-II and III). The first column ('Solution') indicates the correctness or incorrectness of students' written response to the task. The second column ('Probe') describes the students' response to the additional probe (usually a wrong solution if the student had correctly solved or vice-versa). The third column ('Changes') records the number of changes and the type of changes that the student made to the solution during the interview. The table suggests that the expression with a product term (item 1 in both trials) was the easiest and the least confusing for students to evaluate and explain in both the trials. Also, for expressions with only simple terms (item 2 in both trials), all the students, except one, correctly evaluated the expression in their written responses before the interview. The only incorrect written response during MST-II was corrected during the interview. Looking at the responses for all the items, three students after MST-II (one in item 1 and two in item 2) and one after MST-III (one in item 2) changed their incorrect response to the additional probe to the correct one during the interview. There is only one student in MST-II who changed his correct solution (item 1) to an incorrect one. Two students after MST-II incorrectly judged the alternative solution (one each in items 1 and 2) used in the additional probe and could not give a satisfactory explanation with respect to them.

In the interview after MST-II, thirteen of the fourteen students clearly indicated in these items their knowledge of the fact that an expression has a unique value. This guided their acceptance or rejection of the alternative solutions to the expressions shown to them. Broadly, two reasons were seen in their explanation: one, that the rule applied is correct or wrong; and two, that the value arrived at by the two ways is the same or different. Six of these fourteen students displayed some level of uncertainty, at least in one of the two items being discussed, while identifying the correctness of an alternative solution shown to them or explaining their own solution. In contrast, the students after MST-III, could show different ways of evaluating an expression and could distinguish a correct solution from a wrong solution. They frequently justified their solution methods for evaluating such expressions by stating rules they had learnt during the study. All those students probed with an alternative solution 22-7+9=22-16 confirmed its incorrectness and some others gave another solution in which -7 and +9 were combined to get +2. Most of the responses were clearly articulated; the reasons usually stated were – change in terms (7+9=16 instead of -7+9=2) (7 students in MST-II and 11 in MST-III) or the need for a bracket around 7+9 (two students each in MST-II and MST-III) – for the proposed solution to be correct. All of them were also confident that two different but correct ways of evaluation would always lead to the same answer and a few of them indicated the conditions when the answer/ value of the expression would change, like putting brackets around 7 and 9 or changing the sign of the terms. Some excerpts from the interviews with the students (marked with an asterisk in Table 6.2) will be discussed below[9]. The excerpts from the interview will illustrate the explanations given by the students as well as the coding of the students' responses as seen in Table 6.2.

[9] The interviewer's questions are in normal font and students' responses have been italicized.

	MST-II						MST-III					
Name	Item 1: 15+6×5			Item 2: 25-10+5			Item 1: 5+3×6			Item 2: 22-7+9		
	Solution	Probe	Changes	Solution	Probe	Changes	Solution	Probe	Changes	Solution	Probe	Changes
BP*	C	SEC	IACA	C	UE	NC	C	SE	NC	CE	SE	CEC
PD*	C	SE	NC	C	OC	IACA	C	SE	NC	C	SE	NC
BK*	C	SE	NC	C	SE	NC	C	SE	NC	C	SE	NC
AY	C	SE	CSISCS	I	SE	ISCS	C	SE	NC	CE	SE	CEC
NN	C	SE	NC	C	NE	CAIACA	C	SE	NC	C	UE	NC
SG	C	SE	NC	C	SE	NC	C	SE	NC	CE	SE	NC
NW	C	SE	NC	C	SE	NC	C	SE	NC	C	SE	NC
RG	C	SE	NC	C	NE	NC	C	SE	NC	C	SE	NC
AS	C	SE	NC	C	SE	NC	C	SE	NC	C	SEC	IACA
AN	C	SE	NC	C	SE	NC	C	SE	NC	C	SE	NC
SV	C	SE	NC	C	SE	NC	C	SE	NC	C	SE	NC
MC	C	SE	NC	C	IJ	NC	C	SE	NC	C	SE	NC
AB*	C	IJ	NC, CSIS	C	SE	NC	C	SE	NC	C	SE	NC
BM	C	SE	NC	C	SE	NC	C	SE	NC	C	SE	NC
PG	-	-	-	-	-	-	C	SE	NC	C	SE	NC
JS	-	-	-	-	-	-	C	SE	NC	CE	SE	NC
TJ	-	-	-	-	-	-	C	SE	NC	C	SE	NC

Table 6.2: Responses of the students interviewed in the two items of evaluating expressions after MST-II and III (* indicates students whose interviews are discussed in the text)

Written solution to the evaluation task (Solution):

(C) Correct – the solution given for the expression is correct

(I) Incorrect – the solution given for the expression is incorrect

(CE) Calculation error – solution procedure is correct but contains calculation error

Explanation to the alternative solution (Probe):

(SE) Satisfactory explanation – able to explain correctly own solution as well as the alternative solution and displays the knowledge of rules and concepts (based on terms and combination of terms, brackets)

(SEC) Satisfactory explanation with changes – able to explain correctly own solution as well as the alternative solution but involves a change in the answer during discussion

(UE) Unsatisfactory explanation – creates an ad-hoc explanation suitable for the specific situation at hand and may involve frequent change of answers from one to another

(NE) No explanation – cannot give any explanation for the alternative solution and only knows own solution

(OC) One correct – choosing an answer because only one solution can be correct

(IJ) Incorrect judgment – making an incorrect judgment about the alternative solution

Changes made by the student during the interview (Changes):

(NC) No changes made

(CEC) Calculation error corrected – the subject corrects the calculation error

(ISCS) Incorrect solution to correct solution – the subject changes his/ her incorrect solution for the expression to a correct one

(IACA) Incorrect alternative to correct alternative – the subject changes his/ her incorrect answer for alternative solution to a correct answer

(CSIS) Correct solution to incorrect solution – the subject changes his/ her correct solution for the expression to an incorrect one

(CAIA) Correct alternative to incorrect alternative – the subject changes his/ her correct answer for alternative solution to an incorrect answer

(CAIACA) Correct alternative to incorrect alternative to correct alternative – the subject changes his/ her correct answer for the alternative solution to an incorrect answer to back again to a correct answer

Faced with the task of solving 5+3×6 and finding another solution to it, the student AB (MST-III) explains why he thinks 5+3×6 cannot be solved by any way other than the way he solved it (5+18=23) to the interviewer SN.

> SN: Can it be solved in any other way?
>
> *AB: First if we multiply 5 and 6, then 5 6za 30, it will be wrong.*
>
> SN: If you do 5×6, then it will be wrong?
>
> *AB: Yes. 5 6za 30, add 3 to it 33, this is not possible.*
>
> …
>
> SN: Are these two questions the same [5+3×6 and 5×6+3]?
>
> *AB: No. Because this [×6] is not a product term. Terms are only positive or negative.*

Although he himself proposed the second solution for the expression, he agreed that it was wrong. This realization would have followed his encounter with two different answers/ values of the expression. Later he went on to give a more structural justification as to why his solution is not correct, understanding that the factors of the product term cannot be separated. He used the same argument of 5+3 not being a term to explain why 8×6 is also not a correct solution (coded as satisfactory explanation 'SE').

A typical response in both the trials to justify the solution to an expression like 25-10+5 is illustrated by the following. The student BK (MST-II) explains her reasoning very clearly as to when a simplification 25-10+5 to 25-15 would be correct. This was one kind of reasoning evidenced for this question; another was the reason that -10+5 must be -5 and cannot be -15.

> RJ: One boy did it like this. 25-10+5 this is the question. He did it like this 25-15. So the answer is 10. Is this correct?
>
> *BK: No teacher.*
>
> RJ: Not correct. Why?

BK: Teacher, it is not bracket. If there is bracket [around 10+5] we can do like this.

This student definitely understands the change that will be produced in this expression when the brackets are present and anticipates the result of the computation. Further, she shows her ability to solve the expression by flexibly combining terms leading to the same answer (solving the expression 25-10+5 as 30-10=20, collecting the positive terms first). The student AN (MST-II) took another line of argument for the above situation and explained why 25-15 cannot be the solution for 25-10+5: '*Writing 25 for 25 is correct but -10 and +5 can never be added… It has to be subtracted, -15, -5 should be the answer*'. She insisted that -10 and +5 cannot be 'added' and that they should be 'subtracted'. Even though she is using the idea of terms and is well versed with the idea of combining terms, she uses an operational language to explain it, which could be due to the influence of school instruction. Her own solution to the expression was 30-10=20.

On the other hand, a student BP (MST-II) gave a correct solution to 25-10+5 by sequentially moving from left to right and was further asked by the interviewer RJ whether 25-15 is a correct solution for it.

RJ: …Ok, so you have got +15. So if someone did it like this, that 25-10+5, so he did it like this 25-10+5 is 15, then he got the answer as 10. So is this correct?

BP: No

RJ: Why?

BP: Teacher, because you have compared this [10+5].

RJ: Combined these.

BP: Teacher, there is –10, and it is +25, so it is bigger, so you have to combine these first.

…

RJ: So you are sure this is right [-10+5=-15] or this is wrong?

BP: Teacher, wrong.

RJ: This is wrong. Why? You don't know?

BP: (in Marathi) 25 is bigger, we do not have to combine –10 and 5, but combine 25-10, +25 is bigger, so combine 25 and –10.

She was unclear about why the alternative solution is wrong although she was sure that it is wrong. The interviewer guided her to use the correct language 'combine' and not 'compare'. Her explanation did not indicate any reasoning based on rules of integer operations; rather she created ad-hoc reasoning for the specific situation in hand (coded unsatisfactory explanation 'UE'). Four students of the fourteen in MST-II could not offer any explanation for why the proposed solution was not correct but they were sure that it is incorrect. This situation did not occur in MST-III.

In MST-III as well BP began explaining why 22-7+9 cannot be solved as 22-16 hesitantly; saying 22 is bigger number and therefore has to be done first. But then she clarified her answer soon.

RB: Another student wrote it as 22-16, 7+9 is 16, equal to 6. Is this correct?

BP: No.

RB: No. Why?

BP: Because +22 is bigger number. So we have to first do this.

RB: Acha. What do we have to do first?

BP: First we do 22-7 or we put bracket and do -7+9.

RB: What will come then if we put -7+9 in bracket?

BP: Put 7+9 in bracket.

RB: What will happen if we put 7+9 in bracket?

BP: 16.

RB: So if here [22-(7+9)] there is a bracket then only it will be 16. If bracket is not there then what will happen?

BP: +2.

She had improved her understanding and could now explain the correct and multiple ways of solving the expression. She could also clarify that in the presence of the bracket around 7+9, the answer would be 16.

Another student PD (MST-III) tried to give an explanation for the same question as above. She had solved the expression as 22-7+9=22+2=24 and showed another way of solving it as 15+9=24 during the interview.

> RB: Now what you have done. 22-7 and then added 9. So you got the same answer. Hmm. Would you always get the same answer if you solve it by different ways? Every time you would get the same answer when solved in different ways?
>
> *PD: No.*
>
> RB: No? When is it that the answer will not be the same? Because you have just seen by solving in two different ways -7+9 and 22-7. You have got the same answer. Can you give me an example when you will not get the same answer?
>
> *PD: Here -22 and here +7.*
>
> RB: Acha. If we do like this, you think we would not get the same answer. If I do it like this 22-16, how did we get 16, 7+9, equal to 16. Is this a correct way of solving this?
>
> *PD: No*
>
> RB: What have I done?
>
> *PD: You got 16 by doing -7+9. But -7 is a smaller number, if we take 9 positive cards and -7 positive cards, negative cards, and make pairs then we are left with -2, +2.*

She displayed her awareness of different ways of solving the expression and also conditions under which the answer will change. For her, changing the terms is one criterion which changes the answer of the expression. She also shows her understanding of integer operations and was the only student who explained computation on integers using cards. A few more students (3) gave such examples to illustrate when answers change, to answer the question whether two different ways of evaluation would give the same answer. They mostly identified changing terms or putting brackets with negative sign outside as reasons for change in value of the expression. Some others pointed out that different but correct ways of evaluation would lead to the same answer.

During MST-II, for some students explaining their own solutions and alternative solutions to these items was quite hard and they repeatedly changed their explanation, not always successful in the end (see Table 6.2). At times they made incorrect judgments regarding the alternative solution without feeling the need to change it. The rules for evaluating such simple expressions were not as clear and transparent and they were on shaky grounds, also because of their weak knowledge of integer operations. This could be explored only as a result of exposing the students to multiple solutions and allowing them to reflect on their solutions and the procedures they had learnt. These students can be thought to be in the 'participatory' phase, restricted to the situation they are working in, and have not moved up to the 'anticipatory' phase which requires them to reflect on the procedures and being able to construct a more generalized understanding of rules and procedures and validity of transformation (Tzur and Simon, 2004). These issues were probed further in the next trial MST-III.

The situation after MST-III is slightly different. Table 6.2 shows evidence of their comfort with such simple expressions and their ability to mentally think about processes and actions and to anticipate the consequences of these. Compared to the interview sessions with students after MST-II which saw them changing their responses often, the students after MST-III were more confident and less hesitant in stating their judgments and their arguments for or against a solution. Barring a single case where an unsatisfactory explanation was given, all others gave well articulated and convincing arguments to support their case. Also, once the students realized their error, it was easy enough for them to correct their responses. In the due course more examples will be given to substantiate these results as well as to show situations when this breaks down.

6.1.3 Bracket opening rules and evaluating expressions with brackets

Brackets, as a concept and as a parsing device explicating the structure of expressions form an important aspect of expressions as has been explained in Chapter 3 (Section 3.3.2). The teaching approach focused on both static and dynamic use of brackets. The static use of brackets dealt with treating them as precedence operation on the part of the expression enclosed in the bracket ('do brackets first') and the dynamic use requires one to use equivalence relations embedded in such expressions with brackets, that is, equality of a-(b+c) with a-b-c or a×(b+c) with a×b+a×c (Linchevski and Livneh, 1999). Although brackets can easily be understood as a precedence operation, it is not always possible to use this conception in the context of reasoning *about* expressions and manipulation of algebraic expressions. Therefore bracket opening rules were given much importance in the teaching learning sequence. These rules were reformulated using the concept of terms: for a negative bracketed term, signs of all the terms change on removing the bracket (e.g. 23-(8+9)=23-8-9); and distributive property is applied for a product term with a bracketed factor (e.g. 3×(7-5)=3×7-3×5). Students worked on two kinds of tasks: writing an equal expression for a bracketed expression and evaluation of bracketed expressions.

Students' understanding of bracket opening rules

The students were given a task in which they had to write an expression equal to an expression containing a bracket (e.g. 23-(7-3)=?) (for example, Q.25 in Appendix IIB, Q.16 in Appendix IIIA, Q.14 in IIIB). The performance of the students in this task in the three trials is shown in Figure 6.4. Due to the prevalent inconsistency in responses seen among the students in the first two trials with respect to bracket opening rules, the open format of the question (write an expression equal to a given one with brackets) was changed to a multiple choice format in which the students had to mark one of the two expressions equal to the given one (see Q. 11 in Appendix IVA and Q.13 in Appendix IVB).

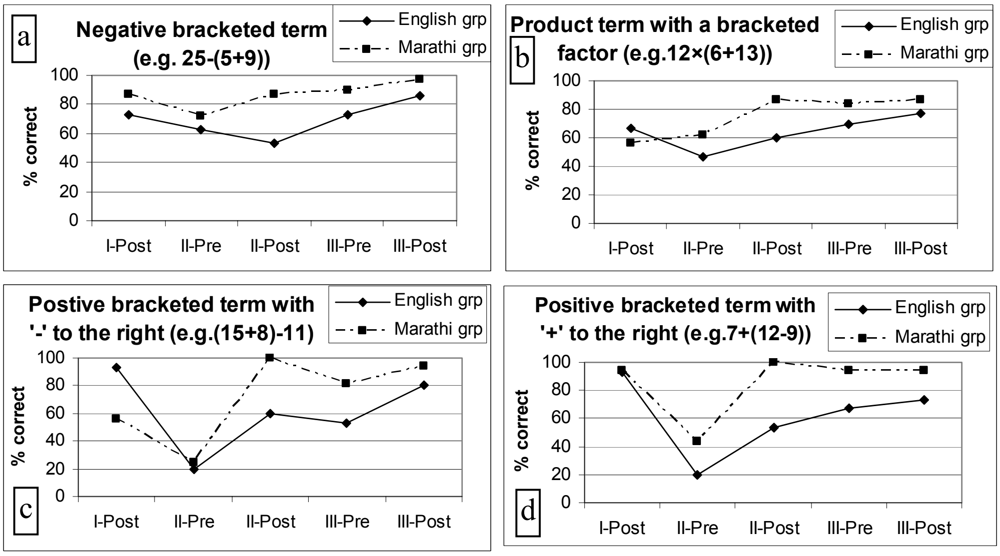


Figure 6.4: English and Marathi medium students' performance in bracket opening rules in the three trials with respect to the different kinds of expressions ($n_{english}$=15, $n_{marathi}$=16)

Note. There were two items each of the types 'negative bracketed term' and 'product term with a bracketed factor' and one item each of the other two types. This task was not posed in the pre-test of MST-I. I-Post = Post-test (MST-I), II-Pre = Pre-test (MST-II), II-Post = Post-test (MST-II), III-Pre = Pre-test (MST–III), III-Post = Post-test (MST-III).

The graphs show that the Marathi medium students performed better than the English medium students. The Marathi medium students learnt these rules by the end of the second trial (MST-II) and then maintained their performance. The English medium students gradually learned these rules over the three trials. Both the groups recorded a low performance in the pre test of MST-II which was conducted after some months of MST-I. The rules were over-generalized to expressions when it was a positive bracketed term (see Figure 6.4 (c) and (d)) not only in the pre-test of MST-II but persisted in the subsequent tests as well. The errors in Figure 6.4 (a) and (b) are the expected ones: not changing the signs of all the terms inside the bracket after it is removed or not distributing the common factor over both the terms. The structural difference between the kinds of terms (negative or positive bracketed term or a product term with a bracket factor) in the expressions was probably not captured by the students and the rules inappropriately applied. This has implication for evaluating expressions with brackets, although in this study one does not find any direct relation between students' performance in the above task and the use of these rules while evaluating expressions.

Evaluation of bracketed expressions

Students' performance in evaluation of expressions which contained brackets also improved significantly in the post test compared to the pre test in all the trials. Students were asked to evaluate a simple expression with brackets (e.g. 5×(9+3)) in all the pre and post tests, except in the post test of MST-II (see Q.3B in Appendix IIA, IIB, IIIA, Q.2(4) in Appendix IVA, Q.5(4) in Appendix IVB). Table 6.3 shows the performance of students for a simple expression with brackets. In the post test of MST-II and MST-III, some more complex items of this kind were added.

Sample item	MST-I				MST-II				MST-III			
	English		Marathi		English		Marathi		English		Marathi	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
5×(9+3)	7	73	44	69	60	-	81	-	67	80	81	94

Table 6.3: Performance (in percentage) of the students in the trials in correctly evaluating expressions with brackets ($n_{english}$=15, $n_{marathi}$=16)

Note. No such item was posed in post test of MST-II and in the other tests there was only one item of this kind.

Nearly half of the Marathi students knew the solution of such an expression when they came for the course for the first time (pre-test MST-I). But most of the other students in this group as well as the students in the English group ignored the bracket completely and computed the value of the expression from left to right. These students either did not know the significance of the bracket or were using a wrong rule for opening the bracket, the possibility of the latter being low as these students had not as yet been introduced to such rules. In the post test of the same trial, only four students ignored the bracket or forgot the correct rule of removing the bracket ('incomplete distribution') with the remaining errors being calculation mistakes. But in the pre test of MST-II the performance of the students was reasonably good and all the students who succeeded solved inside the bracket first, using it as a precedence operation. A majority of the errors in this simple item with brackets were due to calculation mistakes. Similarly, in MST-III, students performed well in the post test, the errors being in computation. Error in applying the distributive property was rare. In the first two trials, most students preferred to solve inside the bracket first (BF – 61% in post test of MST-I and 84% in pre test of MST-II) than opening the bracket using distributive property (BO – 29% in post test of MST-I and 10% in the pre test of MST-II). In the last trial, during the pre test almost equal number of students used each of the strategies (BF – 42% and BO – 39%) but in the post test most students opened the brackets in this simple expression also (BF – 19% and BO – 81%).

	BF	BO	BO+BF	Other	Total
English	6 (*4)	7 (*6)	1(*0)	1 (*1)	15 (*11)
Marathi	8 (*3)	5 (*2)	3 (*2)	0	16 (*7)
Total	14 (*7)	12 (*8)	4 (*2)	1 (*1)	31 (*18)

Table 6.4: Number of student responses by type of strategy in evaluating the expression 27-3×(5+7) in Post-test of MST-II (*Numbers in the bracket denote the number of incorrect responses, $n_{english}$=15, $n_{marathi}$=16)

Note. BF = Solving the bracket first, BO = Opening the bracket first, BO+BF = Using a combination of both these strategies, Other = Other strategies.

The item being considered very simple for students was dropped from the post test of MST-II. Instead, the item was replaced by a more difficult expression with brackets: 27-3×(5+7) (see Q. 3B in Appendix IIIB). The performance of the students in this item was poor with only 27% of English medium students and 56% of Marathi medium students succeeding in this task. This expression required more than one rule of opening bracket to solve the expression correctly, if the student decided against solving the bracket first. Compared to the pre test, where most students preferred to solve the bracket first, here many students tried to remove the bracket first before simplifying and in the process committed errors by not changing the sign of the terms inside the bracket. Table 6.4 shows the strategies used by students to evaluate the expression 27-3×(5+7). Each of the strategies – solving the bracket first (BF) or opening the bracket (BO) – was used by almost equal number of students, with slightly more errors in the case of opening bracket (BO). Some of the students started solving by removing the bracket but midway changed the strategy to solve the bracket first (BO+BF in Table 6.4).

	Correct	ID	Sign not changed	Sign not changed + ID	Calculation error	Others
English	4	0	4	1	5	1
Marathi	9	0	2	0	4	1
Total	13	0	6	1	9	2
Percentage	42	0	20	3	29	6

Table 6.5: Number of student responses by type of error in evaluating the expression 27-3×(5+7) (Pos-test MST-II) ($n_{english}$=15, $n_{marathi}$=16, N=31)

Note. ID = Incomplete distribution.

The number of students who made particular kinds of error in this item is given in the Table 6.5. The ‘incomplete distribution’ error (multiplying only the first term) was rare (one instance) and some made calculation errors which include errors in signs after computation. Many errors are due to calculation mistakes followed by the mistake of not changing the signs after removing the bracket. The new learning and emphasis on rules of removing brackets seemed to have interfered with the knowledge of brackets that they already possessed.

Some examples of students' solutions from the post tests of the first two trials are given in Figure 6.5. Figures (a) and (b) are from MST-I and (c) to (f) are from MST-II. Both (a) and (b) describe solutions based on precedence of brackets but in (b) the bracket is first removed and then the solution is carried out by adding inside the bracket. This could be an indication of the ambiguity students sense in the expressions; for them an expression with and without bracket could mean the same thing. Figure (c) shows a simple straight forward solution to the expression 27-3×(5+7) and the others (d) to (f) are based on opening bracket. They are more complex and have more scope for errors, whether in applying the distributive property (f) or in changing the sign of the terms inside the bracket (e). Figure (d) shows a correct solution.

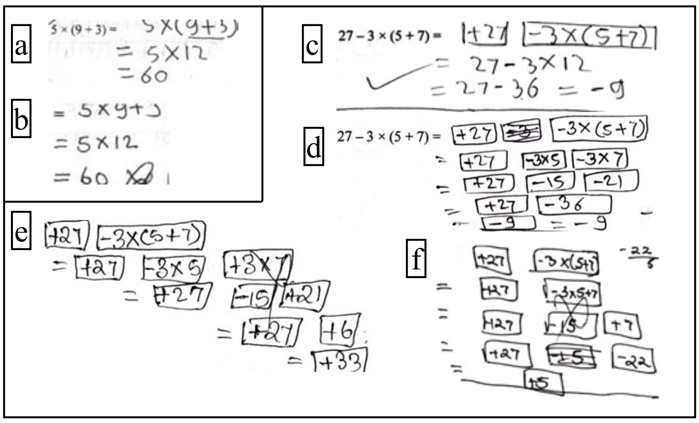


Figure 6.5: Sample of evaluation of bracketed expressions by students in post test of MST-I and MST-II (a-b: MST-I; c-f: MST-II)

S.No.	Item	English		Marathi	
		Pre	Post	Pre	Post
1.	25-(4+3×5)	0	47	44	87
2.	34-6×(9-5)	53	53	62	75
3.	28-(13-7+5)	27	27	56	87
4.	19-2×(3+6×7)	-	40	-	37

Table 6.6: Performance (in percentage) of students in correctly evaluating expressions with brackets in MST-III ($n_{english}$=15, $n_{marathi}$=16)

Note. The item 19-2×(3+6×7) was not posed in the pre-test of MST-III and there was only one item of each type.

Items similar to the above requiring more than one rule of removing bracket were used in the post test of MST-III (see Appendix IVB, Q. 5(5-8)). The performance of the students in such items had a large variation. A table showing the performance in each of the items is given above (Table 6.6). In MST-II, post test performance on an item closest to 34-6×(9-5) was 27% for English medium and 56% for Marathi medium students.

	BF	BO	Not done	Total
25-(4+3×5)	12 (*7)	19 (*3)	0	31 (*10)
34-6×(9-5)	9 (*3)	22 (*8)	0	31 (*11)
28-(13-7+5)	11 (*9)	20 (*4)	0	31 (*13)
19-2×(3+6×7)	10 (*3)	19 (*14)	2	31 (*17)
Total	42 (*22)	80 (*29)	2	124 (*51)

Table 6.7: Number of student responses by type of strategy in evaluating bracketed expressions in Post-test of MST-III (*Numbers in the bracket denote the number of incorrect responses) (N=31)

Note. BF = Bracket solved first, BO = Bracket opened first, Not done = number of instances of not attempting an item.

As one may expect, the least level of performance were seen in the more complex expressions 28-(13-7+5) (the English group) and 19-2×(3+6×7) (both groups). The performance of the English medium students was much lower than the Marathi group, except for the last expression. Most students preferred to open the brackets to evaluate such expressions than to solve the bracket first. Some students used the two methods flexibly (solving brackets as precedence rules or using bracket opening rules) while evaluating the expressions. Table 6.7 shows the strategies students used to evaluate expressions of this kind – solving the bracket first (BF) and opening the bracket (BO). The number of students choosing a strategy varies slightly across the items. Students who solved inside the bracket were successful only half the time; they made structural, computational and other errors which are not easy to classify. The rate of error (36%) among students who opened the bracket was less than those who solved inside the bracket. In contrast, in the item in MST-II (comparable to item 2 in MST-III, Table 6.4), the use of the two strategies BO and BF were almost equal, with slightly more errors while opening bracket. The overall performance in MST-III has improved compared to MST-II. Moreover, the percentage of bracket opening errors and calculation errors has slightly reduced in MST-III but has led to resurfacing of structural errors in this context which is a result of the items used in MST-III (see Table 6.8).

Table 6.8 shows the predominant errors made by students while evaluating such expressions. It is clear from the table that a majority of the students committed errors either due to incorrect application of bracket opening rules or incorrect parsing of the sub-expression inside the bracket. Most errors while opening brackets had to do with the sign changing rules rather than applying distributive property. The errors in evaluating inside the bracket, largely by the English medium students, were ‘LRB’ (left to right inside bracket) and the ‘detachment’ errors. So for expressions embedded inside brackets, they were unable to perceive the separation of the simple term and the product term. Calculation errors and some other errors which were a combination of the above mentioned structural and bracket opening errors accounted for another large majority of the errors. The Marathi group of students did not commit the structural errors like ‘LR’ and detachment and displayed a better understanding of the bracket opening rules.

Item	Correct	Sign not changed	Sign changed inside bracket	ID	LRB	De-tach-ment	Calc. error	Others	Not done
25-(4+3×5)	21	2	2	0	4	0	1	1	0
34-6×(9-5)	20	5	0	1	0	1	2	2	0
28-(13-7+5)	18	2	2	0	0	4	1	4	0
19-2×(3+6×7)	12	3	1	2	0	0	6	5	2
Total	71	12	5	3	4	5	9	13	2
Percentage	57.25	Bracket opening errors: 16			Structural errors: 7.25		7.25	10.5	1.75

Table 6.8: Number of student responses by type of error in evaluating bracketed expression in Post-test of MST-III (N=31)

Note. ID = Incomplete distribution, LRB = Left to right inside bracket, Others = Combination of structural and bracket opening errors, Not done = instances of not attempting an item.

Figure 6.6 shows some solutions by students of expressions with brackets in MST-III and instantiates the errors as discussed above. The figures demonstrate some of the typical errors committed by students. Students evidently preferred to open the brackets using the rules than to solve inside the bracket. Although all students used the concept of term to analyze the expressions, not all of them were successful. Whereas Figure 6.6(c) is an example of deeper analysis of the sub-expression, Figures 6.6(b) and (d) are examples of structural error in the sub-expression embedded inside the bracket. Some students were also found to change the sign of the terms inside the bracket while retaining the bracket as in Figure

6.6(d): 28-(13-7+5)=28-(-13+7-5) (figure contains correction made by the teacher; the answer was correct). A sample of correct solutions is also shown (Figures 6.6(a) and (e)).

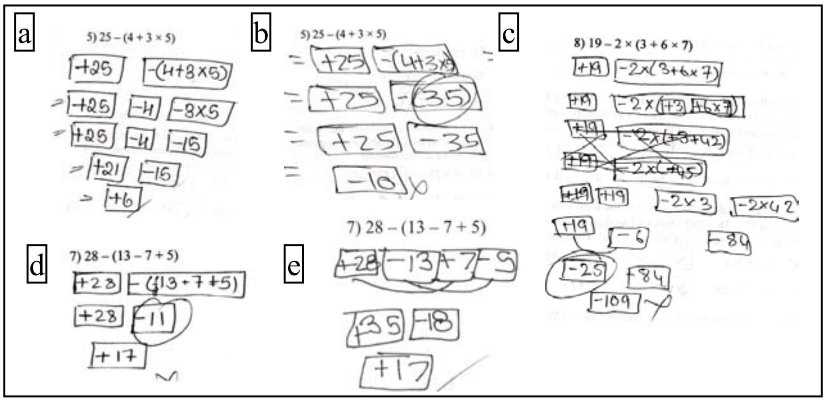


Figure 6.6: Sample of students' solution of expressions with brackets in post-test of MST-III

Students' understanding of evaluating expressions with brackets as revealed through the interview

Students were also interviewed on the evaluation of arithmetic expressions with brackets. Two such items were posed after both the trials: one had a negative sign to the left of the bracket (a negative bracketed term) and the second had multiplication sign outside the bracket (product term with a bracket factor) (see Q.A (3 and 4) and Interview schedule (Tasks 4 and 6) in Appendix VA), Arithemtic test Q1. (C and D) and Interview schedule: arithmetic (Tasks 3 and 4) in Appendix VB). Students had more trouble in these expressions compared to the simple arithmetic expressions discussed earlier – first, in evaluating simple expressions with brackets and then in justifying their solution, and further explaining the correctness or the incorrectness of an alternative solution. Table 6.11 summarizes the students' responses to the two items in this task in the interviews after the two trials MST-II and III. As before, this table also describes correctness or incorrectness of students' written solution ('Solution'), their response to the alternative solution of the same expression (‘Probe’) and finally, the changes they made in the process (‘Changes’). The table shows that four students out of fourteen made error at least once in evaluating such expression after MST-II but they succeeded in correcting their solutions. In six instances students made incorrect judgments, with or without an appropriate explanation, for the alternative solution. Only two instances could be satisfactorily resolved during the interview. Out of the remaining four, one student could not correct the judgment and another used the unique value of the expression as the criterion for changing her judgment, however not appreciating the equality of the expressions. Two students held the belief that removing the bracket always results in change of sign, irrespective of what sign is before the bracket.

After MST-III, twelve students (out of 17) were readily seen to use both ways of evaluating the expressions with brackets, that is, either solving the bracketed part first or removing the bracket by using an appropriate rule indicating that they understood the equivalence of these two procedures. These students also knew the rules of opening bracket well. The other five students could not successfully solve the expression with the negative sign outside the bracket, but the item with multiplication sign was error free. Except for two students who failed to be convinced about the incorrectness of their solution, the other three corrected their solution easily when given an opportunity to do so. These students also knew the correct rules, but two of them were unsure about the equality of the two ways of evaluation (that is, evaluating inside the brackets, or opening the brackets). One of them stressed that although such bracket opening rules can be used to evaluate the expressions, the expression enclosed by the bracket should be given precedence. Interview excerpts of students marked with an asterisk in Table 6.9 will be discussed below.

	MST-II						MST-III					
Name	Item 1: 25-(10+5)			Item 2: 6×(2+3)			Item 1: 22-(7+9)			Item 2: 5×(3+8)		
	Solution	Probe	Changes	Solution	Probe	Changes	Solution	Probe	Changes	Solution	Probes	Changes
BP*	C	SE	NC	I	SE	ISCS	C	SE	NC	CE	SE	CEC
PD	I	SE	ISCS	C	UE	IACA	C	SE	NC	C	SE	NC
BK	C	SE	NC	C	SE	NC	C	SE	NC	C	SE	NC
AY*	I	SE	ISCS	C	IJ, NE	NC	C	SE	NC	C	SE	NC
NN	I	UE	ISCS, CAIACA	C	SE	NC	I	SE	ISCS	C	SE	NC
SG*	C	SEC	IACA	C	UE	IACA	I	SE	ISCS	C	SE	NC
NW	C	SE	NC	C	SE	NC	C	SE	NC	C	SE	NC
RG	C	SE	NC	C	SE	NC	C	SE	NC	C	SE	NC
AS	C	UE	NC	C	NE	NC	I	SE	ISCS	C	SE	NC
AN	C	SE	NC	C	SE	NC	C	SE	NC	C	SE	NC
SV	C	SE	NC	C	SE	NC	C	SE	NC	C	SE	NC
MC	C	SEC	IACA	C	NE	NC	C	SE	NC	C	SE	NC
AB	C	SE	NC	C	SE	NC	I	IJ	ISCS, IACA	C	SE	NC
BM	C	SE	NC	C	SE	NC	C	SE	NC	C	SE	NC
PG	-	-	-	-	-	-	C	SE	NC	C	SE	NC
JS*	-	-	-	-	-	-	I	IJ, UE	ISCS	C	SE	NC
TJ	-	-	-	-	-	-	C	SE	NC	C	SE	NC

Table 6.9: Responses of the students interviewed in the two items of evaluating expressions with brackets after MST-II and III (* indicates students whose interviews are discussed in the text)

Written solution to the evaluation task (Solution):

(C) Correct – the solution given for the expression is correct

(I) Incorrect – the solution given for the expression is incorrect

(CE) Calculation error – solution procedure is correct but contains calculation error

Explanation to the alternative solution (Probe):

(SE) Satisfactory explanation – able to explain correctly own solution as well as the alternative solution and displays the knowledge of rules and concepts (based on terms and combination of terms, brackets)

(SEC) Satisfactory explanation with changes – able to explain correctly own solution as well as the alternative solution but involves a change in the answer during discussion

(UE) Unsatisfactory explanation – creates an ad-hoc explanation suitable for the specific situation at hand and may involve frequent change of answers from one to another

(NE) No explanation – cannot give any explanation for the alternative solution and only knows own solution

(OC) One correct – choosing an answer because only one solution can be correct

(IJ) Incorrect judgment – making an incorrect judgment about the alternative solution

Changes made by the student during the interview (Changes):

(NC) No changes made

(CEC) Calculation error corrected – the subject corrects the calculation error

(ISCS) Incorrect solution to correct solution – the subject changes his/ her incorrect solution for the expression to a correct one

(IACA) Incorrect alternative to correct alternative – the subject changes his/ her incorrect answer for alternative solution to a correct answer

(CSIS) Correct solution to incorrect solution – the subject changes his/ her correct solution for the expression to an incorrect one

(CAIA) Correct alternative to incorrect alternative – the subject changes his/ her correct answer for alternative solution to an incorrect answer

(CAIACA) Correct alternative to incorrect alternative to correct alternative – the subject changes his/ her correct answer for the alternative solution to an incorrect answer to back again to a correct answer

Most students (except those mentioned in the above two paragraphs who were not able to resolve the issue) in the interview supported their solution or their judgment on the alternative solution by stating the correct rules for bracket opening and pointing out the equivalence of the two ways of evaluating an expression with bracket. The transcripts below show the range of understanding that the students had at the end of the second and the third trial with respect to evaluating expressions with bracket. The following transcript gives an indication of the extent of over generalization of rules which was seen among two students after MST-II. However, these students did not display this misconception after MST-III. The student BP (MST-II) quickly agrees to the alternative solutions and agrees that her solution is wrong but decides the correctness of the last solution by appealing again to the unique value property.

> RJ: Now this question. 6×(3+2). You first did 6×3 and then you did –2 from +2, and then +18-2. Where did you get –2 over here?
>
> *BP: Opened brackets na.*
>
> RJ: You opened the bracket. So you got –2. +18 and –2. But one boy, he did it like this. 6×(3+2). He did it like this 6×3+6×2= 18+12=30. Is this correct?
>
> *BP: This is correct.*
>
> …
>
> RJ: Why?
>
> *BP: Because, 6 is product term, 6 'into', it is common.*
>
> RJ: 6 is common, and this is a product term. …But one boy did it like this 6×5 =30. Is this correct?
>
> *BP: Yes, these two answers are same. And he has added 3 and 2 and did 6×5.*
>
> RJ: Can we add like this?
>
> *BP: Yes.*

The following excerpt also shows the confusion students had with the meaning and use of bracket. The student AY (MST-II) changed his answer and explanation many times but obviously without much knowledge of his own acts. He solved the expression 25-(10+5) initially as 25-10+5=25+5=-30 and when pushed further tried to solve it as +25+10+5=15+5=+20 during the interview, failing once again.

RB: Ok, if someone does this like this 25-(10+5)=25-15=10. Is this correct?

AY: Yes teacher.

RB: This is correct? Sure correct?

AY: Yes.

RB: Can this question have two different answers?

AY: Teacher no.

RB: Then what should be done? Are both of these correct or one of them is correct?

AY: This is correct [points to 25-15].

RB: This is correct. Why?

AY: Because they are in one bracket and we can add them. And we can do 25-15.

RB: So we could have as well removed the bracket and done. But then why is this [+25+10+5=15+5=+20] not correct?

AY: Teacher, because here I have added [10+5], here it should have been subtracted.

RB: Where should it be subtracted?

AY: 15+5, I should subtract it and write 10.

RB: But is 15+5 10? If you add 15 and 5 would you get 10?

AY: Subtract here [-5 instead of +5].

RB: Here it should be -5. But where will -5 come from?

AY: The sign inside bracket will change, so it will be -5.

RB: This will be -5 and here it will be +10.

AY: Teacher, yes.

His poor knowledge of integer operations interferes with his other learning and this issue was troubling throughout the whole interview. He was not aware of the different answers he arrived at after each of his attempts to solve the expressions 25-(10+5). He could not also use the bracket opening rule correctly. He agreed completely with the alternative correct solution, explained it satisfactorily and started to correct his own steps. He tried to fix the problem in an ad-hoc manner in bits and pieces without ever gaining full control of the situation and taking stock of the necessary rules to be followed. However, in MST-III, he had no trouble in solving a similar expression and explaining the alternative solution displaying his understanding of the equality of the two ways of evaluation.

Even after MST-III, some students faced difficulties in responding to expressions with brackets. The overall situation was better than after MST-II, with the expression 22-(7+9) being an exception evoking all kinds of responses. One student JS explains her solution for 22-(7+9) and her confusion with regard to the use of brackets is evident. She solved it as 22+7-9=22-3=19. She was one of the few students who did not think the solution 22-16=6 was a correct solution because '*there is a minus sign outside the bracket, so the sign inside would change*'.

> RB: Acha. There was another student who did it as 22-7-9= 22-7 is 15-9=6. Is this student correct? This student also said that there is a minus sign outside the bracket and the signs change.
>
> *JS: May be it is correct.*
>
> RB: It could be correct? Why can it be correct?
>
> *JS: Because he has also written minus sign, -7-9, later he has solved this.*
>
> RB: What is the difference between this solution [22-7-9=15-9] and the one you have solved [22+7-9]?
>
> *JS: I have first solved the bracket and then the part outside bracket, and he has solved first the part outside the bracket and then inside the bracket.*
>
> RB: What do you think, can this [22+7-9] and this [22-7-9] both be correct? What is your answer, +19 and what is the answer of this student, +6. Can both the answers be correct? One of the two has to be correct. Which of these is correct?
>
> *JS: This one [22+7-9].*

JS insisted that the solution 22-7-9=15-9=6 was not correct as '*... you have to first solve inside the bracket and then outside. But he has first solved outside*'. Although she agreed that different ways of solving the expression should lead to the same answer, she could not find the error in her solution and checked all other solutions with respect to her solution. She too, like AY in the previous transcript, did not have adequate knowledge of integer operations and probably did not see the difference between 7-9 and 9-7. Her understanding of the meaning of bracket was confounded with bracket opening rules she had learnt, making it difficult for her to attend to the important aspects of the solution process. It was only by the end of the interview, through heavy prompting that she could understand her mistake. She is not very articulate in her explanation but one can understand what she means to say in the context. For both these students (AY and JS), integer operations together with shaky understanding of brackets led to the confusion.

Another student SG (MST-III) understood the meaning of brackets as precedence operation and was also aware of the bracket opening rule.

> RB: Is there any other way of solving it?
>
> *SG: Teacher, I do not know.*
>
> RB: Now what have you have done is 7+9=16. There was a student who did it like this. You have to tell me whether he did it correct or wrong. 22-7-9=15-9=6. Is this correct?
>
> *SG: Teacher, this is in the bracket, so this has to be done first. We can do it this way also. But it is in the bracket so we should do it first.*

He clearly understood the difference in the procedures of evaluating an expression with and without brackets. But he was not comfortable in using the bracket opening rules to evaluate the expression, though he reluctantly agreed to such a procedure. He appreciated the various ways of evaluating expressions when the expression did not contain bracket. He clearly maintained this difference between solving expressions involving brackets and without any brackets in other instances also during the interview ('*If there is bracket we must do the bracket and if there is no bracket then there is a different way of doing it.*').

More students made errors in evaluating expressions with a negative bracketed term compared to a product term with brackets. All students could correctly explain the mechanism of the distributive property (without explicitly stating it) and the precedence of the brackets but this understanding was very procedural and no effort was made to explore students' understanding about the truth of the rule. Neither did any student try to explain it. Below is the conversation with a student RG (MST-III) during the interview about evaluating 5×(3+8).

> SN: Explain the solution to this.
>
> *RG: Here 5×(3+8) is one term. And here I have made them two different terms +5×3 and +5×8 and then 5 3za 15 and 5 8 za 40 and then added them.*
>
> SN: And what have you done here [pointing to his other solution 5×11=55]?
>
> *RG: Here I added inside the bracket and solved.*

RG was aware of both ways of evaluating the expression and was convinced that the two ways of evaluation are equivalent and would lead to the same answer. These discussions during the interview also indicated their better grasp of the distributive property than using the idea of inverse for a negative bracketed term.

The analysis of strategies used for solving the expressions with bracket and the nature of errors in the post test shows a trend of moving from giving precedence to the brackets to opening the bracket. Although over the trials more number of students used bracket opening as the strategy for solving such expressions, less of them made errors in applying the correct rules. Also, students were not using a single strategy across the expressions. The structural errors resurfaced as the students failed to attend to the structure of the expression and the sub-expression. Some of them probably failed to grasp the equality of the expressions while removing the bracket, the rules mechanically applied, and this seems to be the cause of confusion for a few of the students (as seen in the interview and probably true of some others who were not interviewed). Whether they solved inside the bracket first or opened the bracket apparently displaying the dynamic 'structural' use of brackets, in both cases, it seems to have remained at the level of procedures, without appreciating the duality – the process and the product – that is, understanding that the bracketed (sub-)expression can be replaced by a number or an equal expression, which is a demonstration of 'proceptual' thinking (Sfard, 1991; Gray and Tall, 1994) inherent in the use of the bracket. The bracketed expression till the end did not become a 'process' (as in APOS) in the minds of the students which they could run through mentally without actually carrying it out (Dubinsky and MacDonald, 2001). This flexible understanding of brackets is crucial not only for evaluating the expressions as above but also in generating representations.

6.1.4 Easy ways of evaluating arithmetic expressions

In order to focus the attention of students on the structure of expressions and to de-emphasize precedence rules, even though these formed the basis of evaluation of arithmetic expressions during MST-I, students were encouraged to look at the relationships between terms of the expression and find easy ways of computing them (e.g. 29-7+11+7 or 14×3+10×8+14×7). Two main objectives of this task were to complement procedures with structure sense and to introduce non-sequential computation in arithmetic itself rather than postpone it to algebra. Table 6.10 shows the performance of the students in successfully evaluating complex arithmetic expressions in the three trials of the main study (e.g. see Q.30 in Appendix IIB, Q.19 in Appendix IIIB, Q.12 in Appendix IVA, Q.19 in IVB). In all the trials, the performance of the Marathi group is better than the English group, in both cases of expressions with only simple terms and only product terms. Whereas in MST-I, the errors were due to the difficult calculations they carried out by adhering to the precedence rules including two cases of 'LR' error; in MST-II, most students identified terms which could be combined to make computation easy but they made numerous errors in integer operations, leading to low performance. One of the items in the MST-II post test was 12×9+16×5-17×9, where many students made a mistake in writing the result of the sub-expression 12-17 obtained after extracting the common factor 9 (most wrote 5 instead of -5). Even those who wrote the correct answer for this sub-expression, attached the wrong sign to the factor (wrote 5×(16+9) instead of 5×(16-9)) while extracting the common factor in the next step of evaluation. In the second item, which had a common factor among all the terms, most of the errors were found to be non-systematic like changing the sign of a term or calculation errors. Similarly, in MST-III, most of the errors were due to calculation mistakes, errors in sign, changing the sign of the term or the term itself. There were seven instances of 'detachment' and three of 'LR' in the pre test and three instances of 'detachment' in the post test of MST-III.

Sample item	MST-I		MST-II		MST-III			
	English	Marathi	English	Marathi	English		Marathi	
	Post	Post	Post	Post	Pre	Post	Pre	Post
-28+49+8+20-49	53	62	67	100	40	73	69	81
48-56+17+9	33	44	-	-	40	87	44	94
3×16+16×12-16×7	-	-	47	81	27	73	62	87
7×18-6×11+4×18	27	56	0	25	47	33	69	94

Table 6.10: Performance (in percentage) of students correctly evaluating complex expressions across the trials ($n_{english}$=15, $n_{marathi}$=16)

Note. There was only one item of each type. Blank entries against items not posed in the tests.

Analysis of the strategies used to simplify these expressions would reveal the extent to which students actually identified and found easy ways of evaluation. The expectation was that the improved performance in such tasks would also be reflected in the strategies chosen. Table 6.11 and 6.12 below shows the number of students in each category of strategy chosen by them, relational (RN), precedence rules (PR), unknown strategy (US) or not attempted (NA), not necessarily leading to correct solutions in the end. A strategy was coded 'relational' when the students noticed relationships in the terms and combined them flexibly, using properties of numbers and operations to make the computation task simpler. Further, a strategy was coded 'precedence rule' when the student used precedence rules to simplify the expression, not noticing any relationship between the terms. Some of the students did not show any working for their responses, and therefore such responses are classified under unknown strategy. The tables show the change in strategies of the students. There is a large difference in the way the students approached the problem between MST-I and II and some minor difference between MST-II and III. Surprisingly, many students could use relational strategies for evaluating such expressions in the pre test of MST-III, which was conducted after many months of MST-II.

Sample item	MST-I (Post-test)				MST-II(Post-test)			
	RN	PR	US	NA	RN	PR	US	NA
29-7+11+7	13(*4)	11(*4)	3(*1)	4	30(*4)	1(*1)	0	0
11×4+9×11-7×11	-	-	-	-	21(*7)	8(*3)	0	1
14×3+10×8+14×7	6(*1)	15(*8)	2(*1)	8	18(*17)	9(*5)	0	5

Table 6.11: Number of student responses by type of strategy in solving expressions using easy ways in the first two trials (N=31, *the number in the bracket denotes number of incorrect responses)

Note. RN = Relational strategy, PR = Precedence rules, US = Unknown strategy, NA = Not attempted the item. Entries for items not posed in trials in blank.

During the first trial of the main study, students found it difficult to reconcile the two different instructions for evaluating expressions: one, to apply precedence rules to evaluate expressions and the other, to explore relationships between terms to make computation easy. This was reflected in the performance of the students in the post test as well where students could not find useful relations to make calculations easy. Instead, they computed the expressions using precedence rules, and in the process committed calculation mistakes due to tedious long calculations.

Sample item	Pre test				Post test			
	RN	PR	US	NA	RN	PR	US	NA
48-56+17+9	24(*5)	3(*3)	2(*2)	2	19(*3)	11(*3)	1(*1)	0
69-26-11+26-8	18(*7)	9(*7)	1(*1)	3	28(*3)	3(*0)	0	0
3×16+16×12-16×7	14(*7)	14(*7)	1(*1)	2	21(*3)	10(*3)	0	0
7×18-6×11+4×18	19(*5)	6(*2)	1(*1)	5	22(*7)	9(*4)	0	0

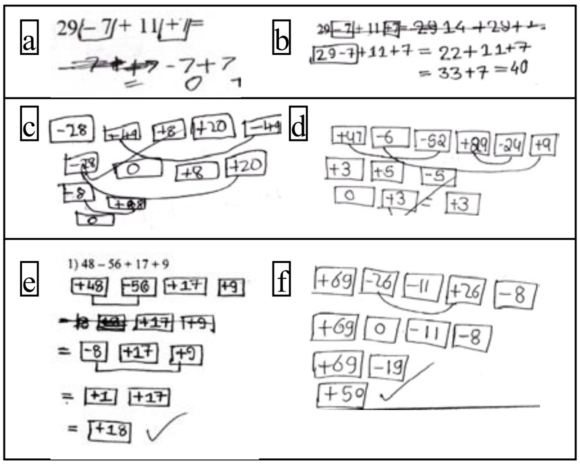
Table 6.12: Number of student responses by strategy in solving expressions using easy ways in MST-III (N=31, *the number in the bracket denotes number of incorrect responses)

Note. RN = Relational strategy, PR = Precedence rules, US = Unknown strategy, NA = Not attempted the item. Entries for items not posed in tests is blank.

The items in MST-II and MST-III were more complex, but more number of students chose relational strategies than precedence. They looked for terms which when combined could lead to easy computation of the expression. The 'terms approach' helped students to identify relationships between terms and combine them flexibly. Also, these solution procedures, at least in the case of simple terms, were spontaneously generated by the students during classroom discussions. The number of students efficiently using the strategy of combining terms is much higher in expressions with simple terms (where some terms cancelled each other or combined to form multiples of 10) than in the case of expressions with product terms, where distributive property could be used once or twice. The knowledge of distributive property is necessary but not sufficient for using such strategies while evaluating expressions. Students (except two instances in three trials) who used the distributive property in this task also could write the equal expression for

3×(4+5)=__ (another task in all the tests) after removing brackets but not vice-versa. Those who could not do so, preferred to convert each product term into simple term.

The above tables indicate students' increasing ability to perceive structure of expressions and use it in carrying out tasks like the one being discussed here. This is not to say that they always found the most efficient way of computing the answer, especially in the case of expressions with simple terms where there were multiple ways possible. But the non-sequential computing of expressions requires an awareness of the fact that combining terms in any order does not change the value of the expression. Such understanding and using this to one's advantage is the first step towards developing a stable structure sense and would be helpful in reasoning *about* expressions. This would also connect with the simplification procedures in algebra which essentially require the flexibility as the sequential left to right computation is no longer possible.



MST-I

MST-II

MST-III

Figure 6.7: Sample of students' evaluation of expressions containing only simple terms using easy ways in the post test of the three trials

Figures 6.7 and 6.8 show students' solutions to some expressions using easy ways. As has been pointed out earlier, the solutions to expressions, both with simple and product terms, during MST-I were not as flexible as they were in MST-II and III. Although students tried to see the relationships between terms and use them in their computations, they failed in their attempts (Figure 6.7(b), 6.8(b)). The reasons for this have been discussed above. Further, the nature of the expressions used in MST-II and III allowed more ways of manipulating them. Some examples of such solution in the case of expressions with only simple terms are seen in Figure 6.7(c, e, f).

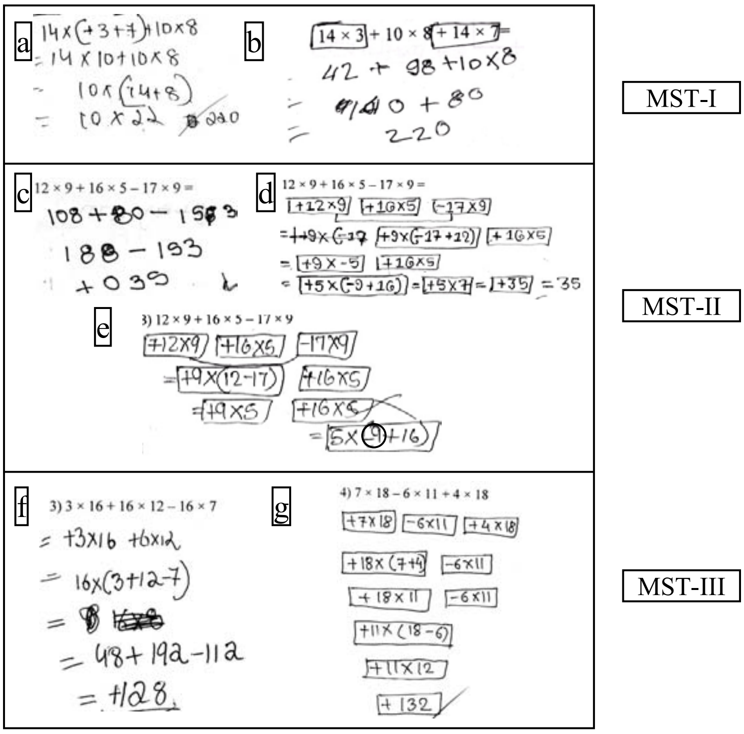


Figure 6.8: Sample of students' evaluation of expressions containing product terms using easy ways in the post test of the three trials

In expressions with product terms the students extracted the common factor using the distributive property. Some students applied this procedure twice, simplifying the computation even further (Figure 6.8(g), MST-III). Also, the solutions to expressions with product terms in MST-II indicate the type of the errors which were responsible for the low overall performance in the task (see Figure 6.8(e)). To avoid mistakes, some students carefully chose not to use distributive property the second time when one of the factors had a negative sign, and instead solved the product terms. Occasionally responses of the kind shown in Figure 6.8(f) were seen when a few students used the property to extract the common factor and not knowing what to do further, opened it again to evaluate the expression.

In a slight variation of the above item, students were asked to show 47-6-52+29-24+9 = 3 in the post test of MST-II. 40% of the English medium students and 56% of the Marathi medium students were successful in completing the task. Most of the students who succeeded in the task found efficient ways of combining terms, reducing the chances of errors. One such solution can be seen in Figure 6.7(d). The unsuccessful students made non-systematic errors, like changing the sign of the terms or a term itself midway through the solution process and a few systematic errors like incorrect sign after integer operations and 'detachment' error were also seen. The rationale behind the use of such tasks was to guide students to manipulate expressions with respect to a goal which is a very useful skill in algebra problem solving (Mason et al., 1985; Arcavi, 1994; Boero et al., 2001, discussed in Chapter 2, section 2.7.3). A similar task was also used in the context of algebra.

The discussion on evaluating expressions in the previous sections reveal that the concept of terms was being used by the students in all computation tasks and was no longer restricted to the context of comparing expressions and judging equality of expressions. Students improved their performance in evaluating expressions using flexible means in MST-II and III over MST-I. It is important to understand that correct identification of terms is an essential pre requisite for evaluating expressions flexibly but is not sufficient. Although most students could identify terms correctly in MST-II and III, not all could successfully complete the tasks (evaluation of simple and complex expressions), many errors being in integer operations as well as non-systematic errors. Many of these students also made errors while solving simple two termed expressions (e.g. -5+8=?), designed to check their skills in integer operations. Their performance with respect to bracketed expressions was not satisfactory and the interview transcripts show non-clarity regarding the meaning and purpose of brackets. The competing demands made by these expressions requiring the students to analyze the terms in the expressions as well as apply rules on them, were too much for them to handle. It needs a certain degree of automaticity with the parsing of expressions to be able to move ahead. For a few students, identifying terms remained at a mechanical level; they were unable to understand the purpose of parsing and could not benefit from the approach. Many others gradually moved from the sequential left to right processes to focusing on relations among terms. This was governed by an implicit understanding that terms of an expression could be combined in any order without changing its value, which students had learnt while identifying and generating equal expressions tasks and will be discussed in the next chapter.

6.2 *Understanding of simplification of algebraic expressions*

Students' understanding of evaluating arithmetic expressions – simple as well as the more complex expressions has already been discussed. The simplification of algebraic expressions follows the same rules as of transforming arithmetic expressions (that a product term needs to be simplified to a simple term to combine with another simple term or else two product terms can be combined if they have a common factor), even though there are subtle differences in the evaluation procedure and notations and conventions as discussed in Chapter 2. One can be successful in evaluating arithmetic expressions using procedural precedence rules as well but such an understanding is often not helpful in simplification of algebraic expressions where the knowledge of structure of expressions plays an important role in appreciating the rules for manipulating algebraic expressions. It requires knowledge of rules and properties of operations, that is, some knowledge of both systemic structure and surface structure but the goal is to apply this knowledge to arrive at a compact expression in its simplest form. Simplification of algebraic expressions was a problematic area for quite some time; students finding it difficult to understand the rules of simplification. The teaching approach and the continuous effort to modify to make this accessible to students have been discussed in Chapter 5.

Across the trials, students performed these tasks with various degrees of success. The performance of the students in the simplification task will be discussed in this section along with the relation of the knowledge of the rules and procedures for transforming arithmetic expressions to their understanding of the procedure of simplifying algebraic expressions.

6.2.1 Simplification of algebraic expressions

Sample item	MST-I		MST-II		MST-III			
	English	Marathi	English	Marathi	English		Marathi	
	Post	Post	Post	Post	Pre	Post	Pre	Post
$5\times x+16+7\times x-11$	26	31	-	-	37	87	44	78
$x+15-13\times x-9$	-	-	0	25	-	80	-	90

Table 6.13: Students' performance (in percentage) in the simplification task of algebraic expression ($n_{english}$=15, $n_{marathi}$=16)

Note. $5\times x+16+7\times x-11$: 2 items each in post-test of MST-I, pre-test of MST-III and post-test of MST-III; $x+15-13\times x-9$: 1 item in post-test of MST-IIand 2 items in post-test of MST-III.

Students' performance in simplifying algebraic expressions in the post test in the first two trials was much below their performance in any of the tasks of manipulating arithmetic expressions. There were two such items in the post test of MST-

I, one in MST-II and four in MST-III (see Q. 21 in Appendix IIB, Q.19(C and D) in Appendix IIIB, Q.18 in Appendices IVA and IVB). Average performance has been taken in cases of multiple items with similar structure (post test of MST-III). Table 6.13 shows the performance of the students in the task over the three trials.

The analysis of the responses to the simplification task reveal qualitative shift in their understanding. Table 6.14 shows the frequency of different errors in the three trials made by the students. A large chunk of the errors is caused due to 'conjoining' error (e.g. $3+4\times y=7\times y$) in both MST-I and II, which is made not just in the penultimate step but also along the simplification process, and due to non-systematic errors (NSE, e.g. change of sign, change of term, arbitrary solution process). In MST-II, the performance of the students in the algebraic simplification task was equally poor, the error rate being even higher. The item was slightly more complex than the last trial ($x+15-13\times x-9$), including a 'singleton' and a negative answer for a part of the manipulation ($x-13\times x= -12\times x$). Some of these students (16%) identified $x+15$ to be a single term and changed it to $x\times 15$, a form which they were familiar with (considered NSE while coding). The reason could be deeper than just carelessly perceiving a structure which is more familiar. Using the approach which was used in this study to build an understanding of algebraic expressions and transformations on them, it is much easier to understand the meaning of $3\times y$ than of 'y'; the former can be understood as 'three times any number' but the latter is harder; operating on or with it subsequently also becomes harder. In MST-III, the students were largely able to simplify algebraic expressions. In this trial, all students, except one, used the idea of extracting the common factor to simplify such expressions.

Types of error	MST-I (2 items)	MST-II (1 item)	MST-III (4 items)
Conjoining	31	26	0
NSE	18	45	10
Integer operations	3	3	5
Calculation error	3	0	1
Not done	16	6	0

Table 6.14: Percentage of incorrect responses by type of error in simplifying algebraic expression in the post test of the three trials (N=31)

Note. NSE = Non-systematic errors, Not done = not attempted the item/ question

The classroom discussions during MST-I showed that students faced difficulties in understanding the notational, structural and procedural similarities and differences between manipulating arithmetic and algebraic expressions (see Chapter 5 for detailed discussion). Even if some students learnt to do it during the trials, they could not perform successfully in the test at the end of the trial. The classroom performance of the students in MST-II indicated that they had understood the similarity between the procedures in the two kinds of expressions, arithmetic and algebraic, well. The students were able to simplify algebraic expressions and the low performance in the post test could be attributed to the 'tricky' expression involving a singleton.

Figure 6.9 gives a glimpse of how students simplified algebraic expressions in the post tests of the three trials. In MST-I students rewrote the algebraic expression arranging the like terms together (Figure 6.9(a)) or explicitly identified the like terms (Figure 6.9(c)) and simplified by extracting the common factor. Failing to appreciate the need for separating the like terms or distinguishing the terms, many students were seen to simplify the expression as shown in Figure 6.9(b) (conjoining error). In MST-II, together with conjoining errors, students committed errors in integer operations as well (Figure 6.9(d)), and other errors of changing the terms (e.g. considering x+15 as x×15, Figure 6.9(f)). Figure 6.9(e) is a correct solution and this was rare in MST-II.

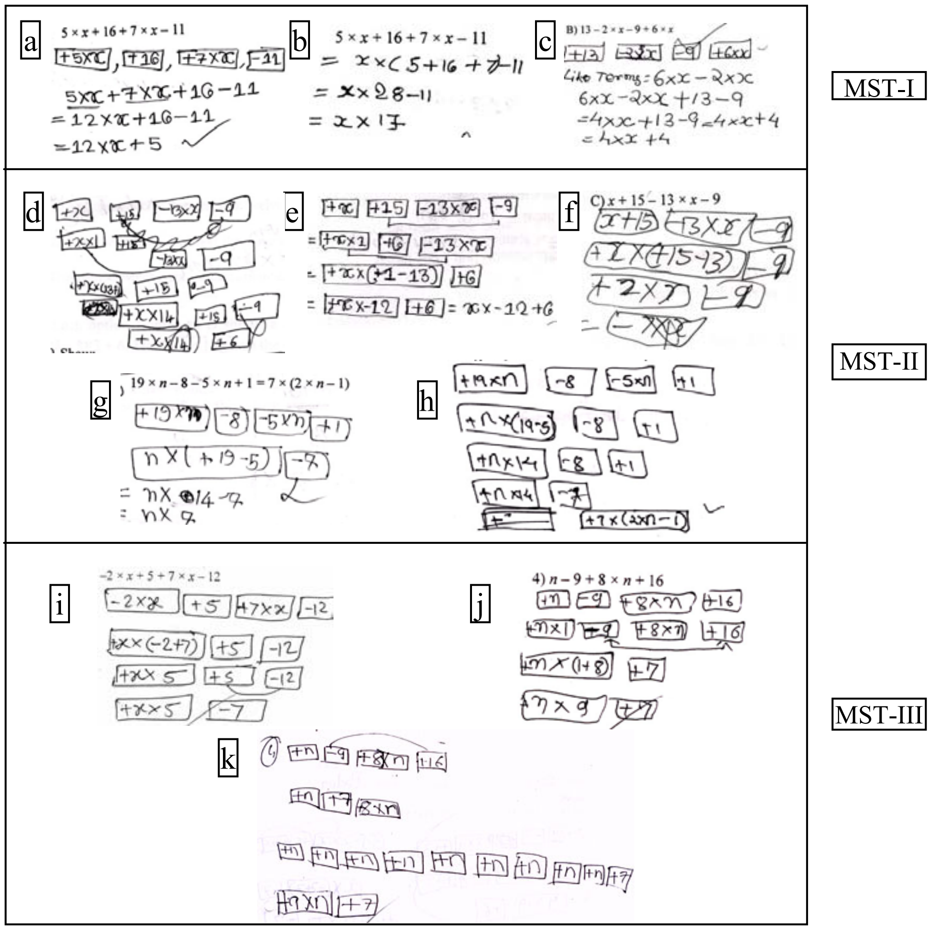


Figure 6.9: Example of students' work on simplification of algebraic expressions in the post test of the three trials

In a slight variation of the simplification task and similar to the task in arithmetic, the students were asked to show that the expression 19×n-8-5×n+1 is equal to the expression 7×(2×n-1) in MST-II. The performance of the students was very poor in this task as well with only 33% of the English medium and 6% of the Marathi medium students being able to successfully complete the task. Many of the students did not attempt the task and very few were successful. Figures 6.9(g) and (h) show two solutions by students in this task in the post test. The solution in 6.9(g) indicates that the student could simplify the expression but could not see the desired goal in it and instead ended up conjoining. The other solution (Figure 6.9(h)) is complete and the student meticulously derived the expression from the given one. The classroom discussions indicate that although the students could begin well and get the simplified expression but manipulating it further to reach the desired outcome was difficult for them. The final expression was mostly arrived by intervention from the teachers and it is hard to say whether students appreciated that the final expression, which had very different surface structure, was equivalent to the original expression. The purpose behind the use of such tasks was not successfully met with as the students, devoid of any context, did not see the need for it. This was more successful in the arithmetic context. These aspects were not dealt with in detail at this time. Their understanding of simplification process of algebraic expressions at the end of MST-III is discussed later in this section.

In contrast to the students' responses during MST-I and MST-II discussed thus far, many of the solutions in MST-III were correct and Figure 6.9(i, j, k) shows some of the solutions in this trial. One of the students used a basic principle of decomposing the product term into 'singletons' and seeing the effect of adding some or taking away some 'singletons' from the existing ones (Figure 6.9(k)). Some others followed the more formal way of using the distributive property arriving at the simplified expression (Figure 6.9(i, j)). Almost all the students could also think of 'n' as a product term $1\times n$ and apply the distributive property to such cases. This strategy of converting the simple variable term into a product term had its limitation. The students failed to arrive at the result of $n+8\times n$ orally without recourse to such a long process. This formal strategy was long and cumbersome to handle when the algebraic expression was embedded in a context and quicker way of seeing the result was required.

There were some other indications that students were making a connection between the procedures in arithmetic and algebra. For example, 18 out of the 25 students in MST-I, who could not find easy ways of evaluating arithmetic expressions with product terms requiring the use of distributive property also failed to simplify algebraic expressions or made errors. The other seven simplified correctly, two of them making calculation errors. Further, 5 out of 19 instances in MST-I and all in MST-II who made the 'conjoining' error while simplifying algebraic expressions, also made the 'LR' error. An example of this pattern is, solving the expression 8+2×7 as 10×7=70 and simplifying the algebraic expression $5x+6+7x-11=7x$, computing sequentially from left to right. There seems to be a transfer of misunderstandings and misperceptions of structure of expressions between the two domains, leading to similar procedural errors in both domains. Only by the end of MST-III, students could work on the algebraic expressions comfortably, treating the smaller units (terms) as entities (not requiring computations), which could be combined using the properties of operations (thus making the shift from 'interiorization' phase to the 'condensation' phase).

6.2.2 Evaluation of algebraic expressions

Even though students by the end of MST-III learnt to simplify algebraic expressions, the performance in evaluating an algebraic expression given the value of the letter (e.g. $5+3\times x$, $x=2$) was not very good for the English medium students all through the trials (see for example, Q.18 in IIA, Q.17 in Appendix IIB). Results of the students' performance in this task in the various trials are given in Table 6.15.

	MST-I				MST-II				MST-III			
	English		Marathi		English		Marathi		English		Marathi	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
n-14, *n*=7	-	-	-	-	-	40	-	81	53	47	69	75
5+8×*n*, *n*=7	0	33	12	75	40	53	87	94	40	47	75	81

Table 6.15: Performance (in percentage) of students in evaluating algebraic expression ($n_{english}$=15, $n_{marathi}$=16)

Note. There was one item of each type in the pre and the post-tests of MST-I, II and III. Entries blank against items not posed.

In the post test of MST-II, three such items were included compared to one item in the previous trial. In the third item -7+2×*k*+12 for *k*=6, students' performance dropped to 33% (English) and 75% (Marathi), due to errors in integer operations that needed to be carried out. A list of the nature of errors made by the students in this task together with their frequency in the three trials is given below (Table 6.16). Few instances of the structural error 'LR' (e.g. 7+3×*x* = 10×*x* = 20 for *x*=2) continued to recur but reduced considerably by MST-III. Some of the 'LR' errors were however accompanied by no substitution of the letter by the number ('LR+No substitution', 9+5×*k*=14×*k*). In one instance, a student completely ignored the letter and evaluated the expression as if it was an arithmetic expression. Errors due to integer operations ('sign errors') were also a major contributor to the low performance. The category of errors called 'others' consisted of errors which are non-systematic and random or incomplete, (e.g. *k*-12 for *k*=6, *k*+6+6=*k*×(6+6) or -7+2×*k*+12=5+2×*k*=10×*k*). Some students did not attempt one or more items ('not attempted'). Over the three trials, there is not much improvement in the students' performance except for a slight reduction in the number of errors per item (from 14/ item in MST-I to 11.3/ item in MST-II, Table 6.16) and lesser number of structural errors. The performance is a bit surprising as students performed fairly well in the simplification of algebraic expressions task in the post test of MST-III and also as will be seen later, explained quite clearly the process of simplification in the interview after MST-III. This task was not discussed much in the classroom after MST-I; thus their unfamiliarity with the task could have led them to not attempt it. Some did not understand the requirement of the task and led to the strange solutions (esp. seen in the 'others' category).

	MST-I (1 item)	MST-II (3 items)	MST-III (2 items)
Correct	17 (55%)	59 (63%)	39 (63%)
LR	5	6	2
Detachment	0	3	0
LR+Detachment	0	1	0
LR+No substitution	3	3	2
Sign errors	0	9	6
Letter ignored	0	1	0
Calculation error	0	1	1
Others	2	7	4
Not attempted	4	3	8
Average error	14/ item	11.3/ item	11.5/ item

Table 6.16: Number of student responses by type of error in post-test in evaluating algebraic expressions (N=31)

Note. LR = Left to right, LR+Detachment = left to right together with detachment error, LR+No substitution = left to right, without substituting the value of the letter.

Students who made the 'LR' error (and its other variants) in this task also failed to simplify algebraic expressions, either conjoining terms or making non-systematic errors (random transformation) in the process in MST-I and II. However, in MST-III the above error did not affect their performance in the simplification of algebraic expression task, barring sign errors and change of terms while simplification. Few of them made the 'LR' error while evaluating a similar arithmetic expression as well (e.g. 3+4×7=7×7=49; 3/8 in MST-I and 2/10 in MST-II, 1/4 in MST-III). However, in MST-II and III, these students did not make a similar error while evaluating arithmetic expressions using easy ways. The Marathi medium students seem to be more consistent than the English medium students in their appreciation of the structure of expressions in the various tasks and understanding of the similarity in the rules of transformation in arithmetic and the algebraic expressions.

In another task in the post test of all the trials, students were required to find the value of an algebraic expression given the value of a related expression (e.g. y+35=72, y+34=?) (see Q.14 in Appendix IIA, Q.13 in Appendix IIB, Q.10 in Appendix IIIA etc.). This task has a slightly better performance than the evaluation of the algebraic expression task but it did not necessarily require them to understand the letter as a number. The answer could be found by looking at the pattern in the expressions. Students' performance in this task is given in Table 6.17.

	MST-I				MST-II				MST-III			
	English		Marathi		English		Marathi		English		Marathi	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
y+35=72, y+34=?	7	33	62	69	47	60	81	87	47	33	50	62

Table 6.17: Performance of students (in percentage) in deriving the value of an algebraic expression from a related expression ($n_{english}$=15, $n_{marathi}$=16)

Note. There was only one item of this type in all the tests.

The students who were successful found the answer by looking at the change in the expression, rather than finding the value of the letter and substituting it in the later expression, except for two students in MST-I and three in MST-III. Increasing number of students learnt to see the relationship between the expressions in the later main study trials (17 in MST-I, 23 in MST-II and MST-III). The performance of the students in both the groups dropped in MST-III as the expressions involved a minus sign (*u*-53=26, *u*-54=?), the students making the error by simply comparing the numbers without considering the operation sign (10 students). Few students from the English medium group in each of the trials were seen to add all or some of the numbers to arrive at the answer, although it decreased over the trials (6 in MST-I, 1 in MST-II, 2 in MST-III), again showing a misunderstanding about the '=' sign. Some did not attempt the question in each of the trials (6 each in MST-I and II, 3 in MST-III). The responses of a couple of students in these two tasks across the trials indicate a shaky understanding of '=' sign and the letter.

Students' understanding of simplification of algebraic expression and the letter revealed through the interviews

The responses of students on the evaluating algebraic expression task, that is, knowledge of substitution procedure, were reasons enough to question their understanding of the meaning of the letter and the procedure of evaluation. Some questions were included in the interview after MST-III to be used as a probe for furthering the understanding of students' knowledge of simplifying algebraic expressions and the meaning of the letter (see Algebra test Q.1 and Interveiw schedule: algebra (tasks 1 and 2) in Appendix VB). In the interview, the students were asked to first explain their solution to the simplification of algebraic expression. Next they were required to predict the value of the original expression for a value of the letter, when the value of the simplified expression for the same value of the letter was given. Table 6.18 summarizes students' responses to the task of simplifying two algebraic expressions. The table shows their initial written solutions to the simplification task before the interview began ('Solution'), followed by their explanation for equality in value of the original and the simplified expression

(‘Probe’) and finally the changes they made while answering the questions (‘Changes’).

In the interview, all the students successfully justified the procedure of simplification of algebraic expressions by drawing on their knowledge of evaluating arithmetic expressions. One student in fact made the conjoining error while simplifying the expression but while explaining she corrected it. Students predominantly explained the simplification process by repeating the procedure they carried out, accompanied by statements like only product terms can be combined in the presence of a common factor and a simple and a product term cannot be combined. As an explanation to the second part of the task about equivalence of the expressions in the simplification process, many of the students indicated the similarity in procedures of computing an arithmetic expression and an algebraic expression, hinting at the structural similarity between the two expressions and thus procedural similarity. Eleven out of the seventeen students interviewed, knew without resorting to calculation that the given expression and the final simplified expression are equal (see Table 6.18, Probe column). They affirmed that each step in the simplification procedure yields equal expressions and hence their values would be the same for any value of the letter. They drew on their knowledge of simplifying similar arithmetic expression to arrive at this generalized understanding that valid transformations keep the expressions equivalent. This is a very important idea for algebra. Six students were not so confident and actually calculated the values of the original and the simplified expressions for a given value of the letter reaching the same conclusion as above. However, one student among these could not get a feel for the generality of the result, even after computation. She was working on a case by case manner and, most likely would have again computed if she was faced with another task of the same kind. Interview transcripts of students marked with an asterisk in Table 6.18 will be discussed below as earlier.

	Item1: 5×a+6-2×a+9			Item 2: b+9+6×b-5		
Name	Solution	Probe	Changes	Solution	Probe	Changes
BP	I	SE	ISCS	I	SE	ISCS
PD	C	SE	NC	C	SE	NC
BK*	C	SE	NC	C	SE	NC
AY	C	SEC+Cn	IACA	CE	SE	CEC
NN	C	SEC	IACA	C	SE	NC
SG	C	SE	NC	C	SE	NC
PG	C	SECn	NC	C	SECn	NC
JS	C	UECn	NC	C	UECn	IACA
NW*	C	SE	NC	C	SE	NC
RG	C	SE	NC	C	SE	NC
AS	C	SE	NC	C	SE	NC
AN	C	SE	NC	C	SEC+Cn	IACA
SV*	C	SE	NC	C	SE	NC
MC	C	SE	NC	C	SE	NC
AB*	C	SECn	NC	C	SE	NC
BM	C	SECn	NC	C	SECn	NC
TJ	C	SEC+Cn	CAIACA	C	SE	NC

Table 6.18: Responses of the students interviewed after MST-III in simplifying algebraic expression (* indicates students whose interviews are discussed in the text)

Written solution to the simplification task (Solution):

(C) Correct – the solution given for the expression is correct

(I) Incorrect- the solution given for the expression is incorrect

(CE) Calculation error – solution procedure is correct but contains calculation error

Explanation for equality in value of the original and the simplified expression (Probe):

(SE) Satisfactory explanation – able to explain correctly own solution as well as the alternative solution and displays the knowledge of rules and concepts (based on terms and combination of terms)

(SEC) Satisfactory explanation with changes – able to explain correctly own solution as well as the alternative solution but involves a change in the answer during discussion

(SECn) Satisfactory explanation based on calculation – the subject explains the solution satisfactorily but takes help of calculations

(SEC+Cn) Satisfactory explanation with changes and based on calculation – able to explain correctly own solution as well as the alternative solution but involves a change in the answer during discussion and uses calculation in the explanation

(UE) Unsatisfactory explanation – creates an ad-hoc explanation suitable for the specific situation at hand and may involve frequent change of answers from one to another,

(UECn) Unsatisfactory explanation based on calculation – the subject explains the solution unsatisfactorily taking help of calculations

Changes made by the student during the interview (Changes):

(NC) No changes made

(CEC) Calculation error corrected – the subject corrects the calculation error

(ISCS) Incorrect solution to correct solution – the subject changes his/ her incorrect solution for the expression to a correct one

(IACA) Incorrect alternative to correct alternative – the subject changes his/ her incorrect answer for the probe to a correct answer

(CAIACA) Correct alternative to incorrect alternative – the subject changes his/ her correct answer for the probe to an incorrect answer to back again to a correct answer

Below are two transcripts to illustrate the ways in which students justified themselves. SV is a student who understood algebraic expressions and their manipulation process. He had simplified the expressions using the distributive property. In the following interview the expression $5\times a+6-2\times a+9$ is being discussed with the interviewer SN.

> SN: Okay. Now suppose I put a=4, then what is the value of $a\times3+15$, 27. Then what is the value of this [$5\times a+6-2\times a+9$]?
>
> *SV: If the value of this [$a\times3+15$] is 27, then the value of this [$5\times a+6-2\times a+9$] also will be 27.*
>
> SN: Why?
>
> *SV: Because it is an equal expression.*
>
> SN: Which expressions are equal?
>
> *SV: This expression [$5\times a+6-2\times a+9$] and this expression [$a\times3+15$] are equal.*
>
> ...
>
> SN: Okay, now if I decide to simplify further, then can I write $18\times a$ as the answer?
>
> *SV: No.*
>
> SN: Why?
>
> *SV: Because $a\times3$ is the product, you should not do 15+3 and write. The product term is to be done first.*

He was very clear that the simplified and the original expression are equivalent for all values of the letter. Not only that, he also thought that all the steps in the simplification process yield equal expressions and therefore their values would be the same for a given value of the letter. He was aware of the constraints on transformations, thus accepting the non-closure of algebraic expressions.

Another student NW's interview transcript on this task is given below. She simplified the two expressions correctly and gave satisfactory explanation. The same expression $5\times a+6-2\times a+9$ as above is being discussed here.

SN: Now suppose that the value of '*a*' is 4, then what will be the value ($a\times3+15$) of this expression? 4 three za 12 and 15, 27. Then what will be the value of this expression [$5\times a+6-2\times a+9$]? If a=4, then original expression?

NW: 27.

SN: How did you find it?

NW: 5×4.

SN: Means you substituted the value?

NW: Yes.

SN: What is the value of this [$a\times3+15$]?

NW: 27

SN: What is the value of that [$5\times a+6-2\times a+9$]?

NW: 27

SN: Why?

NW: This expression [5×a+6-2×a+9] has been written in a simpler form.

SN: Then is this expression [$a\times3+15$] the same as this expression [$5\times a+6-2\times a+9$]?

NW: Same.

She too was sure that the given original expression and the simplified expression are equal and that each step in the process yields a simplified equivalent version of the original expression. But while predicting the value of the original expression as well as the intermediate expressions for the given value of '*a*', she substituted the value of '*a*' in them to be sure of her conclusion. She supplemented her understanding of the equivalence of the expressions by calculations only to confirm her judgment, which was a quick mental process than detailed step-by-step process. In some other cases, students were found to be calculating because they had probably never thought about the relationship of the steps in the simplification process and did not know that every expression was equivalent to every other expression. The interview situation appeared to be the first occasion when they tried to think about the issue.

The student AB also simplified the expression correctly, but the conversation with him shows that he was not confident of what he did. The interview tried to explore what he understood of the procedure.

SN: How did you do in the first question [$5\times a+6-2\times a+9$]?

AB: First I wrote the terms.

SN: What after that?

AB: Then I first combined the easy forms simple terms, product term then I took simple term, no I did not take simple terms, first equal …

SN: What is this $(5-2)\times a$?

AB: Look here madam, 5×a and -2×a is there, so I did (5-2)×a.

SN: How come?

AB: This is same, therefore 5-a.

…

SN: If I say a=4 then? Then what will be the value of this expression [$3\times a+15$]? 3×4, 4 three's 12, 12 and 15, 27 is the value of this expression.

AB: 27.

SN: What will be the value of this expression [$5\times a+6-2\times a+9$], if a=4?

AB: This [5×a+6-2×a+9]?

SN: The original expression?

AB: … 27.

SN: How is it that the value of this [$3\times a+15$] is 27 and this also is 27 [$5\times a+6-2\times a+9$]?

AB: Madam, 5×4=20+6, 26-2×4 means -8, - 26+9, that is why 26-8, means 18, subtract +9, 18 and 9 is 27.

He could not clearly explain his simplification process but he was aware of the process and also knew that the simplified expression can be unclosed stating properties of operation. The interviewer constantly intervened and guided his responses. Many of his sentences were broken and he rapidly changed his sentences, at times referring to the wrong term or the sign. Even though he predicted the correct value of the original expression, he did not explain why the answers should be same, other than calculate the value of the expression given the value of '*a*'. He continued to calculate while explaining a similar case with another expression but seemed to know by then that the values would be the same and the calculation was only for confirmation. He did not invoke the idea of equivalence of expressions as a reason for the same answer.

A couple of more cases like the above were seen. One of the students JS could not get the general idea, after the complete discussion in two expressions, that the value of the original and the simplified expression will always be equal for a given value of the letter. She repeatedly calculated, not necessarily correctly, and got confused. In fact, at times she manipulated the expressions in a way which led her to accept the wrong answer the interviewer had used as a probe. For example, when the interviewer asked her if for a=4, 5×a+6-2×a+9 can be equal to 33, she manipulated it as follows: '*5×a/ 4×5… 20/ a×2… -2/ 20-2… 18/ 6+9 is 15, 15+18 is 33*'. She missed out on the term -2×a and included only part of it (-2) in her calculation.

In another case, the student BK was very clear that the result of the original and the simplified expression will be the same, the reason derived from the similarity in the computation process, in contrast to the earlier justifications which were based on the idea of equal expressions. She showed this spontaneous understanding in both the items that were posed to her.

> RB: What have you done in the first one [5×a+6-2×a+9]?
>
> *BK: It is +5×a+6-2×a+9. These two [5×a-2×a] are same therefore +a×(5-2).*
>
> RB: What is '*a*' here?
>
> *BK: It is same in both.*
>
> RB: Ok. If I put a=4, then it is 12+15 here [a×3+15], means the answer is 27. Ok. What would be the answer of this expression 5×a+6-2×a+9?
>
> *BK: It will be this only.*

RB: It will be this. Why?

BK: Because this is a product term and we do not know what the number 'a' is. So we have to do it like this only.

In the last sentence BK points out the inability to convert the product term into simple term which is one way in which the term can be combined with other simple terms and thus simplify the expression, and therefore the need to extract the common factor for purposes of simplification. Not all students could spontaneously give such answers as described above. As mentioned earlier six students appeared to have not thought about the relation between the given expression and the simplified expression. They discovered it while answering the questions during the interview. They relied on calculation and having found the answer some tried to search for a reason for the equality of the answers of the original and the simplified expression. A quick review of the simplification process in the case of arithmetic expressions with product terms and common factors led them to generalize the process to algebra. Thus, it led to the conclusion that all the steps in the simplification process are equivalent to each other and hence the answers will be equal for all values of the letter. Two of these students to be doubly sure of their conclusion, checked by computing both the expressions they were posed, trying best not to over generalize the statement. One of them further pointed out that just as the original expression could be simplified, similarly the simplified expression could be again converted into the original expression. Although, he took some time to articulate what he understood in general of the simplification process from the concrete cases in front of him, he was able to articulate the reversible process of stripping down the original expression to the simplified expression and building it up again to the original expression.

This understanding is important for recognizing the non-arbitrariness of the rules of simplification. Many other students who understood the equivalence of the expressions in the simplification process did not articulate it but worked in this reversible manner to answer the probe. Students who made these responses appear to have connected the procedures for manipulating arithmetic expressions and algebraic expressions and the perception of structure of the expressions was an important tool in the process. Also, these students had no difficulty in substituting a value for the letter and mentally computing the result. This is another place where one can see the complementary nature of procedure and structure sense, although it began with the procedural goal of arriving at a simplified expression. The task on generating equal expressions which occupied an important place in the teaching approach was created to enhance this complementarity and will be discussed in the next chapter. Further, the responses of the students to some other questions in the interview, like whether $a+b=b+a$ or $a+b-b=a$, clearly indicated their understanding of the letter as a number and the generality of the above statements.

6.3 *Students' overall understanding of procedures*

Comparing the results from the three trials of the main study, overall improvements in their performance, especially between the first and the last trial, is seen. Moreover, there are some subtle shifts in students' abilities in understanding and performance on tasks requiring knowledge of rules and procedures of manipulating arithmetic and algebraic expressions. In all the trials, students improved significantly in the post test over their pre test scores in the items that have been discussed in the sections above at .01 level (based on t-test). Table 6.19 shows the average score of the students in the post test of the three trials (all questions on procedures and rules included), and then a comparison between the pre test average and the post test average on the same items. The students not only improved over the pre tests in the post tests, they also gained over the trials in tasks requiring procedural skills and knowledge of rules.

	MST-I	MST-II	MST-III
Post test Average (all items)	53.5% (10.7 out of 20)	66.7% (16 out of 24)	79.6% (22.3 out of 28)
Pre test Average – items common to pre and post test	20% (1.2 out of 6)	62.8% (8.8 out of 14)	65.6% (16.4 out of 25)
Post test Average – items common to pre and post test	56.7% (3.4 out of 6)	79.3% (11.1 out of 14)	80.8% (20.2 out of 25)
Std. dev. Pre test	1.2	2.7	5.4
Std. dev. Post test	1.8	3.2	4.2
df	30	30	30
Difference between means	2.2*	2.3*	3.8*
t-value (paired-samples)	8.523	4.356	6.347

Table 6.19: Comparison of students' performance in the three trials in procedural questions and knowledge of bracket opening rules

* $p < .01$

The changes in their ability from MST-I to MST-II and then to MST-III and limitations of their abilities in terms of consistency in manipulating arithmetic and algebraic expressions and within each of these domains have been already discussed. The interview responses also indicate the extent to which students could use the structure of the expressions to anticipate the correctness of a solution and the result of evaluating/ simplifying the expressions. The analysis of the qualitative responses indicates a gradual progress, with a big quantitative and qualitative shift from MST-I to MST-II and some subtle and deeper shifts made between MST-II and MST-III.

The analysis presented in this chapter shows that the performance of the students in the evaluation/ simplification tasks improved consistently over the three trials together with a major shift in strategies for working on them from the first to the second trial. Students became gradually proficient in evaluating simple expressions flexibly with a reduction in structural errors. Structural errors re-emerged in the more complex situations. They could also evaluate the more complex expressions finding easy ways of solution through a careful analysis of the expression and finding the relation between the terms. Students used the same techniques as of evaluating arithmetic expressions while simplifying algebraic expressions. They displayed a robust understanding of the rules and procedures of combining terms, both in the case of arithmetic and algebraic expressions, in the process of justifying their own solutions and while accepting or rejecting an alternative solution during the interviews, at the end of MST-III. However, throughout the various tasks in the trials, the presence of brackets and integer operations seemed to cause problems for the students. The importance of both these concepts for algebra has already been noted.

The purpose of defining a set of concepts (terms and equality), naming the terms in certain ways (simple and product term, bracket term, variable term etc.) and making the concept of terms visually salient was to capitalize on students' prior knowledge and build on it to develop an extended set of meanings and uses for the same symbols that they were familiar with. In the procedure tasks, students' initial understanding of operations on expressions as being sequential or based on precedence rules was gradually converted into structure oriented understanding based on the properties of operations and conventions. It took the students some time to learn to use this system and till then they continued to work in their old world. Once this was accepted, students moved to another stage where it was possible for them to see the relationships between arithmetic and algebra and work with the expressions in both the domains similarly. The shift from the 'inventive-semiotic stage' to a period of 'structural development' (Goldin and Kaput, 1996) was a slow process for most of the students and only by the end of the three trials, students were comfortable with the concepts and the ideas and were able to apply them in various situations. But this does not mean that there were no errors later, some errors due to mis-perception of structure of expressions resurfaced in the more complex situations, suggesting the vulnerability of the students to slip to an older and more automatic solution process, not necessarily leading to a correct answer. In the next chapter, students' understanding in tasks which predominantly exploit the structure of expressions will be discussed, with a focus on the use of the same concepts and ideas in contexts of judging equality.

Chapter 7: Analysis II: Understanding of structure of expressions and equality

7.0 *A brief overview of the chapter*

The previous chapter described students' understanding of procedures and rules concerning arithmetic and algebraic expressions, their understanding of similarity and differences in the transformational rules in the two domains and the ways the students used these concepts and rules to evaluate/ simplify expressions. The goal of the tasks discussed was to arrive at a numerical answer or more compact expression. Although a sense of the structure of expressions played an important facilitating role in completing some of the tasks successfully and efficiently, the students largely implemented rules and procedures that they were exposed to.

In contrast, the tasks to be discussed in this chapter predominantly use a sense of the structure of the expressions rather than computation or implementation of a procedure, to make a response. They are more open-ended, potentially have multiple solutions and may need the use of the procedures and rules discussed in the previous chapter in a flexible manner. Initially, in some of the tasks the students could use calculations to find the answer, but a majority of the tasks discussed in this section had to be completed without computation. There were four kinds of such tasks, all focusing on the equality relation: (i) comparing two expressions using the signs $<$, $=$, $>$ with and without computation, (ii) filling in the blank with a number or a term so that the expressions on both sides of the '=' are equal, (iii) judging which expressions from a list are equal to a given expression, and (iv) generating expressions equal to a given expression. Data from the pre and the post tests together with interview transcripts, classroom discussions and daily practice sheets across the trials will be used to explicate students' understanding of equality.

In the process, the chapter will explore students' understanding of the structure of expressions, not just 'surface structure' but also 'systemic structure', which allows one to understand the relation between the terms and the whole expression and between two expressions, centered around the idea of 'equality'. It is this mature sense of structure which enables students to understand the possibilities and constraints on transformations on expressions; structure sense and procedures thereby become complementary in nature, rather than remaining two separate skills. A glimpse of this has already been seen in the previous chapter in the discussion of students' flexible ways of evaluating expressions and in some of the interview responses. Attending to the structure of expressions is one way to tie students' expectations of the result of operating on numbers to formal arithmetic and algebra. Further, the chapter will explore students' understanding of the equality relationship which is the crux of formal algebraic manipulation and it was one of the goals of the teaching approach to enhance their understanding of equality and the related '=' sign, from the narrow meaning of a sign separating the question from the answer to a symbol of equality.

7.1 *Understanding of equality with computation*

7.1.1 Comparing two expressions

Comparing a pair of expressions with calculations was the easiest task for the students which almost all the students were able to answer correctly (Q.6 in Appendices IIA and IIB). The performance of the Marathi medium students was almost perfect (87% to 100%) and the English medium students (67% to 93%) also performed reasonably well in MST-I. Pairs of expressions, where one expression contained a division operation (24÷4 and 6+2) were the ones with the least performance. Students' responses in this task showed only two instances of systematic error, the students interpreting the first term on the R.H.S. to be the value of the expression in the L.H.S. These items, containing a single binary operation were subsequently dropped from the tests.

7.1.2 Filling the blank

The situation was, however, not the same while filling in the blank with a number so that the expressions on both sides of the '=' became equal (e.g. 21+8=__-1) (see for example, Q.5 in Appendices IIA, IIB, Q4 in Appendices IIIA, IIIB). The earlier task is simpler in the sense that it requires one to fill in the box with one of the signs <, =, > after comparing the expressions, whereas this task tempts one to make the error by putting the answer in the blank space after the '=' sign. Instead of the automatic response, one needs to consciously pay attention to the expressions on both sides of the '=' sign and make them equal, choosing the appropriate number in the blank. In the pre and post tests conducted after each of the three trials, the question consisted of four items, with the blank in varying positions. In the pre test in MST-I, the English medium students had very limited understanding of the '=' symbol and the Marathi students were better than these students as seen from their responses to such items (see Figure 7.1). Students' responses revealed their misconceptions about the sign. The performance of the students in the English group was quite poor in the post test as well, even though they had, in the overall, improved significantly over the pre test (at .01 level, t-test). However, the performance of the English group, in the more tempting positions of the blank, like a blank at the first position after the '=' sign did not improve. On the other hand, the Marathi group improved its performance (at .05 level) in the overall and performed reasonably well in the task irrespective of the position of the blank. Figure 7.1 shows the performance of the students in various items of the task. Students from both the groups performed better in the pre-test of MST-II than during the post-test of MST-I, in spite of the fact that there was a long gap between the two trials. The Marathi medium students maintained their high performance to a large extent after the pre test of MST-II but the English group was inconsistent and never reached the level of the other group. The reason is hard to speculate but the emphasis on non-computational tasks exploiting structure sense of expressions could be one reason why they performed inconsistently on such tasks, which are more computational. The inconsistency even in the later stages is surprising, since they were capable of much more sophisticated reasoning with respect to equality as evidenced in other tasks like identifying and generating equal expressions. These tasks will be discussed shortly.

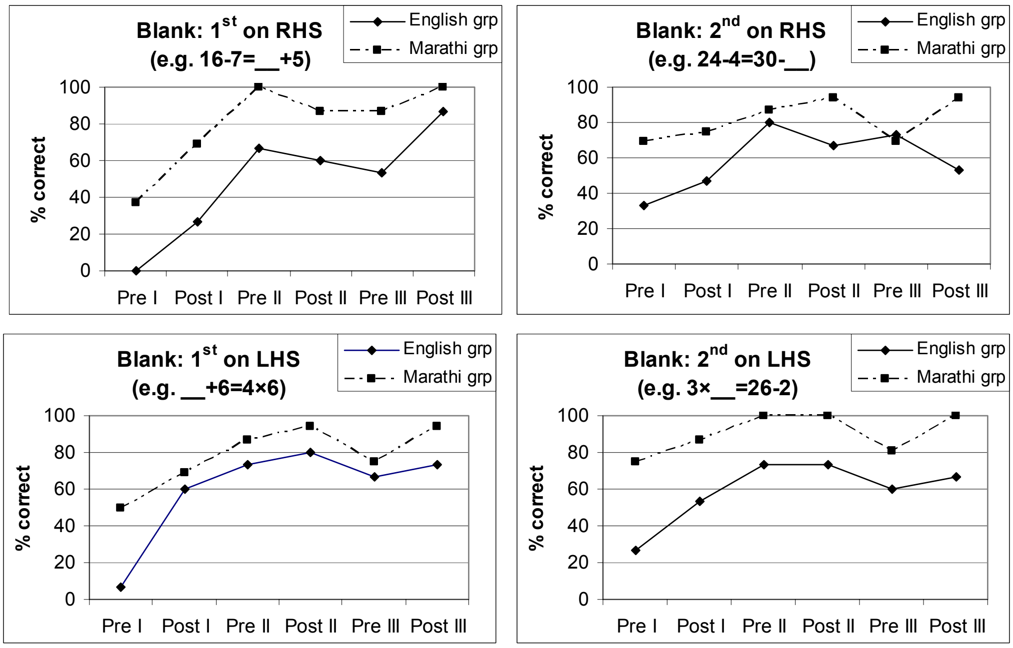


Figure 7.1: English and Marathi medium students' performance in making two expressions equal by filling in the blank in the three trials ($n_{english}$=15, $n_{marathi}$=16)

Note. There was one item of each kind in each of the tests. Pre I = Pre-test (MST-I), Post I = Post-test (MST-I), Pre II = Pre-test (MST-II), Post II = Post-test (MST-II), Pre III = Pre-test (MST-III), Post III = Post-test (MST-III).

Analysis of the nature of the responses in these tasks indicates the reasons for the difference in the performances and especially the deterioration in the performance. Many students wrote the answer/ value of the expression in the blank. For example, 27 – 3 = 24 + 10. In other positions of the blank also, some of the students showed a similar tendency, for example, 16 + 5 = 25 – 21 or 20, the blank being filled by evaluating the left hand side (21) or computation of the RHS leading to 5 (25-20) on the LHS. All these kinds of responses can be considered as indicative of an 'answer' conception of '='. These students do not see the equivalence between two expressions but only look at parts of the expression. Most of the students who have not been able to fill the blank correctly, display such a conception of '=' barring a few students who made calculation errors or any other error, like writing the sum of all the numbers in the blank. Although students improved their understanding of '=' by MST-II and this was visible through their participation in the numerous other tasks requiring the use of '=' sign and idea of equality, it is not very clear why some students persistently committed the same mistakes. Table 7.1 gives the number of cases of 'answer' conception in the three trials. All other errors are due to calculation mistakes or random numbers (for which it is difficult to find any reason) and some items were not attempted. This issue was not sufficiently probed in the trials due to the emphasis which was placed on non-computational tasks which involved 'relational/ structural thinking'.

	MST-I	MST-II	MST-III
Pre test	54	7	19
Post test	31	14	5

Table 7.1: Number of students' responses displaying 'answer' conception of the '=' sign in the three trials (N=31)

Note. The analysis is done over four items.

In the classroom, except on the very first encounter with such tasks, students understood the meaning of the '=' sign as indicating the equality relation. However, another misconception was found even among those who took the '=' sign to indicate equality in the context of non-computational tasks: understanding '=' as a sign of association (also reported in Herscovics, 1989). An example of such a task during MST-III and the subsequent discussion that takes place highlights how students who have a proper understanding of the '=' symbol struggle with this misconception. The students were working on the task 45+29 = 47+28__ where the blank had to be filled without computation. The many responses for the blank were +1, -1, +2, -3. Having found the change in the terms on the RHS with respect to LHS to be +2-1=+1, one student explained that putting +1 in the blank on the right hand side, which is already one more than left hand side, would make it even bigger, so the blank should be filled by -1. Solutions like +1 shows an 'association' conception of '=', which means that 47+28 is one more than 45+29. The term filled in the blank by the students often stated the magnitude of the difference, rather than compensating the expression to bring about equality. Students in these circumstances correctly found the amount by which one of the expressions was greater/ smaller than the other and could also verbalize this relationship but had difficulty symbolizing correctly.

Some other tasks like comparing two termed simple expressions (e.g. 234+487 and 235+486), filling the blank without computation so that two expressions are equal (e.g. 35 + 26 __= 35+ 25) or finding the value of an expression given the value of a related expression (given 234+487=721, find 235+488) revealed students' understanding of the meaning of the '=' sign and were part of the post tests in MST-I and II. The students' performance on these tasks is discussed in Naik, Banerjee and Subramaniam (2005). The purpose of these tasks was to direct students' attention to relationships between numbers and operations, and develop a sense of expectation for results without recourse to computation, that is, to build structure sense. The above article reported students' strategies and ability to communicate the reasons for their answers verbally as well as symbolically (with little help from the teacher) to tasks of the above kind and the move towards symbolic justifications with increasing complexity of the items. Students were more successful in items consisting of only positive terms than items which involved a negative term (e.g. 85 – 38 and 86 – 39). The article also pointed out the errors in students' understanding of '=' sign as has been described in this section.

In yet another conversation during MST-III centered on eliciting students' understanding of the '=' symbol, students provided two answers for the blank in 347+285=349+__: 283 and 287. Following is the excerpt from the classroom discussion for the above example.

> Teacher: How did you get 287, Atul?
>
> *Atul: You have to add 2, no no, it should be 283. There [RHS] is 2 more, so reduce 2.*
>
> Teacher: Why should doing -2 make the expressions equal?
>
> *Reema: 349 is 2 more, so you have to less 2 from 285.*
>
> Teacher: If I do -3, then?
>
> *Prathamesh: RHS will become 1 less.*

This example is a simpler one as the sign involved in the expression is a '+' sign, the students had more trouble when '–' sign was involved in the expressions. These results together with the performance on some more tasks to be discussed later suggest that, although the students understood the meaning of '=' sign as can be seen from their verbal explanations, their written work does show errors in choosing the signs and filling the blank with the correct number or term, that is, in symbolization.

7.2 Students' judgment about equality without computation

Two of the tasks (comparing expressions and filling in the blank with a number to make expressions equal) which were discussed in the previous section required calculations and checked for students' understanding of '=' sign directly. Some other tasks based on the notion of equality were also created to enhance structural thinking among students by encouraging students to link procedures of evaluating/ simplifying expressions to the structure of expressions. It was important to examine to what extent the students could use the concept of terms to compare expressions, having already seen its use in the procedural tasks as discussed in Chapter 6. These tasks are different from the routine tasks students usually work on and do not require calculations but anticipations and expectations of results and reflection on their actions. They form an important part of reasoning *about* expressions where students essentially justified their responses based on rules and properties of syntactic transformations.

Students worked on two kinds of tasks without computation: (a) identifying expressions equal to a given expression from a list (usually containing four options, with more than one equal), (b) generating equal expressions for a given expression. The purpose of these tasks was to enable the students to explore transformations and to learn to anticipate how they change the value of an expression or keep it invariant. Although, during MST-I transformations were restricted to rearranging terms, later this task required the students to be aware of the rules of evaluating expressions and of handling brackets.

The tasks were gradually increased in complexity with the growing experience of the students. Table 7.2 shows examples of the kind of items that were chosen for the task of identifying equal expressions from a list. Three kinds of expressions were used: (1) arithmetic expression with only simple terms, (2) with simple and product term and (3) algebraic expressions. Type (a) items in each of the three categories only consist of rearranging terms/ numbers and Type (b) items consist of other transformations, like increasing and decreasing terms by same or slightly different amounts, rewriting a term as sum, difference, product, using brackets, adding and subtracting the same number. Question items containing Type (b) options occasionally included one option with only rearrangement of terms or numbers. Usually, large numbers were used to form the expression to discourage students from calculating.

Type	(a) Only rearrangement of terms	(b) Other transformations
Type 1 – Simple terms	**127+284-195** (1) 127+195-284 (2) 284+127-195 (3) 195+284-127 (4) 127-195+284	**87-38+26** (1) 87-(38+26) (2) 26+87-38 (3) 87-30-8+26 (4) 87+13-38+26-13
Type 2 – Product terms	**27+17×32+14** (1) 27+14+17×32 (2) 27+17×14+32 (3)17+27×32+14 (4) 17×32+27+14	**18-27+4×6-15** (1)18-(27+4×6-15) (2) 19-15-28+4×(2+4) (3) 6×(3+4)-27-15 (4) 18-20+7+4×6-10+5 etc.
Type 3 – Algebraic expression	**9×x+12-6×x-17** (1) 17+9×x+12-6×x (2) 21×x-6×x-17 (3) x×9-6×x+12-17 (4) 9×12+x-6×x-17	**7×w-19-11×w+21** (1) 7×w-(19+11×w)+21 (2) w×(7+11)-19+21 (3) 7×w-19-9×w-2×w+21 (4) -19-11×w+7×(w+3) etc.

Table 7.2: Type of items used in the task of identifying equal expressions

7.2.1 Students' performance in identifying equal arithmetic expressions

Type	MST-I				MST-II				MST-III			
	English		Marathi		English		Marathi		English		Marathi	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
1(a)	0	53	25	69	73	-	87	-	-	-	-	-
2(a)	13	40	25	69	47	-	62	-	60	60	87	94

Table 7.3: Percentage of students correctly identifying equal expressions with terms rearranged in the three trials ($n_{english}$=15, $n_{marathi}$=16)

Note. 1(a) = Expressions with only simple terms and transformation restricted to rearranging terms, 2(a) = Expressions with product terms and transformation restricted to rearranging terms. The task consisted of one expression followed by four options in all the tests where it was posed. A response was considered correct when all the four options were correctly judged.

In the pre-test of MST-I, tasks of only types 1(a) and 2(a), that is, equal arithmetic expression with only rearrangement of terms, were used (see Q.7 and Q.8 in Appendices IIA and IIB). A score of '1' was given if a student got all the four options correct, else it was marked '0'. As Table 7.3 shows, the students significantly improved (at .01 level in t-test: paired sample, df =30) their performance in the post test with respect to their pre test in MST-I. Items of this type were not used in the post test of MST-II (where more complex items were used as discussed below) but in the pre test of the same trial they performed better than the earlier (Q.6 and Q.7 in Appendix IIIA). No doubt, the students had increased familiarity with task. While the Marathi medium students improved to reach a high level of performance, the English medium students made steady progress. Students used the concept of terms quite readily in these tasks. Even though the performance of the students was not very high in judging equality of all the items taken together as seen in Table 7.3, the performance of the students in judging the equality of individual expressions in a list was reasonably good for both the types of expressions (Types 1 and 2) across the trials. More than 70% of the students were successful in each of the options except for one option (the pair 34+21×19+28 and 21+34×19+28) where some students (6-8) repeatedly split the product term by commuting 34 and 21 or 19 and 28.

Based on classroom observations where students worked quite comfortably on items of the above kind, the task was made difficult by changing the nature of transformations applied to the expressions to keep the value invariant. This was made possible due to the flexibility incorporated in the 'terms approach' in MST-II compared to the rigidity in the rules in MST-I (discussed in Chapter 5, section 5.2.2). It allowed the use of procedures and structure together to judge equality of expressions. Students required to know that only changed parts of the expressions need to be compared with corresponding parts of the given expression. To be equal, the computation of sub-expressions must lead to a term in the given expression. In a later section this issue will be revisited when discussion will be undertaken on students' ability to generate equal expressions for a given expression.

	Given expression: **18-27+4×6-15**		
		English (Post)	**Marathi (Post)**
1	18-(27+4×6)-15	60	75
2	4×6-(27+15)+18	80	75
3	19-15-28+4×(2+4)	47	37
4	6×(3+4)-27-15	40	50
5	18-20+7+4×6-10+5	53	62
6	8×4-15+18-2×4-27	13	37

Table 7.4: Percentage of correct responses in the post test of MST-II in identifying expressions equal to a given expression with transformations other than rearranging terms (Type 2(b)) ($n_{english}$=15, $n_{marathi}$=16)

In the post test of MST-II and in MST-III, students were required to identify equal expressions where more complicated transformations were applied, that is, transformations of the type (b) as shown in Table 7.2 (see Q.6 in Appendix IIIB, Q.9 and Q.10 in Appendix IVA, Q.11 and Q.12 in Appendix IVB). In contrast to the task of type (a), in this case, each option was marked '0' or '1' depending on whether it was wrongly or correctly judged. Students' performance varied with the expression which was being compared and they (especially English medium) were found to be inconsistent in their judgments. The tables below (Table 7.4 and 7.5) show the performance of students in such items in MST-II and III. Students were able to work on these complicated tasks with a reasonable degree of success, which could be considered to be an indication of progress made by them compared to the first trial. In the expressions containing bracket with a negative sign outside, the performance of students is better when the given option is equal to the expression than when it is not in MST-II (Table 7.4). The performance is lower for the third, fourth and the last item. The third item involved multiple transformations: equal compensation and splitting a factor of the product term, the fourth one required extracting a common factor between a simple and product term and the last item involved decomposing the product term into difference of two product terms and these were extremely difficult for the students to perceive.

The options were made simpler in the MST-III post test based on the feedback from MST-II. The performance of the students was very high in judging pairs of the type 87-38+26 and 87-(38+26) (in the case of both simple and product term). Most of the students (73% of English and 100% of Marathi students) avoided the detachment error in the pair 87-38+26 and 87-30-8+26 but many fell into the trap in the pair 23-4×6-9 and 23-4×6-8+1. The number combination used in the option might have been too tempting for the students (see Linchevski and Livneh, 1999, 2002, discussed in Chapter 2, section 2.3.4). The options marked 4 for both expressions 1 and 2 in Table 7.5 have the same underlying principle: equal compensation among two terms by increasing one and decreasing the other (Expression 1) and adding and subtracting the same number (Expression 2). The first one was found to be quite difficult by the students and the second one also was not satisfactorily attempted. Students had successfully used this form of expression and cancelled the equal and opposite terms while evaluating expressions using easy ways. Also, as will be seen in the discussion of the interview results, students quite comfortably judged equality/ inequality of the expressions like item 4 in Expression 2.

	Expression 1: **23-4×6-9**					Expression 2: **87-38+26**				
		English		Marathi			English		Marathi	
		Pre	Post	Pre	Post		Pre	Post	Pre	Post
1	23-(4×6+9)	53	87	56	100	87-(38+26)	80	87	81	100
2	23-4×6-8+1	47	27	31	75	87-30-8+26	93	73	100	100
3	23-(7-3)×6-9	60	60	31	56	26+87-38	67	93	75	100
4	22-4×6-8	40	33	37	50	87+13-38+26-13	60	60	50	75

Table 7.5: Percentage of correct responses in identifying expressions equal to a given expression in the post test of MST-III with transformations other than rearranging terms (Type 1(b), 2(b)) ($n_{english}$=15, $n_{marathi}$=16)

In the tasks which explored students' understanding of equality, their responses across the trials and across various types - computational and non-computational, are not consistent. In the tasks that required computation, students did well in comparing simple expressions with one binary operation using the signs <, =, >, but made errors in filling the blanks to make two expressions equal. However, they seemed to possess an adequate understanding of '=' sign as revealed from similar tasks which did not require computation (section 7.1.2). Of course, they had trouble in symbolizing their understanding. Similarly, in the task of identifying equal expressions from a list where only terms or numbers were rearranged (Types 1(a) and 2(a) in table 7.2), students' performance was better for expressions involving only simple terms (Type 1(a)) compared to those involving a product term (Type 2(a)). This could be attributed to the difficulty in internalizing the 'product term' as a unit in contrast to a 'simple term' as a unit and the subsequent implications for transformations. Further, students' ability to work on these non-computational tasks of identifying equal expressions from a list did not ensure their success in the tasks based on computation, especially filling the blank to make two expressions equal. This is a bit surprising as success in a complicated task like identifying equality of expressions from a list definitely implies a sound understanding of equality and '=' sign. Two explanations may be proposed. The first is that the students were using the concept of terms while identifying equal expressions as a short-cut to find the answer, rather than having a deeper understanding of equality of expressions as having equal value. Alternatively, it may be the case that the two tasks place slightly different demands on students. The 'fill in the blank' task tempts students to use a more automatic response to the '=' sign leading to errors; the 'identifying equal expressions' task is challenging, requiring careful attention to structural features of the expression and the transformations used thus capturing their understanding of equality better. The classroom discussions indicated that the students were not using the idea of terms mechanically while identifying equal expressions or generating equal expressions, but the written test responses do not clarify this issue. The students calculated the value of the expression to confirm their judgment whenever in doubt about the equality of two expressions in the classroom. This issue will be further explored through the interviews, especially after MST-III.

7.2.2 Exploring students' understanding of equality of arithmetic expressions through interviews

As has been discussed earlier, the purpose of the interviews taken two months after the end of the second trial (MST-II) and four months after the end of third trial (MST-III) of the main study was to explore students' understanding of procedures and equality and their use of the concepts and ideas taught as part of the study. In this section, students' responses in the interview to the tasks of identifying arithmetic expressions equal to a given expression will be discussed. The students were given an expression followed by three to four expressions which had to be judged equal/ non-equal to the given one (see Q. B, Q.D, and Interview schedule (Tasks 2 and 7) in Appendix VA, Arithmetic test Q.2, Q.3 and Interview schedule: arithmetic (Tasks 5 and 6) in Appendix VB) . The expressions chosen can be classified into: (a) expressions with only simple terms and (b) expressions with a product term. The transformations used to change the form of the expressions were rearranging terms, using brackets and adding and subtracting the same number. Some of the expressions in the list were equal and some were unequal.

Expressions with simple terms only

Tables 7.6 and 7.7 summarize the responses of the students in the interviews after MST-II and MST-III in the case of expressions with only simple terms. The columns in the Table 7.6 (MST-II) indicate their written judgment on the equality or inequality of the listed options with respect to the given expression ('Judgment') prior to the interview, the nature of explanation the students gave to support their judgment ('Explanation') and the changes that were made in their response during the interview ('Changes'). After MST-III, they were further asked to state whether the expression in the list would be bigger or smaller with respect to the given one if it was not equal ('Explanation/ Comparison'). When the expressions were equal, students were to state whether their values will be the same. This probe was added to see whether the students were mechanically checking terms, or whether they could anticipate how the value of the expression changed as it was transformed, which was a new task for students. These explanations were coded as satisfactory, unsatisfactory, incorrect judgment, whether based on calculation etc. and will be illustrated below with the excerpts from the interview. Interview excerpts of students marked with an asterisk in the two tables (7.6 and 7.7) will be discussed.

In the case of expressions with only simple terms, five instances of error in judging the equality of the listed expressions with 49-37+23 were seen after MST-II compared to none in MST-III (see Tables 7.6 and 7.7). Four of these errors were with respect to expressions with extra terms (compare 49-37+23 with 49-5-37+5+23 or 49-5-37-5+23). All students, except one, changed and corrected their judgment during the interview. This one student tried to compute the expression orally and matched it with the given expression, the terms being different, the expression was judged to be not equal. On a couple of more occasions in the next expression (37-49+23), students complemented their explanation with calculation, to be doubly sure. The last option was straightforward and no errors or changes in responses were seen. Students' performance was better in this question after MST-III than in the earlier interview. The incorrect judgments in MST-III were largely seen in the context of comparing the two expressions for more/ less.

Name	Option 1: 49-5-37+5+23 (#) OR 49-5-37-5+23			Option 2: 37-49+23			Option 3: 23+49-37 (#)		
	Judgment	Explanation	Changes	Judgment	Explanation	Changes	Judgment	Explanation	Changes
BP	C	SE	NC	C	SE	NC	C	SE	NC
PD	C	SE	NC	I	SECn	ISCS	C	SE	NS
BK	C	SE	NC	C	SE	NC	C	SE	NC
AY*	C	SEC	CSISCS	C	SE	NC	C	SE	NC
NN	C	SE	NC	C	SE	NC	C	SE	NC
SG	C	SE	NC	C	SE	NC	C	SE	NC
NW*	I	ECn	NC	C	SE	NC	C	SE	NC
RG	C	SE	NC	C	SE	NC	C	SE	NC
AS	I	SEC	ISCS	C	SE	NC	C	SE	NC
AN	C	SE	NC	C	SE	NC	C	SE	NC
SV	C	SE	NC	C	SECn	NC	C	SE	NC
MC	C	SE	NC	C	SE	NC	C	SE	NC
AB	I	SEC	ISCS	C	SE	NC	C	SE	NC
BM*	I	SEC	ISCS	C	SE	NC	C	SE	NC

Table 7.6: Responses of the students interviewed after MST-II in identifying equal expressions for the expression 49-37+23 (# indicates equal expression to the given one, * indicates students whose interviews are discussed in the text)

Name	Option 1: 48-23-2+59+2 (#)			Option 2: 48-59+23			Option 3: 48-(23+59)		
	Judgment	Explanation/ Comparison	Changes	Judgment	Explanation/ Comparison	Changes	Judgment	Explanation/ Comparison	Changes
BP	C	-	NC	C	IJ	NC	C	SE	NC
PD*	C	SE	NC	C	SE	NC	C	IJ	NC
BK	C	-	NC	C	SE	NC	C	SE	NC
AY	C	SE	NC	C	SE	NC	C	UE	CSIS
NN	C	-	NC	C	SE	NC	C	SE	NC
SG	C	-	NC	C	SE	NC	C	SE	NC
PG	C	-	NC	C	SE	NC	C	SE	NC
JS	C	SE	NC	C	SE	NC	C	SE	NC
NW	C	SE	NC	C	-	NC	C	SE	NC
RG	C	SE	NC	C	SEC	CSIS, CAIACA	C	SE	NC
AS	C	SE	NC	C	-	NC	C	-	NC
AN	C	SE	NC	C	SE	NC	C	SE	NC
SV	C	SE	NC	C	SE	NC	C	SE	NC
MC	C	SE	NC	C	SE	NC	C	SE	NC
AB	C	-	NC	C	SEC	IACA	C	-	NC
BM*	C	SE	NC	C	SE	NC	C	SE	NC
TJ	C	SECn	NC	C	UECn	IACA	C	-	NC

Table 7.7: Responses of the students interviewed after MST-III in identifying equal expressions for the expression 48-23+59 (# indicates equal expression to the given one, * indicates students whose interviews are discussed in the text)

Judgment on equality/ inequality of expressions (Judgment):

(C) Correct – the solution given for the expression is correct

(I) Incorrect- the solution given for the expression is incorrect

Explanation for the judgment or comparing two expressions (Explanation/ Comparison):

(SE) Satisfactory explanation – able to explain satisfactorily own judgment as well as the probe and displays the knowledge of rules and concepts

(SEC) Satisfactory explanation with changes - able to explain satisfactorily own solution as well as the probe but involves a change in the answer during discussion

(SECn) Satisfactory explanation together with calculation – the subject gives a satisfactory explanation to judgment and the probe but uses calculation

(ECn) Explanation based on calculation – the subject gives an explanation based on calculation only

(UE) Unsatisfactory explanation – creates an ad-hoc explanation suitable for the specific situation at hand and may involve frequent change in answers from one to another

(UECn) Unsatisfactory explanation based on calculation – the subject gives an unsatisfactory explanation based on calculation

(IJ) Incorrect judgment – subject making an incorrect judgment with respect to the probe

(NE) No explanation – cannot give any explanation for the probe and only knows own solution

Changes made during the interview (Changes):

(IACA) Incorrect additional probe to correct additional probe – the subject changes his/ her incorrect answer for probe to a correct answer

(ISCS) Incorrect solution to correct solution – the subject changes his/ her initial incorrect judgment about equality/ inequality for the expression to a correct one

(CAIA) Correct additional probe to incorrect additional probe – the subject changes his/ her correct answer for probe to an incorrect one

(CSIS) Correct solution to incorrect solution – the subject changes his/ her initial correct judgment for the expression to an incorrect one

(CAIACA) Correct additional probe to incorrect to correct additional probe – the subject changes his/ her correct answer for the probe to an incorrect answer to back again to a correct answer

(CSISCS) Correct solution to incorrect solution to correct solution – the subject changes his/ her initial correct judgment for the expression to an incorrect and back to correct judgment

(NC) No changes made

An interview with the student NW (MST-II) is given below. While comparing the expressions 49-37+23 and 49-5-37+5+23 she did not see the complete expression but looked at parts of it. In the process, she lost track of the relationships between the parts and ended up making a wrong judgment by relying on her oral calculations and surface structure of the expressions.

> PB: You had to identify the expression equal to 49-37+23. You said the first expression [49-5-37+5+23] is not equal to it. Why?
>
> *NW: No.*
>
> PB: Why?
>
> *NW: Here +49 is correct and here +, -5 and -37 is there. Therefore these two will get added and the sum is +42 and the sum of these two [+5+23] is +28 and the answer of this, and this [49-37+23] expression is a little smaller.*
>
> PB: Means, did you calculate and see?
>
> *NW: No, but 49-42+28 would be like this, therefore this expression [49-37+23] is not equal to this expression [49-42+28].*

Another student BM (MST-II) was unsure about the equality of the expressions 49-37+23 and 49-5-37+5+23.

> SN: Ok, tell me why this [49-5-37+5+23] is not equal?
>
> *BM: Teacher, here 49-5 and minus 37, teacher here 5 is more, teacher here [49-37+23] -5 and +5 were not there.*
>
> SN: These were not there.
>
> *BM: Yes.*
>
> SN: Then is this expression [49-5-37+5+23] equal or not equal?
>
> *BM: They are equal but here I …*
>
> SN: Is the expression equal?
>
> *BM: If we cut +5 and -5 then the answer comes the same.*
>
> SN: And if we keep +5, -5 there, then what is the answer?
>
> *BM: [long pause]*
>
> SN: If both of them [-5 and +5] are taken off then you say that they are equal.
>
> *BM: Yes.*

SN: But both of them are left there, then?

BM: Teacher, these [the two expressions being compared] will not be equal.

Her initial conclusion was drawn by looking at the surface structure of the expressions where the two expressions differed in length. With the interviewer's help she finally managed to focus her attention on relationships between the terms and arrive at the correct conclusion, that the expressions are equal. In the case of the student BM, the interviewer SN had to invest a lot of effort to make her notice the relationships between the terms. It was not a normal practice in the interviews to guide students towards the correct answer but was done in this particular case. Her inability to anticipate the effect of adding +5 and -5 to the expression is visible. She was not able to combine the terms flexibly, regardless of order. However, in interviews after MST-III she could confidently explain this item. All the other students, besides these two, figured out quite easily that -5+5=0 and hence would not make any difference to the value of the expression.

For the pair of expressions 49-37+23 and 49-5-35-5+23, student AY (MST-II) began hesitantly. The excerpt is given below.

RB: Now let's go to the last question 49-37+23. You said this [49-5-35-5+23] is not correct. Why?

AY: Teacher, because in this, this is correct.

RB: Why correct?

AY: Because, not correct.

RB: Not correct. Means what you have done is correct.

AY: Yes teacher.

RB: Why?

AY: Because here -5-5 is extra. Had it been -5+5 then subtracting would have given us 0 but here it is both -5.

A few of the students fumbled in the beginning while expressing the reasons for their choices but most of them ended with a satisfactory explanation. Two students calculated and checked the inequality of 49-37+23 and 37-49+23 before concluding that they are unequal as the terms are different in the two expressions. It is also clear from their responses that most of them understand that two expressions are equal primarily because their values are same (barring a few like BM). The value of two expressions can remain same due to certain transformation: re-ordering terms, putting brackets, adding and subtracting the same number. The interview at this time probed a few transformations and it is not possible to comment on their understanding of general rules of transformations. But it is unlikely that they used the concept of terms in these situations mechanically or meaninglessly, an issue that was more explicitly probed after MST-III. Some of them were struggling with the idea and did not achieve the expertise and stability as some others. They were also inconsistent in their judgment in that they did not doubt their judgment in one situation but wanted to clarify for themselves by computation in another situation where the same property or rule was being applied.

In a similar task of comparing expressions with simple terms after MST-III, PD was one student who confidently answered the first two parts of the question and successfully identified the smaller/ bigger of the two expressions. She used terms to judge the equality/ inequality of expressions as well to compare them. For example, while comparing the expressions 48-59+23 and 48-23+59 she said that 48-23+59 is more because '*when you plus 48 and plus 59 you will get answer and -23 you will get [in 48-23+59], but when you plus 48 plus 23 you will get another answer and minus 59 here [in 48-59+23] will give smaller answer*'. However, she was not so clear about the last option 48-(23+59).

> RB: Ok. What about this one 48-(23+59)? You said this is not equal. Why?
>
> *PD: Because when you open the bracket you will get minus sign.*
>
> RB: Ok. Which one is more out of these two?
>
> *PD: This [48-(23+59)].*

RB: This one is more. Why?

PD: Because there is 48 negative, you add 23, and add 59, minus.

RB: Han, ok.

PD: You will get the answer here [48-(23+59)] more.

RB: You will get answer more. Because 23 and 59 are to be added.

PD: 48 negative cutz [subtract] hai, if 23 negative more is cutz [subtracted] and 59 negative is cutz [subtracted], then answer will be more.

She was sure that equal expressions have equal value. She was also articulate in her reasoning about when the expressions are equal and why one of the expressions should be bigger or smaller than the other. Even in the last part of the question she correctly judged the expressions (48-23+59 and 48-(23+59)) to be not equal but made an error while identifying the bigger of the two expressions. In the first two parts, she had correctly attached the signs with the numbers and anticipated the result of rearranging the numbers only. But in the last part she did not use the idea of terms and in fact read it sequentially from left to right, in the process making an error.

Some others correctly concluded that since both the numbers 23 and 59 are being subtracted from 48, therefore it is less than the original expression 48-23+59. In response to the last part of the question, this was another line of reasoning which allowed students to correctly identify the bigger expression as exemplified by the excerpt below by student BM (MST-III).

SN: This is more or less [48-(23+59)]? Compared to this original expression [48-23+59]?

BM: It is smaller.

SN: This also is less?

BM: Because, the negative signs are next to each other, their sum would be more. And here it is positive [+48] and here it is negative [-(23+59)], so subtraction will be carried out on these, and the answer will be less.

She said that the expression 48-(23+59) would be less than 48-23+59 because the value of the former would be less than the latter. She used both the structure of the expression as well as procedures to consolidate her answer. In the group of students interviewed, some of the students used such rules as '*sign would be of the bigger number*' (probably learnt at school) while working out the procedures and some others used a qualitative sense of operations to justify their answer. For example, one of the students commented that the original expression is bigger than 48-(23+59) because '*here everything is minus*', in essence comparing the effect of each of the terms. This was being facilitated by students' strong sense of structure of expressions. All students could give reasons for their judgment about equality/ inequality but made occasional mistakes in comparing them for more/ less, though they had the strategies in place to approach the task.

Expressions with a product term

Tables 7.8 and 7.9 summarize students' performance in the two trials in the interview task requiring them to identify equal expressions from a list consisting of three to four options in the case of expressions with a product term. The columns in the Table 7.8 (MST-II) are similar to the ones in table 7.6 and indicate their written judgment on the equality or inequality of the listed options with respect to the given expression ('Judgment') prior to the interview, the nature of explanation the students gave to support their judgment ('Explanation') and the changes that were made during the interview ('Changes'). In MST-III (Table 7.9), students' responses were coded for Judgment and Changes together with their explanation on the 'comparison task' ('Explanation/ Comparison') similar to Table 7.7. The coding of the responses will once again be clarified through interview excerpts.

In the interviews, although most students used the concept of terms appropriately, there were a few students who used it differently and often incorrectly while identifying equal expressions, especially after MST-II. Most of the students after MST-III could satisfactorily explain their responses to the questions and the probe (that is, compare the given expression with the ones in the list for which is greater) and showed a deeper level of understanding than using a mere shortcut procedure. They used the concept of terms not only to check for equality of expressions (arithmetic or algebraic) but also to judge which expression is greater/ lesser of two unequal expressions. In both the interviews after MST-II and MST-III, incorrect responses were found. A majority of these errors had to do with brackets (4 errors in MST-II and 2 in MST-III respectively, option 3 in Table 7.8 and Option 4 in Table 7.9). Although all students (MST-II) had correctly judged the inequality of the expression 18-15+13×4 with the given expression 18-13+15×4, few students (5) were found to split the product term while explaining the reason for the inequality. All students, except one, corrected this pattern of reasoning by refocusing their attention on terms when posed with another option 18+15-13×4. Even in MST-III, two students were found to incorrectly judge the equality of a pair of expressions similar to the above. In both the trials, all others, except student SG, corrected their responses when given the opportunity to do so. Two of these students depended on calculation to be sure of their answers.

In MST-III, the additional probe consisted of judging whether the expression was greater/ lesser if the expressions were unequal. In this regard, nine instances of incorrect judgment were seen, five of which were easily corrected during the interview. As has been seen in many other items with brackets, here also students changed their answers more when the expression contained a bracket indicating their discomfort with brackets and lack of confidence in interpreting the negative sign before the bracket. Excerpts from students' interviews (marked with an asterisk in the tables below) will be discussed to elucidate the nature of their reasoning after the two trials MST-II and MST-III.

	Option 1: 18-15+13×4			Option 2: 4×15+18-13 (#)			Option 3: 18-(13-15×4) (#)		
	Judgment	Explanation	Changes	Judgment	Explanation	Changes	Judgment	Explanation	Changes
BP*	C	UE	NC	C	SE	NC	C	SE	NC
PD	C	UE	NC	C	SE	NC	C	SE	NC
BK	C	SE	NC	C	SE	NC	C	SE	NC
AY	C	SE	NC	C	SE	NC	I	SEC	ISCS
NN	C	SE	NC	C	SE	NC	I	SEC	ISCS
SG	C	UE	NC	I	UE	NC	C	SE	NC
NW	C	SE	NC	C	SE	NC	C	SE	NC
RG	C	SE	NC	C	SE	NC	I	SEC	ISCS
AS	C	SE	NC	C	SE	NC	C	SE	NC
AN	C	UE	NC	C	SE	NC	C	SE	NS
SV	C	SE	NC	C	SE	NC	C	SE	NC
MC*	C	SE	NC	C	SE	NC	C	SE	NC
AB	C	UE	NC	C	SE	NC	I	SEC	ISCS
BM	C	SE	NC	C	SE	NC	C	SE	NC

Table 7.8: Responses of the students interviewed after MST-II in identifying equal expressions for the expression 18-13+15×4 (# indicates equal expression to the given one, * indicates students whose interviews are discussed in the text)

Name	Option 1: 24-18+13×6			Option 2: 24+18-13×6			Option 3: 6×18-13+24 (#)			Option 4: 24-(13-18×6) (#)		
	Judgment	Explanation/ Comparison	Changes	Judgment	Explanation/ Comparison	Changes	Judgment	Explanation/ Comparison	Changes	Judgment	Explanation/ Comparison	Changes
BP	C	SE	NC	I	IJ	ISCS	C	SE	NC	I	IJ	ISCS
PD	C	SE	NC	C	IJ	NC	C	UE	NC	C	SE	NC
BK	C	SE	NC	C	SE	NC	C	SE	NC	C	SE	NC
AY*	C	SE	NC	C	SE	NC	C	SE	NC	C	UE	CSISCS
NN	C	SE	NC	C	SE	NC	C	SE	NC	I	SE	ISCS
SG	C	SE	NC	I	-	NC	I	IJ	NC	C	UE	CSISCS
PG	C	SEC	IACA	C	SEC	IACA	C	SE	NC	C	-	NC
JS*	C	SE	NC	C	SE	NC	C	SE	NC	C	SE	NC
NW	C	SE	NC	C	SE	NC	C	-	NC	C	-	NC
RG	C	SE	NC	C	SE	NC	C	-	NC	C	-	NC
AS	C	SEC	IACA	C	SE	NC	C	-	NC	C	SE	NC
AN	C	SE	NC	C	SE	NC	C	SE	NC	C	SE	NC
SV	C	SE	NC	C	SE	NC	C	-	NC	C	SE	NC
MC	C	SEC	IACA	C	SE	NC	C	-	NC	C	SE	NC
AB	C	SE	NC	C	UECn	IACA	C	-	NC	C	SE	NC
BM	C	SE	NC	C	SE	NC	C	-	NC	C	SE	NC
TJ*	C	SECn	NC	C	UECn	NC	C	SE	NC	C	SE	NC

Table 7.9: Responses of the students interviewed after MST-III in identifying equal expressions for the expression 24-13+18×6 (# indicates equal expression to the given one, * indicates students whose interviews are discussed in the text)

Judgment on equality/ inequality of expressions (Judgment):

(C) Correct – the solution given for the expression is correct

(I) Incorrect- the solution given for the expression is incorrect

Explanation for the judgment or comparing two expressions (Explanation/ Comparison):

(SE) Satisfactory explanation – able to explain satisfactorily own judgment as well as the probe and displays the knowledge of rules and concepts

(SEC) Satisfactory explanation with changes - able to explain satisfactorily own solution as well as the probe but involves a change in the answer during discussion

(SECn) Satisfactory explanation together with calculation – the subject gives a satisfactory explanation to judgment and the probe but uses calculation

(ECn) Explanation based on calculation – the subject gives an explanation based on calculation only

(UE) Unsatisfactory explanation – creates an ad-hoc explanation suitable for the specific situation at hand and may involve frequent change in answers from one to another

(UECn) Unsatisfactory explanation based on calculation – the subject gives an unsatisfactory explanation based on calculation

(IJ) Incorrect judgment – subject making an incorrect judgment with respect to the probe

(NE) No explanation – cannot give any explanation for the probe and only knows own solution

Changes made during the interview (Changes):

(IACA) Incorrect additional probe to correct additional probe – the subject changes his/ her incorrect answer for probe to a correct answer

(ISCS) Incorrect solution to correct solution – the subject changes his/ her initial incorrect judgment about equality/ inequality for the expression to a correct one

(CAIA) Correct additional probe to incorrect additional probe – the subject changes his/ her correct answer for probe to an incorrect one

(CSIS) Correct solution to incorrect solution – the subject changes his/ her initial correct judgment for the expression to an incorrect one

(CAIACA) Correct additional probe to incorrect to correct additional probe – the subject changes his/ her correct answer for the probe to an incorrect answer to back again to a correct answer

(CSISCS) Correct solution to incorrect solution to correct solution – the subject changes his/ her initial correct judgment for the expression to an incorrect and back to correct judgment

(NC) No changes made

The student MC (MST-II) gave short and well articulated responses to explain his judgment. This way of reasoning was typical of those who succeeded in the task.

> SN: Ok, so you say that this [18-13+15×4] and this expression [18-15+13×4] are not equal, the first one, why aren't they equal?
>
> *MC: Here it is given +15×4 and here +13×5 is given, the terms are wrong.*
>
> …
>
> SN: Ok, what do you say about the second expression [4×15+18-13]?
>
> *MC: It is correct.*
>
> SN: Why?
>
> *MC: [Inaudible]*
>
> SN: Have you calculated the answer of anything?
>
> *MC: No, its terms are same.*
>
> SN: And the third [18-(13-15×4)]?
>
> *MC: It is equal.*
>
> SN: Why?
>
> *MC: There is a minus sign before the bracket, here [*18-(13-15×4)*] it was 13, here it was +13, here it will become -13 and 15, it was -15×4 there [*18-(13-15×4)*], it will become +15×4. Therefore the expressions are equal.*

This student consistently used the concept of terms for identifying whether a given expression is equal or not. Further, he used the bracket opening rule correctly to anticipate the result of the expression after it is removed and again matched the terms to decide about its equality. In the following interview excerpt, student BP (MST-II) is judging whether the two expressions 18-15+13×4 and 4×15+18-13 are equal to 18-13+15×4. She had given the correct written judgment for this pair but her reasoning is interesting and was marked 'unsatisfactory (UE)' for the first option but 'satisfactory (SE)' for the second option.

> RJ: Which of the following expressions is equal? Ok, you have written first [18-15+13×4] is wrong and 2nd [4×15+18-13] and 3rd [18-(13-15×4)] is right, ok. So, how you got this, that these two are correct [2nd, 3rd] and this is wrong [1st]?

BP: Teacher because, here [1st] +18 and here [original expression] also +18, here [1st] –15 and here +15 [original expression].

RJ: Ok, so that is wrong. So why second one is correct?

BP: Because +4×15 and here is +15×4, +18 and +18 and –13 [comparing the two expressions].

Her response to the first expression suggests that she was not looking at terms but she managed to give the correct answer. In response to the next expression, she changed the reasoning style and pointed out the correct terms, thereby displaying correct conceptions about transformations which keep the value of an expression same. In all, five students made such alterations while answering, of which only one student (SG) failed to get a crucial item (4×15+18-13) and one more option made for him (18+15-13×5) correct, indicating his use of terms was not stable and indeed was splitting the product term. He also could not correct his response in the interview after MST-III.

The interviews conducted after MST-III were longer and included the additional probe to see if the students could apply the idea of terms to compare the expressions when they were unequal. The student JS showed quite clearly her ability to use the concept of terms to not only judge whether two expressions are equal or not but also used it quite confidently to judge which among them is greater. Many students displayed such a robust understanding of terms and expressions.

RB: … What do you think, which of these is bigger [24-13+18×6 and 24-18+13×6]?

JS: this [24-13+18×6]

RB: This is bigger. Why?

JS: Because here it is +18×6 and here +13×6. Here it is 24-18 and here 24-13.

…

RB: … Now which is bigger among these two [24-13+18×6 and 24+18-13×6]?

JS: (pause) This [24-13+18×6].

RB: This is bigger. Why is this bigger?

JS: Here +18×6 is there which would give more answer, and here if we do -13×6 it will give less answer.

RB: You will get less answer. Now what about this third one 6×18-13+24? You said that this is equal. Why is it equal?

JS: +6×18 is a product term, -13 and here also -13, here +24 and here also +24.

RB: If you are saying that both of these are equal, would their answers be same?

JS: Yes.

RB: Han, pucca? If its answer is 115, then how much will be the answer of this?

JS: 115

She combined the operation sense (her language displays a sense of procedures) with structure sense (indicated by the way she used the concept of terms) to correctly identify which among the two expressions was bigger/ smaller. At one point initially, she split the product term but corrected it the moment she realized that the product term would make a difference to the value of the expression and that matching the operation signs attached with the numbers is not a sufficient criterion for expressions to be equal. Also, she did not have any difficulty in this task to compare the two expressions where one of them had a bracket, in contrast to the situation where she had to evaluate an expression with a bracket (see section 6.1.3, p. 217). She was able to use terms to understand the effect of each of the parts/ units to the whole expression and how changing one component changes the value of the expression.

The student TJ also judged the equality/ inequality of two expressions confidently and knew the conditions when two expressions are equal as well as the meaning of equal expressions. But he was not so confident while comparing the two expressions 24-18+13×6 and 24-13+18×6. An excerpt from the interview is given below.

SN: Now you tell me whether the value of this expression [24-13+18×6] is more or less than this expression, 24-18+13×6?

TJ: [pause]

SN: Is it more or less than the original expression [24-13+18×6]?

TJ: It is less.

SN: Why is it less?

TJ: Here the sign has also changed. Here it is +18×6. And there it is -18+13×6, 78.

SN: Yes.

TJ: And the difference of these is 6.

SN: Yes.

TJ: And if you add the two you get 84.

SN: Did you calculate and check?

TJ: Yes.

…

SN: Can you tell me without calculations? That the answer of this would be greater?

TJ: [Pause] Yes.

SN: How?

TJ: Here it is 18×6, and there 13×6. It has become less and here [24-13+18×6] also -13 is there, and -18 is here [24-18+13×6]. Only here subtraction, the product is more, therefore.

He used the concept of terms to identify the equality/ inequality of the expression and he used it thereafter as well for comparing the expressions. He was aware of the fact that these two expressions cannot have equal value as they are unequal which is evident from his statement '*If they were equal, then only their value would have been equal. And these expressions are different, the terms have been changed, therefore the answers will also be different*'. He often took recourse to calculations for comparing the expressions, although when pressed hard he could give reasons about the expressions without calculations.

In the above instances students could arrive at the correct conclusions. But a few students were not so successful. One such example is the following from the interview with the student AY who is comparing the expression 24-13+18×6 with the expression 24-(13-18×6).

RB: Let us see the next 24-(13-18×6). You said this is equal. Why is it equal?

AY: Not equal

RB: Not equal. Why?

AY: Because three numbers are in the bracket, so the answer for these two [13-18×6] have to be found inside the bracket and whatever answer comes that has to be kept inside bracket and then do it with this [24] then you would get it not equal.

RB: You would get not equal.

AY: Yes.

RB: Why did you write equal? What did you think? This [24-13+18×6] is an expression and this [24-(13-18×6)] is another expression. You are comparing these expressions.

AY: I thought that if we open the bracket first then we get +18×6.

RB: Ok. You think that if we open the bracket then we get equal and if we do not open the bracket and solve inside it then it is not equal. Are you thinking this?

AY: Yes.

RB: Is this possible?

AY: I do not know.

He could correctly evaluate expressions with brackets and showed various ways of evaluating expressions leading to the same value. He had definitely improved his understanding with regard to evaluation of bracketed expressions compared to his performance in the earlier interview after MST-II. But while comparing expressions, brackets continued to be a problem for him as the excerpt shows. This confusion persisted in all the tasks involving brackets (even when the expression in brackets had only simple terms) which required a ‘proceptual’ understanding rather than a simple understanding of rules of opening bracket. From very little knowledge of brackets after MST-II, he had moved to a ‘participatory’ phase where he could carry out the complete operation himself but could not ‘anticipate’ the results in the two situations – solving the sub-expression inside the bracket and removing the bracket – mentally without physically carrying it out (Tzur and Simon, 2004, discussed in Chapter 2, section 2.4), and was hence not confident of the equality of the two expressions.

The excerpts above show the successes and failures of the students in answering questions based on structure of expressions. The students were sufficiently oriented by the end of MST-III to use the structure of expressions and terms to complete the tasks of judging equality of expressions and comparing them if unequal and understood how each term contributed to the value of the expression. Some of them still made errors in the case of expressions with a negative bracketed term. In comparison to MST-II, they were more stable and consistent at this time in the use of concepts and rules. Comparing expressions of the kind that are discussed above is a more demanding task than judging equality of expressions where one can use the concept of terms as a short-cut procedure without really developing a sense of structure. The confidence of students in their sense of operation and structure as well as lack of confidence in computing large numbers for some, dissuaded them from calculating values of the expressions except for a couple of them who invariably took support from their skills of computation.

7.2.3 Equivalence of algebraic expressions

Having discussed students' understanding of equality of arithmetic expressions in various cases as well as from various data sources (written test and interviews), let us now turn our attention to algebraic expressions and explore the impact of the earlier understanding in the context of algebra. In the post test of the first trial (MST-I), students were asked to identify algebraic expressions from a list which were equivalent to the given expression. The only transformation used to generate the options for the task at this time was of rearranging terms (e.g. identify if $9\times x+12-6\times x-17$ is equivalent to $17+9\times x+12-6\times x$), that is, of Type 3(a) (see table

7.2). Tasks of type 3(a) was also used in the pre test of MST-II and in the pre and the post tests of MST-III, but not in the post test of MST-II (where other types such as Type (b) shown in Table 7.2 were used) (see Q.22 in Appendix IIB, Q.13 in Appendix IIIA, Q.8 in Appendix IVA, Q.10 in Appendix IVB).

Type	MST-I				MST-II				MST-III			
	English		Marathi		English		Marathi		English		Marathi	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
3(a)	-	53	-	69	53	-	75	-	47	73	81	100

Table 7.10: Percentage of students correctly identifying equivalence of expressions of Type 3(a) in the three trials ($n_{english}$=15, $n_{marathi}$=16)

Note. 3(a) = Algebraic expression with transformations restricted to rearranging terms. One item of this type was posed in the tests which consisted of an algebraic expression followed by four options. Each question consisted of an expression followed by four options. Answer considered correct only when all four options judged correctly. Entries blank against tests where item not posed.

As in the case of arithmetic expressions, in the case of algebraic expressions as well students appropriately used the concept of terms to judge equivalence of two expressions. Students' performance in identifying equivalent algebraic expressions of Type 3(a) in the three trials is given in Table 7.10. Similar to the case of arithmetic expressions, there is an improvement in their performance through the three trials. In fact, students' performance in identifying equivalent algebraic expressions is slightly better than in similar arithmetic expressions (compare tables 7.3 and 7.10). Again, like in the case of arithmetic expressions, their performance in identifying equivalence of individual options with the given expression is better than in the overall. More than 80% of the students could avoid the conjoining error while judging the equivalence across the trials. But some students' decision, especially in the first two trials, for considering an algebraic expression to be equivalent to another seems to be influenced by their understanding of procedure of simplifying algebraic expressions, like first collecting the like terms together, not considering $9\times x$ to be same as $x\times 9$. For example, some students did not consider the given expression $13\times x–9–3\times x+15$ to be equivalent to $15–9–3\times x+13\times x$ and some considered $13\times x+3\times x–9+15$ to be equivalent to the given expression even though the signs of the terms have been changed. In MST-III, two students were also seen to split the product term when they judged $9\times 12+x-6\times x-17$ to be equivalent to $9\times x+12-6\times x-17$.

In the post test of MST-II, Type (b) tasks of the identifying equal expression for algebraic expression were posed, which included transformations such as adding brackets, splitting a term, extracting a common factor, etc (see Q.7 in Appendix IIIB). The options for the algebraic expressions were slightly simpler than the arithmetic expressions, (which at times involved more than one transformation in an option), a reason for the better performance in the case of algebra (compare tables 7.4 and 7.11). Just about half the students could identify an expression which was transformed by extracting a common factor (Table 7.11, items 2 and 4). Their performance in a few items like 1, 3, 5 is very good which dealt respectively with brackets, equal compensation and decomposing a product term into two product terms.

S.No	Expression: **$7\times w$-19-11$\times w$+21**	**English**	**Marathi**
1	$7\times w-(19+11\times w)+21$	93	81
2	$w\times(7+11)-19+21$	47	44
3	$20-20+7\times w-11\times w$	87	87
4	$-19-11\times w+7\times(w+3)$	40	44
5	$7\times w-19-9\times w-2\times w+21$	80	81
6	$2\times(-2\times w+1)$	40	19

Table 7.11: Percentage of students correctly identifying equivalent algebraic expressions of Type 3(b) in the post test of MST-II ($n_{english}$=15, $n_{marathi}$=16)

Note. 3(b) = Transformations other than rearranging terms.

The post test of MST-III contained items only of type 3(a), that is, rearranging of terms, and the issue of students' understanding of more complex transformations of algebraic expressions was dealt in the interviews. Their understanding about equivalence of algebraic expressions has already been discussed in the earlier chapter in the context of simplifying algebraic expressions. Students could compare algebraic expressions with terms rearranged for equivalence with ease at the end of MST-III, although a few of them were at times tempted to split the product term. Judging the equivalence of algebraic expressions with only terms rearranged was not a point of difficulty for them since the beginning as could be seen from their performance in this task in the earlier trials. Identifying and matching the terms in the two expressions is all that is required while comparing the expressions.

In the interview after MST-III, students were asked to first judge the equivalence of the expressions listed with the given expression (See Algebra test Q.2 and Interview schedule: algebra (Task 3) in Appendix VB). Further they were asked if the values of two such algebraic expressions will be equal for a given value of the letter, for example if $13\times m-7-8\times m+4$ will be equal to $-7+4+13\times m-m\times 8$ for $m=2$. A summary of the performance of the students is shown in Table 7.12. The table shows students' responses to judging equivalence of the listed options to the given expression ('Judgment'), their explanations for whether the value of two algebraic expressions will have the same or different when the letter is substituted by a number ('Comparison on substitution') and changes made during the interview ('Changes').

The students judged the equivalence of the algebraic expressions by comparing terms. Four instances of errors in judging equivalence of algebraic expressions were found. Three of these were easily corrected but one could not be corrected. With respect to the second part of the question, ten students were clear that if two algebraic expressions are equal, then they are equal for all possible values of the letter. Four others substituted the value of the letter (SEval) in each of the expressions just to confirm if they were equal arithmetic expressions and thereby concluded the values of the two algebraic expressions being compared to be the same. One of them could not draw the correct conclusion about the equality/ inequality of values of the two algebraic expressions in some cases. The student SG continued to focus on numbers and operations separately (not terms) while reasoning about these algebraic expressions (like $13\times m-7-8\times m+4$ is not equal to $-7+4+13\times m-m\times 8$ for $m=2$ as -8×2 and -2×8 are not same). Three more calculated (SECn) parts of the algebraic expressions after substituting the value of the letter to see if the pairs of expression were equal or unequal. These categories of coding will be elaborated using the excerpts from the interview of students marked with an asterisk in Table 7.12.

Name	Option 1: 13×*m*-7-8×4+*m*			Option 2: -7+4+13×*m*-*m*×8 (#)			Option 3: *m*×(13-8)-7+4 (#)			Option 4: 13×*m*-(7-8×*m*)+4		
	Judgment	Comparison on substitution	Changes	Judgment	Comparison on substitution	Changes	Judgment	Comparison on substitution	Changes	Judgment	Comparison on substitution	Changes
BP	C	SEC	IACA	C	SE	NC	I	-	NC	I	SEC	ISCS, IACA
PD	C	SE	NC	I	SE	ISCS	C	SE	NC	C	SE	NC
BK	C	SE	NC	C	SE	NC	C	SE	NC	C	SE	NC
AY	C	SEval	NC	C	SEval	NC	C	SEval	NC	C	SEval	NC
NN	C	SE	NC	C	SE	NC	C	SE	NC	C	SE	NC
SG	C	SEval	NC	C	UE	CAIA	I	UE	ISCS	C	SE	NC
PG	C	SEval	NC	C	SEval	NC	C	SEval	NC	C	SE	NC
JS	C	SEval	NC	C	SEval	NC	C	SEval	NC	C	SE	NC
NW	C	-	NC	C	SE	NC	C	SE	NC	C	-	NC
RG	C	-	NC	C	SE	NC	C	SE	NC	C	-	NC
AS	C	SE	NC	C	SE	NC	C	SE	NC	C	SE	NC
AN	C	SE	NC	C	SE	NC	C	SE	NC	C	SE	NC
SV	C	SE	NC	C	SE	NC	C	SE	NC	C	SE	NC
MC	C	SE	NC	C	SE	NC	C	SE	NC	C	SE	NC
AB	C	SECn	NC	C	SECn	NC	C	-	NC	C	SECn	NC
BM	C	-	NC	C	SECn	NC	C	SECn	NC	C	-	NC
TJ	C	SECn	NC	C	SECn	NC	C	SECn	NC	C	SECn	NC

Table 7.12: Responses of the students interviewed after MSC-III in identifying equivalent expressions for the expression 13×*m*-7-8×*m*+4 (# indicates equivalent expressions, * indicates students whose interviews are discussed in the text)

Judgment on equality/ inequality of expressions (Judgment):

(C) Correct – the solution given for the expression is correct

(I) Incorrect- the solution given for the expression is incorrect

Explanation for the judgment or comparing two expressions (Explanation/ Comparison):

(SE) Satisfactory explanation – able to explain satisfactorily own judgment as well as the probe and displays the knowledge of rules and concepts

(SEC) Satisfactory explanation with changes – able to explain satisfactorily own solution as well as the probe but involves a change in the answer during discussion

(SECn) Satisfactory explanation together with calculation – the subject gives a satisfactory explanation to judgment and the probe but uses calculation

(SEval) Satisfactory explanation based on substituting the value of the letter – The subject substitutes the value of the letter in the given pair of expressions and decides if they are equal arithmetic expressions

(ECn) Explanation based on calculation – the subject gives an explanation based on calculation only

(UE) Unsatisfactory explanation – creates an ad-hoc explanation suitable for the specific situation at hand and may involve frequent change in answers from one to another

(IJ) Incorrect judgment – subject making an incorrect judgment with respect to the probe

(NE) No explanation – cannot give any explanation for the probe and only knows own solution

Changes made during the interview (Changes):

(IACA) Incorrect additional probe to correct additional probe – the subject changes his/ her incorrect answer for probe to a correct answer

(ISCS) Incorrect solution to correct solution – the subject changes his/ her initial incorrect judgment about equality/ inequality for the expression to a correct one

(CAIA) Correct additional probe to incorrect additional probe – the subject changes his/ her correct answer for probe to an incorrect one

(CSIS) Correct solution to incorrect solution – the subject changes his/ her initial correct judgment for the expression to an incorrect one

(NC) No changes made

Below are two examples from students' interviews describing their understanding about the equivalence of algebraic expressions. The student BK was very clear about the conditions for equivalence of expressions which she explained by comparing the terms of the expressions and using the rules of bracket opening. She also displayed a well developed understanding that equivalent algebraic expressions are equal for all values of the letter.

> RB: Ok. If I put m=2 in this first expression [13×m-7-8×4+m] and I put m=2 in the original expression [13×m-7-8×m+4], would I get the same value?
>
> *BK: No.*
>
> RB: It will not be. Why?
>
> *BK: Because it is 8×4 [13×m-7-8×4+m], if it is 4 [that is, the value of m] here then it would be the same value for both.*
>
> RB: Acha. Ok. If I put m=2 in the second expression [-7+4+13×m-m×8] and m=2 in the original expression [13×m-7-8×m+4] then would they be the same?
>
> *BK: Yes.*
>
> RB: Why?
>
> *BK: Because, m is any number, if we put any number for that then they would be the same.*

She very clearly knew that two equivalent algebraic expressions would always be equal for any value of the letter and could see the equality in forms of the algebraic and the arithmetic expressions without explicitly carrying out the verification process. This same confidence and clarity in understanding about the letter as well as about the expression was also seen in her responses to the simplification of algebraic expressions task. She was one of the two students who could articulate the possibility of the expressions 13×m-7-8×m+4 and 13×m-7-8×4+m to be equal for m=4. All others pointed out their inequality as -8×2 (-16) is more than -8×4 (-32). A few students, during their conversation, even went on to argue that two non-equivalent (which cannot be transformed into each other) algebraic expressions can never be equal, missing the point that solving an equation is to find the values of the variable for which two expressions are equal. Since this topic was not dealt with in the study, this point was not pursued further during the interviews. However, in the last expression in the list with brackets preceded by a negative sign, she incorrectly predicted the value of the expression to be same with the opposite sign (‘*The sum will be same but the sign will be wrong*’), not noticing that the full expression is not the inverse of the given expression.

The student AB was also able to judge the equivalence of the given expression with each option using the concept of terms but he was not so sure about the equality of values for equivalent algebraic expressions where he resorted to calculations of some parts of the expression. He was first asked to find the value of the original expression 13×*m*-7-8×*m*+4 for *m*=2 which he found to be 7. He was then asked which of the expressions in the list would have the same value as the original expression.

> SN: Then out of the expressions 1, 2, 3, 4, which of them would have the value 7?
>
> *AB: This 2nd [-7+4+13×m-m×8] and 3rd [m×(13-8)-7+4].*
>
> SN: Why?
>
> *AB: Because here also 13×m=26 and here -16 and here all the other signs are the same. Then here the number is the same. That is why the answer will be same.*
>
> SN: The first one cannot have the answer 7?
>
> *AB: No. because here [13×m-7-8×m+4] 2 is there and here [13×m-7-8×4+m] 4. Then here if we had done 2, then the product would have been more than this, means 8 4’s is 32. And here 16 is there. Then it would have been double and here is m=2.*

He understood the contribution of each term to the value of the whole expression but was not very confident. He did not have a clear generalized idea that equivalent algebraic expressions would have the same value for any value of the letter. A similar point was noticed in his interview for the task of simplification of algebraic expressions. It may be noted that the students interviewed after MST-III on their understanding of simplification process or equivalence of algebraic expressions did not show any trouble in substituting the letter by a number, which was seen among a few students in the written test.

These instances indicate students' understanding of algebraic expressions as well as the meaning of the letter. It is quite evident from the responses such as those made by the student BK that many of them have a robust understanding of expressions - arithmetic and algebraic. They appreciated the structural similarity of the two kinds of expressions and they were aware of the common rules of transformation that are applicable in both places. An additional question posed after the 'judging equivalence' task required the students to tell whether two occurrences of '*m*' in the expression $13\times m$-7-$8\times m$+4 could take two different values. Ten students straightaway denied the above possibility citing '*m*' to be the common factor between the two terms, and its use in the simplification process, to be an important reason. Some of them mentioned that for different numbers, one has to use different letters. Four agreed to the possibility and suggested appropriate changes in the solution procedure where it will not be possible to extract the common factor. The other three were unsure as they regarded the question as an extension of 'equal value' task and thought of the possibility of changing the *m*'s similarly in all the other listed expressions, thereby maintaining their equivalence/ inequality to the given expression as well as adjust the solution procedure. The students indeed saw the expressions as solvable entities and related them to arithmetic expressions.

Some of these students can be thought to have moved to the phase of structural development governed largely by the structural features of the arithmetic system (Goldin and Kaput, 1996). Some others still needed to go back to the numerical referent and establish the results. They continued to work at the inventive-semiotic stage (ibid); for them the letter simply replaced the numbers. The fact that the students engaged in such acts meaningfully is an indication of the unity in the two worlds: arithmetic and algebra, in contrast to the apprehension researchers had expressed with regard to the generalized arithmetic approach to algebra (cf. Lee and Wheeler, 1989; Linchevski and Livneh, 1999; Cerulli and Mariotti, 2001). The ready use of the numerical reference for the letter gives hope that their understanding is less likely to be 'pseudo-structural'.

Further, this connection allowed most students to see the letter as a placeholder for a number and some students to see it as a 'general number'. These can be interpreted from their justification for the simplification process of algebraic expressions and their understanding of the equivalence of algebraic expressions. In both cases the students appealed to the structural similarity between arithmetic and algebraic expressions and their understanding of arithmetic expressions. However, the vocabulary used by students often describes the letter as 'a number in the mind' or 'an unknown number'. The first of these ascriptions comes from another context, the 'Think-of-a-number' game where the letter stood for a number in someone else's mind. The second of the ascriptions does not come from the usual source of solving equations as that was not part of the study. Instead it comes from the various situations the students were exposed to while introducing algebra where the letter could take one or more values like guess-the-number game, open sentences ($x+y=18$) and letter-number line. This instructional program was instrumental, to some extent, in giving a sense of the letter as generalized number, instead of an object or specific number, without taking recourse to modeling situations algebraically. In the next chapter, an effort would be made to elucidate the contexts in which the letter was used and the nature of students' understanding with regard to usage of letters and algebraic expressions in those contexts.

7.3 *Generating equal expressions*

Although the post test responses of the students to the structural tasks were not clearly indicative of the capacity and the understanding the students had developed with respect to arithmetic and algebraic expressions, the interview data did show the extent of their understanding. Another task which revealed their understanding of equality/ equivalence of expressions and ability to use both structural and procedural aspects of expressions was generating equal expressions to a given expression. This was an interesting activity for the students as it gave them the autonomy to generate expressions rather than respond to given expressions. The expression needed to obey a constraint - that the value of the expression remains unchanged. It enabled them to work according to their own level and gave them the opportunity to use the concepts, rules and procedures in conjunction, learnt during the trials by making connections between them.

In MST-I, the students only made expressions where the terms were rearranged. The fact that rearranging terms of an expression does not change the value of the expression was quickly found by them by checking the values of the expressions with only numbers reordered and terms reordered. This helped them firstly in identifying equal expressions and then in generating such expressions themselves. Subsequently, students identified the terms and then changed the position of the terms by carefully moving the boxes containing the terms to generate equal expressions. Students rarely made errors while doing this. In MST-II, they were again asked to generate expressions equal to the given one, not only by rearranging terms but also by using other transformations which when applied would leave the value of the expression same. Both the groups responded very enthusiastically to this challenge. All the students did not work at an equal level in the classroom, some tried harder transformations and some restricted themselves to rearranging terms. Students' lack of confidence in the more difficult transformations as well as lack of anticipatory skills with respect to operations and rules, rather than lack of knowledge of rules of transformation could have been one reason for some of the students to make simpler equal expressions by rearranging terms. Students took time to understand the requirement of the task: the expression needed to be transformed in ways so that the value remained same, but they were not allowed to completely change the expression.

Strategy \ Item	30-14+16 (154 responses)	11×4-21+7×4 (141 responses)	12×x+5-6×x+8 (144 responses)
Splitting a term	56 (*8)	31 (*6)	35 (*3)
Simplifying/ combining terms	0	41 (*0)	12 (*2)
Rearranging terms	26 (*3)	20 (*0)	30 (*1)
Using bracket	4 (*2)	8 (*1)	23 (*2)
Others	14 (*3)	0	0

Table 7.13: Proportion (in percentage) of strategies used for generating equal expressions in the classroom in MST-II (*numbers in the bracket indicates percent of incorrect expressions)

Note. All percentage calculated on total number of responses given in the top row.

Analysis of the students' written expressions for three kinds of expressions in the classroom during MST-II: expression with simple terms (e.g. 30-14+16), expression with two product terms and a simple term (e.g. 11×4-21+7×4) and an algebraic expression (e.g. 12×x+5-6×x+8), revealed the predominant strategies used by the students. The strategies used for generating equal expressions and the percentage of expressions in each category, for one item of each kind as mentioned above, are given in Table 7.13. Of course not all these expressions were correct and the incorrect expressions gave rise to fruitful discussions in the classroom with regard to integer operations, use of brackets and bracket opening rules and other syntactic transformations which keep the value of the expression invariant.

The category *splitting a term* involves rewriting the term as sum, difference, product or quotient (e.g. rewriting 30-14+16 as 20+10-14+10+6). In the case of expressions with product terms or algebraic expression it includes splitting the simple term in the expression or splitting the numerical factor or converting the product term into a simple term and subsequently rewriting it as sum, difference, product and quotient (e.g. rewriting 11×4-21+7×4 as 40+4-20+1+28 or 12×x+8-6×x+5 as (6+6)×x+5-3-3×x+8, both of which are incorrect). The second strategy of *simplifying/ combining terms* was seen in the case of expressions with product terms or algebraic expression. It involves simplifying the product term into simple term with or without rearranging them (e.g. 44-21+28 as an expression equal to 11×4-21+7×4) or combining the simple or the variable terms, also rearranging the terms in the process (e.g. 6×x+8+5 as being equivalent to 12×x+8-6×x+5). *Rearranging terms* was a strategy which was used in all the three kinds of expressions (e.g. rearranging 30-14+16 as16+30-14). *Brackets* too were used to make an expression equal to a given one, sometimes in the obvious places and some other times very innovatively (e.g. expressions equal to 30-14+16 are 30-(14-16) and 2×(15-7)+16; an expression equal to 11×4-21+7×4 is (11+7)×4-21; an expression equivalent to 12×x+5-6×x+8 is 5-(-8+6x-12x)). Some *other strategies* were used in the case of expressions with simple terms only and consisted of using a combination of rules (e.g. -2×7+2×8+6×5) such as rearranging the terms as well as splitting the terms as products, making an altogether new expression with the same value (e.g. 28+4) and equal compensation between terms (e.g. 31-14+17, an incorrect expression). Students made errors while splitting a negative term indicating the detachment error (e.g. -39=-30+9) and in using the bracket with a negative or a multiplication sign, the English medium students making more errors than the Marathi medium students. Also, slightly more complex expressions were seen in the Marathi group. Figure 7.2 gives some examples of the expressions generated by the students during this trial. It must be noted that the complex looking expressions were not the result of incremental changes to the original expression. Rather, the students rewrote each term or parts of the expressions using brackets and other transformations, and then checked if it would lead to the terms of the original expression.

$\mathbf{25-18+9}$	$\mathbf{11\times4-21+7\times4}$	$\mathbf{8\times x+12+6\times x}$
1) $25-(18-9)$	1) $44-21+28$	1) $12+6\times x+8\times x$
2) $-18+9+25$	2) $40+4-21+20+8$	2) $4+4\times x+6\times x+12$*
3) $25-(18-3\times3)$	3) $4\times(7+11)-21$	3) $(4+4)\times x+(3+3)\times x+12$
4) $26-19+3\times3$	4) $25+3+40+4-25-4$*	4) $(8+6)\times x+4\times3$
5) $26-17+3\times3$*	5) $26+46-21$	5) $(4+4)\times x+6+6+(4\times3)\times x$*
6) $5\times5-3\times6+3\times3$	6) $30-2+50-6-20+1$*	6) $(10-2)\times x+12+(7-1)\times x$
7) $25-3\times(6+3)$*		

Figure 7.2: Examples of equal expressions generated by students during classroom discussion in MST-II (expressions marked with an asterisk are not equal to the given expression)

Even in MST-III, some students used only simpler transformations like rearranging terms or rewriting a positive term as sum, difference and product. The emphasis in this trial was not so much on recording of the various strategies which students use to generate the equal expressions but to see if they understood whether the transformations could be applied on any expression. Students gave indications of their capability to reverse the processes of combining/ splitting (which were used to generate the equal expressions), putting and removing brackets, anticipating the results of the transformations and check for their equality. Figure 7.3 shows some typical examples of equal expressions that students generated for a given expression in the classroom during MST-III.

The complexity of the expressions generated by the students was no doubt greater than the previous efforts by them in MST-II. More occurrences were seen of operating on the signed number like $-16 = -10-6$ instead of $-(10+6)$ or complex splitting of simple term like $+17$ as $2\times8+1$ (e.g. 5^{th} and 7^{th} expressions in Figure

7.3(a), 7th in Figure 7.3(b) and 1st in Figure 7.3(c)). Detachment errors could still be seen (e.g. 6th expression in Figure 7.3(c)). The Marathi group of students were more innovative due to their greater comfort with integer operations and brackets and flexible and simultaneous use of different rules of transformation (e.g. an expression equal to 48-11×6+17 is -2×(-24+33)+17). But the more algebraic approach of rewriting the product term as sum or difference of product terms was not still seen (e.g. 12×7 as 4×7+8×7). The English group also continued to make efforts in making the task interesting and led to many fruitful discussions in the classroom whenever the individual effort had errors and the group suggested corrections. Algebraic expressions were not discussed at this time and were left to be explored through the interview, which has already been discussed in the earlier section.

49-58+67 a
1) 49-(58-67)
2) 49-2×29-67*
3) 49-(50-8)+67*
4) 67+49-58
5) -19×3-1+24×2+1+67
6) -2×25-8+7×7+67
7) 2×24+1-3×39-11×6+1*

72-12×7+19 b
1) 72-(12×7-19)
2) 17+2-(72+12×7)*
3) 12×(6+7)+19*
4) 8×9-80-4+19
5) 2×(36-42)+19
6) 2×(36-42+8)+3
7) 8×9-84+9×2+1

38-16+29/ 32-16+29 c
1) -4×4+4×8+14×2+1
2) 4×(4-8)+29*
3) 29-2×(8-16)
4) 64÷2-32÷2+29
5) 31-15+29
6) 28+10-6+10+19+10*
7) 3×10+8-2×5-6+2×10+3×3
8) +29-(8×2-19×2)

48-11×6+17 d
1) 3×(16-22)+17
2) -2×(-24+33)+17
3) 24×2-15×4-6+8×2+1
4) 50-66+15
5) 7×(7+2)-1-22×3+3
6) -66+(4×10+8)+(16+1)
7) 6×(8-11)+2×(8+1)*
8) 40+8-(3+3)×(10+1)+17

Figure 7.3: A sample of students' responses from the classroom in MST-III: writing equal expressions for a given expression (expressions marked with an asterisk are not equal to the given expression)

Through the discussions, students learnt to generate expressions by compensating terms equally (e.g. finding which of the expressions 36-44+13 or 36-46+13 or 32-46+9 is equal to 35-45+13). Similar discussions were seen in the case of use of brackets for generating equal expressions. Poor knowledge of integer operations and the confusion between bracket opening rules and the need for brackets was evident in their efforts, for example, whether -25-4 is -21 or not and how it can be corrected. Discussions with respect to the use of brackets was a common feature in the classroom where a student tried to defend his or her solution and others tried to convince him or her of the error. These situations were a good occasion to discuss the various rules and procedures for evaluating expressions, opening brackets and the need for brackets. One such discussion is given below.

Students generating equal expressions for the expression 49-58+67 (episode lasts for a little over 2 minutes during MST-III).

> *Savitri: +49-(50-8)+67*
>
> Tr: Is this correct?
>
> *Prathamesh and some others: Teacher, no.*
>
> Tr: Right, Jayashree?
>
> *Jayashree and Navya: Teacher, yes. Before bracket minus sign is there, so it is plus inside the bracket and it is 50+8 is equal to 58.* [Many other students voice their agreement to this argument]
>
> *Prathamesh: Teacher, but -8 is +8, how can it be 58?*
>
> *Saurabh: Teacher, teacher first we have to do that, bracket first we have to solve. That is why the answer will be 42.*
>
> Tr: What Saurabh is saying is that we have to first solve the bracket, means we will get 42.
>
> *Saurabh: -42.*
>
> Tr: We get -42. But what is the term we want?
>
> *Prathamesh: We want -58.*
>
> *Saurabh and few others: If you put plus sign inside the bracket then you would get the correct answer.*

Tr: What Saurabh is saying is that if there were a plus sign inside the bracket then we would get the right answer.

Prathamesh: Yes.

Tr: What do you say, Bhagyashree? Which is correct? Is this correct [+49-(50-8)+67] or should we put plus sign inside?

Bhagyashree: Plus sign inside is correct.

Navya: This is correct [+49-(50-8)+67]

Tr: Is this correct?

Saurabh: No, teacher.

Joel: Teacher, how can it be correct?

Navya: Before the bracket, minus sign is there, so inside the bracket minus sign is there.

Tr: Let us see this. Before the bracket minus sign is there. Let us remove the bracket. What will we get?

Navya and others: Teacher, it will be 49-50+8.

Tr: +67

Prathamesh: So it is -50+8

Tr: What will be -50+8?

Saurabh, Joel and many others: -42. [Some continue to say -58]

Joel: Answer will be [inaudible]

Prathamesh: There are more negative cards.

(The class further discussed the solution for -50+8 using positive and negative cards.)

The above discussion and the whole task of generating expressions itself was made possible by the fact that students had some degree of confidence in parsing expressions together with flexible knowledge of combining terms, that is, rules of transforming expressions. This activity depended on students' understanding of equality and operations on numbers and further helped in deepening that understanding by using them in situations which were challenging enough but which made sense to them. The aim of the task was to help build structural understanding by coordinating knowledge of procedures and anticipation of the result of the procedure. It is evident from the discussion in the preceding paragraphs that their performance is not error free, even after enough experience in the tasks and the requisite skills; but they had strategies in place to deal with conflicting situations. In the above episode the students implicitly knew that a term can be split and re-written as a sum or difference or using brackets which would require some surface changes in the expression but leave the value invariant. They were also aware that only the changed components need to be compared to judge the equality of the two expressions as the presence of the same terms ensures equal expressions. A necessity to work mentally without necessarily computing sequentially was imposed in the process. Also this activity was instrumental in situating the rules and procedures together with students' implicit knowledge and expectations within a mathematical context of equality/ equivalence which was meaningful for students. Participating in the activity gave a chance to make the implicit understanding of operations explicit, turning the properties of operations and constraints on them into rules of transformation (see Cerulli and Mariotti, 2001, section 2.3.5). It encouraged students to create and transform expressions, compose and decompose expressions, which according to Mason et al. (1985) is an important activity to understand symbolic algebra and one of the routes to learn formal algebra.

Another point to be noticed is that even though many of these students did not perform well in the direct task of '=' symbol which required them to fill in the blank to make two expressions equal, they could do tasks which were more open like identifying and generating equal expressions. The written responses together with interview data and classroom discussions indicate sound understanding in the case of many of the students of the concept of equality. However, as has been discussed earlier, a few of the students were found to have difficulty using the brackets and the procedural rules for computing the sub-expression and failed to anticipate the result of carrying out such an operation. They thought that the solutions of the expressions in two different ways: using brackets as a precedence rule and using the bracket opening rules, could lead to different values (see discussions in sections 6.1.3, especially pp. 215-216 and 7.2.2, p. 281). They had misinterpreted the rule and were using it with a superficial understanding of brackets and sign changing, without really grasping the equality of the expressions.

7.4 *Overall performance of students in structure tasks*

This chapter was devoted to discussion of students' understanding of the structure of expressions and of equality together with the '=' sign, in the context of tasks requiring computation and in purely structural tasks. The data in the table below (Table 7.14) substantiates the earlier discussion in the sections above restating the fact that students came to the study with very little understanding of '=' sign or equality (Pre test average, MST-I) and made considerable progress in the three trials, even when the tasks became gradually more complex. In the predominantly structure oriented task of judging equality of expressions, items were most complex in the post test of MST-II and the least in the post test of MST-I, the transformations being restricted in the case of the latter to only 'rearranging terms' (Type (a)). When only the task of judging equality of expressions are compared, the students gained in performance between the post test of MST-I and the pre test of MST-II (from 60% to 66.6%) but slipped to 57.5% in the post test of MST-II when the transformations were more complex of Type (b). Overall, taking all the structure tasks together, the students performed slightly better in the post test of MST-II than in MST-I (as seen from Table 7.14). The larger difference between the post tests of MST-II and MST-III could be due to the reduction in complexity in the nature of transformations used, although they were still of Type (b).

	MST-I	MST-II	MST-III
Post test Average	60% (4.2 out of 7)	63.75% (10.2 out of 16)	77.8% (10.9 out of 14)
Pre test Average – items common to pre and post test	30% (1.8 out of 6)	82.5% (3.3 out of 4)	64.3% (9.0 out of 14)
Post test Average – items common to pre and post test	60% (3.6 out of 6)	63.75% (3.3 out of 4)	77.8% (10.9 out of 14)
Std. dev. Pre test	1.7	1.2	2.6
Std. dev. Post test	2.2	1.3	2.9
df	30	30	30
Difference between means	1.8*	0	1.9*
t-value (paired-samples)	4.831	-	4.983

Table 7.14: Comparison of average scores across the trials on structure based tasks (N=31)

* $p < .01$

7.4.1 The procedure-structure relationship

Students' improvement in performance in the procedural tasks over the three trials was discussed in Chapter 6. Since one of the main concerns of the study was to develop a strong sense of both procedures and structure among students and also to understand their interrelationship, it is important to analyze the performance in the two kinds of tasks together. Table 7.15 below shows the average scores in the post test of the three trials for the procedural and the structural tasks.

The average of the students in the procedural and structural tasks together has gradually improved over the trials. The items in both kinds of tasks (procedural and structural) became increasingly complex with the trials but the students' overall performance did not deteriorate. By MST-III, students became more consistent in their reasoning styles both in the context of procedures and structure sense, which is reflected in the improved performance of the students. The written responses of the students in the post test, especially in the tasks of evaluating expressions using easy ways and simplification of algebraic expressions together with the interviews demonstrated their ability to use both procedure and structure sense of expressions to deal with evaluation tasks as well as the identifying equal expressions task. The differences in students' capabilities in working with the symbols, manipulating them and reasoning about them over the trials have been discussed in these two chapters (Chapters 6 and 7). The discussions brought forth the nature of the tasks, the scope of the interplay between procedure and structure and the subtleties involved in carrying them out. There is not much difference between the students' average scores in procedure and structure tasks across the trials with a slightly better average score in the procedure tasks than in the structure tasks. This suggests that the two skills progress simultaneously and are complementary in nature.

	MST-I	MST-II	MST-III
Average – procedural and structural tasks	61.8% (16.7 out of 27)	65.25% (26.1 out of 40)	79% (33.2 out of 42)
Average procedure tasks	62.5 (12.5 out of 20)	66.7% (16 out of 24)	79.6% (22.3 out of 28)
Average structure tasks	60% (4.2 out of 7)	63.75% (10.2 out of 16)	77.8% (10.9 out of 14)
Correlation – structure and procedure	0.8*	0.6*	0.8*
df	29	29	29

Table 7.15: Comparison of average scores in the post test of the three trials on Procedural and Structural tasks (N=31)

Note. r = .456 is significant at .01 level for df = 29.

* $p < .01$.

Further, a high correlation between the performance of students in the procedural and structural tasks can be seen. This is maintained when the two groups (English and Marathi) are treated separately (r = 0.6 to 0.8). The graphs in Figure 7.4 show the high correlation trend together with the nature of the change in the three trials. The number of students moving to the upper right corner of the graph increases with the trials. Many students in MST-I could not successfully attempt the structure tasks. Students with very high scores in procedure were the ones who got many of the structure questions also correct. The situation improved in the next two trials, with many more students moving up on both the axes in the graph, displaying some evenness in their performance in both kinds of tasks. The students had become more consistent and also flexible in using the different concepts and ideas to work on the tasks in the later trials. The absence of students in the upper left corner of the graphs indicate the need for some familiarity with procedural knowledge before being able to perceive and use structure of expressions in tasks. But after that, these competencies may complement each other – a rise in procedural understanding leads to an appreciation for structure of expressions and vice versa.

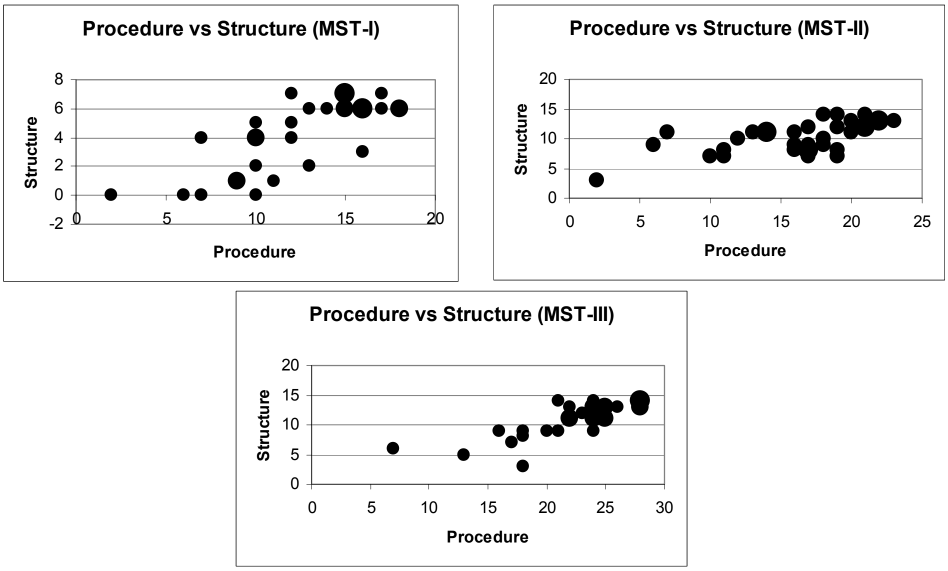


Figure 7.4: Distribution of students' performance in the procedural vs structural tasks in the three trials

Note. MST-I: Structure tasks:7, Procedure tasks: 20; MST-II: Structure tasks: 16, Procedure tasks: 24; MST-III: Structure tasks: 14, Procedure tasks: 28.

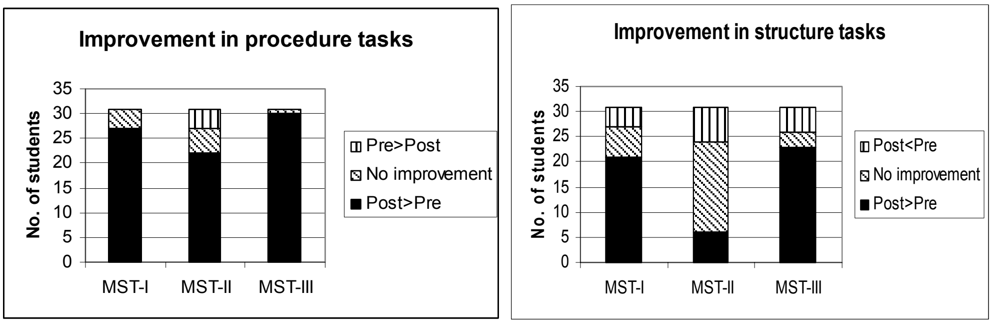


Figure 7.5: Number of students improving in the post-test over the pre-test across the trials in the procedure and the structure tasks (N=31)

The graphs in Figure 7.5 show the number of students who improved their performance in the post test in procedure and the structure tasks over the pre test in comparable items across the three trials. The number of students who improved over their pre test in the post test is always slightly more in the procedure tasks than the structure tasks. This lag in performance in the structure tasks too is indicative of the interaction between the procedure and the structure tasks and the necessity to have some procedural understanding before being able to abstract the structure of expressions. With each of the trials more number of students improved in both procedures and structure tasks, except in MST-II. This could be a manifestation of the nature of items in the post test of MST-II which did not sufficiently test students' understanding, being too complex for them to handle.

7.4.2 The arithmetic-algebra relationship

It is important to examine the relationship between students' responses to the arithmetic and the algebra tasks, the transition from arithmetic to algebra being the purpose of the study. The various tasks for the arithmetic part (evaluation of expressions, '=' sign and equality, rules for bracket opening) were so designed, that they allowed for transfer of these capabilities to the context of algebra, through an internalization and abstraction of properties of operations and understanding the possibilities and constraints on transformations.

	MST-I	MST-II	MST-III
Average – Arithmetic (Procedure+Structure)	64.1% (14.1 out of 22)	71.1% (19.2 out of 27)	75.9% (25.8 out of 34)
Average – Algebra (Procedure+Structure)	52% (2.6 out of 5)	57.5% (6.9 out of 12)	81.25% (6.5 out of 8)
Correlation – Arithmetic and Algebra	0.8*	0.6*	0.8*
Average – Arithmetic (Procedure tasks only)	61.1% (5.5 out of 9)	68.2% (7.5 out of 11)	76.7% (11.5 out of 15)
Average – Algebra (Procedure tasks only)	50% (2.0 out of 4)	53.3% (3.2 out of 6)	80% (5.6 out of 7)
Correlation – Arithmetic and Algebra (Procedure tasks)	0.6*	0.6*	0.6*
df	29	29	29

Table 7.16: Comparison of average scores in Arithmetic and Algebra in the post tests of the three trials (N=31)

Note. r = .456 is significant at .01 level for df = 29.

* $p < .01$.

Table 7.16 shows the average scores of the students in the various tasks in the post tests in the domain of arithmetic and algebra. The algebra tasks were fewer in number than the arithmetic tasks but combined many of the skills developed in the context of arithmetic. The first set of averages includes both procedural and structural tasks in each of the two domains. There is an incremental difference between the performances of the students in the three trials, the items also getting more complex in each trial. As per the expectation, students' scores on arithmetic and algebra are highly correlated in the trials but the correlation is slightly lower in MST-II than in MST-I and III due to the difference in the nature of the items and their complexity. Also, the English and Marathi groups separately show a high correlation between the scores in arithmetic and the algebraic tasks (r = 0.6 to 0.8). Thus students' developing understanding of arithmetic seems to influence their understanding of algebra. The graphs below further illustrate these statements (Figure 7.6). A trend similar to the procedure-structure tasks (Figure 7.4) is seen here, with students performing well in the algebra tasks only when they had acquired some understanding of various aspects of arithmetic. Even in the first trial (MST-I), the students who scored sufficiently high in arithmetic, scored well in algebra. This becomes prominent in the second trial (MST-II) when many more students do better in arithmetic enabling them to also perform better in algebra. In the third trial (MST-III), most students managed a reasonable degree of success in both the domains.

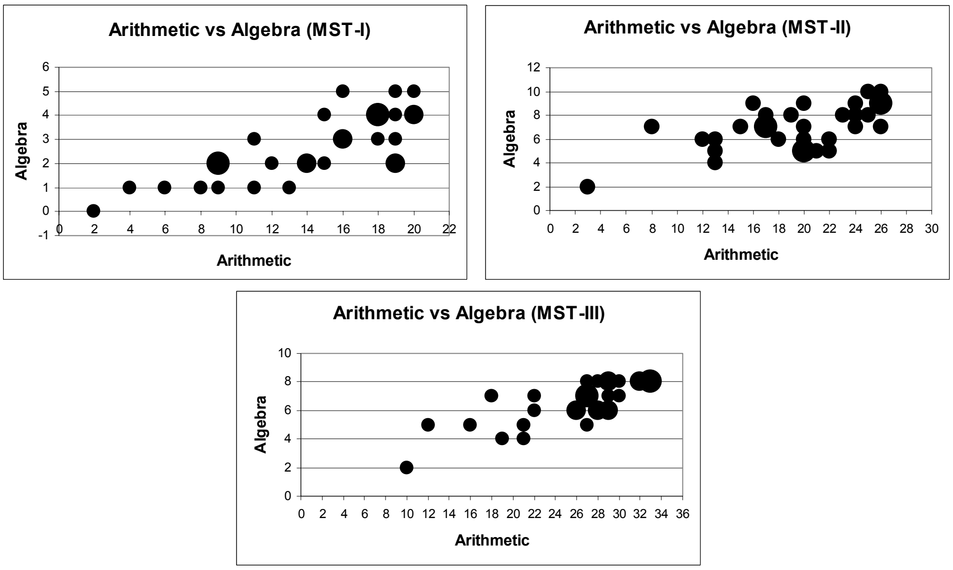


Figure 7.6: Distribution of students' performance in Arithmetic vs Algebra tasks in the three trials

Note. MST-I: Arithmetic tasks: 22, Algebra tasks: 5; MST-II: Arithmetic tasks: 27, Algebra tasks: 12; MST-III: Arithmetic: 34, Algebra tasks: 8.

The average score in algebra is below the average score in arithmetic except for MST-III where the students performed slightly better in algebra, largely due to their better performance in the procedural tasks in algebra. The low success in algebra in the earlier trials has been evident through the discussion with regard to their errors in manipulating algebraic expressions and the possible reasons for these. However, the scores in the two domains with respect to procedural understanding are also correlated. The scores in the structure tasks in arithmetic and algebra are not amenable to such an analysis due to very few items (usually only one item) in algebra which checked for structure sense. Interview responses of the students and classroom discussion in the structure tasks showed their grasp of the structural similarity between the two domains and also their consistency in using the rules across the tasks/ items. In both the contexts of interview and classroom, students were seen to be able to argue about their judgments and correct them whenever given the opportunity to do so. At times the judgments were corrected spontaneously and other times they engaged in long discussions in order to do so. The written responses had indicated some inconsistency with respect to the structure tasks where they correctly responded to one task and not the other, and they performed slightly better in algebra than in arithmetic. However, there are many evidences from interview and classroom discussion which indicate that most students developed strong understanding of the concept of equality, even though individual performance was not error free.

The discussion in this chapter revealed students' understanding of structure of expressions and their ideas of equality, together with the use of the concept of terms, through written responses, detailed interviews and classroom discussions. Although most students learnt to use the concept of terms meaningfully to check for equality and comparing two expressions by the end of the three trials, some showed inconsistencies and misinterpretation of structure leading to incorrect responses. The tasks helped them to consolidate their understanding of both procedures and structure and use them in a complementary manner, pushing them to the ‘structural phase’ (Goldin and Kaput, 1996) where they could appreciate the similarity between arithmetic and algebraic expressions and facilitate the transition. The symbols had to be necessarily seen as ‘processes’ and ‘objects’ or flexible ‘procepts’. They learnt the strategies of keeping the value of an expression invariant by using various transformations, leading to the understanding that the change in surface structure still keeps the value same. The quantitative analysis of the data also indicates the growing understanding of the students through the trials, the performance in structure tasks lagging slightly behind procedures and algebra lagging behind arithmetic in all trials except MST-III. The increasing comfort with arithmetic and algebra as well as structure and procedure sense for expressions could be a result of more coherence in the teaching approach itself and students’ ability to use the linkages appropriately. The results support the suggestions made by Lichevski and Livneh (1999, 2000), Malara and Iaderosa (1999) and Liebenberg et al. (1999b) about the fruitfulness of focusing away from computational tasks and engaging in a reflective discussion about the computational procedures and rules, which is required for a consistent performance in arithmetic as well as in algebra.

Chapter 8: Analysis III: Understanding of use of algebra in contexts

8.0 *A brief overview of the chapter*

A large part of the teaching program was devoted to developing meaning for the syntax of algebraic symbols by generalizing from similar contexts in arithmetic. The results of this part of the programme which involved reasoning *about* expressions have been discussed in the previous two chapters (Chapters 6 and 7). Engagement of the students in these tasks allowed them to move from the 'inventive-semiotic stage' to the phase of 'structural-development'. The movement to the next level of 'autonomous' stage was made possible by initiating the students to a culture of using algebra in rich situations (discussed in Chapter 3, sections 3.2.1 and 3.4) where the symbols could represent entities in the problem world. This requires careful building up of basic understanding of the contexts where algebra can be used: problem solving, generalizing, verifying, justifying and proving (see section 3.4). Various contexts and tasks, namely the letter-number-line, representing common situations like the relationship between heights, lengths, objects or dimensions of rectangle, think-of-a-number game and pattern generalization, were created in the study which required students to represent, generalize, justify and prove (that is, reason *with* expressions). These tasks were also supposed to bridge arithmetic and algebra, by making it possible to use the knowledge developed about syntactic transformations in the context of reasoning *about* expressions in situations, where algebra is used as a tool. This was to lead to a change in the style of reasoning: from arithmetic reasoning focusing on specific situations to general reasoning about a range of situations. Whereas 'reasoning *about* expressions' dealt with making generalized rules of procedures, exploring properties and transformations of expressions with a focus on the vital relationship of equality, ‘reasoning *with* expressions’ deals with using the symbolic expressions as a tool in situations of generalization, proving and justifying.

The tasks which required students to reason *with* expressions were part of each trial and they were changed or modified as and when needed. By the end of the study, many students had demonstrated a robust understanding of knowledge of procedures and rules of transforming expressions as well as equality of expressions as has been discussed in the preceding chapters. In this chapter, students’ understanding of algebra in contexts as reflected in their post tests of the main study trials and in the interviews at the end of MST-III will be discussed. These will also be supported and elaborated, whenever appropriate, by using instances from the classroom discussions in the three trials. Students worked on various tasks: (i) simple situations of representation using the letter, (ii) letter-number line, (iii) exploring patterns in the calendar and explaining them, (iv) think-of-a-number game and (v) pattern generalization from growing shapes. The analysis of the students’ performance in these five contexts will focus on their understanding of the notion of letter, representation using letters of relations between quantities, sequences of operations and of generalized rules for patterns, appreciating the need for simplification of expressions, drawing valid conclusions and the idea of substitution. In the process, the efficacy of the tasks in nurturing algebraic thinking and bridging arithmetic and algebra will also be discussed.

8.1 *Understanding of representing simple situations*

Representing situations using the letter is a major leap forward in the realm of algebra, a step that is not encountered in the context of arithmetic. In the tasks given to the students, they were required to represent using algebraic expressions simple situations, which were either depicted diagrammatically or verbally. Some of these items were similar to the CSMS test items and some were modified versions of them. The items can be found in the pre and the post tests in each of the trials (see Q.12, Q.13, Q.16 in Appendix IIA, Q.11, Q.12, Q.15, Q.24 in Appendix IIB, Q.21, Q.22, Q.23, Q.24, Q.25 in Appendix IIIB, Q.17 in Appendix IVA and Q.16, Q.17 in Appendix IVB). All the trials had a few items of this kind. The purpose of using these tasks was to engage students in simple symbolic representations which could reveal their understanding of the letter, notations and conventions in algebra, a concern shared by both CSMS (Kuchemann, 1981) and SESM (Booth, 1984) studies. A few examples of the task are: finding the combined length of a rod which is made by joining a rod of length *t* cm and another of 3 cm; finding the area of a rectangle with a part of or one of its dimensions given by a letter.

The average performance of the students (Table 8.1) in these tasks was not very good till the end of the study, with a somewhat better performance by the Marathi medium students than the English medium students. Some of the Marathi medium students were already aware of representation of simple situations with a letter as seen from their pre-test performance in MST-I. The number of items ranged from one to seven in each of the tests. The performance of both the groups dropped in the post test of MST-III compared to the pre test of MST-III, which is a surprising result.

MST-I				MST-II		MST-III			
English		Marathi		English	Marathi	English		Marathi	
Pre	Post	Pre	Post	Post	Post	Pre	Post	Pre	Post
0	22	29	40	30	65	27	3	75	56

Table 8.1: Average performance (in percentage) of students in the tasks requiring representation of situations ($n_{english}$=15, $n_{marathi}$=16)

Note. Pre-test MST-I had 3 items, Post-test MST-I had 6 items, Post-test MST-II had 7 items, 1 item in Pre-test of MST-III and 2 items in the Post-test of MST-III. No such items were posed in the pre-test of MST-II.

Students continued to make notational and conventional errors in all the trials, especially the English group. Although students were not seen to use the letter as a 'label/ object', some ignored the letter or assumed a specific value for the letter. Further, students' inappropriate knowledge of measures of length (multiplicative relations, $3\times y\times 5$ as the total distance traversed) and area (combination of additive and multiplicative relations, $x\times 5+4$ as the area of a rectangle with dimensions $x+5$ units and 4 units) made it difficult for them to complete these tasks. Such situations were discussed to a limited extent only in the first two trials (MST-I and II) in the classroom. The responses of the students in the classroom and the tests revealed that they failed to make sense of the task and the need to use the letter in such contrived situations. Thus, these were gradually replaced by tasks where the use of the letter was more natural and had a purpose in being able to draw an inference about the situation.

8.2 *The letter-number line*

The letter-number line (Carraher et al., 2001) is a generalized representation of the number line. It involves representations using simple algebraic expressions like $x+1$, $x+2$, $x-1$, $x-2$ (Figure 8.1), denoting both the relationship between successive numbers and their distance from x (discussed in Chapter 5, sections 5.1.2 and 5.2.1). This was a simple context in which students could accept the non-closure of algebraic expressions and which also gave the students the understanding that the letter stands for a number (detailed discussion in Chapter 2, section 2.2). It allowed students to interpret expressions both as a process and as the result of the process (see discussion in section 5.1.2). For example, $x+2$ is 'the number which is two more than x' and is the process of 'adding two to any number or moving two steps to the right of x'. Three tasks were posed in the context of the letter-number-line: (a) constructing the letter-number-line, (b) letter-number-line journeys and (c) finding the distance between two points on the letter-number-line. The performance of the students in the tests on reproducing the letter-number line and marking a few points on it was almost perfect (94%) in all the trials (see Q.28 in Appendix IIB, Q.15 in Appendix IIIA, Q.12 in Appendix IIIB, Q.13 in Appendix IVA).

They could also complete what was called a 'portion of the number-line' (a question posed in MST-II, Figure 8.2) in which students had to write three numbers to the left and right of any given positive or negative number with 90% success. This task was found to be an interesting one for the classroom as it unearthed many misconceptions which students held about the number line, like numbers to the left of any number are negative and to the right are positive and confusion about the order relations among integers. Students did not exhibit the same misconception in the post test as were observed in the classroom.

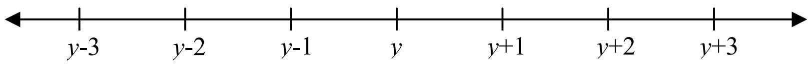


Figure 8.1: The letter number line

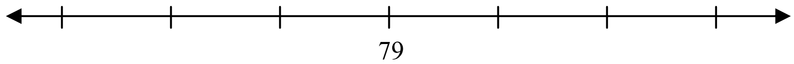


Figure 8.2: Portion of a number line (MST-II)

In another activity on the letter-number-line during MST-I, students learnt to see relationships between the expressions on the number line, like $x-3\underline{+5} = x+2$. Whereas the English medium students restricted to counting the number of jumps from x-3 to x+2 and occasionally reading it as 'x+2 is five more than x-3', the Marathi students went much ahead to understand and verbalize all the complementary relationships that could be deduced from this mathematical sentence. Students pointed out all the three relationships: 'x+2 is five more than x-3', 'x-3 is five less than x+2' and 'the difference between x-3 and x+2 is five'. They could express the meaning of expressions like m+2-5 as 'five less than m+2' or 'five less than two more than m'. These students gave indications of treating the expressions x-1, x+2 etc. as objects which represented a number and could be compared with other such numbers, rather than mere operations on the number line. Both the groups had exposure to similar tasks in the case of the ordinary number line.

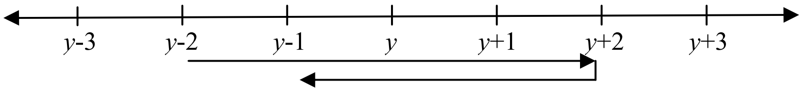


Figure 8.3: Number line journey

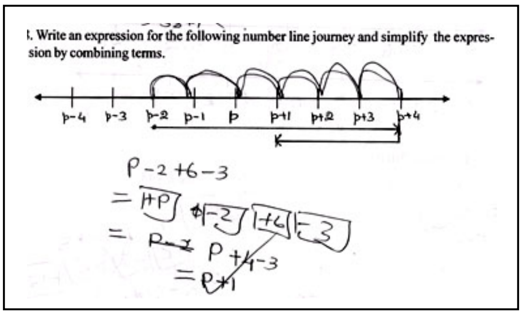


Figure 8.4: Sample of a students' solution to the journey on the number line task in the post test of MST-II

This activity was further extended to 'number-line-journeys' (Figure 8.3) in MST-II which was enjoyed by the students due to its simplicity and the sequential nature of representation (See Q.18 in Appendix IIIB, Q.14 in Appendix IVA). The students had to create a representation of the journey shown in the figure and then by manipulating it show that the end point is the same as shown in the figure. For example, in Figure 8.3, students would have to show the expression y-2+4-3 is equal to y-1 which is the end point of the journey on the letter-number line. Initially, students found representing and following the conventions in even the simple one step number line journeys (similar to fill in the blanks as above) to be difficult, but soon they were comfortable with the task in multi-step journeys as well. They represented a journey by an algebraic expression and then by computation verified that it would really lead to the end point shown on the journey. Figure 8.4

shows a typical solution to this task and the use of terms to manipulate the expression. In the post test at the end of MST-II, 80% and 50% of the English and the Marathi medium students respectively could write an expression for the situation but only 67% and 25% of them respectively verified the correctness of the end result by computation. It is possible that they did not feel the need to show such an obvious thing by computation since it could be directly verified by counting.

Correct/ Types of errors	MST-I	MST-II
Correct	31	35
Representational errors	29	29
Simplification errors	13	29
Not attempted	23	0
Others	4	7

Table 8.2: Percentage responses by types of error in finding the distance between two points on the letter-number line in the post test in two trials (N=31)

Note. Others = Strategies which could not classified into any of the other ones. The task was not posed in MST-III. There was only one item of this kind in any test.

In contrast, another task on the number line which required finding the difference between two points on it, by counting the number of steps as well as by representing the difference as an expression and manipulating it to get the result, was harder for the students (see Q.23 in Appendix IIB and Q.13 in Appendix IIIB). Table 8.2 shows the performance of the students in the task at the end of MST-I and II. The hardest thing for the students was writing the correct expression representing the situation although they could find the difference orally by counting or mentally calculating the number of jumps. Students made representational errors by writing the sum of the two points, subtracting the bigger from the smaller number or not using the brackets when necessary. They also made errors in simplification (e.g. k-k=k, 2×k) including errors in bracket opening rules and integer operations. For example, students' solutions for finding the distance between k-13 and k-7 were: k-7-k+13=k-6 or k-7-(k-13) = 2×k-6 or k-7+k-13=k×2-(7-13)=+6. Each of these solutions has errors in representation, simplification and sign errors. The increase in simplification error and decrease in 'no attempts' indicates a small gain in performance in MST-II with regard to students' capability to at least represent the situation using an algebraic expression.

This task was introduced to provide a simple context for working with algebraic expressions. The task of writing an expression for the distance between two points (e.g. distance between x-4 and x+3) is quite challenging for students not exposed to symbolic algebra. However, students could take recourse to the fall back strategy of simply counting the intervals between the two points on the letter-number line. It was expected that this would provide feedback on whether the expression and the simplification was correct. But, this task did not work as expected. Some of the difficulties were (i) it involved a complicated representation using the bracket; (ii) the representation for the distance between the two points does not follow the same order as given in the question (that is, distance between a and b is given by b-a, if b is more than a) and (iii) the motivation for solving the problem using algebra was reduced since they could obtain the solution directly by imagining the letter-number line or by drawing it and thereby counting the number of jumps required to go from one point to another.

The classroom discussions during MST-II, especially in the English group show that although the students thought that the distance between two points obtained through simplification and through direct counting must be the same, they lacked the resources to convincingly secure this. Simplification itself seemed to be a hurdle and many of them were not surprised at two different answers obtained for the same problem. They gave ad-hoc justifications for the discrepancy and randomly attempted to correct the solution without displaying any understanding of the symbolic representation or its transformation. Students' reasoning in this situation did not match their capabilities seen in the context of reasoning *about* expressions. The Marathi students performed better in the classroom, systematically identifying the bigger number and writing the expression and simplifying. However, their test performance was not very different. This was not the case in the more sequential activity of number-line journey which was much simpler in two senses: requiring step-by-step representation involving only two operations of addition and subtraction, and the operations required were only on numbers, the letter did not need to be operated upon (even the students were aware of this). Hence this task was closer to arithmetic than algebra.

8.3 *Calendar pattern*

During MST-II, a context of exploring and justifying calendar patterns (Bell, 1995) was used. Students were required to find relationships between the numbers in the rows and columns of a calendar and represent those using letters (see Q.26 in Appendix IIIB). Further, they were asked to find patterns in the arrangement of the numbers and also justify a given pattern (e.g. in Figure 8.5, A+H = C+F = B+G = D+E = 2×*x*). The post test results for this task are not very satisfactory with none of the English medium students able to either fill all the blank cells with appropriate relationships or able to justify the pattern (Figure 8.5). In contrast, 44% of the Marathi medium students could fill the blank cells correctly and 19% could justify the given pattern.

Sun		7	14	21	28
Mon	1	8	15	22	29
Tue	2	9	16	23	30
Wed	3	10	17	24	
Thurs	4	11	18	25	
Fri	5	12	19	26	
Sat	6	13	20	27	

	10	17
	11	

A	B	C
D	x	E
F	G	H

Figure 8.5: Calendar task

However, the classroom performance was better than the test performance, although the nature of the difficulties and errors students made during the post test were the same as those seen during the classroom discussions. Students in both the groups understood simple relations between the numbers in the calendar: relation among numbers in rows, among numbers in columns and among numbers diagonally arranged. They could represent these relationships using a letter (Figure 8.6), but many students (esp. English medium) made errors while filling cells using a letter in a 3×3 grid (as in Figure 8.5) due to their poor knowledge of integers. Figure 8.6(a) shows a student's first exploration on the calendar task, in which all the relations are correctly identified and represented in a generalized fashion. The last 'plus' shaped grid is interesting as none of the cells contain '*g*', the student's choice of the letter to make the representation. This indicates a situation specific use of the letter as specific number drawn from the letter-number-line context and was observed in many other instances. In the classroom, students pointed out that 'the number to the left of *m* is *m*-1, so the cell just before *m* is *m*-1' which indicates the influence of the letter-number line. Figure 8.6(b) shows increase in complexity of the pattern. The first pattern is correctly identified; represented and proved (the sum of the two outer numbers on the diagonal is equal to twice the middle number). But, the cells of the 'plus' grid are not correctly filled and the pattern which is proved does not follow from the figure. Figure 8.6(c) shows another student's attempt to prove a pattern with the same shape (sum of the outer numbers in a row/ column to be twice the number in the middle).

Students also worked with the more difficult shapes, like the 3×3 grid and tried to fill the cells by carefully analyzing the relations between the numbers in the rows and the columns. However, not all students could fill the cells correctly. It was a difficult task and there were many chances of errors, the requirement being to fill nine cells. Figure 8.7(a) shows a student's work on filling the cells and then proving a very simple pattern of constant difference between the first and the last number across the rows.

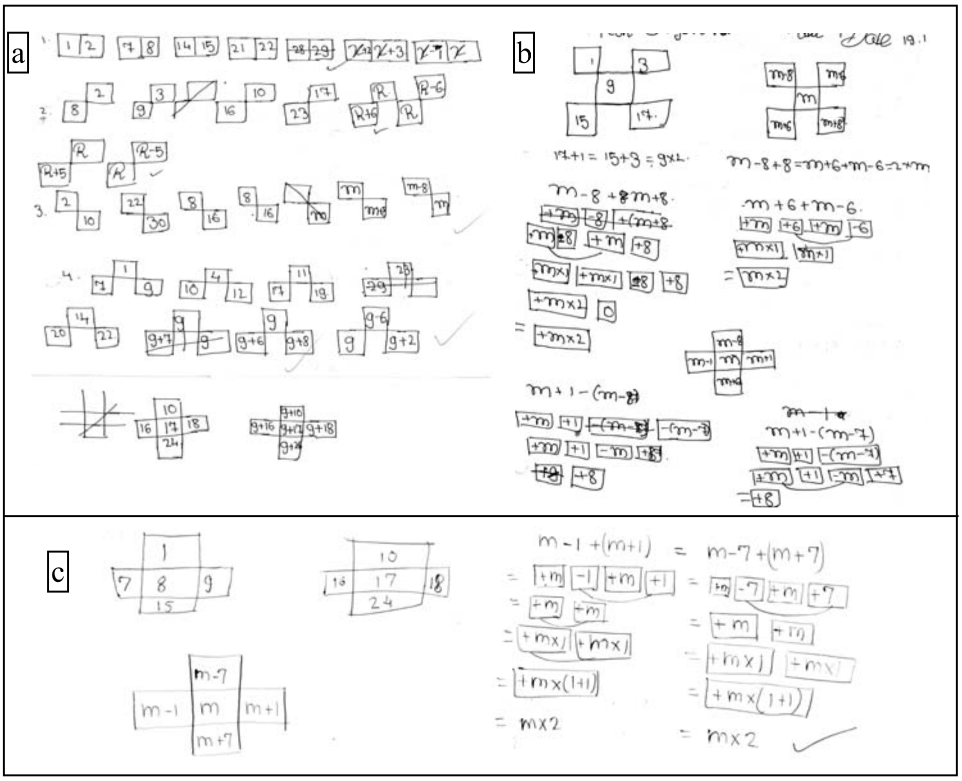


Figure 8.6: Students' initial work on the 'Calendar' task in the classroom in MST-II

In Figure 8.7(b), there are two errors in filling the cells in the top row, the student writing *a*-8 above *a*+1 instead of *a*-6 and *a*-9 above *a*+2 (corrected by the teacher), perhaps maintaining the symmetry with *a*+8 which is below *a*+1 and *a*+9 which is below *a*+2 or due to incorrect knowledge of order relation among integers. The relations involving only '+' sign in the last row are correct. Although the student had identified a constant difference of 21 between the sums of the adjacent rows, it could not be proved because of errors in filling the cells. The errors were less in both the groups of students when they operated on only positive numbers. Figure 8.7(c) is a correct representation of the general relationships and shows a reasonably successful attempt to prove the constant difference of 21 between rows.

Students' responses in another task, of proving among three consecutive numbers 'the sum of the first and the last is equal to twice the middle', again showed the interference from the letter-number line context. They chose the three consecutive numbers in many ways like *n*-1, *n*, *n*+1 or *n*+9, *n*+10, *n*+11 or *n*+98, *n*+99, *n*+100. However, they appreciated the combination *n*-1, *n*, *n*+1 to be the easiest representation of three consecutive numbers which makes the proof simple.

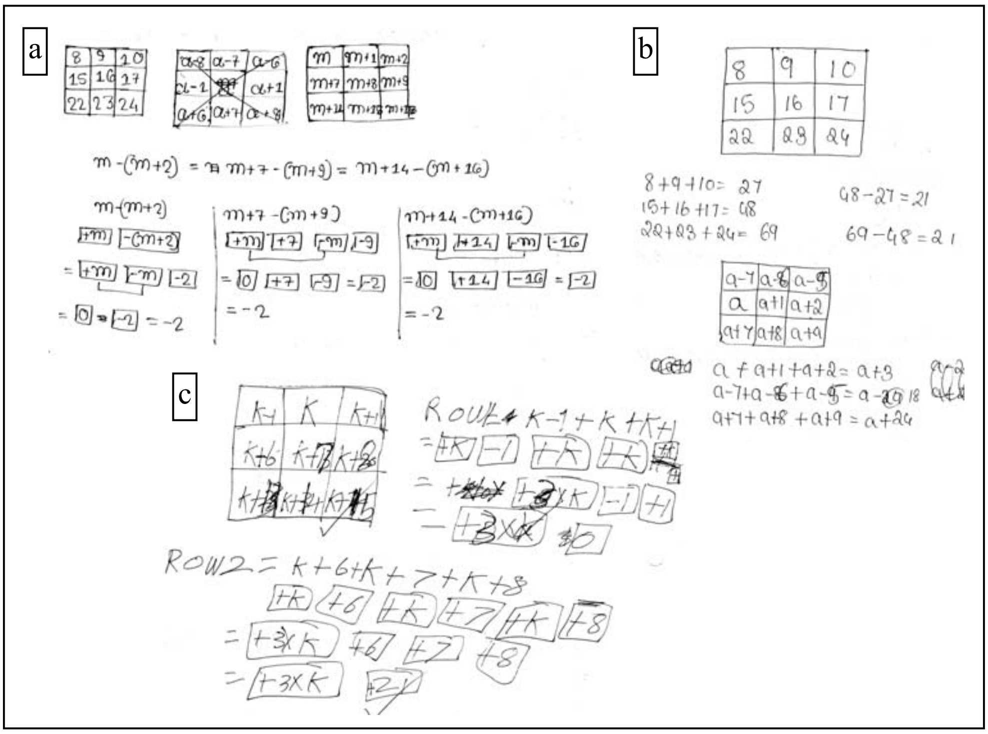


Figure 8.7: Students' work on the 'Calendar' task – exploring patterns in 3×3 grid from the classroom worksheets in MST-II

Justifying the pattern for consecutive numbers and for calendar patterns was difficult for the students due to their shaky understanding of simplification of algebraic expressions at this time, more so due to the presence of 'singletons' in all these expressions. Errors like *m*+*m*=0 or *m*, *m*-*m*=*m* or 2×*m* appeared often, in spite of writing the terms. Many were satisfied by checking the pattern with numbers and did not seek to prove it for a general case, lacking the understanding of justification/ proof. The task proved to be too difficult for the students due to the many expectations of the task: generalizing number relations and representing them using the letter, exploring and making sense of patterns among these numbers and justifying them. Not enough time was spent to create understanding of these requirements or to explore patterns in the calendar or discuss issues regarding proof or justification. It was only by the end that the students seemed to make sense of the task when a few students could explicitly state what the result of the simplification should be. This new evolving understanding of the task did not immediately lead to independent correct representation and proving. But this task did give them the opportunity to discuss symbolic representations, like the need for putting brackets, in their attempts to express the pattern to be justified.

8.4 *Think-of-a-number game*

The multiple requirements of filling the cells with generalized numbers, identifying, representing and proving the pattern in the calendar task made it difficult for students to work on the task, leading to the exploration of other tasks which looked at only single aspects like generalization or proving. The capacities which would develop as a result of engaging with these tasks can then be combined to work on tasks like calendar patterns. It is in this context that the activity 'think-of-a-number' (Mason et al., 1985) game was introduced. The task required the students to operate on a number following a sequence of instructions and then explain why everyone would get the same answer. The following set of instructions is an example of the game: 'Think of a number. Add 2. Multiply it by 3. Subtract 3. Subtract the original number. Add 10. Subtract twice the original number. Add 2. What is the answer you have got? Why is it that each one of you have got the same answer?'. This task is more complicated than the sequential letter-number line task but simpler than the calendar patterns task, requiring a sequential representation of operations on a number followed by interpretation of the end result, and also incorporating the possibility of proof.

Post test data

In MST-I, all except eight students in the study sample who belonged to group B_1, were introduced briefly to the task. The ideas of proving and justifying were new for the students. The post test contained a question on this task (see Q.26 in Appendix IIB). Responses that provided a proof of the generality of the result obtained by numerical computations, that is, generated the algebraic expression for the situation and manipulated it correctly to arrive at the conclusion were scored as successful. The post test result showed that the Marathi students (44% success) were better than the English medium students (one student out of the seven). A closer look at the post test written responses and the classroom discussions indicate the reason for the low performance. The teacher dominated the discussions and initiated the use of letter to represent the number which could be considered by any person. These ideas were not always accepted by the students, more because they failed to appreciate the need to justify the answer using algebra.

Further, the representation was easily learnt by the students when the sequence of instructions were simple but needed constant support when they were more complex, especially when brackets were required. Manipulating the expression to reach the valid conclusion was also difficult for most of them. Some of the students were found to do arbitrary calculation in the middle and write the correct answer in the end (both in the post test and the classroom). For example, for the problem in the post test in MST-I (Think of a number. Add 5 to it. Subtract 2 from it. Subtract original number.), one of the students solved it as n+5-2=12-n=3 and further wrote n+n+5-2=0+5-2=3. In this solution both the representation and further manipulation is incorrect but the student is aware that the solution of the simplification has to be 3. Others did not know the goal of the exercise and were not surprised by a discrepant answer, neither in the class nor in the test. These issues were not discussed in the classroom which made the problem beyond reach for most of the students. Their poor knowledge of simplification procedures of algebraic expressions and integer operations contributed to their lack of success.

This task was not taken up in MST-II, but was introduced again in MST-III. When it was reintroduced, the students did not reveal any familiarity with the task. Students worked on the task as before, and also made similar problems for their peers. In the post test they were asked to respond to the problem 'Think of a number. Subtract 1 from it. Multiply the result by 2. Add 5. Subtract the original number. Add 4. Subtract the original number. What do you get? Show that everyone will get the same answer' (see Q.1 in Appendix IVB). Here the Marathi group again performed better (62%) than the English group (13%).

	Only arithmetic	Only algebra	Arithmetic and algebra
English	8	0	7
Marathi	1	2	13
Total	9	2	20

Table 8.3: Number of student responses by strategy for solving the think-of-a-number game in MST-III ($n_{english}$=15, $n_{marathi}$=16)

Table 8.3 shows the nature of students' solutions to the task. Students who had proposed only an arithmetic solution to the problem (step-by-step computation or arithmetic expression) belong to the category 'Only arithmetic', students who had proposed only an algebraic solution but no arithmetic solution belong to the category 'Only algebra' and those who had offered both kinds of solution belong to the category 'Arithmetic and algebra'. Almost all the Marathi students attempted an algebraic solution in contrast to the English group where only half the students did so. The rest of the students were satisfied with an arithmetic solution, either a step-by-step one or an arithmetic expression.

Table 8.4 further explores the correctness or otherwise and the nature of the solutions proposed by the students in an effort to prove the universality of the result. 'ARseq' denotes the category where students had solved the problem by operating sequentially on the number as per the instruction. 'ARexpn' denotes the category where students only made an arithmetic expression in response to the problem and 'ALexpn' denotes the category where students made an algebraic expression in response to the problem. Since it was not enough to generate an algebraic expression but also to manipulate it correctly (solutions were considered to be correct when the algebraic expression was correct and correctly simplified), another subcategory 'ALsimp' was created to identify the correctness/ incorrectness of the simplification procedure on the correct algebraic representations as marked in 'ALexpn'. It can be seen that many Marathi medium students not only wrote an algebraic expression for the situation but also successfully simplified the expression to show the generality of the solution.

	Not done	ARseq		ARexpn		ALexpn		ALsimp	
		C	I	C	I	C	I	C	I
English	4	1	0	2	1	4	3	2	2
Marathi	0	0	0	1	0	14	1	10	4
Total	4	1	0	3	1	18	4	12	6

Table 8.4: Distribution of number of correct and incorrect responses for various strategies in the think-of-a-number game ($n_{english}$=15, $n_{marathi}$=16)

Note. ARseq = Sequential operation on number, ARexpn = Arithmetic expression, ALexpn = Algebraic expression, ALsimp = Simplification of algebraic expression (a subcategory of ALexpn), C = Correct, I = Incorrect, Not done = not attempting the solution.

The Marathi medium students readily accepted the formal way of proving in this context and adopted it in their attempts, visible in the post test as well. Solutions as in Figure 8.8 (a) could be seen only among the Marathi medium students in the post test. This student first wrote the algebraic solution followed by the arithmetic solution, clearly writing below the arithmetic solution that the answer will be the same irrespective of the number chosen. Some of the English medium students made the expression with one particular number (Figure 8.8(b)) and some did not use bracket in the appropriate place. A few replaced the general number represented by the letter with the particular number they had chosen for the numerical computation in the middle of the expression (see Figure 8.8(c)), where instead of subtracting ‘x’ the student subtracted ‘2’. In contrast, most of the Marathi medium students could make the correct representation but a few made errors while simplifying the expressions.

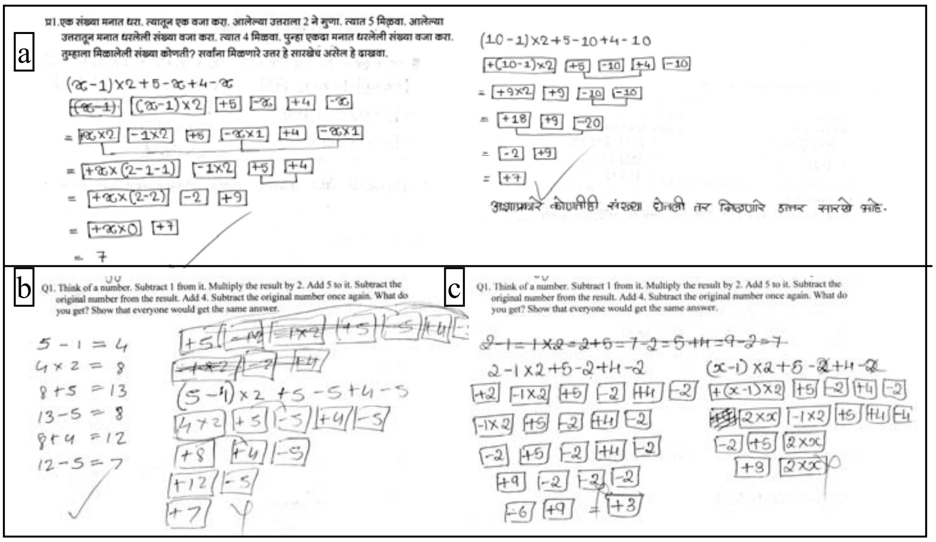


Figure 8.8: Solutions for the think-of-a-number-game in the post test of MST-III

Classroom discussion

The classroom discussions gave a glimpse of the students’ developing algebraic thinking and their ability to use symbolic expressions in the process. The students from both the groups actively participated in proving the patterns in the answer for the task given by the teacher as well as designing similar problems for their peers. Although students began with inductively generalizing the result from numerous examples, they were soon found to give verbal justification for the pattern, keeping track of the transformations mentally. For example, an explanation given by a student for the problem 'Think of a number. Add 6. Subtract 2. Subtract the original number. Subtract 3.' was the following: '50-50=0, 6-5=1'. Although she was using a specific number 50 for her explanation, what this student was actually pointing out was that the starting number cancels out in the transformations and the only thing left is 6-5 which is equal to 1 irrespective of what the starting number was. This situation could be easily exploited to begin the use of the letter to denote the original number on which the sequence of operations could be carried out. Students' ability to see an expression as composed of terms and flexible combination of terms might have helped them to understand the transformations on the starting number without much difficulty and use it in the context of algebraic expression. In a very simple situation, 'Think of a number. Add 2. Subtract 2. Add original number. Subtract original number', NN remarked 'She told to add 2, then subtract 2. So we will get 0. And again add original number and subtract original number, so it becomes 0. $x+2-2+x-x = x$ $(+2-2=0, +x-x=0)$'. She was able to see the parallels in the two solutions: arithmetic and algebra.

However, students began the manipulation of the algebraic expressions in this context randomly, making similar errors as has been discussed earlier in the context of the letter-number line and calendar patterns: writing $x+x=x$ or $x-x=x$ or $2x$. They tried to fix the discrepancy in an ad-hoc manner and misconceptions regarding the letter were revealed in the process, like 'letter is equal to 0 and we do not know its value'. The misconception could have developed due to the emphasis on the letter-number line and the understanding that the letter-number line is symmetric about 'x' and so 'x' is like '0', which is the origin in the number line. Discussions such as the one described in the previous paragraph were important in making the students realize that following the valid rules of transforming the algebraic expression would lead to the same answer/ relation as the arithmetic solution. Further, the discussions with regard to representation and syntax were fruitful in developing an understanding of the need for brackets and reinforced the dual interpretation of brackets as precedence operation and resulting in equal expressions as a result of bracket opening rules.

Another task on which the students worked in pairs was to make similar problems for their peers and also predict the pattern in the answers everyone would obtain. All the students enthusiastically participated in constructing the questions but the Marathi group students were seen to make slightly more number of complicated problems (e.g. Think of a number. Add 2. Multiply by 2. Add 10. Subtract 8. Subtract double the original number. Add 1. Subtract 7. The answer for everyone is 0.) than their English medium counterparts, who avoided the use of brackets and multiplication of a previous result (e.g. Think of a number. Add original number. Subtract 3. Add 4. Subtract 1. Subtract original number. Everyone gets back their original number). The strategy of relying on mental tracking of the transformations followed by verbal explanations did not allow them to easily make the transition to more complicated operations on the number, where it was essential to make the symbolic representation to remember the transformations. This exercise could have contributed to the Marathi students' better performance in the post test as well.

Figure 8.9 shows some examples of students' efforts to generate questions for the Think-of-a-number game during classroom discussions. The examples display that students were working at all levels and also that some students understood the purpose of the task and the need for symbolization better than others. Some students made the problem, checked it by numerical instances and used algebraic expression to show the pattern in the answer (for example Figure 8.9(c)). Many of these were not seen in the English group. At times they did make a mistake in either inducing the pattern from the numerical instances or in the manipulation of the algebraic expression, thus not being able to arrive at a definite conclusion (Figure 8.9(a)). There were also others who continued to work with the numerical instances and not feeling the need to prove or find the pattern in the answers (Figure 8.9(b)).

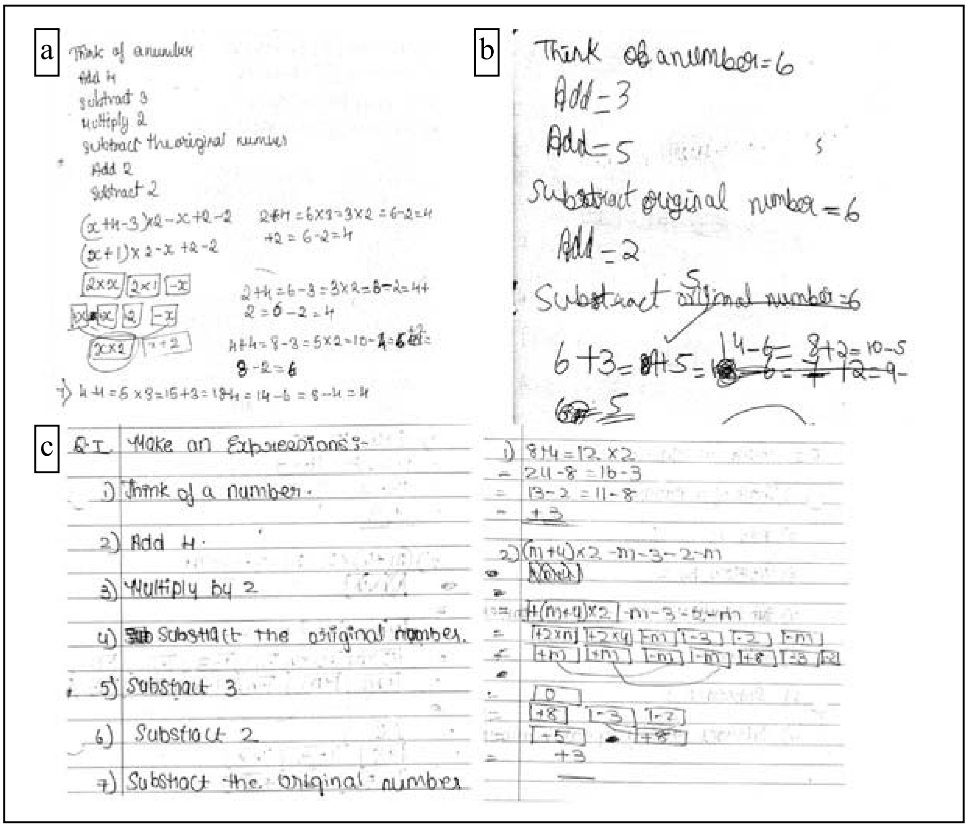


Figure 8.9: Examples of students' efforts to generate problems for 'Think-of-a-number' game during classroom discussion

Students posed their question made in pairs to the whole group who worked on them and that acted as an assessment of the quality of the question and the designers' knowledge about their question and the result. The discussions largely revolved around the representation of the problem, that is, the algebraic expression, the need and meaning of brackets, integer operations and manipulation of expressions. Although, students' experience with reasoning *about* expressions in the context of syntactic based transformations was helpful in allowing the students to think of these situations with the help of expressions, the emphasis on bracket opening rules overshadowed the meaning and purpose of brackets in enclosing parts of expression to be given precedence in operation (repeated attempts described in Chapter 5). This was also a situation where the students were spontaneously seen to decompose a product term with a variable factor as sum of 'singletons', which when combined with a term of opposite sign led to zero (as in Figure 8.9 (a and c)). The rewriting of a product term as the sum of 'singletons' was seen in only one student's response to the procedural task of simplification of algebraic expression and not seen while generating equal expressions to a given expression. It is likely that this strategy came in response to this particular situation and perhaps due to their engagement with verbalizing the explanation where they had experienced the cancellations in their mental calculations. Of course, it was helped by their knowledge of 'terms' of an expression.

Results from individual interviews

The difference between MST-I and III was that students had understood the goal of the task which made it possible for them to anticipate the result of the manipulation of the algebraic expression. But the important point to explore here is whether the students found the algebraic representation more useful compared to the arithmetic representation and the step-by-step solution process. The interviews conducted with the students after MST-III delved into these matters. The task posed in the interview was 'Think of a number. Add 2 to it. Subtract 5 from it. Subtract the original number. Add 4. Write an expression for this instruction' (see Algebra test Q. 3 and Interview schedule: Algebra (Task 4) in Appendix VB). Students were first told to write an expression for the situation prior to the interview ('Solution'). Students were then shown a card with an algebraic expression representing the situation (x+2-5-x+4) and asked to identify if it was the correct and would lead to the same value as their expression ('Expression for given situation'). Further, two questions were asked: (i) students were shown a card showing the expression x-5+2+4-x and asked whether it was a correct representation for the above situation ('Another expression'), (ii) they were asked if the values of the above expression and the initial expression (x+2-5-x+4 or an arithmetic expression made by the student) will be the same ('Value of expressions'). Another question was concerned with their ability to interpret an expression or its simplified form in the context of the game: make a similar problem for the expression x×2-4+5-1-x = x ('Making problem'). Finally they were asked to explain the utility of algebra in this situation ('Use of algebra').

Table 8.5 summarizes the responses of the students to this task during the interview. The table shows that out of seventeen students, five wrote an arithmetic expression as the solution for the task. Four others did not write an expression at all and worked out the solution step-by-step. Two of these students when asked to write an expression for the situation failed to write one. The rest of the students (8) wrote an algebraic expression. But all of them correctly identified the algebraic expression (x+2-5-x+4) shown as representing the situation, thereby demonstrating their ability to understand symbolic representation in this context; some were not able to produce one independently. Although, the students considered the other equivalent expression (x-5+2+4-x) to be representing the same situation, they showed awareness of the change in order of the instructions compared to the original situation. Their prior experience of judging equivalence of algebraic expressions led them to easily infer the value of this expression to be equal to the original expression, except four students who were unsure about the equality in value.

Four students simplified the original algebraic expression representing the situation with difficulty ('Simplification', not specifically asked in the interview). They also were the ones who were not so sure about the equality of the answers of the original arithmetic/ algebraic expression (x+2-5-x+4) and the equivalent algebraic expression (x-5+2+4-x). The other students (13) had no trouble in simplifying the expression and many times they worked it out orally or inferred from the solution to the arithmetic expression or the step-by-step numerical solution. The interview also threw light on the reasons for the low performance in the post test in this task. Not many students thought algebra to be useful for the situation and therefore displayed resistance to use algebra as a tool for representing or drawing a conclusion. On being asked to explain the utility of the algebraic expression as a representation in this context, one student BK said '*a number represents a general number and if the same operations are carried out on the number, it can be shown that everyone would get the same answer*'. These students were treating the number as a quasi-variable (Fujii and Stephens, 2001) and treated the two kinds of numbers (number in the mind and the other numbers appearing explicitly in the expression) used in the situation differently. Very few of them saw the use of the letter as standing for any number, succinctly representing the situation and thus helping to prove the pattern in the answer for the general case. But when given an algebraic expression they could 'create' the corresponding 'think-of-a-number' game. They could also interpret the simplified expression correctly, for example, that 'x' in the simplified expression denotes the number originally thought by an individual. The coding of the students' responses will be explained as earlier by some interview excerpts of students marked with an asterisk in the table below (Table 8.5).

Name	1. Solution	2. Expression for given situation	3. Simplification	4. Another expression	5. Value of expressions	6. Making problem	7. Use of algebra
BP	NUMSoln	IAL	SSAL	IAL	SEV	SMPCI	UCAL
PD	AR	IAL	SALD	IAL	USEV	SMPCI	-
BK	AR	IAL	SSAL	IAL	SEV	SMPCI	NUMQV
AY	NUMSoln	FWE, IAL	SSAL	IAL	SEV	SMPCI	SEAL
NN	AR	IAL	SSAL	IAL	SEV	SMPCI	NUMQV
SG	NUMSoln	FWE, IAL	FSAL	IAL	USEV	SMPCI	-
PG*	AR	WAR, IAL	SSAL	IAL	SEV	SMPCI	SEAL
JS*	NUMSoln	WAL, IAL	SSAL	IAL	SEV	SMPCI	NUMQV
NW	AL	IAL	SSAL	IAL	SEV	SMPCI	-
RG	AL	IAL	SSAL	IAL	SEV	SMPCI	-
AS	AR	WAL, IAL	SSAL	IAL	SEV	SMPCI	UCAL
AN*	AL	IAL	SSAL	IAL	SEV	SMPCI	SEAL
SV	AL	IAL	SSAL	IAL	SEV	SMPCI	NUMQV
MC*	AL	IAL	SSAL	IAL	SEV	SMPCI	NUMQV
AB	AL	IAL	SALD	IAL	USEV	SMPCI	SEAL
BM	AL	IAL	SSAL	IAL	SEV	SMPCI	-
TJ*	AL	IAL	SALD	IAL	USEV	SMPCI	UCAL

Table 8.5: Responses of the students interviewed after MSC-III for the Think-of-a-number game (*indicates students whose interviews are discussed in the text).

Solution to the task – Representation of the problem (Col. 1):

(AR) Arithmetic – the subject has written an arithmetic expression as a representation of the situation

(AL) Algebraic – the subject has written an algebraic expression as a representation of the situation

(NUMSoln) Numeric solution – the subject has evaluated the problem step by step using a specific number instead of writing an expression

Identification of the algebraic expressions (Col. 2 and 4):

(WAL) Writes an algebraic expression – the subject on being asked to write an expression, writes an algebraic expression for the situation

(WAR) Writes arithmetic expression – the subject on being asked to write an expression, writes an arithmetic expression for the situation

(FWE) Fails to write an expression – the subject on being asked to write an expression, fails to write one

(IAL) Identifies algebraic expression – the subject identifies the algebraic expression shown to him/ her as a correct representation of the situation

Simplifying the algebraic expression (Col. 3):

(SSAL) Satisfactorily simplifies algebraic expression – the subject satisfactorily simplifies or demonstrates the ability to simplify the algebraic expression to arrive at the conclusion

(SALD) Simplification of algebraic expression with difficulty – the subject simplifies the algebraic expression with difficulty and lot of effort

(FSAL) Fails to simplify algebraic expression – the subject fails to simplify the algebraic expression or does not understand the process of simplification

Value of the two algebraic expressions and the arithmetic expression (Col. 5):

(SEV) Sure about equality in value – the subject understands that the values of all the expressions would be same

(USEV) Unsure about equality in value – the subject is not sure that the values of all the expressions would be same

Making a problem for the given expression (Col. 6)

(SMPCI) Satisfactorily makes a problem with correct interpretation – the subject satisfactorily makes a problem for the given expression and also correctly interprets it

(SMP) Satisfactorily makes a problem – the subject satisfactorily makes a problem for the given expression and fails to interpret it

Use of algebra (Col. 7)

(SEAL) Satisfactorily explains the use of algebra – the subject satisfactorily explains the use of algebra for purposes of proving

(UCAL) Unclear about the use of algebra – the subject does not have enough understanding/ clarity about the use of algebra for the situation but gives vague explanation

(NUMQV) Numbers as quasi variable – the subject explains the situation using numbers and thinks the use of numbers as same as that of letters

Students mainly thought in four different ways: (i) wrote/ recognized an algebraic expression and were aware of its purpose (4 students), (ii) wrote an arithmetic/ algebraic expression and thought of the number as a generalized number or quasi-variable (5 students) and (iii) wrote an arithmetic expression or solved sequentially to get the answer and were unclear about the use of algebra (2 students). One student although used an algebraic expression, was unclear about the use of algebra and explained with the help of numbers repeatedly without commenting on the generality of the solution. Five students were not asked the question of purpose of algebra explicitly. Two of them were uncomfortable with this task, were not very sure about the equality of the values of the original arithmetic/ algebraic expression and the equivalent algebraic expression and had trouble simplifying the algebraic expressions. The other three could answer all the questions satisfactorily but that is not sufficient to conclude about their 'belief' in the algebraic approach.

The student TJ on being posed with the task wrote an algebraic expression. TJ's responses indicate that he could not think of the generality of the solution and the purpose of algebra.

> SN: What have you done, explain me.
>
> *TJ: Think of a number, so I thought of 3. Add 2 to it. Subtract from it means that from the sum of these two subtract 5. That is why I have used a bracket and we would get one answer after combining these two and from that 5 is subtracted. Again to subtract the original number means it would be three here [pointing to the x].*
>
> SN: But you have written '*x*' here.
>
> *TJ: Yes. Here it would be 3. And then it would be equal to +4.*
>
> SN: But you have written the whole expression with *x*, so if the number you thought in the mind is *x*, then how will this expression be?
>
> *TJ: Wherever there is the original number, write x there.*
>
> SN: How, show by writing?

TJ: [He rewrites the expression (x+2)-5-x+4]

Although he had written an algebraic expression, while explaining his answer to the interviewer SN, he repeatedly thought in terms of a number. The algebraic expression is probably an outward performance for him to satisfy the teacher, even though he shows enough awareness of the possibility of replacing the letter by a number or vice-versa. He evaluated the algebraic expressions for two values of x: 2 and 3 and found the same answer '1', but was unsure of the value of the algebraic expressions itself, guessed it would be x-1 and found on simplification x+1. After much effort from the interviewer, he could be convinced that x-x=0, which he seemed unable to connect with all his earlier arithmetic experience. According to him +x and –x cancelled and 'x' remained. It is not easy to induce a cognitive conflict when students lack conviction about the utility of a procedure or a tool; for them computing with arithmetic expressions and algebraic expressions are two different worlds and the answer can differ in these worlds. Some of the students had not moved to the algebra world but had a more sophisticated understanding of arithmetic expressions which was helpful in solving many of the tasks they were working on. The students knew what the representations meant and how inferences could be drawn in these situations, but the purpose of algebra was still not clear to them.

The student SG was similar and he even failed to write an arithmetic or algebraic expression and found the answer by sequential computation. He could recognize the correct representation for the situation, probably not understanding the significance of the algebraic way of solving the task. He too was not perturbed by the discrepancy in the answers in the arithmetic and the algebraic expressions (1 and x+1), but knew that equivalent algebraic expressions would have the same value (in this case it was x+1). The step-by-step computation on the number was not sufficient to see the overall change in the initial number at the end of the sequence of transformations and thus he was unable to connect his explanation with the algebraic expression. He tended to search for ad-hoc explanations for the discrepancies (like, the value of x is 1), not paying attention to either meaning or logic of the syntax. He repeatedly checked his solution with the number but was not able to spot the error in the simplification of the algebraic expression. He had difficulty in understanding the meaning of the letter and the algebraic expression as representing the general case. With a lot of support he could simplify the expression in this situation although he had correctly simplified the two expressions which he was asked in the beginning of the interview.

Another student MC when asked the same question also wrote an algebraic expression but he did not have any specific reason for the choice of the letter '*m*' in comparison to a number. His responses show that, unlike TJ, he thought of the number as a generalized number.

> SN: Why did you take '*m*'?
>
> *MC: That is the main/ original expression, that is why.*
>
> SN: But why did you take a letter? You could have worked with a number as well? 5, 10, but why did you take ' *m*'? Why did you take a letter?
>
> *MC: The number in the mind.*
>
> SN: It is the number in the mind, therefore *m*.
>
> *MC: Yes.*
>
> SN: But can't the number in the mind be numbers 5, 10, 15?
>
> *MC: It can be.*
>
> SN: But then why did you take '*m*'?
>
> *MC: Just like that.*
>
> …
>
> SN: What do you think, is it better to use number or letter?
>
> *MC: Number.*
>
> SN: But why did you use letter? Is it better to use number? Why?
>
> *MC: We can check the answer.*

SN: Can't we check with letter?

MC: We can.

SN: But you have to put back a number for the letter.

MC: Yes.

SN: How do you check with letter?

MC: +m-m the answer is 0, that is how.

MC agreed that both letter and the number could be used in this task but preferred to use the number which allowed him to check the answer. He was comfortable with simplifying the algebraic expression (he had done so before the interview began) and was sure that inferences could be drawn from the simplification but this did not give him any more sense of 'truth' than the one derived from the numerical world. He was sure of the equivalence of the two algebraic expressions and also indicated the awareness of invariance in value of the expression due to a change in the letter from 'm' to 'x'. For many of these students writing an algebraic expression was part of the norm, it does not necessarily indicate an appreciation for the generality and the power of the algebraic expression.

Another student JS had worked out a solution in a step-by-step manner but wrote an algebraic expression when asked by the interviewer. She demonstrated a fairly good understanding of equivalence of algebraic expressions and the task itself, knowing well that the values of all these expressions would be equal, but she simplified the first algebraic expression to see the result. Some students among those who were interviewed were aware that the arithmetic or the step-by-step solution and the original algebraic expression should have the same solution and did not simplify whereas some others simplified to be sure. A few had great difficulty in this regard like TJ and SG. This task once again checked for their understanding of equivalence and simplification of algebraic expressions. JS did not have any trouble making the question for the expression and interpreting the result of the simplification. She made a problem for the expression 2×x-4+5-x-1 as follows: '*Think of a number. Multiply 2. Minus subtract 4, add plus add 5, subtract original number, subtract 1*'. She also was able to interpret the final 'x' as getting back the original number. Like MC, she too thought expressions with numbers were sufficient to draw conclusion about the situation, displaying the 'quasi-variable' idea.

PG wrote an arithmetic expression (5+2-5-5+4) for the task. But he recognized the algebraic expression (x+2-5-x+4) to be denoting the situation and went on to answer all the other questions satisfactorily. He was one of the few students who understood the purpose of using the 'x' in the expression and put it quite articulately.

> RB: So what is the difference between this [x+2-5-x+4] and your expression [5+2-5-5+4]?
>
> *PG: Teacher, only the number is different which I have thought. And here it is 'x'.*
>
> …
>
> RB: Now tell me is there any advantage of writing these x's compared to the 5 you have written? Is there any advantage of doing this? By writing 'x'?
>
> *PG: You are giving 5 of the students for writing this. So all the 5 students will come the same. Some of them takes 7, some of them takes 2, it will come the same.*
>
> RB: So is there any advantage of using the 'x'? That is the question?
>
> *PG: 'x' means any number. Any number can be done like this. That is why it is used.*

PG did not see any difference between his expression and the algebraic expressions shown to him due to his ability to perceive the general structure of the expression. He would have seen the 'truth' of the result in the arithmetic expression but he knew to convince others of the universality of the result the letter had to be used. The student AN on the other hand wrote an algebraic expression in response to the situation and knew the purpose of algebra but was not very articulate about it. She remarked that '*because which is that number we do not know. That is why writing any letter would do, but not a number.*' She insisted on using the letter for writing the expression as the number is not known but was sure that the same answer will be arrived at by solving it with any number. The idea of proof or justification was not very clear to her. It is not the case that the number is not known but that one needs to show the result for all values of the letter.

Overall, most of the students did not appreciate the use of algebra in this problem. This is not to say that they did not understand the representation or they were not thinking algebraically. Almost all of them understood the processes involved in such problems but were satisfied to understand or explain the problem numerically. Their problem does not lie in understanding a chain of deductive logic or in accepting the 'truth' that is established by the algebraic method (cf. Healy and Hoyles, 2000; Liebenberg et al., 1999b, Cerulli and Mariotti, 2001, discussed in Chapter 2, pp. 90-91, p. 73, p. 43 respectively). The interviews revealed that most of them were capable of identifying the representation, manipulating the expression and could anticipate the equality in value of the various expressions encountered during the task. Representing the problem using an algebraic expression is not sufficient to demonstrate the acceptance of the algebraic way of approaching the problem. Nor is it the case that arithmetic representation was accompanied by non-appreciation of the generality of the solution. These students were comfortable in explaining and convincing themselves, peers or the teacher using a mix of verbal language and symbols but many of them did not appreciate the need to communicate in a manner which would convince everyone without any ambiguity. This could be one reason why the students time and again fell back on their numerical understanding and reasoning based on it. The students seem to progress from a 'quasi-variable' approach to a complete algebraic approach while developing the idea of proving and justifying. This gradual progression is essential to build the need for an algebraic approach (Hanna and Jahanke, 1993), which lends the unambiguous symbolic representation and also the understanding that manipulating the algebraic representation by conforming to the rules of transformation would lead to conclusions which would be true and certain (Dettori et al, 2001). It is possible that if the focus is on more complex problems, then the need for algebra would be more evident. The dilemma in such a situation is whether to use students' intuitive understanding or expectations and begin the process of representation in simple situations or use more complicated tasks to make it challenging enough where this kind of representation would be essential. This is evident also from the questions which the students made for their peers, the simpler ones did not require any representation but the harder ones could not be found without the support of the representation.

8.5 *Pattern generalization*

In this section, discussion of a task, used during MST-III, specifically focusing on generalization in the context of patterns in shapes, will be taken up. This is a highly studied task in the research literature and embeds algebraic expressions in the context of prediction and writing a general rule. In this study, students worked on patterns of shapes such as those shown in Figure 8.10 where they were asked to answer questions about specific positions in the pattern and make a rule as indicated in the first two problems in the figure. This task is different from the think-of-a-number game in certain respects. It required some understanding of generalization to be able to abstract the relation between the starting number and the final result by a process of 'seeing the general in the particular', which many of the students were capable of. Subsequently, this understanding needed to be transformed into representation and manipulating the representation. As the analysis of the task in the previous section revealed, students' beliefs about the usefulness of the algebraic approach are an important factor in the successful completion of context based algebra tasks.

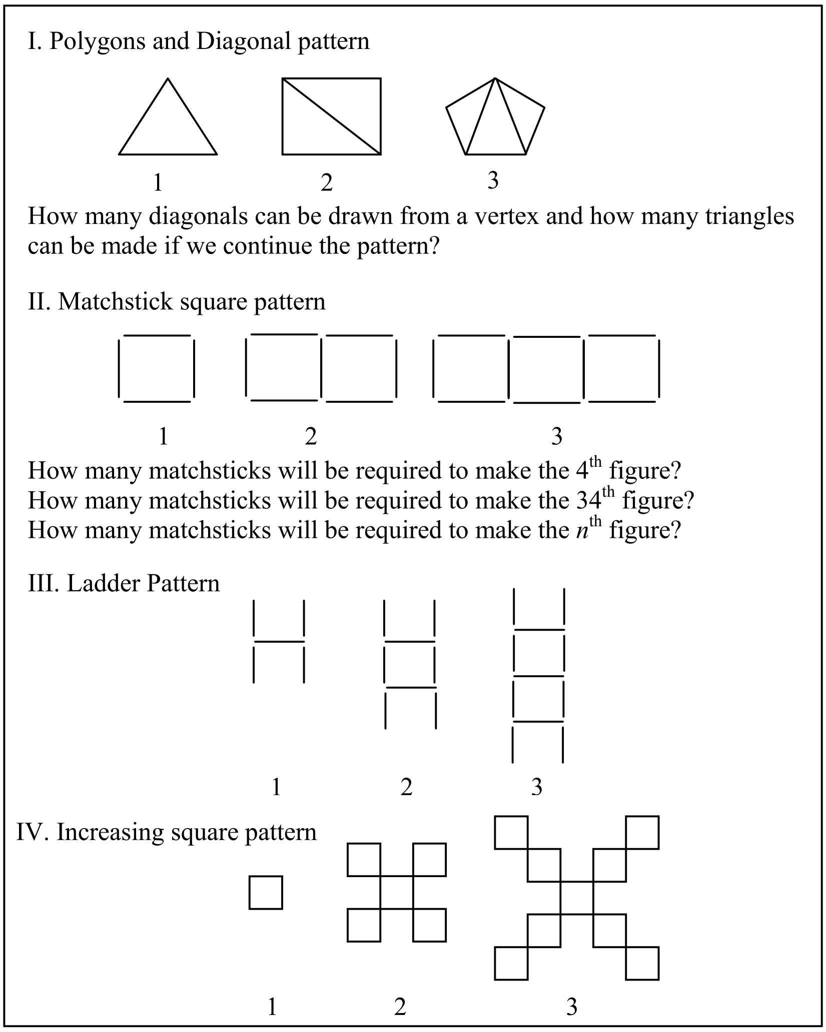


Figure 8.10: Pattern generalization tasks used in MST-III during classroom discussion

Post test data

The performance of the students in the post test in generating and representing a rule for patterns of shapes using expressions (a matchstick pattern of triangles, Figure 8.11) was also far better for the Marathi medium students compared to the

English medium students, both for specific numerical positions as well as for the general case (for example, predicting the number of matchsticks required to make triangles for the 4th, 5th, 17th and 59th positions and then write a rule for mth position) (Table 8.6). Most of the English medium students could not successfully write the generalized rule, even though some could correctly write the expressions for the specific positions. The emphasis on generating the rule and writing arithmetic expressions for specific positions perhaps distracted some of these students away from looking at the number pattern and extending it at least for the two immediately successive positions, which is a simple task.

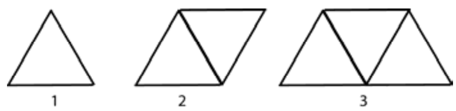


Figure 8.11: Matchstick pattern used in the post test of MST-III (see Q.2, Appendix IVB)

Positions	4th	5th	17th	59th	mth
English	47	53	33	33	13
Marathi	81	81	81	75	69

Table 8.6: Performance of students (in percentage) in the post test of MST-III in finding the number of matchsticks/ rule for various positions in the pattern ($n_{english}$=15, $n_{marathi}$=15)

The students' written responses were categorized by the strategies that they used to arrive at the number of matchsticks in each of the specific positions as well as the general rule. Table 8.7 reports the number of students following particular strategies as can be interpreted from their written response. It also shows the break-up of students into those who got all the responses to the specific numerical positions correct and the correct general rule. A strategy is called 'Expression for specific positions' (ExpnSP) when the students have written arithmetic expressions for the specific positions before converting that into an algebraic expression representing the rule for the pattern. The expressions display a pattern which can be abstracted by appropriate shift of attention, stressing and ignoring certain aspects of the expression. One has to separately treat the numbers in the expression which are unchanging and numbers which are connected to the index number, thus feeding into the general rule. The possibility of connecting the general rule or the arithmetic expression for the specific positions to the structure of the figure will be discussed later when the classroom discussion will be explored. Also, one finds expressions which begin as a recursive relation but are soon converted into an explicit functional rule. For example, one student wrote the expressions 2+7, 2+9 for the 4th and the 5th positions respectively but derived the rule 2+2×*m*-1 by subsequently connecting 7 to 4 by the relation 2×4-1, 9 to 5 by the relation 2×5-1. A few students wrote erroneous arithmetic expressions, which nevertheless contained a growing pattern different from the given one. Students were also found to use incorrect methods like writing expressions or values randomly for particular positions and/ or the rule, not maintaining a pattern across the positions (RandExpn/Val) and recursive relation (RCRSN), that is, adding 2 to the preceding output, both of which did not lead to any success.

Strategies		English	Marathi	Total
ExpnSP	No. of students	7	14	21
	All specific positions correct	5	11	16
	Correct Rule	2	12	14
RandExpn/Val	No. of students	5	2	7
RCRSN	No. of students	1	0	1
Not done		2	0	2

Table 8.7: Number of students following various strategies in the pattern generalization task in the post test of MST-III ($n_{english}$=15, $n_{marathi}$=16)

Note. ExpnSP = Expression for specific positions, RandExpn/Val = Random expressions or values for specific positions, RCRSN = Recursion strategy, Not done = instances of not attempting the task.

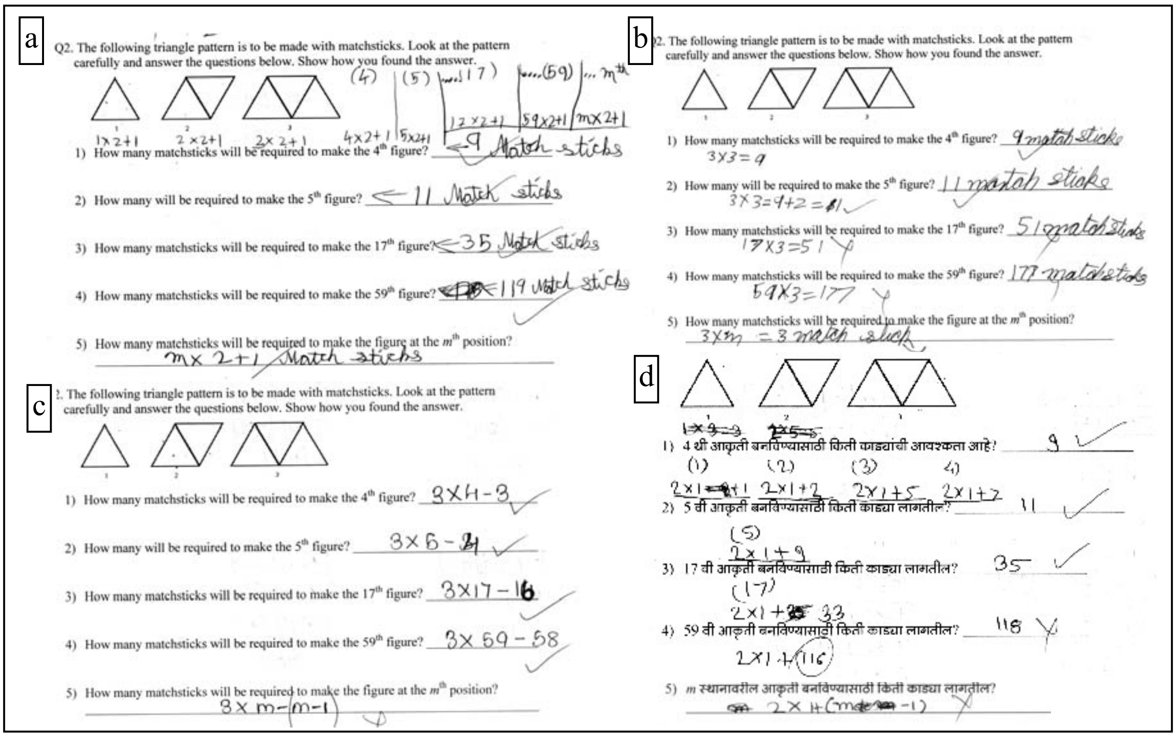


Figure 8.12: Sample of students' responses in the pattern generalizing task from the post test of MST-III

Figure 8.12 shows some varied responses of the students in the pattern generalization task in the post test of MST-III. Figure 8.12(a) shows a correct response to the pattern where the student systematically wrote arithmetic expressions and extended the pattern for the specific positions as well as wrote the general rule. This is in contrast to 8.12(b) where the student found the number of matchsticks for 4th and 5th position by recursion. He obtained the number of matchsticks for the later positions by simply multiplying the numerical position by 3, making the linearity error (since one figure needs three matchsticks, k figures would need k×3). He further showed a misunderstanding about the letter in interpreting 3×m as '3 matchsticks' for m^{th} position. The solution in Figure 8.12(c) is correct but for the missing bracket around (m-1), again by writing arithmetic expressions for each position before generalizing. The last solution in Figure 8.12(d) begins with the recursive relation which has been interestingly put as an arithmetic expression (2×1+1, 2×1+3, 2×1+5 etc, 2 being added to the result of the previous one). However, the student also abstracted the relation of the numbers 3, 5, 7, 9 with the index number for purposes of prediction to larger values and the general rule. But the student made errors for the 59th position and the general rule.

Classroom discussion

Even though the students did not perform very well in the post test, many of their written responses were interesting. They worked on various patterns, like Polygons and Diagonals, Matchstick square pattern, Ladder pattern and lastly, Increasing Squares pattern (see Figure 8.10). Students carefully wrote the arithmetic expressions for each position before generalizing them to the algebraic rule. Some of the arithmetic expressions could be easily connected to the structure of the pattern as they were created by counting the number of matchsticks/ dots/ squares by grouping them in some visually salient ways. Some others could not be similarly connected as they were generated by only focusing on the number pattern and not the growing shapes. A few others were initiated by focusing on the recursive relation, which could be easily converted into an explicit rule since the patterns were linear functions. Table 8.8 summarizes and gives a glimpse of the rules generated by the students in the classroom for the pattern generalization activity. They are categorized as rules which the students could connect to the given pattern of growing shapes and those they could not.

A majority of the rules generated in the classroom could not be physically connected to the pattern as they were quite complicated and challenging, made with only the number pattern in mind and required a group effort to find the generalized rule. Several researchers have suggested that the rule for generalization be kept close to the structure of the pattern and encourage strategies based on counting, the argument being that students should be able to make sense of the algebraic expressions they form in this context (e.g. Healy et al., 2001).

Rules / Pattern	Rules connected to the pattern	Rules not connected to the pattern
Polygon and Diagonal	Diagonals = k-3 Triangles = k-2	
Matchstick Square	4+(m-1)×3 3×m+1 2×m+(m+1)	4×m-(m-1) 4×(m+1)-(m+3) 5×m-(m×2-1) 3×(m+1)-2
Ladder	(n×3)+2 3×(n+1)-1 2×(n+1)+n	5×n-(n-1)×2 6×n-(n×2+n-2) 5×n-(n×2-2)
Increasing Squares	1+4×(k-1)	1, 2×2+1, 2×3+3, 2×4+5, …2×23+ (abandoned due to difficulty in generalization) 4×k-3 5×(k-1)-(k-2) 2×(k+k-1)-1 or 2×(2×k-1)-1 2×(k×2-2)+1 or 2×(k+k-2)+1 5×k-(k+3)

Table 8.8: Rules generated by students for the pattern generalization task in the classroom during MST-III

In contrast, students' classroom responses in this study indicate that it is also possible for students to meaningfully abstract patterns from numbers alone in the form of complex expressions and arrive at the functional rule. Most of them were able to perceive the relation between the position/ index number of the figure with the numbers in the arithmetic expressions to be able to symbolize them algebraically. They appreciated the need to make expressions in a pattern for each of the specific positions so that they can be easily generalized for the n^{th} position and also the predictive value of a general expression, that is, its use in determining the value for any position. In fact, a few students' attempts to make random expressions for specific values or equivalent expressions to the already generated ones were discouraged by their peers because such expressions could not be generalized and thus did not serve any purpose. Linearity error (i.e. if $f(n)=k$, then $f(mn)=mk$), while generating the rules in the classroom, was minimum (only one instance) due to carefully writing the arithmetic expressions for the particular numerical positions and checking if it satisfied the other given positions in the pattern before writing the general rule. Also, the specific positions for which the students had to predict the value were rarely multiples of each other (e.g. finding the values for $f(k)$ and $f(nk)$, see 'seductive numbers', Sasman et al., 1999 discussed in Chapter 2, section 2.7.1) and so less likely to induce linearity error.

Like in the previous task of think-of-a-number game where verbalization played a role in representation and understanding transformations on them, in the pattern generalization task verbal explanations paved the way for symbolization. To describe the rule for the n^{th} position or n squares in the Matchstick Square Pattern, a student gave the rule $4+n-1\times3$. The explanation given was 'n-1 is 61 (pointing to the 62^{nd} position) and like that n-1'. Another said 'whichever number minus 1 and in n it is minus 1'. The rule is not completely correct since the brackets are missing (which they went on to later discuss and rectify), but the basis of the expression was in the verbalization of the rule they had made.

To correct expressions like the above, both the groups seemed inclined to find syntactic based reasons focusing on equivalence of expressions without paying much attention to the meaning of the symbols. This could be a result of the influence of the instruction in tasks of 'reasoning *about* expressions'. While trying to understand the need for bracket around n-1 in the above rule, students gave reasons like '*3 is the common factor*' or '*we get the right answer*'(checked by substituting a value for the letter). This is inappropriate; and the correct reason was rightly pointed out later by one student: that the bracket around n-1 is needed to give precedence to the operation of subtraction before multiplication. In another instance for the same pattern, students wrote the arithmetic expressions 4, 4×2-1, 4×3-2, 4×4-3 for the first, second, third and fourth figures respectively from which the rules n×4-n-1 and n×4-(n+1) were generalized for the n^{th} position. The incorrectness of these rules was not recognized from the second expression (n×4-(n+1)) and its equivalence with the first, but from the fact that –n-1 leads to subtracting a number more than the figure number, where as one less than the figure number needed to be subtracted for the rule to work. This led to the corrected rule n×4-n+1 followed by n×4-(n-1) which is more transparent and maps the arithmetic expressions closely. This problem was faced repeatedly during the teaching sequence and especially with respect to the use of brackets where there was interference between bracket opening rules and the need for brackets.

Students were also expected to find if two or more rules generated for the same pattern were equal, in order to create the need to work on the representations they had made. Surprisingly, not many students spontaneously thought of substituting the letter by a number and checking the value and many were not sure whether the different rules for the same pattern are equal. On the other hand, they were aware that the rules ought to be equal as they were generated from the same pattern and involves the same input-output numbers. Most students chose the strategy of simplifying the expressions and bringing them to the simplest form, which is clearly a result of the extensive activities with respect to judging equivalence of expressions throughout the trials. For example, to show that the rules $3\times n+1$ and $4\times n-(n-1)$ are equivalent, students simplified the latter expression and showed it to be equal to $3\times n+1$. Those who chose the substitution strategy soon accepted the simplification strategy, as in the former method there was no way of arriving at the 'certain' knowledge that the rules are equivalent for all values of the letter. All the same, it is important for students to realize that even substitution of the letter by a number in the two rules/ expressions also should give the same value. The fact that some of them were interested in simplifying the expressions and checking if they could be brought to the same form is worth noting. But this did not imply an understanding that once it is shown that two expressions are equivalent using transformations, then they are equal for all values of the letter. For example, one of the students said that 'unless the value of the letter is known, it would not be possible to judge the equivalence of the expressions'. The understanding of the letter in an expression was still incomplete for such students.

The responses of the students noticed in the final trial of the study (MST-III) in these tasks were facilitated by students' ability to reason *about* expressions that helped them to separate the context from the expression and engage in discussions about the expression, appreciate other forms of the expressions and draw conclusions based on syntactic transformations. Most students did not have any difficulty in generating functional relationships (cf. English and Warren, 1998; Stacey, 1989; Sasman et al., 1999). However, the gap between syntactic based understanding of symbols and transformations on them and using them meaningfully in contexts needs to be bridged, that is, the students need to undergo another transition between 'reasoning *about* expressions' to 'reasoning *with* expressions'. With repeated discussions of the kind mentioned above, students started to appreciate at least the role of brackets while representing contexts.

Results from the interview data

A pattern generalizing task was included in the interview at the end of MST-III, which required the students to generalize a pattern as shown in Figure 8.13 and find the number of dots for various positions (5^{th}, 11^{th}, 58^{th} and k^{th}) (see Algebra test Q.4 and Interview schedule: Algebra (Task 5) Appendix VB).

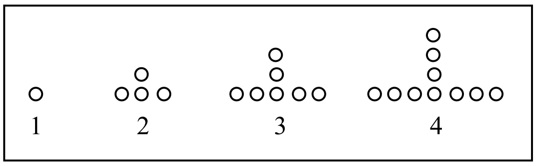


Figure 8.13: Pattern generalization task used in the interview after MST-III

Further, the students were shown another rule (algebraic expression) and asked if the rule was equivalent to the one they had generated. In the first part of the task which was about finding the number of dots for the specific positions and the general rule, many of the students needed help in obtaining a general formula for the n^{th} pattern. The interviewers directed the students' attention to ways of counting the dots and helped them to verbalize the rule before they could symbolize them. Table 8.9 gives a summary of the students' performance in the task. The students' responses are coded for their ability to write the value/ expression for specific positions and the general rule (Columns 1 to 4), the strategy used by them to arrive at these ('Strategy', Column 5) and their ability to show the equivalence or otherwise of two rules for the above pattern ('Equivalence', Column 6).

Name	1. 5th position	2. 11th position	3. 58th position	4. *k*th position	5. Strategy	6. Equivalence
BP	CH	CH	CH	CH	CNTG	SIMP
PD*	CH	CH	CH	CH	ExpnSP→CNTG	SIMP
BK*	C	C	C	C	CNTG	NE – SIMP
AY	C	C	C	CH	ExpnSP	NE – SIMP
NN	C	C	C	C	RCRSN/ExpnSP	NE – SIMP
SG	C	CH	CH	CH	ExpnSP	NE – FEqv
PG	C	C	C	C	ExpnSP	SIMP
JS*	C	CH	C	CH	RCRSN/ExpnSP	NE – SIMP
NW	C	C	C	IG	ExpnSP	SIMP – FEqv
RG	C	C	C	CH	ExpnSP	NE – SUBST – SIMP
AS	C	C	CH	CH	RCRSN→CNTG	NE – SUBST – FEqv
AN*	C	C	C	C	RCRSN/ExpnSP	SIMP
SV	C	C	C	C	ExpnSP	SIMP
MC*	C	CH	CH	CH	ExpnSP	NE – SUBST – SIMP
AB	CH	CH	CH	CH	RandVal→CNTG	NE – SUBST – NE – FEqv
BM	C	CH	CH	CH	RCRSN→CNTG	SIMP
TJ	CH	CH	CH	CH	ExpnSP→ExpnSP	SIMP

Table 8.9: Responses of the students interviewed after MST-III for the pattern generalization task (*Asterisk indicates students whose interviews are discussed in the text).

Prediction for positions (Col. 1 to 4):

(C) Correct – the subject wrote the correct value/ rule to find the number of dots

(CH) Corrected with help – the subject writes the correct value/ rule with help from the interviewer

(IG) Incorrect generalization – the subject incorrectly generalizes the rule for the k^{th} position

Strategy used for generalizing (Col. 5):

(CNTG) Counting – the subject generated the rule based on counting the dots by perceptually grouping them (e.g. k+k-1+k-1 or k+(k-1)×2)

(ExpnSP) Expressions for positions – the subject writes expressions for specific positions in a pattern (e.g. 1+3×(k-1), k×2+k-2, 4×(k-1)-(k-2), k×3-2) without connecting to the figure/ pattern

(RCRSN) Recursive adding – the subject arrives at the value by identifying the recursive relation between two consecutive positions

(RandVal) Random values – the subject writes random values for the positions and the general rule

(A→B) The subject is made to change his/her strategy from A to B during the interview

(A/B) The subject starts with a strategy A but himself/ herself shifts to B to complete the task

Equivalence of expressions/ rules (Col. 6):

(SIMP) Simplification – the subject chooses simplification as the strategy for showing two rules to be equivalent

(SUBST) Substitution – the subject chooses substitution as the strategy for showing two rules to be equivalent

(NE) Not equal – the subject states that two rules are not equal

(FEqv) Fails to show equivalence – the subject fails to show the equivalence of the rules

(A-B) The subject starts by the response A and then shifts to B

Most of the students (12) could find a meaningful strategy of counting the pattern or describing it verbally and were able to connect it with the figure/ index number but not all could symbolize correctly and completely. (These responses are those without the arrow in column 5.) Of the twelve students, three students could correctly find the number of dots for the specific numerical positions but could not write the general rule. Four others could verbally explain the pattern but could not express it as expressions, either arithmetic or algebraic. The remaining five were the ones who required no help from the interviewer and could complete the task on their own, both specific numerical positions and the k^{th} position. However, not many students on their own connected their rules to the spatial or counting patterns in the figures, except two students. The five students other than these twelve, had to be guided to look for efficient ways of counting which can lead to generalization. These five students started with one of the strategies like ExpnSP or RCRSN or RandVal and were directed to the CNTG or a more easily generalizable ExpnSP strategy (e.g. ExpnSP→CNTG), which could help them to complete the task. They failed to check the validity of their rules with respect to the pattern, a problem which was also seen in the post test and has already been pointed out in the literature (Healy et al., 2001).

For most of the students the way to check for equality of the two rules was through showing that the two expressions can be brought to the same form. Seven students straightaway went on to check for the simplification of the two expressions, while six others started by saying that the two expressions are not equivalent and later showed the equivalence of the two expressions by simplification, at times even using substitution of the letter by a number in the process (2 students). The remaining four students failed to show the equivalence of the expressions, even though two of them used substitution to check if the expressions were equivalent. Some of those who attempted to show the equivalence of the expressions using simplification could not do so without help, as they tended to compare surface features of the expressions or were not very sure of what was required to be done to show the expressions to be equivalent.

One student BK while comparing $3\times k$-2 with $k+k$-1+k-1 said that the expressions were not equal as '*in one of the expressions there is 3×k and in the other there is only k, -2 is same in both the places*'. While simplifying she made an error in writing $k+k+k$ as k. In such cases writing the terms and extracting the common factor helped the students see the equivalence of the expressions. Similarly, the student PD hesitantly simplified the two expressions (k-1)×2+k (her rule) and 3×(k-1)+1 to 3×k-2 but was never sure of what she was doing and the purpose of it and asked the interviewer for feedback. She had no trouble in the simplification procedure, although she was required to use distributive property here.

The student AN, on the other hand, was confident and knew what needed to be done to show the expressions to be equivalent. She had generated the expression k×3-2 for the given pattern. When asked if the expression 3×(k-1)+1 could be equal to her expression, she immediately chose to simplify the expression 3×(k-1)+1 and showed them to be equivalent: '*Both the numbers are to be multiplied by 3, then 3×k and 3×-1, therefore 3×-1 means -3 will be the answer here. And 3×k, -3 and +1, -2 will be the answer and +3×k*'.

The student JS found the rule $k+k+k$-2 for the pattern with some guidance. She adopted the method of simplifying the expressions to show they were equivalent but committed some calculation errors for which she again needed help. When asked to test the equivalence of the expressions $k+k+k$-2 (her rule) and 3×(k-1)+1, she simplified the expressions as 2×$k+k$-2 and 3×k-2 respectively. Comparing the surface structures of the expressions (2×k with 3×k and k-2 with -2, probably another interference from the letter-number line context) she concluded that they are not equal. She could not see the equivalence of $2 \times k+k$ with $3 \times k$ till she wrote the terms on the insistence of the interviewer and then immediately concluded them to be equivalent.

MC was the only student who successfully used both substitution by a number as well as the simplification process to verify the equivalence of his rule with $3 \times k$-2. He had found the rule $4 \times (k-1)-(k-2)$. He effortlessly showed the equivalence of the two expressions. The following is the excerpt from the interview on this task.

> SN: Now tell me you have got this rule for k [$4 \times (k-1)-(k-2)$], if someone else has got the answer $3 \times k$-2, then is that correct?
>
> *MC: It is correct.*
>
> SN: How do you know?
>
> *MC: I replaced k by one value.*
>
> SN: Then this and this are equal?
>
> *MC: Yes.*
>
> SN: How? Can you show me? Please try.
>
> *MC: The answer of this [*$4 \times (k-1)-(k-2)$*] expression is this [*$3 \times k-2$*].*

Although students needed help in the task, many were able to complete it with that support. The interviewer guided the students' attention to the structure of the expressions, prompting them to use the idea of terms to judge the equivalence of the expressions when they matched the surface features of the expressions concluding the expressions to be not equivalent. Many of these students had successfully explained the simplification of the algebraic expressions in the interview, but they hesitated in using the same strategies and principles in the context of justification. Repeatedly, this gap or divide between reasoning *about* expressions and reasoning *with* expressions could be seen.

The two contexts of think-of-a-number game and pattern generalization used during MST-III gave slightly different opportunities for engaging in algebraic thinking and using algebra as a tool to the students. In the context of think-of-a-number game, especially when the situation was simple involving only addition and subtraction, students could write algebraic expressions representing the situation and manipulate it due to the sequential operations in the task mirroring the structure of the expression. But, the issue that came to the fore in this context was the need for using algebra as a tool for proving and justifying and the belief in the efficacy of the method. Think-of-a-number game as a task allowed for algebraic thinking and reasoning but did not necessitate the use of algebra. The present task of pattern generalizing is closely tied with the idea of prediction. Verbalizing different ways of describing the pattern helps in arriving at the generalized rule and is an important step but is not sufficient to exploit the potential of the task. The generalized algebraic expressions are not only useful for prediction (in fact, a clearly articulated verbal rule would be sufficient for prediction) but also to appreciate the equivalence of the various rules for the same pattern, which requires manipulating the expressions. Thus, it also gets connected with reasoning *about* expressions. The task has the potential to create a context for solving linear equations as well.

Understanding of substitution

Pattern generalization task was also one context where students could display preliminary understanding of the very important idea in algebra of 'substitution'. This was amply demonstrated by the students in the task of generating equal expressions for a given expression when they rewrote a term as a sum, difference or product or a combination of two operations. In the pattern generalizing activity, the idea of substitution was required when they had to write the rules for not only the n^{th} position but also the $(n+1)^{th}$, $(n+2)^{th}$ positions which they accomplished with not much difficulty. For example, in the Square Pattern, a student remarked that the rule for the m^{th} position $4+(m-1)\times 3$ is same as $4+n\times 3$ (an indication of the letter as belonging to the set of alphabets, mistakenly concluding that $m-1=n$ instead of l). Another student responded by pointing out that $4+n\times3$ is the rule for the $(n+1)^{th}$ position and m-1 is not equal to n. She understood that in the generalized rule for the $(n+1)^{th}$ position, one less than the figure number needs to be multiplied by 3 (1^{st} - $4+0\times3$, 2^{nd} - $4+1\times3$, 3^{rd} – $4+2\times3$, $(n+1)^{th}$ – $4+n\times3$), and also that the choice of the letter is immaterial and that 'm' and 'n' have no quantitative relation as far as algebra is concerned. Except during the first exposure to the letter-number line context, similar misinterpretation of the letter was not generally observed. Henceforth, students wrote the rules for $(n+1)^{th}$ and $(n+2)^{th}$ positions with ease. They carefully analyzed the relations between the index number and the various components of the algebraic expressions describing the pattern to arrive at the rules for the $(n+1)^{th}$ or $(n+2)^{th}$ positions, but never by substituting $n+1$ or $n+2$ in place of n. This is not to say that the students did not make any error in the process but that they had schemes in place to complete the task as a group. Individual ability for substitution of this kind was not assessed during the trials. The two rules for the n^{th} position in the context of the Increasing Squares pattern were $4\times n-3$ and $5\times(n-1)-(n-2)$. The corresponding expressions for the $(n+1)^{th}$ position were found to be $4\times(n+1)-3$ and $5\times n-(n-1)$ respectively, the justification for the latter being 'one has been subtracted from the expression' in the rule for the n^{th} position (that is, n-1 is one less than n and n-2 is one less than n-1). A similar analysis of the relationships between the index number and the various components of the general rule led to the correct expression $5\times(n+1)-n$ for the $(n+2)^{th}$ position, subtracting one from $n+2$ and then one from $n+1$. The students naturally treated $n+1$, $n+2$ as numbers having relation with n, which is noteworthy. This could be as a result of their exposure to many of the earlier tasks, especially on the letter-number line.

These observations indicate that they have an implicit understanding of substitution which is not still formal and suggest that the idea of substitution appears to develop in stages of growing abstraction and difficulty. The students first learn to substitute a number for an expression (in evaluating an expression within brackets, for example) followed by an expression for a number (as seen while generating equal expressions). They then grasp the generality in the procedures embodied in an expression where it is possible to replace a number by another one without changing the essence of the procedure (this is the idea of the quasi-variable). Further, they learn to substitute a letter by a number to understand the connection between arithmetic and algebraic procedures and syntax. They also generalize patterns replacing the particular numbers by a letter to encode the general rule suitable for prediction. Finally, they learn to substitute a letter by an expression or vice-versa and an expression by another expression. The responses of the students in the study pointed to several examples of the earlier kinds of substitution but very few instances of the last kind were found. The students were not exposed to many situations where they would have required substitution of the last kind and probably they were not yet ready for it.

8.6 *Summarizing the contexts and students' understanding of algebra in contexts*

Contexts where algebra could be used as a tool were an important part of the teaching learning sequence. Situations, which lead naturally to the use of algebra giving meaning to the letter and the expressions, have been valued in research as they demonstrate the purpose of algebra and situate the rules and procedures meaningfully. The choice and design of tasks that make algebra meaningful to students is governed by many subtleties, like the nature of thinking required to solve the problem – algebraic or arithmetic, the extent to which algebraic symbols are required, whether the solution can be induced from specific cases or general solutions are needed, whether the general solution resembles actions on specific cases or it is different (Booth, 1989b; Radford, 1996; Mason, 1996; Kieran, 1989b; Arcavi, 1995, discussed in Chapter 2). In this study, several tasks were used to give the students a sense of using algebra for generalized representation as well as the need for developing a sound knowledge of syntactic transformations. Besides representing situations in a general manner, the tasks were also chosen to develop among students the ability to generalize, prove and justify. The purpose of choosing such tasks was to use their previously developed knowledge of concepts, rules and procedures and skills in reasoning *about* expressions in the context of syntactic based transformations and to move them towards reasoning *with* expressions. The tasks were gradually made more challenging with respect to the nature of representation required and the degree of manipulation that was needed to arrive at the conclusion. From the simple situations only requiring representation, to the letter-number line with a little more complexity and then to justifying and proving patterns in number arrangements in calendar and think-of-a-number game requiring not just representation but also manipulation, and finally the pattern generalization where symbolic representation was the key issue, students were exposed to many contexts in algebra.

The results indicate that many of the students could not use algebra for representing simple situations. As pointed out earlier, this could be attributed to their not seeing the purpose of the exercise. The symbolic representations they made often were not correct, and showed ignorance of the conventions of algebraic notations and misinterpretations of the letter. Subsequently, these tasks were dropped/ de-emphasized in the teaching. Students' performance in the subsequent tasks indicates that most of them, barring a few instances, correctly interpreted the letter as standing for a number. Through their experience in using symbols in different ways in the context of reasoning *about* expressions, they had enough exposure to making simple representations and also to unclosed expressions. They could choose/ identify an appropriate representation for different tasks such as the letter-number line, think-of-a-number game, understand the small changes made on them, even when a few of them failed to generate one themselves. An exception to this was the task of finding the distance between two points on the letter-number line. Some students did not make progress in this task, probably because they had recourse to simpler ways for finding the distance. Also, only some students were convinced about the need for an algebraic representation and this was evident from their interviews after MST-III, especially in the think-of-a-number game. The students were able to use algebraic thinking on numbers and conclude correctly, without necessarily using algebraic symbols. Since many of the problems chosen for the students were not very complex, this might have had a role to play in the students not choosing the reasoning based on symbolic algebra. Most of these tasks required general solutions but were close to the solutions of particular cases, thus making it possible to engage in algebraic thinking using verbal means without recourse to algebraic symbols.

It was only in the pattern generalizing task that symbolic representation was integral to the exercise, first in writing a rule for the n^{th} position and second in comparing the rules for equivalence. Although the students were more successful in it while working in the classroom, they did not achieve enough competence to work independently. Most of the time they were able to find meaningful ways of arriving at the generalized rule, to see functional relationships connecting the index number to the output but had trouble in writing the algebraic expression representing the rule while working alone. A few did not check the correspondence of their rules and arithmetic expressions with the given pattern. Throughout the study, students were engaged in some form of generalization and in a sense were already initiated into a 'culture of generalization' (Lee, 1996). They generalized the various constraints and possibilities on transformations and used them for identifying and generating equal expressions. They also generalized the rules of transformation learnt in the context of arithmetic expressions to manipulating algebraic expressions as well as evaluating more complex arithmetic expressions. Even while using arithmetic expressions to describe the pattern, many students were able to see the general in the particular by separating the numbers that were changing from the non-changing ones and relating these to the figure/ index numbers for the generalized rule. Students in many studies (see Chapter 2, section 2.7.1) have been found to be unable to understand the task requirements, are able to verbalize the patterns they see in a figure but cannot symbolize it, their focus being on recursive relations and not functional ones which can be generalized (English and Warren, 1998; Lee, 1996; Stacey, 1989; Sasman et al., 1999). However, the students in this study did not face many of these problems probably because of their prior experience with working on expressions and reasoning about syntactic aspects of expressions. An important point to be noted is that the algebraic expressions which describe the pattern are not derived from mere empirical induction but involve deliberate acts on the part of the students to focus their attention on various parts of the pattern, capture the essence in the form of an arithmetic expression for each position and from it abstract the algebraic expression, by ignoring and emphasizing certain parts of the arithmetic expression. Both the contexts of think-of-a-number game and pattern generalization help develop the ability to see the general in the particular and also the general solutions are close to solutions for particular instances.

Further, many students found it difficult to manipulate the algebraic expressions in the context even when they showed sufficient grasp of the rules and procedures and the similarity in arithmetic and algebraic structure (both surface and systemic). This issue is not simply a matter of transfer from the syntactic world to contexts but also deeper issues of warrant or need. To be able to successfully manipulate a symbolic expression in the context, students needed anticipatory skills and visualization of the goal, as other researchers have pointed out (Arcavi, 1994; Boero et al., 2001). Only when this was in place, could they also appreciate better the purpose of manipulating the expressions. Students' failure in the task of finding the distance between two points on the letter-number line and the calendar task can be attributed to their not being able to make sense of the expectations from the task, either finding it not worthy of such a complicated method or being too complex to handle with their limited capacities and understanding. This is also seen in the initial resistance to use algebra for purposes of justification in the think-of-a-number game and instead relying on verbal descriptions/ explanations in the classroom, and then a gradual shift towards symbolic expressions with increase in complexity of the problem. Their interview responses in the think-of-a-number game further elaborated this observation and gave evidence of the extent of their understanding of such manipulation and its interpretation. In the context of the pattern generalization task, students appreciated the need to simplify the algebraic expressions to show their equivalence as well as displayed a preliminary understanding of substitution. These tasks led to fruitful discussions in the classroom with respect to syntax and the semantics of the situation but did not always result in successful completion of the task. In these discussions, their prior knowledge obtained through reasoning *about* expressions was frequently drawn upon, with regard to rules and procedures of transforming expressions and understanding of equality of expressions.

From the limited experience of this study, it can be said that developing ideas of proof and generalization among students requires time and exposure and has to be developed like any other concept or skill, in the course of which students pass through stages. Stage 1: Verbalization and articulation of one's understanding of processes and relationships play a key role in the process of reasoning *with* expressions and can be considered to be the first step. Stage 2: This can be followed by the ‘quasi-variable’ understanding derived from the use of arithmetic expressions in a way that the truth for all possible cases can be revealed. This will require a shift in attention from the specific case represented to the general situation (‘delicate shift of attention from the specific to the general features, appropriately stressing and ignoring features’, Mason (1996)). Stage 3: The stage when students engage with manipulating expressions to predict or convincingly prove and justify the result to be true for all possible cases that they enter the domain of reasoning *with* expressions. The choice of tasks is very crucial in enabling students to move from one stage to the next.

Chapter 9: Discussion and Conclusion

9.0 *Background*

In the teaching approach that was adopted, algebra was approached through the generalized arithmetic route, that is, with an understanding that algebra encodes the general properties and rules of arithmetic, and then complemented with contexts where algebra was used as a tool for generalizing, proving and justifying. The study aimed to develop a teaching-learning sequence which could help the students move from arithmetic to algebra smoothly by capitalizing on their intuitive understanding of numbers and operations on them. The teaching sequence exploited the structure inherent in the arithmetic expressions and tried to build a sequence which strengthened both procedural and structural knowledge of arithmetic and algebraic expressions among students of grade 6.

The teaching intervention, which was a design experiment with multiple trials and multiple groups spread over two years (2003-2005), had two interconnected goals: to develop the teaching learning sequence and to observe and characterize the changes in the students' understanding as they interacted with the instructional material and attempted to move from arithmetic towards algebra. It enabled students to shift from computational understanding of expressions to understanding based on properties of operations, with a focus on structure of expressions and relationships between terms in an expression, so that one could operate with the expressions, rather than operate on the numbers. The approach identified a set of concepts, namely terms and equality, which could explicate the structure of expressions and subsume the rules, procedures and conventions of operating on expressions. These concepts could in the first place be used for reasoning about syntactic transformations of expressions. The understanding that the students developed could then, in turn, support reasoning with the expressions, applying the knowledge of symbols and their manipulation developed till now to solve problems. The teaching intervention also led to the operationalization of some principles of teaching like capitalizing on students' intuitions and articulation/ verbalization capabilities to situate the new learning as well as to take them forward to developing new knowledge.

The two concepts of 'terms' and 'equality' thus supported the transition from arithmetic to algebra and acted as 'bridge concepts'. The identification of these concepts and their elaboration with the teaching sequence was a significant part of the teaching trials. The bridge concepts allowed the students to build on their previous knowledge and intuitions/ expectations with respect to operations and generalize the properties and rules of working with them to the new algebraic symbols. These two concepts also gave the students visual support to perceive the structure of expressions, especially correct unitizing/ parsing of expressions and understanding the crucial idea of equality, and a language for communicating their understanding. The approach afforded a meaning for the symbols in algebra by creating the number as a referent for the letter. Further, it was important that this learning be used in rich contexts where algebra is a tool and that students must explore the ways the algebraic symbolic representation and transformation can be applied. In this way not only would the students grasp the purpose of algebra but they will also be using the letter for representing entities in the problem world, in the process objectifying it.

In the chapters 6, 7 and 8, students' performance in the various tasks was discussed in order to understand their grasp and use of the different concepts, rules and procedures taught during the study. The analysis was carried out on the pre and the post test data, individual interviews, as well as supported by evidences from classroom discussions, daily worksheets of the students and teachers' logs. Chapter 6 elaborated students' understanding of procedures and rules with respect to arithmetic and algebraic expressions and the connection which students made between the two domains. Chapter 7 explicated students' sense of structure of the expressions and their understanding of the important idea of equality based on the perception of structure. Chapter 8 looked into students' understanding of different aspects of algebra such as the meaning of the letter or the expression, representation and manipulation of expressions as a way to arrive at conclusions or substitution, while working on contexts of generalization and justification. In this chapter the research questions which this thesis attempted to address will be revisited and the findings of the previous chapters will be summarized and interpreted in order to throw light on the questions.

9.1 *Answering the first question: Arithmetic necessary for transition to symbolic algebra*

The first set of research questions that the thesis addressed are the following:

- What kind of arithmetic understanding would help in learning symbolic algebra?
    - How should the teaching of arithmetic expressions be restructured to prepare for a transformational capability with algebraic expressions?
    - How effective is such a teaching learning sequence in understanding beginning syntactic algebra?
    - Which tasks of the ones identified are more effective in making the shift possible from arithmetic to symbolic algebra?

9.1.1 Previous research on the nature of arithmetic leading to algebra

The range of studies described in the review of literature point out enormous difficulties which the students face in making sense of conventions and notations of algebra and symbolic expressions. An underlying reason is that in algebra symbols stand for both process and product in contrast to computational arithmetic where these two can be separated. Lack of structure sense of arithmetic expressions, poor understanding of properties of operations and possessing only operational/ computational knowledge of arithmetic rather than representational and relational understanding were found to be the root causes of the trouble (e.g. Chaiklin and Lesgold, 1984; Booth, 1984, 1988; Kieran, 1989a, 1992; Linchevski and Livneh, 1999, discussed in Chapter 2, sections 2.2.2, 2.2.3, 2.3.4). Further, the reasoning styles in the two domains are different which require a change in the attitude of the students, which involves looking for relations, generalizing them and expressing the generality using symbols and manipulating the generalities to arrive at new conclusions. Ideally, the instruction in arithmetic in the primary school years should give students adequate ground to abstract general rules and properties which should initiate them into the domain of algebra. It is precisely because of this reason that algebra follows arithmetic in many traditional curricula. However, the transition to algebra does not happen unless one explicitly focuses on arithmetic in the primary grades that is aimed at moving away from rigid algorithmic procedures. Thus one does not see any connection between arithmetic and algebra in students' thinking or understanding when they reach the middle school.

Some of the efforts which have been made to give meaning to the symbols have utilized and exploited the connection between arithmetic and algebra. Despite the opposition and reservation of some researchers (e.g. Lee and Wheeler, 1989; Balacheff, 2001) to such an approach, there exists empirical and theoretical support for the arithmetic-algebra connection (Chapter 2, sections 2.3 and 2.4). Algebraic expressions and symbols get the referent and the meaning and a way for validating the results when situated or generalized from the arithmetic context (Linchevski and Livneh, 1999). Kirshner (2001) suggested that learning algebra is not about learning rules but about perception based patterns. He argued that a structural approach to teaching algebra, using symbols and rules rationally and reasoning with them would be better than a referential approach to algebra. The studies by Liebenberg et al. (1998, 1999a, b), Malara and Iaderosa (1999), Linchevski and Livneh (1999, 2002) and Livneh and Linchevski (2003, 2007) (see discussion in Chapter 2, section 2.6.2) also indicated the influence of students' arithmetic learning on algebra learning: errors in arithmetic leading to similar errors in algebra or knowledge of 'better' arithmetic, paying attention to the structure of expressions, leading to improved performance in algebra. Some of these studies further pointed out that it is the transformational arithmetic rather than computational arithmetic which will be of more help in making the transition to algebra as students fail to abstract the procedures and structure unless they engage in a reflective, "meta-cognitive" (Malara and Iaderosa, 1999) learning of arithmetic. This study takes the above as important lessons and begins to make the ground for an approach to teaching algebra building on arithmetic, rather than making it redundant.

9.1.2 Restructuring arithmetic teaching and its effectiveness in learning beginning syntactic algebra

In the present study arithmetic teaching was restructured so that it can be connected with symbolic algebra. The study did not start with a fully developed sequence for teaching but a sequence evolved over the trials. The decisions for changing some parts of the teaching learning sequence were based on the students' performance in the classroom, especially as revealed in the discussion sessions, from their practice exercises and from the written tests at the end of each trial. As has been discussed earlier in Chapter 5, it was the '*radicalized*' structural treatment of arithmetic, with a deeper understanding of expressions and constraints and possibilities of transforming them that enabled the transition to algebra.

First and foremost, the students were moved away from their habit of computation to look at relations expressed in an expression by verbalizing the meaning of the expression using phrases ‘more than’, ‘less than’, ‘sum’, ‘difference’, ‘product’, ‘times’. Further, the teaching and learning of arithmetic was refocused away from precedence rules and learning of procedures to a deeper analysis of the structure of expressions. The concept of ‘term’ was identified which, through visual cues, allowed correct parsing of expressions. The precedence rules or sequential order of evaluation of expressions were replaced by structural rules, replacing the vocabulary of addition and subtraction by the vocabulary of combining positive and negative terms (which implicitly meant no subtraction operation on terms), thereby making it possible to combine terms in any order. The structure of the expression, made explicit by marking the terms of the expression, was the only factor which allowed or constrained transformations on them. Brackets were an important part of the teaching-learning sequence due to the need for multiple interpretations (static precedence operation and dynamic bracket opening rules) that are required while working in algebra. The bracket opening rules were also reformulated using the concepts of term and equality. The ideas of ‘inverse’ and ‘multiple’ were found to be useful to understand the bracket opening rules. This approach further made it possible for students to judge equality/ inequality of expressions, and compare pairs of expressions by analyzing the terms and anticipate the effect of changing the terms on the value of the expression. This ensured that the structure of the expression will complement the procedures on the expression. By simultaneously carrying out activities of both these kinds, the approach was successful in combining non-transformational tasks, like anticipating results and judging equality with computational activities, like evaluating/ simplifying expressions.

Findings from the analysis: Procedural tasks and rules

Evaluation of arithmetic expressions

The students improved in their overall performance in the procedural and structural tasks and understanding of rules. Students learnt to parse expressions with ease using the idea of terms and use it in many situations. The results after MST-I showed the prevalence of errors in perceiving the structure of expressions as well as inconsistency in applying similar kinds of judgments across the tasks. However, as the trials progressed there was reduction in structural errors (due to faulty parsing, like 'LR' and detachment) while evaluating simple arithmetic expressions but they did resurface in more complex situations, suggesting the lack of automaticity among students in the simpler contexts. They gained flexibility in evaluating the simple (e.g. $3+4\times5$ or $13-5+7$) and the more complex expressions (e.g. $-28+49+8+20-49$ or $7\times18-6\times11+4\times18$) finding easy ways of computing them, indicating their appreciation of the structure of the expressions and the ability to take advantage of it. Over the trials, more number of students chose 'relational strategies' to evaluate the expressions (more in the case of expressions with only simple terms than with product terms), switching from the more tedious left to right evaluations or solving the product terms which also led to more errors. Integer addition/ subtraction was a weak point resulting in low performance in some items, even when this subsumed under the 'terms approach' to evaluating expressions. This subsumption substantially improved students' ability to work on the various tasks flexibly, but errors related to integer operation continued to appear.

The interviews and the classroom discussions further substantiated the findings from the written test and indicated the students' ability to avoid structural errors in the simple situations. They showed that students were aware of uniqueness of the value of the expression even though one could use multiple ways of evaluating them. Most students comfortably explained the evaluation process of simple expressions as well as correctly accepted or rejected an alternative solution posed to them using appropriate structural concepts by the end of the third trial, although they hesitated in some instances at the end of MST-II. The approach enabled students to focus attention on different parts of the expression simultaneously, which has been found to be difficult in the studies reported in the research literature (cf. Liebenberg et al., 1999a; Malara and Iaderosa, 1999). Most students are not aware of the rules of transformation or are swayed by perceptual patterns and do not consistently apply them to evaluate expressions (see discussion in Chapter 2, section 2.3.4). It was in these contexts of evaluating expressions of various kinds that the students explicitly discussed the constraints and the possibilities in transformation of expressions. This was supported by the visual cues given by marking the terms in the boxes as well as the support given by a specific language through the naming convention like simple terms, product terms, bracket term, negative term, positive term, etc. The flexibility in manipulating arithmetic expressions together with correct perception of structure of expressions paved the way for the manipulation of algebraic expressions.

Simplification of algebraic expressions

By the last trial, most students were comfortable with simplifying algebraic expressions (e.g. $3\times x+4+4\times x-5$), applying the same rules as in arithmetic. Interviews with the students with respect to algebraic expressions after MST-III revealed their awareness of equivalence of all the steps in the process of simplification. For example, the expressions $3\times x+4+4\times x-5$ and $7\times x-1$ are equivalent and so are the steps in between. Although most students were able to evaluate algebraic expressions for a given value of the letter even when they made sign and calculation errors; a few students, however, made structural errors coupled with non-substitution of the letter by a number till the last trial. The students interviewed did not show any such difficulty. The unfamiliarity with the task could be a reason for some students not to understand the requirement of the task.

The appreciation of the similarity between manipulating arithmetic and algebraic expressions was difficult and developed only in subsequent trials when attempts were made to focus away from computation in the context of arithmetic and pay attention to the relational information conveyed by an expression. Consistency in perceiving the structure of expressions and understanding the properties of operations that are used in the context of arithmetic is an important step to move to algebra. The coherence in the teaching-learning sequence which was developed by MST-II (discussed in Chapter 5, section 5.2.2) could be a factor influencing the change as is seen by the end of the last trial. The students successfully generalized their understanding of rules from the context of evaluation of arithmetic expressions to simplification of algebraic expressions, displaying the connection between the two domains in their understanding. The operations on numbers needed to be converted into 'objects', which could be processed mentally without implementing a physical calculation procedure at each step. These could then be combined using the properties of operations.

Evaluating expressions with bracket and bracket opening rules

Evaluation of expressions with brackets, which had an important place in the teaching approach, did not have similar performance levels and it was harder to deal with. The teaching approach laid emphasis on using the brackets as a precedence operation as well as treating it as a structural symbol which can be removed using rules keeping the value of the expression the same. The second conception of bracket is important for manipulating algebraic expressions where it may not be possible to solve the sub-expression embedded in the bracket, while the first one is needed for representation purposes. Although they were quite successful in solving simple expressions with brackets (e.g. $3\times(4+5)$), their performance in the more complex expressions with brackets (e.g. $25-(4+3\times5)$) was not error free. Many of them learnt to use bracket opening rules to evaluate expressions but for some students, this was accompanied by a lack of appreciation of the meaning of the bracket as enclosing parts which have to be given precedence in operation. Also, while evaluating expressions in brackets the structural errors resurfaced (like 'LR' and detachment). Both the written test and interviews revealed that the two notions of bracket were absorbed by some of the students as procedures and not as 'pro-cepts' which did not allow them to easily anticipate the effect of removing and putting the brackets. They failed to simultaneously understand that the bracketed (sub-)expression could be substituted by either a number or another equal expression, which is an indication of an evolved 'proceptual' understanding. Students made more errors when the bracket was preceded by a negative sign rather than the multiplication sign.

In a later development in the teaching approach, the rules for evaluating brackets were reformulated in structural terms using the ideas of 'inverse' of an expression (for negative bracketed term) or 'multiple' of an expression, which operationalize the distributive property. The teaching approach took a while to identify a correct way to deal with the brackets. The initial strategies of embedding bracket opening rules in contexts did not work well due to the presence of many distracters and the difficulty in recalling and applying the rules in a syntactic situation. Inducing the rule from multiple examples also was not very successful as the rules were often wrongly interpreted and over-generalized. The shift to an emphasis on equality in value of the two expressions – with and without brackets, together with analysis of the terms paved the way to a deeper understanding of brackets and bracket opening rules. Some of the ways which were found to be more effective in understanding the different roles of bracket are: (i) students' perception of patterns consolidated through the concepts of inverse value and inverse expression, and multiple of an expression. For example, $13+4-5=12 \Rightarrow -13-4+5=-12$, where -12 is identified as the inverse value of 12 and -13-4+5 is the inverse expression of 13+4-5, symbolized as $-(13+4-5)$. Similarly, $4+5=9 \Rightarrow 8+10=18$ where 18 is twice 9 and 8+10 is the expression (4+5) multiplied by 2 symbolized as $2\times(4+5)$; (ii) embedding them in the context of generating equal expressions where the brackets are used to create new expressions (an expression equal to 32-

24+7+23 is 32-(24-7)+23 or 32-(20+4)+7+23). Discussions of the latter type of examples in the classroom revealed the over-generalizations of rules by students and confusion between the meaning and purpose of brackets, but also led to fruitful and rich explorations. Expressions with brackets also require an adequate structural treatment similar to expressions without brackets and engagement with reasoning about them in the context of syntactic transformations to make sense of the dual interpretation of the brackets.

Structural understanding of expressions and equality

The structural tasks probed students' understanding of the important concept of equality which connects arithmetic and algebra. These tasks revealed their deeper understanding of expressions and a fair degree of understanding of constraints and possibilities of transformations, properties of operations and anticipation of the result of those operations. They understood that terms can be rearranged to keep the value same or they can be changed in ways that the net result does not change, rearranging the signs or numbers changes the value, a positive term increases the value of the expression and a negative term decreases it. Research literature discussed in Chapter 2, both exploratory and classroom interventions, indicate the difficulty students in general have in understanding these ideas. In particular, classroom discussions of how a given expression could be transformed while keeping its value invariant led to significant revelations about students' understanding. Further, these tasks served as better diagnostic and learning tools with respect to the understanding of equality than the more traditional task of filling in the blank.

Students sometimes made errors in equalizing expressions by filling the blank (e.g. 23+4=__-3), displaying the misconceptions which are already reported in the literature. This task required computation and low performance in tasks could be due to the automaticity which these students have with respect to the perception of the '=' sign. However, they could judge the equality/ inequality of a list of arithmetic and algebraic expressions with respect to a given expression without computation and also generate expressions equal to a given one, focusing on the relationships between the terms and the transformations that were applied to it. These tasks require conscious effort on the part of the student to first identify the equality/ equivalence and then to communicate the reason for the judgment.

Interviews after MST-II and III also revealed students' abilities to identify equal expressions from a list and to compare unequal expressions identifying the smaller/ greater expression in a pair. This was accomplished, not through a mechanical short-cut procedure but rather through a meaningful use of the concept of 'terms'. Comparison of such complex expressions was unfamiliar to them and their flexible use of terms to anticipate the change in value which made one expression greater or smaller than the other was an important finding in the interview. They performed reasonably well in the written test in both arithmetic and algebraic expressions, although their performance in expressions with product terms was slightly lower than other expressions, where a few of them consistently failed to use the correct parsing/ unitization to identify the equal expression. This was however not seen when they generated the equal expressions themselves. A few students also faced difficulty in judging equality of expressions when it involved brackets, a problem which was noticed in the evaluation tasks as well. The responses were not error free but the overall nature of responses, as revealed through the different sources of data, indicate that they had strategies in place to deal with these tasks and to rectify their errors and they were clear about equality in value as an essential criterion for two expressions to be equal. Further, most of them were aware that equivalent algebraic expressions (e.g. $3\times x+4+4\times x-5$ and $-5+4\times x+4+3\times x$) will be equal for all values of the letter. Two ways of justifying it were seen: by replacing the letter by a number in both the expressions to arrive at two arithmetic expressions which they know would have equal values or directly inferring that particular cases would hold true since the general case is true.

Overall, students made quantitative and qualitative (in terms of strategies and reasoning) progress in the various tasks over the trials. By the end of the trials many students were able to connect the domains of arithmetic and algebra and move back and forth between them. Moreover, the students' performance in arithmetic and the algebra tasks were correlated and over the trials, with increasing performance in arithmetic, students' performance in algebra also became better. But students needed a basic minimum performance in arithmetic and consistency in their understanding of procedures and structure to be able to work on algebra tasks. The third sub-question dealing with the efficacy of the tasks will be taken up after discussing the performance of the students in the contexts that were used to embed algebra.

9.2 *Answering the second question: Relation between syntactic algebra and understanding purpose of algebra*

- Does understanding the syntax and symbols of algebra support students in understanding the purpose of algebra and in the application of algebra for generalizing and justifying?

The discussion in the previous section dealt with students' reasoning *about* expressions and revealed their understanding of syntactic aspects of arithmetic and algebra, namely manipulation of expressions and the idea of equality/ equivalence. Most of the students, by the end of the trials, showed sufficiently mature understanding of the above aspects of expressions as revealed by their interview responses, their responses in the written test and classroom discussions. The second part of the study dealt with using the same ideas in contexts where expressions could be used to derive conclusions about the situation. In the process, students were challenged to use the concepts, the meanings they had constructed of the symbols in the first part of the study and their knowledge of techniques of manipulating expressions.

Discussions in Chapter 2 revealed the difficulties faced by the students while working on tasks which embed algebra and use it as a tool to solve problems. The performance of the students in this study indicates that their knowledge of reasoning *about* expressions based on syntactic transformations influenced their understanding and approaches to solution while reasoning *with* expressions. Their earlier experience created a predisposition for symbolic representations and thinking with an expression. However, fewer students could convert this understanding to one which could enable them to successfully complete the tasks of reasoning *with* expressions or appreciate the 'purpose of algebra'. The issue is not simply one of transferring the abilities from the syntactic world to the context situations where algebra is to be used as a tool or of creating a situation so that the letter gets embedded in the context and thus creating meaning for the letter or algebra. Two elements that play an important role in these tasks are (i) the culture of generalizing, proving and verifying, with which the students in traditional curricula have very little experience and which needs to be developed and (ii) students' belief about the effectiveness of using algebra in these tasks.

The effect of the above two factors was seen in the students' performance across the trials, especially in their abilities to represent the situation using an algebraic expression and manipulation of the representation. The letter-number line journey and think-of-a-number game were not found to be difficult to represent as they involved a sequential (and a rather arithmetical) step-by-step representation of operations. But in the former context, students did not see the need to manipulate, the result being very obvious from the figure; and in the latter, many did not even see the need to make an algebraic representation when arithmetic expressions and narrative arguments were sufficient to explain the result. More complex versions of the 'think-of-a-number' game did lead to the need for symbolic representation to keep track of the transformations on the initial number. On the other hand, representation for the 'distance' task on the letter-number line was quite difficult for the students as it required describing a relation between the two points and did not necessarily follow the order in which the points appeared in the question. Pattern generalization task was the only one where the representation, although not simple, was found to be necessary to predict the values for larger positions and to write the general rule which satisfied the pattern. The 'calendar' task, which included both the aspects of making generalized representations and justifying/ proving, was a challenging task for the students and they were not adequately prepared for it. The students barely managed to represent the relations between the numbers in the rows and columns and did not succeed much in identifying and proving patterns.

Initial efforts of many students to manipulate the algebraic expressions representing the situations were random, many a time followed by an arbitrary answer, largely due to non-appreciation of the goal of the task. Some others, who were aware of the goal of the task, wrote the expected answer after the end of a random manipulation. Their limited abilities in the initial trials to simplify algebraic expressions contributed to the lack of success in these contexts. Students need to accept manipulation of algebraic expressions to be a valid way of arriving at the conclusion together with an anticipation of the goal of the task toward which the representation can be maneuvered.

In the last trial, with a change in the approach to deal with this issue which encouraged verbalization of the explanations for the answers, some students were seen to engage in algebraic thinking and use narrative arguments, often displaying a quasi-variable approach, to convince others about the generality of a result or to draw conclusions. One must note however, that this did not necessarily require algebraic representation. A few successfully used algebraic representations, could anticipate the goal and accordingly manipulate it to prove the result. Still, a few continued to repeatedly verify the conjecture/ proposition for specific instances, not realizing the limitation of the approach.

This pattern of responses led to the understanding that students' abilities to manipulate algebraic expressions and their knowledge of transformation rules is put into use only after they understand the purpose of the task, the need for algebraic representation and can anticipate the goal. Otherwise, the manipulation of algebraic expressions in the contexts is random or the use of algebra is completely ignored. Possessing the syntactic knowledge of algebraic expressions predisposes students to think in terms of expressions within the contexts but does not guarantee success. Thus, besides the 'push' from arithmetic which lays the ground for initial understanding of algebraic symbols and expressions and reasoning about expressions ('phase of structural development'), one needs the 'pull' from a culture of generalization and the need for general justifications, not restricted to specific instances, to move to the 'autonomous stage'. Eventually it should also lead to an anticipation of the most efficient choice of the representation itself (developing 'symbol sense') which will be crucial for problem solving of this nature (Arcavi, 1994; Boero et al., 2001).

9.3 *Tasks that enable the transfer from arithmetic to symbolic algebra*

- Which tasks of the ones identified are more effective in making the shift possible from arithmetic to symbolic algebra?

Suggestions from the research literature and analysis of the data collected in this study (sections 9.1.1 and 9.1.2) pointed out that the tasks which enable the transition from arithmetic to symbolic algebra are those which move away from computation and engage in reflection about properties of operations. Thus, evaluation tasks based on a thorough analysis of expressions and simultaneously building an understanding of rules of transformation and the conditions under which the rules are applicable are essential to be able to use them consistently in the context of simplification of algebraic expressions.

Further, the above should be complemented by tasks where students do not compute but only judge/ identify equalities and equivalences between expressions and substantiate their responses. Generating equal expressions for given expression is another task which combines both procedural knowledge as well as structural knowledge without the necessity of computing. These tasks tap the important concept of equality and include knowledge of rules and procedures of transforming expressions, and judge students' understanding of possibilities and constraints of transformations. However, a more direct and explicit understanding of the '='sign needs to be developed prior to this for bridging arithmetic and algebra. Engaging students in fill in the blank so that expressions are equal on both sides of the '=' sign as well as comparing small expressions using <, =, > with and without computation are important bridging activities from purely procedural to a structural/ relational understanding of expressions. This study showed that although students displayed a sound understanding of equality in the more structural tasks without computation, some students could not fill the blank correctly with computation; their automatic tendency to write the answer in the blank overpowering their understanding of equality. In fact, the fill in the blank task is very arithmetic in nature as the blank is filled by trial-and-error and does not require any relational understanding.

Students need to be supported by adequate conceptual ideas like 'term', '=' sign, equality, inverse, identity (not used in this study) and symbols like the bracket showing its dual purpose, which will allow the students to communicate their responses and explain them, that is allow them to reason *about* expressions. The study explored various ways of approaching evaluation/ simplification of expressions as well as bracket opening rules. The connection between arithmetic and algebra could be established by first understanding the meaning of the expressions ($3+x$ is a number which is three more than x and $3\times x$ is a number which is three times x) as distinct from the value it stands for, and then explicitly showing structural similarity between the two domains and hence similarity in the rules of transformation. Also, de-emphasizing computation together with an emphasis on reasoning *about* expressions (that is, reasoning based on syntactic transformations), communicating verbally and/ or symbolically the reasons for the choice of the responses created a background to accept symbols as not being arbitrary and not only used for computations but also for representing situations/ change. It further allowed them to accept unclosed expressions, which was helpful while dealing with symbolic algebra.

Efforts were made to situate the bracket opening rules in the story situations and in the context of finding area of rectangle (inspired by CSMS study, 1984) leading to two equivalent but perceptually different representations but were not successful. Students' conception of area and perimeter of figures was very poor and could not support the development of new concepts and symbols. In the story situations, the stories seemed to distract their attention and did not lead to the identification of the essential features of the bracket opening rule. Further, rules were over-generalized from these contexts, indicating insufficient attention paid to the structure of the expressions. Some useful ways of dealing with brackets have already been pointed out in a previous section (section 9.1.2), namely, connecting the bracket opening rules to the concept of equality and analysis of terms and patterns in expressions. An effort was made to give meaning to simple algebraic expressions, like $x+4$, in contexts requiring representation (two rods each of length 4 cm and x cm respectively are joined to form a rod of length ___ cm) but tasks of this sort were not successful with many students. The limitation of such tasks was that the purpose of representing situations with one of the dimensions marked by a letter was not clear to the students and probably made no sense to them. There was no larger goal which such a representation could be used for.

As against these situations, contexts were created where algebra could be used as a tool and students could engage in reasoning *with* expressions. These contexts required students to make algebraic representations and then manipulate them to arrive at a conclusion. The letter-number line was the simplest context of this kind and the least challenging. The purpose of this task was to simultaneously capture the process – the act of operating with the letter, adding and subtracting 1, 2, 3 etc. to the letter, as well as to show each expression (like x-1) to be the result of the process. This also reinforced the idea of lack of closure in algebraic expressions. The context allowed the creation of a few more tasks which not only required representation using an algebraic expression but also necessitated manipulations on it. The letter-number line journey (which was a sequence of actions) and finding the distance between two points on the letter-number line, were two tasks relevant in this context. As pointed out earlier, students found representing the 'journey' simple and the 'distance' very hard. The first 'journey' task is more arithmetic in nature due to the non-sequential representation followed by manipulation which does not require working with or on the unknown. The second 'distance' task, in contrast, requires a complex representation with brackets and it is necessary to work on/ with the letter. Even though the task had the potential to give rise to cognitive conflicts due to the existence of two ways in which one could find the solution (by counting the number of jumps between two points and by representing and simplifying), algebra was hardly used in this situation as there were other easy means of arriving at the answer. The purpose of using algebra need to be clear to the students, for it to be used in the first place.

Tasks like the think-of-a-number game are promising as they involve simple representation and that representation can be meaningfully used to draw inferences about the situation. However, students would need to understand the meaning of verification/ proof to be able to successfully complete the task. In the response to this task, the effect of the students' belief of the efficacy of the algebraic solution was again seen. But it led to fruitful discussions both about representation, explanation of solutions and manipulation of the algebraic expressions as a way to reach the conclusion. The difficulty with this task is one of choosing between a simple problem situation for writing the expression and engaging in algebraic thinking, and a more complex situation challenging narrative arguments making it necessary to use a symbolic representation. In the latter case it might be hard to begin the discussion due to lack of transparency in the situation.

Explorations of number patterns in calendar was found to be very challenging as it included multiple requirements – making general representation of the arrangement of the numbers in the calendar followed by representing patterns among numbers and finally proving them to be true for all similar arrangements. This task needs to build on ideas of generalization, representation and proving. Pattern generalization from growing shapes was another context used, which was fairly able to capitalize on students' capabilities to generalize and write algebraic expressions for the general rule. It was closer to students' experience in reasoning *about* expressions where they had already developed sufficient skills to shift their attention from the particular to the general, carefully identifying and preserving the invariances and generalizing the constantly changing numbers. Students engaged in verbalizing functional rules, either by using a counting strategy leading to a rule or by writing expressions in pattern for specific positions and generalizing it to an algebraic expression. Showing the equivalence of two rules for a single pattern was also not an entirely new idea for them due to their exposure to tasks requiring them to identify equivalent algebraic expressions. This was also a good ground to begin discussion about substitution, although still at the informal level.

9.4 *Answering the third question: Meaning of letter, expression, syntactic rules of transformation*

- 'What meanings do students attach to letters, expressions and syntactic rules of transformations in this learning approach?' was the third question which the thesis aimed to answer.

The emphasis in the teaching approach was on seeing an expression in flexible ways: as a statement expressing relationship and a value. One of the major hurdles in making sense of algebraic symbolism is to understand the meaning of the letter and the duality of the various symbols. Most students understood the letter to be a number; barring a few instances where a couple of students labeled the letter as one in the sequence of alphabets or misinterpreted 'x' to be equal to '0', drawing parallels from the symmetry around '0' in the integer number line. On the other hand, they also showed the understanding that one letter can be replaced by another letter while representing, making no difference to the conclusion drawn.

Expressions were also flexibly understood as expressing relations as well as encoding a sequence of operations leading to a unique value (in the case of an arithmetic expression) and multiple values (in the case of an algebraic expression) depending on the value substituted for the letter. The approach was successful in eliminating the spontaneous tendency of students to compute the value of the expressions. Students could verbalize the meaning of simple expressions like 5+4 or x-3 (four more than five or three less than x) as well as see a statement like x-3+5=x+2 as expressing a relation between x-3 and x+2 (x-3 is five less than x+2) and the fact that subtracting three and adding five to x leads to x+2. They could extend this understanding developed in the context of the letter number line to situations where they had to fill in the blank to make two expressions equal. The discussion quoted in Chapter 7 (section 7.1.2) while students were engaged in the task: 45+29=47+28__, shows students' capacity to think of the blank as not only a number which would lead to the same answer as the left hand side but also see the relationship between the left and right hand of the '=' sign: the expression on the right hand side is one more than the left hand side, and therefore the conclusion that one needs to subtract one from the right hand side. The pattern generalization task also showed students' ability to treat n+1, n+2 as entities for which rules could be extended and written as for 'n'.

Similarly, students knew that equal expressions would have equal values, as only valid transformations are used which completely compensate for any change in a term or a part of the expression. In the process, they developed a sense of reversibility and substitution – an expression or part of the expression could be replaced by a number or conversely, a number could be replaced by an expression. On the other hand, expressions where terms had been changed compared to the given one in such a way that they were no longer equal, were judged unequal and were further estimated to be more or less than the original expression, without computation.

This flexibility in thinking about expressions is often reported to be lacking among students and many of the research studies discussed in Chapter 2 (esp. section 2.3.4) indicated students' poor sense of structure of expressions (e.g. Chaiklin and Lesgold, 1984; Kieran, 1989a, 1992; Wagner and Parker, 1999; Hoch and Dreyfus, 2004), which leads them to believe the rules of transformation to be arbitrary, both in arithmetic as well as algebra. It is probably due to the flexible approach adopted in this teaching sequence for simplifying/ evaluating expressions (rather than precedence rules) that the structure could support their procedural understanding and the possibility arose of using the same rules across the tasks.

9.5 *Answering the fourth question: The procedure-structure connection*

The fourth and the final question asked in the thesis was:

- How do procedural understanding and structure sense of expressions mutually support one another?

Students' scores in procedural tasks and structural tasks are highly positively correlated. But one needs a minimum competence in procedures to internalize and abstract those properties for perceiving structure and further use it in tasks. By the last trial, students made appreciable progress in both procedure and structure tasks. A preliminary analysis of certain questions/ items based on students' perception of structure showed the nature of the interrelationship between procedural understanding and structural understanding to be complex and the necessity of consistently applying the procedures and rules in different situations as a pre-requisite for developing structure sense (Banerjee and Subramaniam, 2005).

Students' understanding as revealed from their written responses as well as their articulations during classroom discussions and interviews while reasoning *about* expressions in the context of syntactic transformations throws light on the connection. In the first trial of the main study, students did not appreciate the connection between the use of terms to check for the precedence rule to be applied and its use for checking the equality of expressions. The rigidity of the precedence rules and lack of clear indication of the purpose of identifying terms led some students not to use terms and in the process they made structural errors like 'LR' and 'detachment'. In the next two trials, one finds the use of a variety of strategies, flexible combination of terms based on the structure of the expressions while evaluating them, gradually also reducing the structural errors. The same complementary use of procedural knowledge and structure sense is also found while substantiating the correctness or otherwise of a procedure for evaluating an expression. An example of such a response is seen while students responded to the probe of whether the expression 25-10+5 could be solved as 25-15: '*it is not bracket. If there is bracket we can do like this*' or '-10+5=-5'. Response for the opposite case of whether 25-(10+5) could be solved as 25-10+5, was also similarly argued; that there is a minus sign outside the bracket, so the signs of both terms must change. The boundaries between procedure and structure begin to become fuzzy in such responses. In the case of simplifying the algebraic expression also, students' understanding of the steps of simplification and their equivalence came from an understanding of the possibilities and constraints of transformations (e.g. '*Because this is a product term and we do not know what the number 'a' is. So we have to do it like this only*' (response of a student while explaining why 5×*a*+6-2×*a*+9 will be equal to 27 when 3×*a*+15 is 27 for *a*=4), which includes an appreciation of both surface and systemic structure. Most of them understood the generality of the whole process, whether one used a common letter variable in the product term or a common numerical factor among two terms (e.g. '*these are equal expressions*'). These students had successfully 'interiorized' (Sfard, 1991) the process and could mentally run through the steps without the need to see the result of each step.

Similarly, in the context of tasks which were predominantly structural, students used both their understanding of structure and operation sense to answer the questions. Students' judgment about an expression being more or less than another required them to correctly parse/ unitize the expression as well as use their anticipations of performing an operation. For example, while comparing the expressions 24-13+18×6 and 24-18+13×6, many students correctly pointed out that the first expression is greater than the second and one of them explained that '*Here it is 18×6, and there 13×6. It has become less and here [24-13+18×6] also -13 is there, and -18 is here [24-18+13×6]. Only here subtraction, the product is more*'. It is the successful combination of correct parsing together with correct anticipation which led to such judgments. An unsuccessful combination of these two can be seen where a student judges the expressions 49-37+23 and 49-5-37+5+23 to be unequal and the former expression to be smaller '*Here +49 is correct and here +, -5 and -37 is there. Therefore these two will get added and the sum is +42 and the sum of these two [+5+23] is +28 and the answer of this [49-42+28], and this [49-37+23]* *expression is a little smaller*'. For many others this was a straightforward situation of equality as -5+5=0. Many of the discussions in the classroom while generating equal expressions for a given expression showed similar complementary use of structure and procedure sense. Also what is worth noting is the comfortable and meaningful use of symbols while reasoning *about* expressions. The use of the symbols in these ways enabled the extension of meaning for the standard symbols in the context of arithmetic and their subsequent use in the domain of algebra. These evidences are promising enough to believe in the complementary nature of procedures and structure and not in the dichotomy between them.

9.6 *Conclusion*

The present study dealt with the issue of transition from arithmetic to algebra, which built on students' understanding of arithmetic and operations. It elaborated on the specific supports, in the form of vocabulary (like specific ways of naming the terms), concepts (terms and equality), rules and procedures (reformulated in structural terms) required for making the transition from arithmetic to algebra, without which it was difficult for students to see the connection between arithmetic and algebra. Further, it pointed out the purpose, strengths and the limitations of the various tasks used at different points of the study. The thesis proposes a teaching guideline on the basis of the trials for making a smoother transition from arithmetic to algebra (see Appendix VI).

The approach which was adopted and evolved during the study has the potential to substantially bridge the gap between arithmetic and algebra. The specific features of the approach which facilitate this connection are:

(i) building on students' understanding of arithmetic operations and intuitions

(ii) moving away from computation and emphasizing structure of the expressions

(iii) fostering an understanding of expressions in terms of information it contains, relationship embedded in it and the value it stands for

(iv) identifying concepts of terms and equality, which are structural and can help in consistently understanding rules of transformation of expressions

(v) reformulating the procedures of evaluating arithmetic expressions in structural terms and using the same rules, terminology, notations and conventions in transforming expressions in arithmetic and algebra

(vi) deepening the understanding of structure of expressions by focusing on invariance of value of expressions, thereby elaborating the understanding of equality and equivalence of expressions

(vii) choosing tasks so that procedures get connected with structure sense

(viii) explicit attention to the number as a referent for the letter

(ix) emphasizing the process-product duality or flexible 'proceptual' understanding through tasks

(x) developing the ability to communicate and reason with symbols

These are important aspects of the arithmetic-algebra transition and have been points of concern in many of the exploratory studies elaborated in the review-of-literature chapter of this thesis.

The approach succeeded in many ways in dealing with the syntactic and the semantic aspects of arithmetic and algebraic expressions. It highlighted the importance of linking procedures with structure in the attempt to connect arithmetic and algebra. Meaning for algebra and algebraic symbols was created through syntactic based reasoning and transformations of expressions (reasoning *about* expressions) and also through an exposure to tasks demonstrating the purpose of algebra (reasoning *with* expressions). Working on these two kinds of tasks probably leads to different kinds of understanding among students, both of which are valuable and are complementary and it is not possible to prioritize one over the other. Thus tasks for algebra have to be carefully chosen so that both kinds of understanding develop among students.

Although students' understanding of rules of transformations and operation sense was visible in the context of syntactic transformations and reasoning *about* expressions, it was not fully used while reasoning *with* expressions. Students could display algebraic thinking by the end of the last trial and convincingly explain their solutions but the transfer to the symbolic mode was not easy, even when they could understand the process of the representation and manipulation to draw conclusions. The unsatisfactory development of the teaching approach with regard to this aspect of algebra, largely guided by the assumption that knowledge of algebraic symbols and manipulation would directly lead to their use in contexts, was probably responsible for many of the effects seen in students' responses, besides their own beliefs about the utility of algebra in these situations. Symbolic proofs/ justifications need to be preceded by developing understanding of the need for algebra and engaging students in verbalizing the process of solution, a point which was realized only in the last trial. It is hypothesized that reasoning *about* expressions may help in reasoning *with* expressions by enabling the students to think in terms of expressions.

At the middle school level, when mathematics begins to become more formal and starts to consolidate the learning that has happened in the primary school years, symbols play an important role in communicating this understanding and leading to further learning. They have to be skillfully developed, together with an associated vocabulary which would allow students to express themselves. Thus, there is a need to make ‘reasoning’ an important feature of the classroom, engaging students in articulating their expectations/ anticipations, strengthening the developing symbol sense and operation sense in the process. This study was only an example of achieving the above.

The study took a single perspective of algebra, as generalized arithmetic. In the process, other approaches to algebra were not explored sufficiently. Further, the study, through a detailed analysis of students’ responses and the teaching intervention, tried to explore and show the potential of the approach in making the teaching and learning of the two domains - arithmetic and algebra, more coherent and connected. It was not designed to experimentally establish the efficacy of this approach with respect to the traditional or any other approach. However, one of the limitations of the study was that the data from later refined teaching trials was confounded by more teaching, thus making it sometimes difficult to ascertain the improvement in students’ performance due to better teaching. The analysis of the data also became difficult when some questions in the tests were frequently changed, resulting in the loss of comparison between two tests on certain items. Interviews at more frequent intervals could have helped in clarifying the nature of learning and understanding students achieved after each teaching trial. These are some points which need to be kept in mind while extending this study. Another direction in which the study can be extended is to include problem solving by framing and solving equations within the scope of the approach and also include rational numbers in the arithmetic expressions and as referents for the letter. Another challenge is to evolve the approach to incorporate non-linear algebraic expressions, multiple variables in expressions and operations on linear and non-linear expressions.

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Appendix – I: List of tasks used in the study

Domain of tasks	Nature of tasks	Example of tasks
Reasoning about expressions	Understanding what an expression is Writing expressions for numbers and verbal sentences and vice versa Identifying terms of an expression	e.g. Writing an expression for ‘a number which is two more than five’, ‘three less than any number’ Verbally describe 18-6, x-5 Identify terms of 13-8+17, 3×6-9+5×4×2, 4-2×x+6
	Evaluating and simplifying expressions	e.g. Evaluate expressions like 7+2×6, 43-8+12 Simplify expressions like 2×x-6+5×x+9
	Tasks based on the ‘=’ sign, with and without computation, requiring explanation of the answer.	e.g. 13+8☐ 21-1 (Put <, =, > in the box) 15-8 = ___ +3 345+487 = 346+488__ 234-148 = 235__
	Comparing expressions without calculation and explaining the responses Judging equality of expressions: arithmetic and algebraic, only simple terms or simple and product terms Generating equal expressions for a given expression using any transformation	e.g. Compare expressions like the following 24+53☐25+52, 65-36☐63 – 37 Identify expressions equal to 23+17×15+12 from the ones given below: 1) 17+23×15+12 2) 23+17×12+15 3) 15×17+23+12 4) 23+3×(17×5+4)
Reasoning with expressions	Letter number line Representing relations between numbers and quantities using a	e.g. Letter-number line journeys, distance between two points on the letter-number line

	letter Calendar patterns Think of a number game Pattern generalization from shapes	Identifying patterns in the arrangement of numbers in calendar, representing the using the letter and proving them Justifying the pattern in the answers of the students with respect to the starting number Finding the generalized rule for describing the pattern, showing if two different rules for the same pattern are equivalent

Appendix – IIA: Pre test (MST-I)

Homi Bhabha Centre for Science Education
TIFR, Mumbai
Summer Camp-2004 (12^{th} April to 1^{st} May 2004)

Name: ____________________ School: ____________________

Q1. Write the expressions corresponding to the following sentences.

A) A number which is two more than five ____________________

B) A number which is three less than seven ____________________

C) A number which is three more than a number k ____________________

D) A number which is seven less than a number n ____________________

Q2. Mark $\boxed{\checkmark}$ against the option which tells you the correct order of evaluating the expression $12 - 2 \times 3 + 4$. Otherwise mark $\boxed{\times}$.

A) (1) $12 - 2 = 10$ (2) $10 \times 3 = 30$ (3) $30 + 4 = 34$

B) (1) $12 - 2 = 10$ (2) $3 + 4 = 7$ (3) $10 \times 7 = 70$

C) (1) $2 \times 3 = 6$ (2) $12 - 6 = 6$ (3) $6 + 4 = 10$

D) (1) $2 \times 3 = 6$ (2) $6 + 4 = 10$ (3) $12 - 10 = 2$

Q3. Evaluate the following expressions.

A) $7 + 3 \times 4$

B) $4 \times (6 + 2)$

C) $15 - 2 + 8$

Q4. $23 - (6 + 7)$ is equal to

A) $23 - 6 + 7$

B) $23 - 6 - 7$

C) $23 + 6 - 7$

D) $23 + 6 + 7$

Q5. Fill in the blanks.

A) $25 - 6 =$ ______ $+ 8$

B) $15 + 3 = 20 -$ ______

C) ______ $+ 4 = 5 \times 2$

D) $6 \times$ ______ $= 16 - 4$

Q6. Comapre the expressions on both sides of the box and write '=', '<' or '>' in the box.

A) $13 + 9$ $\square$ $22 - 1$ B) $18 - 6$ $\square$ 4×3

C) $15 \div 3$ $\square$ $5 + 3$ D) 6×4 $\square$ $24 \div 4$

Q7. Find without calculation which of the following expressions are equal to the expression $323 + 418 - 259$? Mark $\boxed{\checkmark}$ against the options which are equal to the above expression and mark $\boxed{\times}$ if it is not equal. There may be more than one correct answer.

A) $323 + 259 - 418$ $\square$ B) $418 + 323 - 259$ $\square$

C) $259 + 418 - 323$ $\square$ D) $323 - 259 + 418$ $\square$

Q8. Find without calculation which of the following expressions are equal to the expression $23 + 45 \times 16 + 29$? Mark $\boxed{\checkmark}$ against the options which are equal to the above expression and mark $\boxed{\times}$ if it is not equal. There may be more than one correct answer.

A) $23 + 29 + 45 \times 16$ $\square$ B) $23 + 45 \times 29 + 16$ $\square$

C) $45 + 23 \times 16 + 29$ $\square$ D) $45 \times 16 + 23 + 29$ $\square$

Q9. Without calculation, compare the expressions on both sides of the box using '=', '<' or '>' in the box.

A) $28 + 69$ $\square$ $27 + 69$ B) $43 + 89$ $\square$ $41 + 91$

C) $52 - 28$ $\square$ $52 - 27$ D) $71 - 48$ $\square$ $72 - 49$

Q10. Write + or – in the box and write the correct number in the blank so that the expressions on both sides of the "=" are equal.

$32 + 19 = 32 + 17$ $\square$ _____

Q11. If $237 + 489 = 726$, then without calculation, find the value of $239 + 489$

Q12. A strip of paper which is 3 cm long is joined to a strip of paper which is t cm long. What is the total length of the strip?

Q13. Manisha is 10 cm taller than Vikram. If Vikram's height is x cm, what is the height of Manisha?

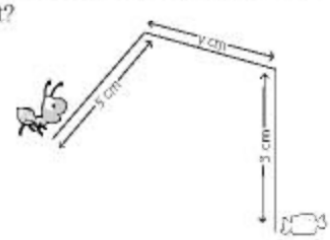
Q14. If $n + 29 = 67$, then $n + 28 = ?$

Q15. 6 cars and some buses are parked on a road side. There are in all 11 vehicles. Which of the following equations represent the above situation? Mark ☑ against the options which are correct, otherwise mark ☒. There may be more than one correct answer.

A) $6c + xb = 11$ ☐ B) $6 + xb = 11$ ☐

C) $6 + b = 11$ ☐ D) $6 + x = 11$ ☐

Q16. An ant goes along the path shown in the figure to get a piece of sweet. What is the total distance travelled by the ant to reach the sweet?



Q17. Find the value of x in each of the following.

A) $x + 6 = 11$ B) $x - 9 = 4$

$x =$ ______ $x =$ ______

C) $5 \times x = 30$ D) $\frac{x}{4} = 7$

$x =$ ______ $x =$ ______

D) $2 \times x + 3 = 15$

$x =$ ______

Q18. Find the value of $5 + 2 \times x$ when $x = 3$.

Q19. Look at the series of numbers

(1) (2) (3) (4) (5) ....

3 5 7 9 11 ....

What is the 40^{th} number in the series? Mark ✓ against the option which is correct, otherwise mark ✗.

A) 100 ☐ B) 90 ☐

C) 81 ☐ D) 83 ☐

Q20. Think of a number. Add three to it. Multiply the result by 2. The number you will get can be written as

A) $n + 3 \times 2$ ☐ B) $(n + 3) \times 2$ ☐

C) $n + 3$ ☐ D) 3×2 ☐

Appendix – IIB: Post test (MST-I)

Homi Bhabha Centre for Science Education
TIFR, Mumbai
Summer Camp-2004 (12^{th} April to 1^{st} May 2004)

Name: ______________________ School: ______________________

Q1. Write the expressions corresponding to the following sentences.

A) A number which is three more than eight ______________________

B) A number which is five less than twelve ______________________

C) A number which is four more than a number m ______________________

D) A number which is two less than a number x ______________________

Q2. Mark $\boxed{\checkmark}$ against the option which tells you the correct order of evaluating the expression $23 - 3\times2 + 5$. Otherwise mark $\boxed{\times}$

A) (1) $23 - 3 = 20$ (2) $20 \times 2 = 40$ (3) $40 + 5 = 45$ ☐

B) (1) $23 - 3 = 20$ (2) $2 + 5 = 7$ (3) $20 \times 7 = 140$ ☐

C) (1) $3 \times 2 = 6$ (2) $23 - 6 = 17$ (3) $17 + 5 = 22$ ☐

D) (1) $3 \times 2 = 6$ (2) $6 + 5 = 11$ (3) $23 - 11 = 12$ ☐

Q3. Evaluate the following expressions.

A) $8 + 2 \times 7$

B) $5 \times (9 + 3)$

C) $19 - 3 + 6$

Q4. $15 - (8 + 4)$ is equal to

A) $15 - 8 + 4$ ☐

B) $15 - 8 - 4$ ☐

C) $15 + 8 - 4$ ☐

D) $15 + 8 + 4$ ☐

Q5. Fill in the blanks.

A) $27 - 3 =$ ______ $+ 10$

B) $16 + 5 = 25 -$ ______

C) ______ $+ 3 = 3 \times 5$

D) $2 \times$ ______ $= 19 - 1$

1

Q6. Compare the expressions on both sides of the box and write '=', '<' or '>' in the box.

A) 15 + 8 ☐ 23 – 1 B) 21 – 3 ☐ 6 × 3

C) 24 ÷ 4 ☐ 6 + 2 D) 4 × 7 ☐ 28 ÷ 2

Q7. Find without calculation which of the following expressions are equal to the expression 127 + 284 – 195? Mark $\boxed{\checkmark}$ against the options which are equal to the above expression and mark $\boxed{\times}$ if it is not equal. There may be more than one correct answer.

A) 127 + 195 – 284 ☐ B) 284 + 127 – 195 ☐

C) 195 + 284 – 127 ☐ D) 127 – 195 + 284 ☐

Q8. Find without calculation which of the following expressions are equal to the expression 27 + 17 × 32 + 14? Mark $\boxed{\checkmark}$ against the options which are equal to the above expression and mark $\boxed{\times}$ if it is not equal. There may be more than one correct answer.

A) 27 + 14 + 17 × 32 ☐ B) 27 + 17 × 14 + 32 ☐

C) 17 + 27 × 32 +14 ☐ D) 17 × 32 + 27 + 14 ☐

Q9. Without calculation, compare the expressions on both sides of the box using '=', '<' or '>' in the box. Write reasons for your answer.

A) 37 + 58 ☐ 36 + 58 ____________________

____

____

B) 54 + 67 ☐ 52 + 69 ____________________

____

____

C) 64 – 37 ☐ 64 – 36 ____________________

____

____

D) 85 – 38 ☐ 86 – 39 ____________________

____

____

Q10. Write + or – in the box and write the correct number in the blank so that the expressions on both sides of the "=" are equal.

35 + 29 = 35 + 27 ☐ ____

Q11. A strip of paper which is 5 cm long is joined to a strip of paper which is p cm long. What is the total length of the strip?

Q12. Rani is 19 cm taller than Vivek. If Vivek's height is n cm, what is the height of Rani?

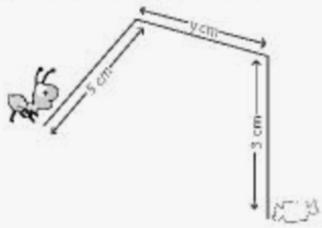
Q13. If $y + 35 = 72$, then $y + 34 = ?$ Give reasons for your answer.

Q14. 6 cars and some buses are parked on a road side. There are in all 11 vehicles. Which of the following equations represent the above situation? Mark $\boxed{\checkmark}$ against the options which are correct, otherwise mark $\boxed{\times}$. There may be more than one correct answer.

A) $6c + xb = 11$ B) $6 + xb = 11$

C) $6 + b = 11$ D) $6 + x = 11$

Q15. An ant goes along the path shown in the figure to get a piece of sweet. What is the total distance travelled by the ant to reach the sweet?



Q16. Find the value of x in each of the following.

A) $x + 5 = 13$

$x =$ ______

B) $x - 7 = 11$

$x =$ ______

C) $4 \times x = 32$

$x =$ ______

D) $\frac{x}{4} = 7$

$x =$ ______

E) $2 \times x + 5 = 19$

$x =$ ______

Q17. Find the value of $7 + 3 \times x$ when $x = 2$.

Q18. Look at the series of numbers

(1)	(2)	(3)	(4)	(5) ....
3	5	7	9	11 ....

What is the 40[th] number in the series? Mark $\boxed{\checkmark}$ against the option which is correct, otherwise mark $\boxed{\times}$.

A) 100 $\square$ B) 90 $\square$

C) 81 $\square$ D) 83 $\square$

Q19. Think of a number. Add four to it. Multiply the result by 3. The number you will get can be written as

A) $n + 4 \times 3$ $\square$ B) $(n + 4) \times 3$ $\square$

C) $n + 4$ $\square$ D) 4×3 $\square$

Q20. Identify the terms in the following expressions.

A) $2 + 3 \times 4$ B) $19 - 6 + 7$

C) $3 \times x - 4 + 6 \times x + 10$

Q21. Simplify

A) $5 \times x + 16 + 7 \times x - 11$ B) $13 - 2 \times x - 9 + 6 \times x$

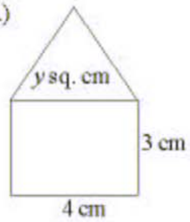
Q22. Which of the following expression is equal to the expression $13 \times x - 9 - 3 \times x + 15$. Mark $\boxed{\checkmark}$ against the option which is correct, otherwise mark $\boxed{\times}$.

A) $15 + 13 \times x - 9 - 3 \times x$ $\square$ B) $4 \times x - 3 \times x + 15$ $\square$

C) $13 \times x + 3 \times x - 9 + 15$ $\square$ D) $15 - 9 - 3 \times x + 13 \times x$ $\square$

Q23. Find the difference between $y - 3$ and $y + 4$

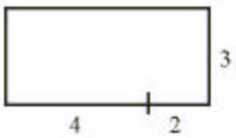
Q24. (A)



Area of triangle = y sq. cm

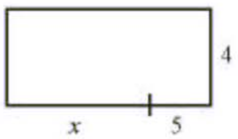
Total area of the figure =

(B)



Area of the rectangle =

(C)



Area of the rectangle =

Q25. Rewrite the expressions by removing the bracket so that they are equal.

A) $25 - (9 + 3) =$

B) $2 \times (6 + 5) =$

C) $(15 + 4) - 7 =$

D) $17 - (8 - 5) =$

E) $7 + (12 - 9) =$

F) $4 \times (9 - 3) =$

Q26. Think of a number. Add 5 to it. Subtract 2 from it. Subtract the original nu
do you get ? Show that everyone will get the same answer as you get.

Q27. If $326 + 598 = 924$, then $324 + 598 = ?$ Give reasons for your answer.

Q28. Mark the points $p - 3,\ p - 1,\ p + 1,\ p + 4$ on the number line and fill in the blank.

$p - 1$ _____ $= p + 3$ $\qquad\qquad$ $p + 2$ _____ $= p - 3$

Q29. Write $<$, $>$ in the boxes.

A) -15 $\square$ -12 $\qquad$ B) -105 $\square$ 11

C) 51 $\square$ -52 $\qquad$ D) -19 $\square$ -81

Q30. Evaluate the expressions so that it is easy to calculate.

A) $29 - 7 + 11 + 7$ $\qquad$ B) $87 - 19 + 13$

C) $14 \times 3 + 10 \times 8 + 14 \times 7$

Appendix – IIIA: Pre test (MST-II)

(1st November to 20th November, 2004)

Homi Bhabha Centre for Science Education
TIFR, Mumbai

Name: ______________________ School: ______________________

Q1. Write the expressions corresponding to the following sentences.

A) A number which is two more than six ______________________

B) A number which is seven less than fourteen ______________________

C) A number which is three more than a number n ______________________

D) A number which is four less than a number y ______________________

Q2. Each option from A to D shows a different sequence in solving the expression $23 - 3\times2 + 5$. Mark $\boxed{\checkmark}$ against the option which shows the correct sequence. Otherwise mark $\boxed{\times}$.

A) (1) $23 - 3 = 20$ (2) $20 \times 2 = 40$ (3) $40 + 5 = 45$

B) (1) $23 - 3 = 20$ (2) $2 + 5 = 7$ (3) $20 \times 7 = 140$

C) (1) $3 \times 2 = 6$ (2) $23 - 6 = 17$ (3) $17 + 5 = 22$

D) (1) $3 \times 2 = 6$ (2) $6 + 5 = 11$ (3) $23 - 11 = 12$

Q3. Evaluate the following expressions.

A) $6 + 5 \times 4$

B) $4 \times (7 + 8)$

C) $24 - 3 + 9$

Q4. Fill in the blanks.

A) $16 - 7 =$ ____ $+ 5$

B) $24 + 4 = 30 -$ ____

C) ____ $+ 6 = 4 \times 6$

D) $3 \times$ ____ $= 26 - 2$

Q5. Which of the following is equal to 21 – (6 + 3)? Mark $\boxed{\checkmark}$ against the option which is correct and mark $\boxed{\times}$ otherwise.

A) 21 – 6 + 3 $\square$ B) 21 – 6 – 3 $\square$

C) 21 + 6 – 3 $\square$ D) 21 + 6 + 3 $\square$

Q6. Find without calculation which of the following expressions are equal to the expression 165 + 328 – 249? Mark $\boxed{\checkmark}$ against the options which are equal to the above expression and mark $\boxed{\times}$ if it is not equal. There may be more than one correct answer.

A) 249 + 165 – 328 $\square$ B) 328 + 165 – 249 $\square$

C) 328 + 249 – 165 $\square$ D) 165 – 249 + 328 $\square$

Q7. Find without calculation which of the following expressions are equal to the expression 34 + 21 × 19 + 28? Mark $\boxed{\checkmark}$ against the options which are equal to the above expression and mark $\boxed{\times}$ if it is not equal. There may be more than one correct answer.

A) 28 + 34 + 21 × 19 $\square$ B) 21 + 34 × 19 + 28 $\square$

C) 34 + 21 × 28 + 19 $\square$ D) 21 × 19 + 34 + 28 $\square$

Q8. Without calculation, compare the expressions on both sides of the box using '=', '<' or '>' in the box. Write reasons for your answer.

A) 43 + 28 $\square$ 44 + 28 ____

____

____

B) 36 + 59 $\square$ 38 + 57 ____

____

____

C) 76 – 48 $\square$ 76 – 47 ____

____

____

D) 64 – 47 $\square$ 65 – 48 ____

____

____

Q9. Write + or – in the box and write the correct number in the blank so that the expressions on both sides of the "=" are equal.

43 + 58 = 43 + 56 $\square$ ____

Q10. If $m + 27 = 65$, then $m + 26 = ?$ Give reasons for your answer

Q11. Find the value of $9 + 5 \times k$ when $k = 4$.

Q12. Identify the terms in the following expressions.

A) $4 + 9 \times 5$

B) $26 - 13 + 18$

C) $5 \times x - 6 + 7 \times x + 11$

Q13. Which of the following expression is equal to the expression $9 \times x - 7 - 4 \times x + 23$. Mark $\boxed{\checkmark}$ against the option which is correct, otherwise mark $\boxed{\times}$.

A) $23 + 9 \times x - 7 - 4 \times x$ $\square$

B) $2 \times x - 4 \times x + 23$ $\square$

C) $9 \times x + 4 \times x - 7 + 23$ $\square$

D) $23 - 7 - 4 \times x + 9 \times x$ $\square$

Q14. If $249 + 567 = 816$, then find the value of $247 + 567$ without calculation. Give reasons for your answer.

Q15. Mark the points $n - 2$, $n - 3$, $n + 2$, $n + 4$ on the number line and fill in the blank.

$n - 2$ _____ $= n + 4$

$n + 3$ _____ $= n - 1$

Q16. Rewrite the following expressions without brackets so that the expressions are equal.

A) $25 - (9 + 3) =$ B) $2 \times (6 + 5) =$

C) $(15 + 4) - 7 =$ D) $17 - (8 - 5) =$

E) $7 + (12 - 9) =$ F) $4 \times (9 - 3) =$

Q17. Solve.

A) $9 - 4 =$ B) $-15 - 6 =$

C) $25 - 15 =$ D) $+17 - 8 =$

E) $-15 + 7 =$

Q18. Write $<$, $>$ in the boxes.

A) $-18 \ \square \ -10$ B) $-91 \ \square \ 19$

C) $47 \ \square \ -48$ D) $-27 \ \square \ -54$

Appendix – IIIB: Post test (MST-II)

(1^{st} November to 20^{th} November, 2004)

Homi Bhabha Centre for Science Education
TIFR, Mumbai

Name: ____________________ School: ____________________

Q1. Write the expressions corresponding to the following sentences.

A) A number which is four more than eleven ____________________

B) A number which is three less than twelve ____________________

C) A number which is five more than a number x ____________________

D) A number which is six less than a number p ____________________

Q2. Each option from A to D shows a different sequence in solving the expression $28 - 8 \times 3 + 6$. Mark $\boxed{\checkmark}$ against the option which shows the correct sequence. Otherwise mark $\boxed{\times}$.

A) (1) $28 - 8 = 20$ (2) $20 \times 3 = 60$ (3) $60 + 5 = 65$ ☐

B) (1) $28 - 8 = 20$ (2) $3 + 6 = 9$ (3) $20 \times 9 = 180$ ☐

C) (1) $8 \times 3 = 24$ (2) $28 - 24 = 4$ (3) $4 + 6 = 10$ ☐

D) (1) $8 \times 3 = 24$ (2) $24 + 6 = 30$ (3) $28 - 30 = -2$ ☐

Q3. Evaluate the following expressions.

A) $13 + 2 \times 8$

B) $27 - 3 \times (5 + 7)$

C) $19 - 4 + 6$

D) $16 - 7 - 9$

Q4. Fill in the blanks.

A) $19 - 6 =$ ____ $+ 4$

B) $28 + 9 = 40 -$ ____

C) ____ $+ 8 = 5 \times 7$

D) $4 \times$ ____ $= 28 - 4$

1

Q5. Which of the following is equal to $18 - (7 + 5)$? Mark $\boxed{\checkmark}$ against the option which is correct and mark $\boxed{\times}$ otherwise.

A) $18 - 7 + 5$ $\square$ B) $18 - 7 - 5$ $\square$

C) $18 + 7 - 5$ $\square$ D) $18 + 7 + 5$ $\square$

Q6. Find without calculation which of the following expressions are equal to the expression $18 - 27 + 4 \times 6 - 15$? Mark $\boxed{\checkmark}$ against the options which are equal to the above expression and mark $\boxed{\times}$ if it is not equal. There may be more than one correct answer.

A) $18 - (27 + 4 \times 6) - 15$ $\square$ B) $19 - 15 - 28 + 4 \times (2 + 4)$ $\square$

C) $6 \times (3 + 4) - 27 - 15$ $\square$ D) $18 - 20 + 7 + 4 \times 6 - 10 + 5$ $\square$

E) $8 \times 4 - 15 + 18 - 2 \times 4 - 27$ $\square$ F) $4 \times 6 - (27 + 15) + 18$ $\square$

Q7. Find without calculation which of the following expressions are equal to the expression $7 \times w - 19 - 11 \times w + 21$? Mark $\boxed{\checkmark}$ against the options which are equal to the above expression and mark $\boxed{\times}$ if it is not equal. There may be more than one correct answer.

A) $7 \times w - (19 + 11 \times w) + 21$ $\square$ B) $w \times (7 + 11) - 19 + 21$ $\square$

C) $7 \times w - 19 - 9 \times w - 2 \times w + 21$ $\square$ D) $-19 - 11 \times w + 7 \times (w + 3)$ $\square$

E) $20 - 20 + 7 \times w - 11 \times w$ $\square$ F) $2 \times (-2 \times w + 1)$ $\square$

Q8. Find the value of the following expressions for $k = 6$.

A) $k - 12$ B) $9 + 5 \times k$

C) $-7 + 2 \times k + 12$

Q9. If $m + 38 = 63$, then $m + 39 = ?$ Give reasons for your answer

Q10. Write down the terms in each expression.

A) $5 + 8 \times 7 \times 3$

B) $18 - 21 + 19$

C) $9 \times x - 12 + 4 \times x + 15$

Q11. If $534 - 256 = 278$, then find the value of the folowing expressions without calculation. Write increase or decrease in the terms below the expression/ Show how you found the answer.

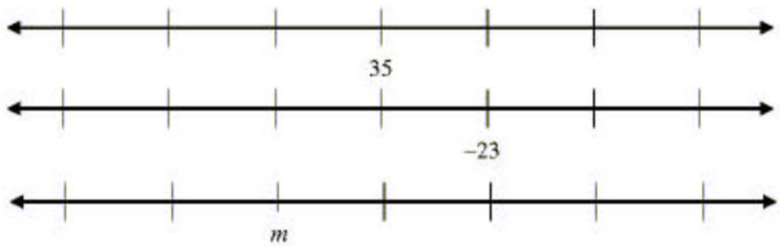
A) $534 - 257 =$

B) $535 - 257 =$

C) $536 - 255 =$

D) $537 - 258 =$

Q12. Complete the following number lines.



Q13. Find distance/ difference.

A) 5 and –8

B) $k - 7$ and $k - 13$

Q14. Open brackets and rewrite the expressions.

A) $25 - (9 + 3) =$ B) $2 \times (6 + 5) =$

C) $(15 + 4) - 7 =$ D) $17 - (8 - 5) =$

E) $7 + (12 - 9) =$ F) $4 \times (9 - 3) =$

Q15. Evaluate:

A) $13 - 5 =$ B) $-21 - 8 =$

C) $26 - 12 =$ D) $+19 - 7 =$

E) $-14 + 8 =$ F) $12 - 17 =$

Q16. Write $<$, $>$ in the boxes.

A) $-13 \;\square\; -9$ B) $-42 \;\square\; 24$

C) $35 \;\square\; -36$ D) $-48 \;\square\; -69$

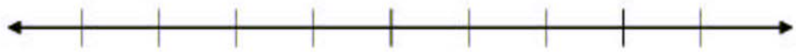
Q17. Without calculation, compare the expressions on both sides of the box using '=', '<' or '>' in the box. Write increase or decrease in the terms under the expressions. Make them equal, if they are not.

A) $54 + 29 \;\square\; 56 + 27$ B) $32 - 19 \;\square\; 32 - 18$

C) $63 + 57 \;\square\; 65 + 56$ D) $74 - 26 \;\square\; 75 - 29$

E) $52 - 37 \;\square\; 53 - 38$

Q18. Write an expression for the following number line journey and simplify the expression by combining terms.



Q19. Find easy ways to solve the following expressions.

A) $-28 + 49 + 8 + 20 - 49$

B) $12 \times 9 + 16 \times 5 - 17 \times 9$

C) $x + 15 - 13 \times x - 9$

D) $11 \times 4 + 9 \times 11 - 7 \times 11$

Q20. Show:

A) $47 - 6 - 52 + 29 - 24 + 9 = 3$

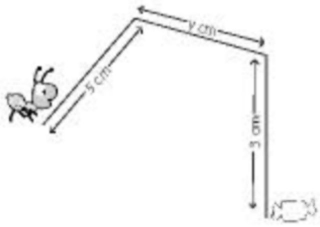
B) $19 \times n - 8 - 5 \times n + 1 = 7 \times (2 \times n - 1)$

Q21. Manali has got 4 marks less than Rupali in mathematics test. Vijaya has got 9 marks more than Rupali in the same test. How many marks has each of them got? Who has scored the maximum?

Q22. Manisha is 10 cm taller than Vikram. If Vikram's height is x cm, what is the height of Manisha?

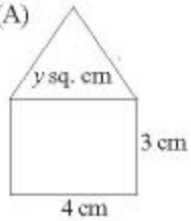
Q23. A strip of paper which is 3 cm long is joined to a strip of paper which is t cm long. What is the total length of the strip?

Q24. An ant goes along the path shown in the figure to get a piece of sweet. What is the total distance travelled by the ant to reach the sweet?



Q25.

(A)



Area of triangle = y sq. cm

Area of the rectangle =

Total area of the figure =

(B) The area of a rectangle is $12 \times h$ sq. units.

Length = Breadth =

(C) Length of a rectangle is four times its breadth.

Length = Breadth =

Perimeter =

Q26. (A) The calendar for the month of November 2004 is shown below. The shape shown below is taken from the calendar. Fill in the blank cells.

	9	16
	10	

	g	

Sun		7	14	21	28
Mon	1	8	15	22	29
Tue	2	9	16	23	30
Wed	3	10	17	24	
Thurs	4	11	18	25	
Fri	5	12	19	26	
Sat	6	13	20	27	

(B) Check that 2 + 10 + 18= 16 + 10 + 4. Using algebra show that this rule is true for any number arranged in the same way.

Appendix – IVA: Pre test (MST-III)

Homi Bhabha Centre for Science Education
TIFR, Mumbai (12.04.05)

Name: ____________________________ School: ______________

Q1. Write the terms for each of the following expressions.

1) $13 + 18 - 21$

2) $14 - 3 \times 7 + 5 \times 8 \times 9$

3) $3 - 5 \times k + 4 + 9 \times k$

Q2. Evaluate.

1) $13 + 4 \times 6$

2) $26 - 13 - 7$

3) $32 - 18 + 8$

4) $6 \times (4 + 9)$

5) $16 - (5 + 2 \times 7)$

6) $27 - 3 \times (4 + 2)$

7) $34 - (17 - 5 + 9)$

Q3. Write the sentences corresponding to the expressions.

1) A number which is 5 more than 9 ______________________________

2) A number which is 3 less than a number p ________________________

3) A number which is 8 less than 15 ______________________________

4) A number which is 6 more than w______________________________

Q4. Fill in the blanks.

1) $24 + 7 =$ ____ $+ 2$

2) $31 - 5 = 16 +$ _____

3) ____ $+ 9 = 6 \times 8$

4) $3 \times$ ____ $= 24 - 6$

Q5. The following are the responses of three students in evaluating the expression $14 + 5 \times 7 - 3$. Mark (√) against the option which you think is correct, otherwise mark (×).

1) $14 + 5 \times 7 - 3 = 19 \times 7 - 3 = 133 - 3 = 130$ ☐

2) $14 + 35 - 3 = 49 - 3 = 46$ ☐

3) $19 \times 7 - 3 = 19 \times 4 = 76$ ☐

Q6. Evaluate.

1) $19 - 8 =$

2) $-15 - 7 =$

3) $24 - 17 =$

4) $+18 - 6 =$

5) $-18 + 6 =$

6) $15 - 28 =$

Q7. Find, without calculation, which of the following expressions are equal to the expression $32 + 24 \times 9 + 21$? Mark (√) against the options which are equal to the above expression, and mark (×) if it is not equal. There may be more than one correct7answer.

1) $21 + 32 + 24 \times 9$ ☐

2) $24 + 32 \times 9 + 21$ ☐

3) $32 + 24 + 21 \times 9$ ☐

4) $9 \times 24 + 32 + 21$ ☐

Q8. Find, without calculation, which of the following expressions are equal to the expression $9 \times x + 12 - 6 \times x - 17$? Mark (√) against the options which are equal to the above expression, and mark (×) if it is not equal. There may be more than one correct answer.

1) $-17 + 9 \times x + 12 - 6 \times x$ ☐

2) $21 \times x - 6 \times x - 17$ ☐

3) $x \times 9 - 6 \times x + 12 - 17$ ☐

4) $9 \times 12 + x - 6 \times x - 17$ ☐

Q9. Find, without calculation, which of the following expressions are equal to the expression $23 - 4 \times 6 - 9$? Mark (√) against the options which are equal to the above expression, and mark (×) if it is not equal. There may be more than one correct answer.

1) $23 - (4 \times 6 + 9)$ ☐ 2) $23 - 4 \times 6 - 8 + 1$ ☐

3) $23 - (7 - 3) \times 6 - 9$ ☐ 4) $22 - 4 \times 6 - 8$ ☐

Q10. Find, without calculation, which of the following expressions are equal to the expression $87 - 38 + 26$? Mark (√) against the options which are equal to the above expression, and mark (×) if it is not equal. There may be more than one correct answer.

1) $87 - (38 + 26)$ ☐ 2) $26 + 87 - 38$ ☐

3) $87 - 30 - 8 + 26$ ☐ 4) $87 + 13 - 38 + 26 - 13$ ☐

Q11. Mark (√) against the options which is equal to the given expressions and mark (×) if not.

1) $24 - (6 + 7)$

a) $24 - 6 + 7$ ☐

b) $24 - 6 - 7$ ☐

2) $16 \times (9 + 8)$

a) $16 \times 9 + 16 \times 8$ ☐

b) $16 \times 9 + 8$ ☐

3) $(13 + 6) - 11$

a) $13 + 6 - 11$ ☐

b) $13 - 6 - 11$ ☐

4) $19 - (13 - 5)$

a) $19 - 13 - 5$ ☐

b) $19 - 13 + 5$ ☐

5) $16 + (12 - 4)$

a) $16 - 12 + 4$ ☐

b) $16 + 12 - 4$ ☐

6) $3 \times (9 - 4)$

a) $3 \times 9 - 4$ ☐

b) $3 \times 9 - 3 \times 4$ ☐

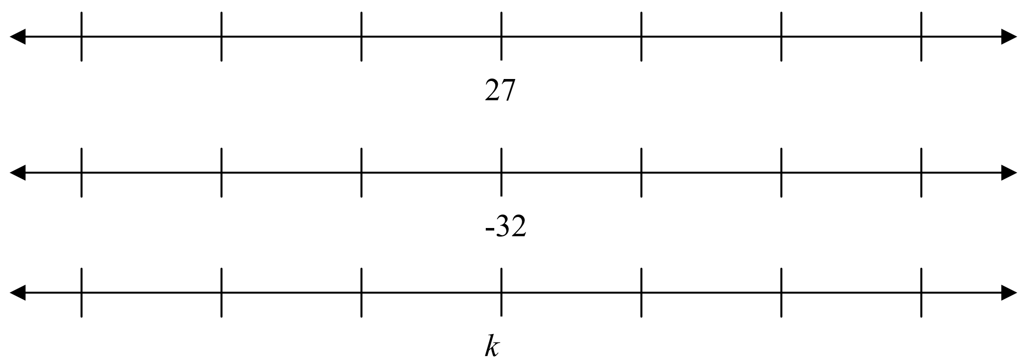
Q12. Find easy ways to solve the following expressions.

1) $23 - 49 + 7 + 19$ 2) $38 - 17 - 12 + 17 - 6$

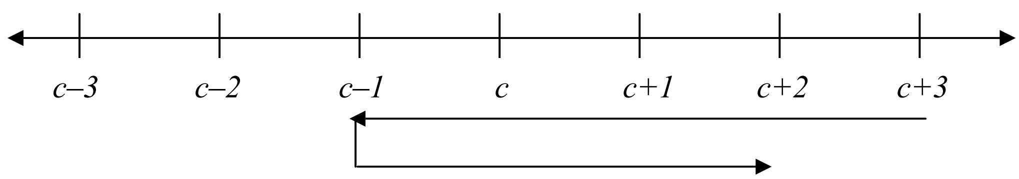
3) $4 \times 12 + 12 \times 9 - 12 \times 5$

4) $5 \times 11 - 6 \times 13 + 8 \times 11$

Q13. Complete the following number lines.



Q14. Write an expression for the following number line journey and simplify the expression.



Q15. Find the value of the expressions for $m = 4$.

1) $m - 6$

2) $6 + 2 \times m$

Q16. If $a - 65 = 34$, then $a - 66 = ?$ Give reasons for your answer.

Q17. Shivani is 5 years older than Lalitha. If Lalitha's age is x years, how old is Shivani?

Q18. Simplify.

1) $2 \times n + 17 + 6 \times n - 9$

2) $14 \times b - 5 - 6 \times b + 13$

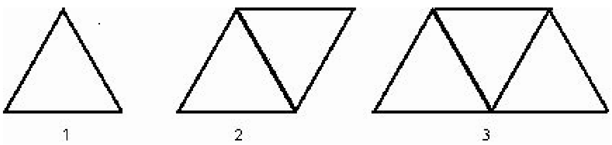
Appendix – IVB: Post test (MST-III)

Homi Bhabha Centre for Science Education
TIFR, Mumbai (29.04.05)

Name: ______________________________ School: _____________

Q1. Think of a number. Subtract 1 from it. Multiply the result by 2. Add 5 to it. Subtract the original number from the result. Add 4. Subtract the original number once again. What do you get? Show that everyone would get the same answer.

Q2. The following triangle pattern is to be made with matchsticks. Look at the pattern carefully and answer the questions below. Show how you found the answer.



1) How many matchsticks will be required to make the 4^{th} figure? ______

2) How many will be required to make the 5^{th} figure? ______________

3) How many matchsticks will be required to make the 17^{th} figure? _____

4) How many matchsticks will be required to make the 59^{th} figure? _____

5) How many matchsticks will be required to make the figure at the m^{th} position? ____

Q3. Take any three consecutive numbers. Show that

1^{st} number + 3^{rd} number = 2 × (2^{nd} number)

Q4. Write the terms for each of the following expressions.

1) $18 + 27 - 14$

2) $15 - 5 \times 9 + 3 \times 6 \times 4$

3) $5 - 7 \times k + 2 + 11 \times k$

Q5. Evaluate.

1) $21 + 4 \times 6$

2) $34 - 19 - 5$

3) $27 - 8 + 3$

4) $8 \times (4 + 9)$

5) 25 – (4 + 3 × 5)

6) 34 – 6 × (9 – 5)

7) 28 – (13 – 7 + 5)

8) 19 – 2 × (3 + 6 × 7)

Q6. Fill in the blanks.

1) 31 + 4 = ____ + 3

2) 24 – 7 = 14 + _____

3) ____ + 9 = 7 × 5

4) 8 × ____ = 48 – 16

Q7. The following are the responses of three students in evaluating the expression 16 + 4 × 8 – 5. Mark (√) against the option which you think is correct, otherwise mark (×).

4) 16 + 4 × 8 – 5 = 20 × 8 – 5 = 160 – 5 = 155 ☐

5) 16 + 32 – 5 = 48 – 5 = 43 ☐

6) 20 × 8 – 5 = 20 × 3 = 60 ☐

Q8. Evaluate.

1) 15 – 3 =

2) –19 – 12 =

3) 19 – 12 =

4) +21 – 16 =

5) –17 + 9 =

6) 12 – 23 =

Q9. Find, without calculation, which of the following expressions are equal to the expression 32 + 24 × 9 + 21? Mark (√) against the options which are equal to the above expression, and mark (×) if it is not equal. There may be more than one correct7answer.

1) 21 + 32 + 24 × 9 ☐

2) 24 + 32 × 9 + 21 ☐

3) 32 + 24 + 21 × 9 ☐

4) 9 × 24 + 32 + 21 ☐

Q10. Find, without calculation, which of the following expressions are equal to the expression $9 \times x + 12 - 6 \times x - 17$? Mark (√) against the options which are equal to the above expression, and mark (×) if it is not equal. There may be more than one correct answer.

1) $-17 + 9 \times x + 12 - 6 \times x$ ☐ 2) $21 \times x - 6 \times x - 17$ ☐

3) $x \times 9 - 6 \times x + 12 - 17$ ☐ 4) $9 \times 12 + x - 6 \times x - 17$ ☐

Q11. Find, without calculation, which of the following expressions are equal to the expression $23 - 4 \times 6 - 9$? Mark (√) against the options which are equal to the above expression, and mark (×) if it is not equal. There may be more than one correct answer.

1) $23 - (4 \times 6 + 9)$ ☐ 2) $23 - 4 \times 6 - 8 + 1$ ☐

3) $23 - (7 - 3) \times 6 - 9$ ☐ 4) $22 - 4 \times 6 - 8$ ☐

Q12. Find, without calculation, which of the following expressions are equal to the expression $87 - 38 + 26$? Mark (√) against the options which are equal to the above expression, and mark (×) if it is not equal. There may be more than one correct answer.

1) $87 - (38 + 26)$ ☐ 2) $26 + 87 - 38$ ☐

3) $87 - 30 - 8 + 26$ ☐ 4) $87 + 13 - 38 + 26 - 13$ ☐

Q13. Mark (√) against the options which is equal to the given expressions and mark (×) if not.

1) $25 - (5 + 9)$

a) $25 - 5 + 9$ ☐

b) $25 - 5 - 9$ ☐

2) $12 \times (6 + 13)$

a) $12 \times 6 + 12 \times 13$ ☐

b) $12 \times 6 + 13$ ☐

3) $(15 + 8) - 11$

a) $15 + 8 - 11$ ☐

b) $15 - 8 - 11$ ☐

4) $29 - (17 - 8)$

a) $29 - 17 - 8$ ☐

b) $29 - 17 + 8$ ☐

5) $21 + (18 - 12)$

a) $21 - 18 + 12$ ☐

b) $21 + 18 - 12$ ☐

6) $8 \times (14 - 5)$

a) $8 \times 14 - 5$ ☐

b) $8 \times 14 - 8 \times 5$ ☐

Q14. Find the value of the expressions for $n = 7$.

1) $n - 14$ 2) $5 + 8 \times n$

Q15. If $u - 53 = 26$, then $u - 54 = ?$ Give reasons for your answer.

Q16. Meena is 7 years younger than Preeti. If Preeti's age is x years, how old is Meena?

Q17. Vikram has t marbles. Shyam has 5 marbles more than Vikram and Yogesh has two times as many marbles as Shyam. How many marbles does each of them have?

Q18. Simplify.

1) $16 \times y - 7 + 3 \times y + 13$ 2) $-2 \times x + 5 + 7 \times x - 12$

3) $6 + 13 \times m - 2 - m$ 4) $n - 9 + 8 \times n + 16$

Q19. Find easy ways to solve the following expressions.

1) $48 - 56 + 17 + 9$

2) $69 - 26 - 11 + 26 - 8$

3) $3 \times 16 + 16 \times 12 - 16 \times 7$

4) $7 \times 18 - 6 \times 11 + 4 \times 18$

Appendix – VA: Interview Schedule and test (MST-II)

Name: ____________________________________ School: ______________

A) Solve/ Evaluate:

1) $15 + 6 \times 5$ 2) $25 - 10 + 5$

3) $25 - (10 + 5)$ 4) $6\times(3+2)$

B) Which of the following expressions are equal to the expression $18 - 13 + 15 \times 4$?

a) $18 - 15 + 13 \times 4$ ☐
b) $4 \times 15 + 18 - 13$ ☐
c) $18 - (13 - 15 \times 4)$ ☐

C) Which of the following expressions are equal to the expression $25 - (10 + 5)$?

a) 25-10+5 ☐
b) 25-10-5 ☐

D) Which of the following expressions are equal to the expression $49 - 37 + 23$?

a) $49 - 5 - 37 + 23 + 5$ ☐
b) $37 - 49 + 23$ ☐
c) $23 + 49 - 37$ ☐

Interview Schedule

Task 1: 15 + 6 × 5	Correct answer 15+30=45	Why do you solve it like this?
		One student did it like this: 21×5=105. Do you think this is correct?
	No	Why?
	Yes	But you are getting different answers? Is that possible? Which one do you think is correct?
	Wrong answer 21×5=105	Someone did it like this: 15+30=45. Do you think this is correct?
	Yes	Why?
	No	That student said that you cannot combine a product term and a simple term. Is that so?
	No	The student said the product term has to be converted to a simple term and then combined. Is he/she correct?
Task 2: 18 – 13 + 15 × 4. Which of these 18 – 15 + 13 × 4, 4 × 15 + 18 – 13 and 18 – (13 – 5 × 4) are equal to the above?	4 × 15 + 18 – 13 and 18 – (13 – 5 × 4)	Calculate and see. Are they equal?
	Yes	Drop it
	No	Why?
	18 – 15 + 13 × 4	Are you sure? Do you want to calculate and check?
	Yes/ No	Why?
Task 3: 25 – 10 + 5	Correct answer 15+5=20	Why like this?
		If someone does it like this 25-15=10, is this correct?
	No	Why?
	Yes?	Can you get two different answers for the same question?
	Wrong answer 25-15=10	One student in an interview did it like this: 15+5=20. Do you think

		this is correct?
	Yes	Why?
	No	That student said you should always move from left to right when the expression has only + and – sign. Is this correct?
Task 4: 25 – (10 + 5)	Correct answer 25-15=10/ 25-10-5=15-5=10	Why this way?
		Is 25-10+5=15+5=20 correct solution of the above problem?
	No	Why?
	Yes	You are getting two answers for the same question. How is that? Which, according to you, is more correct? Why?
	Wrong answer 25-10+5=15+5=20	One person did it like this: 25-10-5=15=5=10. Another person did it like this: 25-15=10. Are they correct? (only one will be given to the student)
	Yes	Why?
	No	The first person said you should always change the signs of the terms inside the bracket. The second one said you should always do bracket first. Are they correct?
Task 5: 25 – (10 + 5). Which of the following are equal to the above 25-10+5 or 25-10-5	25-10+5	Calculate these and see? Are they equal?
	No	Why?
	Yes	Drop it
	25-10-5	Are you sure? Do you want to check by calculations?
	Yes/No	Why?
Task 6: 6×(3+2)	Correct answer 6×5=30/ 6×3+6×2=18+12=30	Why like this?
		Someone did it like this: 6×3+2=18+2=20. Is this correct?
	No	Why?

		Is this equal to 6×3+6×2? If yes, then why?
	Yes	You are getting two answers for the same question. How is that? Which, according to you is more correct? Why?
	Wrong answer 6×3+2=18+2=20	A student solved the problem like this: 6×3+6×2=18+12=30. Another one did it like this: 6×5=30? Are they correct? (only one will be given depending on the student's answer)
	Yes	Why?
	No	The first student said that you have to use distributive property and multiply both the terms inside the bracket with the number outside and the second student said that you do brackets first. Are they correct?
Task 7: 49-37+23. Which of the following is equal to this: 49-5-37+23+5, 37-49+23 or 23+49-37?	49-5-37+23+5 and 23+49-37	Do you want to calculate and check?
	No	Why?
	Yes	Drop it
	37-49+23	Are you sure? Do you want to calculate and check?
	Yes/No	Why?

Appendix – VB: Interview Schedule and test (MST-III)

Arithemtic test

Name: ______________________________________ Date: _________

Q1. Evaluate.

A) $5 + 3 \times 6$

B) $22 - 7 + 9$

C) $22 - (7 + 9)$

D) $5 \times (3 + 8)$

Q2. Which of the following expressions is equal to the expression **$24 - 13 + 18 \times 6$**? Answer without calculation.

1) $24 - 18 + 13 \times 6$ ☐
2) $24 + 18 - 13 \times 6$ ☐
3) $6 \times 18 - 13 + 24$ ☐
4) $24 - (13 - 18 \times 6)$ ☐

Q3. Which of the following expressions is equal to the expression **$48 - 23 + 59$**? Answer without calculation.

1) $48 - 23 - 2 + 59 + 2$ ☐
2) $48 - 59 + 23$ ☐
3) $48 - (23 + 59)$ ☐

Q4. Which of the following is equal to the expression **$22 - (7 + 9)$**? Answer without calculation.

1) $22 - 7 + 9$ ☐
2) $22 - 7 - 9$ ☐

Algebra test

Name: ________________________________ Date: __________

Q1. Simplify.

1) $5 \times a + 6 - 2 \times a + 9$ 2) $b + 9 + 6 \times b - 5$

Q2. Identify which of the following expressions are equal to the expression **$\mathbf{13 \times m - 7 - 8 \times m + 4}$?**

1) $13 \times m - 7 - 8 \times 4 + m$ ☐

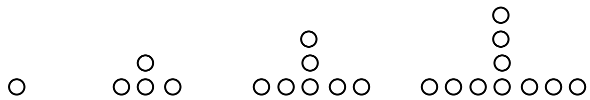
2) $-7 + 4 + 13 \times m - m \times 8$ ☐

3) $m \times (13 - 8) - 7 + 4$ ☐

4) $13 \times m - (7 - 8 \times m) + 4$ ☐

Q3. Think of a number. Add 2 to it. Subtract 5 from it. Subtract the original number from it. Add 4. Write an expression that you get by following this instruction.

Q4. Look at the pattern carefully and answer the following questions.



1) How many dots will be there in the 5^{th} figure? ___________________

2) How many dots will be there in the 11^{th} figure? __________________

3) How many dots will be there in the 58^{th} figure? __________________

4) How many dots in the k^{th} figure? ______________________________

Interview Schedule: Arithmetic

Task 1: Can you solve the expression 5+3×6?	If yes and solves by the terms method correctly, ie, 5+18=23	Why like this? Any other reason for doing it like this? 6×3+5=18+5
	If yes	Give reason.
	If no	Ask if 8×6 is a correct way of solving it?
Are these two ways the same? Would these two ways, if done correctly, always give you the same answer?	If yes	Why? Explain.
	If no	Why? Explain.
	If solves incorrectly as 8×6=48	Explain what you have done. Give hint: one student did it like this 5+18=23, is this correct?
	If yes	Why? You have two answers. Which is correct or both are correct?
	If no	Why? Wait for students' answer. The student said that product term is done first. Is this correct?
Task 2: Can you solve the expression 22-7+9?	If yes and solves it correctly, ie, 24	Do you know any other way of solving it?
	If yes	Show how you can solve it otherwise.
	If no	Ask if 22-16=6 or 22+2=24 or 15+9=24 a correct way of solving it?
Are these two ways the same? Would these two ways, if done correctly, always give you the same answer?	If yes	Why? Explain.
	If no	Why? Explain.
	If solves it incorrectly as 22-	Explain what you have

	16=6	done? Give hint: one student did it like this 15+9=24. Is this correct?
	If yes	Why? You have two answers. Which is correct or both are correct?
	If no	Why? Wait. The student said that one is a negative term and one is positive. You have to combine terms. Is this correct?
Task 3: Can you solve the expression 22-(7+9)	If solves it correctly as 22-16=6 or 22-7-9=15-9=6	Why like this? Explain.
Do you know any other way of solving the expression?	If yes	Show it. Explain.
	If no	Ask if 22-16=6 or 22-7-9=15-9=6 a correct way of solving it?
	If solves it incorrectly as 22-7+9=15+9=24	Why? Give hint 22-7-9=6 or 22-16=6. Is this correct?
	If yes	Why? You have two answers. Which is correct or both are correct?
	If no	You should change the signs of the terms inside the bracket or you should solve inside the bracket. Is this correct?
Task 4: Can you solve the expression 5×(3+8)	If solves it correctly as 5×11 or 5×3+5×8	Why like this? Explain.
Do you know any other way of solving it?	If yes	Show it. Explain.
	If no	Ask if or 5×3+8 or 5×11 or 5×3+5×8 a correct way of solving it?
	If solves it incorrectly as 5×3+8=15+8=23	Why? Give hint 5×11 or 5×3+5×8. Is this correct?
	If yes	Why? You have two an-

		swers. Which is correct or both are correct?
	If no	You should multiply both the terms by the common factor or solve the bracket first. Is this correct?
Task 5: Which of the following are equal to 24-13+18×6	24-18+13×6 and 24-13+18×6	Why? Do you want to calculate and check? Will their answers be same or not? Which would be more/ less?
	24+18-13×6 and 24-13+18×6	Why? Do you want to calculate and check?
	6×18-13+24 and 24-13+18×6	Why? Do you think their answers will be equal? If 6×18-13+24 has value 115, then what is the value of 24-13+18×6
	24-(13-18×6) and 24-13+18×6	Why? Do you want to calculate and check?
Task 6: Which of the following are equal to 48-23+59	48-23-2+59+2 and 48-23+59	Why? Are 48-23-2+59-2 and 48-23+59 equal? Will their answers be equal? Which will be more/ less
	48-59+23 and 48-23+59	Why?
	48-(23+59) and 48-23+59	Why?
Task 7: Which of the following is equal to 22-(7+9)	22-7+9 and 22-(7+9)	Why?
	22-7-9 and 22-(7+9)	Why?

Interview Schedule: Algebra

Task 1: Simplify 5×a+6-2×a+9	Correct answer 3×a+15	Why did you do like this? Explain
What does the letter stand for?	If says number	If you substitute a=4 in 3×a+15 and in 5×a+6-2×a+9, if someone gets 27 for the first expression and 33 for the later, then is he/she correct? Would you get the same answer? Is 3×a+15=18×a? Why
	Incorrect answer 11×a-2×a+9=9×a+9=18×a	Why did you do like this?
What does the letter stand for here?	If says number	If you put a=3, would you get the same answer for 5×3+6-2×3+9 and 18×a
	Solves arithmetic expression correctly and identifies the terms (may be also the common factor of the product terms)	Is it similar to the algebraic expression given to you? Would you like to simplify the algebraic expression again?
	Solves arithmetic expression incorrectly moving from left to right sequentially	Give a hint that the expression contains a product term or when two product terms can be combined?
	If no response to the hint	Drop it
	If the student understands the hint	Ask him/her to solve the arithmetic expression and then the algebraic expression
Task 2: Simplify b+9+6×b-5	Correct answer 7×b+4	Why did you do like this? Explain
		If you substitute b=5 in 7×b+4, you get 39. Then what is the value of b+8+6×b-5? Would you get the same answer? Is 50 a correct value for the expression? is 7×b+4=11×b?
	Wrong answer 9×b+6×b-5=15×b-5=10×b	Why did you do like this?

		If you put b=2 then would you get the same answer for 10×b and b+9+6×b-5?
	Solves it correctly and identifies the terms	Is it similar to the algebraic expression given to you? Would you like to simplify the algebraic expression again?
	Solves it incorrectly moving from left to right	Give a hint that the expression contains a product term. How would we solve such an expression?
	If no response to the hint	Drop it
	If the student understands the hint	Ask him/her to solve the arithmetic expression and then the algebraic expression
Task 3: Identify which of the following expressions are equal to 13×m-7-8×m+4	13×m-7-8×m+4 and 13×m-7-8×4+m.	Are they equal? For m=2, 13×m-7-8×m+4=7, then what is 13×m-7-8×4+m?
	If yes	Why? Explain. Give an arithmetic expression 13×2-7-8×2+4 and 13×2-7-8×4+2 and ask if they are equal
	If no	Why? Explain
		On substituting the same value for the letter, would you get the same answer for both the expressions? Can you put different values of 'm' in the same expression?
	13×m-7-8×m+4 and -7+4+13×m- m×8	Are they equal?
	If yes	Why? Explain.
		Would you get the same value on substitution by a number?
	If no	Why? Explain. Give two

		arithmetic expressions 13×2-7-8×2+4 and -7+4+13×2-8×2 and ask if they are equal.
	13×m-7-8×m+4 and m×(13-8)-7+4	Are they equal?
	If yes	Why? Explain
		Would you get the same value on substitution by a number?
	If no	Why? Explain. Give two arithmetic expressions 13×2-7-8×2+4 and 2×(13-8)-7+4 and ask if they are equal.
	13×m-7-8×m+4 and 13×m-(7-8×m)+4	Are they equal?
	If yes	Why? Explain. Give two arithmetic expressions 13×2-7-8×2+4 and 13×2-(7-8×2)+4 and ask if they are equal.
	If no	Why? Explain
Task 4: Think of a number. Add 2 to it. Subtract 5 from it. Subtract the original number from it. Add 4. Does the expression x+2-5-x+4 match the situation?	If yes	Why?
		Does x-5+2+4-x match the situation? Would they give the same value in the end?
	If yes	How? Why?
	If no	How? Why?
	If no to x+2-5-x+4	Write the expression which you think fits the situation
Is there any advantage of using the letter in the 'think-		

of-a-number' game? Or 6+2-5-6+4 is enough. Explain.		
Frame a question for x×2-4+5-x-1. What does it mean to get 'x' as the answer after simplifying the expression x×2-4+5-x-1?		
Task 5: Give any pattern. Ask the student to find one rule for the pattern.		
Give another rule to the student and ask if they are the same? How do you know they are same?		
Task 6: a+b=b+a. What does 'a' and 'b' stand for? Is it true? a-b=b-a? a+x-x=a? What do these sentences mean?		

Appendix – VI: Teaching Guideline

The following teaching guideline is proposed on the basis of the study undertaken as part of the thesis.

Task 1: Verbalizing task in arithmetic expressions

Looking at expressions as standing for a number and understanding the information conveyed by the expression. In the process, verbalizing the meaning of expressions using relational terms. For example, 10+5 and 5×3 are both equal to the number 15. These two expressions therefore are equal to each other.

Examples:

a. Writing different expressions for a given number. For example, express 15 in different ways (15 = 5×3, 10+5 etc.)

b. Given an expression writing the meaning and vice-versa. For example, 5×3 is “a number which is five times three”. “5 more than 10” is 10 + 5

Task 2: Compare expressions with computation:

Ask students to compare two expressions by using the signs <, =, > by computing their values. This will familiarize them with the signs used for comparing and allow them to look at relations between expressions.

Ask students to fill in the blanks by a number so that the expressions have equal value. They will learn that two expressions are equal if their values are equal and will also understand that ‘=’ sign does not separate the question from the answer but is a relation of equality between two sides.

Examples:

a. Compare the following expressions using signs <, =, >:

12+18 __ 30-1	7+9 __ 9+7
15-5 __ 2 ×5	24÷6 __ 20+4

b. Fill in the blanks:

19-6 =__ +4	4×__ = 28-4
__ +8 = 5×7	28+9 = 40-__

Remarks:

One can see many misconception among students regarding ‘=’ sign and comparing them. Handle them by emphasizing the relational idea of expressions.

Task 3: Introduce number line only with whole numbers

Understand number line in terms of unit distance between any two consecutive points. All numbers can be represented on the number line with 0 as the origin.

Establish relations between numbers and numbers with respect to origin. Moving to the right is equivalent to adding (number of unit distance) and the numbers increase. Moving to the left is equivalent to subtracting and the numbers decrease. Emphasize this vocabulary of 'moving right', 'moving left' on the number line.

Examples:

a. Draw number line starting from 0.

b. Compare numbers on the number line. Given two numbers, which is more/ less? Why?

c. Jumps on the number line: Explore relations between two points on the number line. For example, given the two points 3 and 8, from 3 you have to jump 5 places to the right to reach 8. Reinforce verbalizing like "8 is 5 more than 3 and represent 3+<u>5</u>=8". Do the other way round, 8-<u>5</u>=3.

d. Portions of the number line: Focus on parts of the number line, like a number line starting from 34, mark the three points to the left and to the right.

Task 4: Introduce integers with the help of contexts

Like above and below the sea level, temperatures above and below 0^0C, debit and credit. Use these contexts to emphasize the existence of numbers below/ less than zero.

Then extend the number line to numbers to the left of 0. Reinforce the relations as in Task 3.

Bring following points to students' attention: Any positive number is greater than any negative number (as they are always to the right of the negative numbers), all negative numbers are smaller than 0, all positive numbers are greater than 0.

Examples:

Repeat tasks 3 a, b, c, d with the complete number line.

Remarks:

Revise often order relations between negative numbers and compare with positive numbers. Be cautious against over generalizations, like numbers to the left of any number are negative, e.g. numbers to the left of 27 are the numbers -26, -25 etc. or -28, -29 etc.

Task 5: Guess the number game

Introduce letters in the context of missing numbers with the four basic operation facts. The missing number can be replaced by a 'letter'. Here '*x*' stands for an unknown.

Introduce letters in the context of open number sentences, where letters can take more than one value.

Use letters in the context of perimeter problems with one or more dimensions as not known

Examples:

a. Find the appropriate number which can fill the box so that the relation is true, e. g. 8+□=13.

b. Replace the box by any letter and find the value of the letter.

c. Find all possible values of x and y in $x+y=15$.

d. Find the perimeter of a triangle, rectangle with dimensions given. e.g. if 3 and x are the lengths and breadth of a rectangle then the perimeter is $3+x+3+x$

Remarks:

As this is the first exposure to algebraic expressions, expressions can be left open without any simplification.

Task 6: Verbalizing task with simple algebraic expressions, emphasizing the letter as any number

Use the context of guess the number game to reinforce relational understanding of simple algebraic expressions, emphasizing the letter as standing for a number.

a. Repeat task 1 a, b. For example, $x+4$ is 'a number which is four more than any number x'. Use substitution by any value for the letter to demonstrate this.

b. Find the number, if four more than the number is 15 (that is, $x+4=15$)

Task 7: Introduce letter-number line

Construct it by generalizing the number line already made. This will introduce them to simple unclosed expressions like $x+1$, $x-1$ etc. as well as reinforce their meaning like 'one less than x'.

Examples:

a. Draw letter-number line.

b. Compare the expressions on the letter-number line.

c. Repeat task 3c.

d. Replace the letter by any number and make the number line.

Remarks:

Task 7d will use the ideas of the integer number-line and reinforce order relations among signed numbers. Students can use the letter-number line to verify the correctness of their solutions.

Task 8: Introduction to terms of an expression

Terms are components of an expression demarcated by + and – sign, the + and the – sign attached to the number following it. If no sign is attached to the first number of the expression, then it is considered to be a positive term. For example, in 3+5×2-7, +3, +5×2 and -7. Put terms in boxes. Name the terms +3, -7 as simple terms and +5×2 as a product term. Every term of an expression is either positive or negative, whether simple or product. +3 is a positive term and -7 is a negative term.

Examples:

Identify and write terms of an expression. e.g. 19-7+5, 13+6×4-12, 7+2×x+4

Task 9: Combining terms

In place of the traditional precedence rules of evaluating expressions, combining of terms will be used for evaluating expressions to give flexibility to the procedures. Replace the vocabulary from addition and subtraction to that of positive and negative terms.

How do we combine terms?

For example, in 12+5, there are two simple terms +12 and +5. Both are positive terms. Positive terms can be considered as white cards. Combining 12 white cards and 5 more white cards will yield 17 white cards. Therefore +12+5=+17.

In -12-5, there are two simple terms -12 and -5. Both are negative terms. Negative terms can be considered as black cards. Combining 12 black cards and 5 more black cards will yield 17 black cards. Therefore -12-5=-17.

In 12-5, there are two simple terms, +12 and -5. One is a positive term and the other is negative term. To combine these, we can make pairs of opposite coloured cards which will cancel each other. In this example, we have 5 pairs of black and white cards whose result is 0, and are left with 7 white cards. So the solution for +12-5=+7.

In -12+5, there are two simple terms, -12 and +5. One is a positive term and the other is negative term. To combine these, we make pairs of opposite coloured cards. We get 5 pairs of black and white cards which is 0 and are left with 7 black cards. So the solution for -12+5=-7.

Since every term can be thought of as a set of black or white cards, it is possible to combine them in any order, that is, commutativity holds between any two such terms.

Examples:

a. Evaluate by combining terms:

15-9	-14+6	9-25	23-8	-17-9

Take all combinations of numbers, begin with positive or negative term, bigger number first or smaller number first

Remarks:

Combining terms is nothing but integer addition. Subtraction has been implicitly converted to addition. Here the 'card method' of combining terms can be used. There are many other methods but this seems to be the most suitable in this approach.

The exercise is simple but needs reinforcement for further complex tasks.

Task 10: Evaluation of arithmetic expressions

Expressions with only simple terms: 14-5+7. The expression contains three simple terms, +14, -5+7. To evaluate the expression, combine the terms +14, -5 to get +9 and combine this with +7 to get +16. (card method can be used for the combining procedure)

Expression containing simple and product terms: 3+6×5-8. This expression contains three terms +3, +6×5 and -8. Only simple terms can be combined. Hence the product term +6×5 needs to be converted to a simple term which is equal to +30. Now the expression contains three simple terms, +3, +30, -8. They can be combined as explained above.

Examples:

a. Evaluate expressions:

16-5+8	19+2×3-9	17-3×4-6	15+9-4
21-7+9	-8+5+3×6	-17+8-11	

Remarks:

For the time being the expressions have been evaluated from left to right. But this need not be emphasized as combining terms in any order would lead to the same answer. This they will verify shortly. Some students may be quick to make this observation by themselves. This is not to be discouraged.

Task 11: Simplifying simple algebraic expressions

Explain the similarity between $3\times x$ and 3×2. Explain that $3\times x$ is same as adding x repeatedly three times. It is 3 times x.

Using this understanding, simple algebraic expressions with same letter factor can be combined. For example, $3\times x+2\times x = \underline{x + x + x} + \underline{x + x} = 5\times x$.

For subtraction, the take away model can be used. E.g. $5\times x-3\times x$ can be understood as taking away 3 x's from 5 x's and can be written as $x+x+\cancel{x+x+x} = x+x=2\times x$.

Examples:

a. Expand product terms as sum of 'singletons'.

3×7, $4 \times a$, $6 \times b$, $n \times 4$

b. Write the following expressions as product terms.

4+4+4+4+4

$y+y+y$

$m+m+m+m+m+m$

$g+g+g+p+p$

c. Simplify:

$5 \times 3+6 \times 3$, $5 \times 2+3 \times 4$, $4 \times t+3 \times t$, $6 \times v-4 \times v$, $5 \times s-4 \times r$

Remarks:

In some of these exercises students would be tempted to write a closed answer like for $g+g+g+p+p$ as $5 \times gp$. These kinds of errors can be handled by a careful use of language with the understanding that letter stands for any number and reinforcing the meaning of the expressions.

Task 12: Identifying equal expressions (without computation)

Given a list of expressions, students can guess (just by looking at the expression) and then verify by computation that expressions with the same terms have the same value. Some students think that if the numbers in an expression are the same their solution will be equal. They do not realize the importance of the sign attached to the number. This task emphasizes the utility of the concept of terms in parsing the expression and gives them an understanding that terms can be combined in any order.

a. Which of the following expressions are equal to the given expression?

(1) 27-16+48

27+48-16   16-27+48

48-16+27   27-48+16

(2) $49+4 \times 12+18$

$49+4 \times 18+12$ $4+49 \times 12+18$

$12 \times 4+49+18$ $18+12 \times 4+49$

Remarks:

Students will discover a very useful idea of invariant transformations in expressions (transformations which do not change the value of the expression). They will realize that in product terms, commutativity will be applicable only within the product term. For example 49+4×12+18 is not equal to 4+49×12+18.

Task 13: Making equal expressions:

Rearranging terms would give them practice of moving around the terms in the expression without changing the value of the expression.

Examples:

a. Ask students to make equal expressions by rearranging terms. For example, given the expression 19-7+28 make as many equal expressions as well. Similarly expressions with product terms should be used (e.g. 34+15×9-65).

Task 14: Easy ways of evaluation

To apply the knowledge they have gained in the previous two tasks, ask students to evaluate expressions using easy ways.

For example, in -28+49+8+20-49, -28+8+20 would give an answer 0 and +49-49 will also be zero, so the answer is 0.

Examples:

a. Evaluate:

29-7+11+7	48-56+17+9
69-26-11+26-8	47-6-52+29-24+9

Remarks:

This task is possible because they now know that reordering terms does not change the value of the expression. Precedence rules are not to be emphasized. Longer expressions where some properties like cancellation or when combined would lead to multiples of 5, 10 are to be used.

Task 15: Bracketed terms

In expressions with brackets, like 23-(5+9), there are two terms, +23 and –(5+9). The first term is a simple term. The second term will be called a bracketed term with a negative sign. Some students might call it a negative bracketed term. Similarly in expressions like 19+3×(5+4), there are two terms, +19 and +3×(5+4). The first term is again a simple term but the second term is a bracketed product term.

Solve expressions with bracketed terms. Ask students to evaluate expressions with brackets by solving the bracket first.

Examples:

a. Write the terms for the following expressions. Then evaluate.

13-(3+5)	18-(5+6)

-23+(4+5)　　　　　　　17+(8+4)×3

Task 16: Rules for opening bracket

Rewriting an equal expression for 24-(5+4) by removing the bracket: 24-5+4, 24-5-4. Students can verify by computation which of the expressions will be equal to the one with brackets. They can subsequently generalize that while removing bracket, sign of the terms inside the bracket will change when there is a negative bracketed term.

Similarly demonstration examples have to be taken with a + sign to the left of the bracket and – sign to the right of the bracket. In both these cases they will reach the conclusion that sign of the terms inside the bracket will not change if there is a positive bracketed term.

Students should focus on the equality relation and the sign attached to the bracketed term and the signs of the terms on removing the bracket.

Examples:

a. Write equal expressions by removing the bracket. Identify the terms for the given and the obtained expression.

   45-(23+12), (42-12)-23, 34-(23-7), 45+(34+8), -(5+4)+4×3

b. Make equal expressions for the given expressions (encourage putting brackets and removing brackets for making equal expressions):

   23-(4+5)-9, 45-23-13

Remarks:

The rule for opening bracket can also be introduced by using the concept of ‘inverse’. A negative term is nothing but the additive inverse of the corresponding positive term (combing such terms will give value 0). So the negative bracketed term is the inverse of the corresponding positive bracketed term. The inverse of the expression 5+4 is -5-4. The values of the two expressions are also inverse of each other, that is -9 (=-5-4) is inverse of 9 (=5+4).

Patterns of the following type are useful:

If 5+4=9, then -5-4=?

Or if 4+3-2=5, then _____= -5.

Integer subtraction is taken care of by ‘inverse’ idea.

Task 17: Distributive property of multiplication over addition and subtraction

Similar to the above task, find expressions which will be equal to 3×(5+4): 3×5+4 or 3×5+3×4. Students can verify by computation that the value of the expression remains same in the second case. 3 is the common factor which is distributed over both the terms inside the bracket.

Students can observe that in $3\times5+3\times4$, 3 is the common factor in both the product terms of the expression and can be rewritten as $3\times(5+4)$. This will also pave the way for combining product terms when they have a common factor.

The common factor can be positive or negative, that is, the bracketed product term can be positive or negative, for example, $-3\times(7+5)$. Expansion of this would involve using both the rules of removing bracket (distribute three over each of the terms inside bracket and change the sign of all the terms inside).

Examples:

a. Write equal expressions by removing the bracket. Identify the terms for the given and the obtained expression.

   $4\times(2+3)$, $(12-5)\times3$, $(2\times3)+5$, $-3\times(4+5)$, $-2\times(6-3)$

b. Make equal expressions for the given expressions (encourage putting brackets and removing brackets for making equal expressions):

   $9\times(3+5)-13$, $18-24+12$, $7\times13-8+7\times15$

Remarks:

Distributing common factor over the terms inside the bracket is same as taking 'multiple' of each term inside the bracket. Hence $4\times(2+3)$ is four times the expression $2+3$. Adding four times 2 (4×2 or 8) and four times 3 (4×3 or 12) is same as four times $2+3$ ($4\times(2+3)$). The value of the expression $4\times(2+3)$ is four times that of $2+3$, that is 20 is four times 5.

In case there is any doubt prefer to calculate and ascertain equality of values than reinforcing the rule. Also, one can capitalize on the meaning of the expressions.

Patterns as discussed in the previous task is useful.

If $2+3=5$, then $4+6=$?

Or if $4+5-2=7$, then $8+10-4=$?,

Or________$=14$

Task 18: Comparing expressions without computation

Compare two termed simple related expressions using the signs $<$, $=$, $>$. Students will use various strategies to figure out which expression is bigger, smaller or equal and explain their reasoning for the answers. For example, in $16+19$ __ $17+18$, the two expressions are equal. Students might say that '+17 is 1 more than +16 and +18 is 1 less than +19', where students have compared one of the expressions with the other by parsing the expressions as terms. They might also say that '+16 is 1 less than +17 and +18 is 1 less than +19', in which case they compared terms across the expressions.

Examples:

a. Compare the following expressions:

29+15 __ 30+15	46-18 __ 46-19
42+59 __ 43+58	39-12 __ 38+10
63+57 __ 65+56	74-26 __ 75-29

b. Whenever expressions are unequal, make the above expressions equal.

Remarks:

Examples need to be carefully chosen involving all kinds of order relations between them. Choose numbers which are not easy to calculate. Students should not be forced to reason in a particular way. Let them come out with as many strategies as they can. One would see more errors in expressions with negative sign. Gradually, drive the class towards symbolic reasoning by converting their verbal explanations to symbolic statements. Asking them to find out the magnitude of the difference between the two expressions might be helpful in the symbolization process. More complex expressions force students to use symbols to keep track of the changes in the terms.

Task 19: Find the value of the following expressions given the value of the related expression (without computation)

If 326+598=924, find 324+598.

This task is simple for students and they easily do it. This uses a similar reasoning as the earlier task.

Examples:

a. If 431+127=558, find

430+126	431+128
431+126	431+129

You can similarly make tasks with the negative operation.

Remarks:

This task can also be extended to algebraic expression as m+34=72, find m+35.

In this task also students should be encouraged to provide verbal or symbolic reasoning for their solutions.

Task 20: Fill in the blanks so that the expressions are equal (without computation)

This exercise is meant to reinforce the understanding of the relational meaning of '=' symbol. Students will use their knowledge of comparing expressions developed in the previous tasks to fill in the blank by a term. Using the balance metaphor is useful, that is, add to the side which is less or subtract to the side which is more. E.g. 35+29=35+27__. Here students have to think that 35+27 is smaller than 35+29 by 2. So add 2 to 35+27 to make them equal.

Examples:

a. Fill in the blanks.

27+36 = 25 __	573-378 = 575 __
34-16 __ = 35-15	76-58 = 78-59__

Remarks:

The observations made in task 21 hold true here also. The two different kinds of examples given use the same underlying notion of '=' and can be handled similarly. Some misconception about the '=' sign can be seen here too. Effort should be made to guide the students gradually to symbolic sentences from their verbal ones, justifying the solution.

These symbolic or verbal justifications are creating the ground for reasoning with expressions in the context of algebra.

Task 21: Combining product terms

Rules for evaluating expressions had been stated in task 10. But once students learn distributive property, they can combine product terms if they have common factor. This can be also applied in the context of algebraic expressions with common letter factor. For example, $3\times x+4\times x = (3+4)\times x$.

Use extracting common factors as an additional strategy and encourage students to evaluate/ simplifying expressions using easy ways. Mix arithmetic and algebraic expressions in the task.

Examples:

a. Evaluate/ simplify:

$3\times16+16\times12-16\times7$	$14\times3+10\times8+14\times7$	
$5\times5+8\times3+3\times5$	$12\times9+16\times5-17\times9$	
$4\times x+3\times x$	$12+3\times m+2\times m$	
$2\times a+4\times a+2\times b$	$24+4\times d-12-2\times d$	$x+13-5\times x+4$

Task 22: Evaluate by substitution

Ask students to evaluate algebraic expressions by substituting the value of the letter. This will reinforce the connection between arithmetic and algebraic procedures.

Examples:

a. For m=7, find the values of the following expressions.

$m-9$, $3\times m+9$, $8\times3-2\times m$, $7\times m-3\times10$, $2\times m+8-6\times m$

Task 23: Miscellaneous exercise

Make equal expressions

Examples:

Generate as many equal expressions as possible using all the rules and procedures you know. Use transformations so that they keep the value of the expression equal.

35-16+48 24-12+8 5×3-4×5+10

12×y+5-6×y+8

Remarks:

Here students are expected to explore and use all the valid transformations, like splitting a term as sum, difference, product or quotient, using brackets with a negative sign or by extracting a common factor, which will keep the value invariant. Discuss some responses in the class to see if they understand the generality of these transformations. They should explain their strategies for how and why the expressions are equal.

Task 24: Reasoning with expressions

In contrast to the above tasks where students reasoned about expressions, much of algebra learning is about reasoning with expressions.

Ask students to represent small situations with algebraic expressions. For example: A rod is x cm long and we join another which is 3 cm long, then how long is the new rod?

Examples:

a. Write algebraic expressions to represent the following situations.

Ram is 5 cm taller than Ravi. If Ravi is h cm, how tall is Ram?

Remarks:

These also would serve as a useful ground to prepare students to represent small situations mathematically.

They will reinforce the relational ideas that were established in the beginning.

Task 25: Challenging situations

Take situations which are challenging as well as involve manipulation of the algebraic expressions to arrive at conclusions. Using algebra only for representations is not very worthwhile.

Distance on the letter-number line is one such situation.

Examples:

a. Find the distance between x+3 and x+9, x+5 and x-3, x-5 and x-8

Remarks:

Finding the distance between points on the letter-number-line is difficult not because the manipulation involved on the algebraic expression is difficult but the representation is difficult. It is difficult for students to appreciate the need for the representation when the distance can be easily calculated from the number line or by imagining the number line.

Task 26: Think-of-a-number game

Think of a number. Multiply it by 2. Add 3. Subtract the original number. Subtract 2. Subtract the original number. What is the answer? Get the students to talk about the situation and listen to their arguments for why the whole class should get the same answer. Gradually help them to represent the situation symbolically. Last step is to manipulate the expression to show the desired result.

Examples:

a. Ask students to justify the answers they get in think-of-a-number game.

b. Ask students to make problems of the think-of-a-number game type, they can ask their peers in the class. Explain why everyone who follows the sequence should get the same answer. Discussions regarding think-of-a-number game are very essential before they represent it symbolically and manipulate it to justify their conclusion. They do not get a sense for why they should prove something and what should they prove. The idea of justification/ proof is not natural for students.

Task 27: Pattern-generalizing

Take geometric shapes and patterns which can be generalized.

Examples:

a. Identify as many rules as possible to find the general rule of counting/ predicting dots/ matchsticks for the n^{th} figure.

b. Show that all the rules are equivalent.

Remarks:

Discussions with the students regarding these patterns are essential. They will need help in writing the general rules using brackets appropriately. They will also have to explain verbally first why they think all the different rules should give the same result or are equivalent. There can be various arguments, using examples or showing the equivalence of expressions by manipulating them.

Task 28: Calendar patterns

Identify patterns in the calendar.

Represent relations between the numbers in the calendar and represent them using letter.

Similarly, other patterns existing between numbers and operations should be explored. For example, in three consecutive numbers, 1^{st} number+3^{rd} number=2×2^{nd} number

Examples:

a. Identify different patterns that exist between the numbers in the calendar. See how many of those are always true. Show/ justify.

Remarks:

The earlier tasks would prepare the students to handle this task which has many requirements - using letters to write the generalized relationships between the numbers and then to identify patterns and to prove them.

Appendix – VII: Synopsis

1.0 *Introduction*

Algebra as a domain in mathematics occupies a special position as a major analytical tool leading to higher mathematics and many other branches of science. It provides the symbols and techniques to represent and solve problems, and to reason, justify and prove within mathematics and other areas where mathematics serves as a tool. Many students fail to succeed in algebra and are therefore unable to enroll in advanced mathematics which is a gateway to many prestigious professions as well as academic careers. Thus, it is important to understand the conceptual changes which the students experience while moving to the middle school, especially due to the introduction of algebra, and identify ways to address the problems which arise in the course of its introduction.

In contrast to arithmetic, algebra poses a challenge to most students due to the new symbols it proposes and new ways of acting on those symbols. The notations and the conventions are both problematic and are not easily learnt by students. Further, it takes the students away from operations on numbers to computing with abstract symbols. It is no longer possible to process the symbols in an expression as a strict sequence of binary operations, ending in a numerical answer (Booth, 1988). The symbols need to be reinterpreted in new ways before they can be worked upon. The presence of the letter symbol complicates the situation as students do not understand the meaning of the letter as a number and either ignore it or consider it to have some fixed and arbitrary value or construe its meanings based on common appearances of the letter in many situations outside the domain of mathematics (Kuchemann, 1981; MacGregor and Stacey, 1997).

There are also differences in approaching problems in arithmetic and algebra. While in the arithmetic approach students can work from the known conditions and find intermediate numerical solutions to arrive at the solution to the problem, it is essential in the algebraic approach to use expressions to represent the problem situation using a letter for the unknown (Bednarz and Janvier, 1996; Stacey and MacGregor, 1999). Thus, in the context of arithmetic, students do not appreciate the purpose of recording operation sequences or representing problem situations as well as do not abstract the properties and rules of transformation which can be consistently applied while manipulating expressions (Booth, 1988). They only implement procedures for finding the numerical solution to a problem (posed using symbols or embedded in word problems) which may depend on the context or the numbers involved, and thus do not engage in general solution methods applicable over a range of problems (Ursini et al., 2001). The methods of teaching and learning generally used force the students to rigidly follow algorithms without any space for reflecting on them and for exploring properties and relations between numbers and operations. This is unhelpful to students in understanding the equivalence of different procedures, or their generalizability, making it difficult to shift to algebra. Students' poor skills in representing problem situations and weak understanding of transformation of expressions do not allow the students to move to the step of deducing or inferring about the situation, which is the crux of algebra (Booth, 1989).

In India, teaching of algebra generally follows arithmetic in the curriculum, which also would be the case with many other countries. Research over the last few decades has shown the complexities involved in the transition from arithmetic to algebra as described above and the interference in the learning of algebra from arithmetic. Some studies have cautioned against emphasizing the arithmetic-algebra connection as it leads to many misconceptions and is fraught with pedagogical hurdles (e.g. Lee and Wheeler, 1989). Others, in contrast, have pointed out the promises offered by focusing on the connection (e.g. Linchevski and Livneh, 1999; Carpenter et al., 2003). Although many research studies have explored the arithmetic-algebra connection and have identified the cause of many of the troubles in the teaching and learning of algebra, as will be briefly discussed below, there does not appear to exist a well elaborated model of teaching and learning symbolic algebra in the beginning grades which can help students build the connection between the two domains and handle the problems identified in the literature. This study aimed to develop a teaching approach which could bridge the gap between arithmetic and algebra and create meaning for symbols through two broad sets of activities: working with syntactic transformations and working with contexts that lend purpose to algebra. In the process, the study engaged in analyzing students' responses to the various tasks, and identifying the nature of the support that is required to make the transition. This fed back into the development of the teaching module, thereby evolving and clarifying the approach that facilitates students in making the transition.

1.1 The arithmetic algebra connection

Students' earlier experience in primary school arithmetic is largely one of computing single binary operations and the first exposure to multiple operations is in the context of evaluating arithmetic expressions which encode a sequence of binary operations. This requires following conventions in the form of order of operations so that a unique value is arrived at for each expression, even in the absence of brackets. Such tasks form the first connection between arithmetic and algebra where algebra encodes general rules and properties of operating on these arithmetic expressions, like the commutative, associative and distributive properties, which govern the nature of transformations that are possible on the expressions. Algebra provides the letter symbols to mathematically represent these properties in general terms and it is these properties which determine the rules of transformations for algebraic expressions, and which keep them equivalent. The conventions for operating on the expressions are so designed that they encode the structure of the expressions. Very often students fail to see this connection between arithmetic and algebra and thus are unable to make the required transition to algebra.

1.2 Hurdles in the transition to algebra

Students are habituated through arithmetic to obtain a 'closed' answer or a single number as the result, which leads them to misunderstand notations like $3+x$ and $3x$ as being equivalent. Expressions such as $3+x$ have multiple meanings in algebra (Wagner and Parker, 1999) and it is necessary to treat them as both processes and products/ objects or as flexible 'procepts' (e.g. Sfard, 1991; Tall et al., 2000). For example, $3+x$ can both be understood as a process of adding any number to 3 or the result of this process, namely, the sum of three and any number or three more than any number. Also, the '+' and the '–' signs can be thought of as operations of adding on or taking away (the most common meaning developed in arithmetic), as signs attached to a number used for representing change (increase or decrease) or as encoding a relation of more or less. Similarly, the '=' sign is to be treated as a sign denoting equality or equivalence rather than as an instruction to compute, which is a meaning familiar from the arithmetic context. Whereas in arithmetic, an expression has a fixed meaning and denotation (value), in algebra it is important to separate the denotation of the expression from its meaning which describes the relation embedded in it; because the algebraic expression can be interpreted in various ways depending on the context. One must possess the ability to pay attention to these aspects flexibly, emphasizing one over the other depending on the context, which is a central point in developing algebraic awareness (Mason, 1996; Arzarello et al., 2001).

Many difficulties which students face while manipulating algebraic expressions can be understood by focusing on their understanding of arithmetic expressions and computations in arithmetic. Researchers (e.g. Chaiklin and Lesgold, 1984; Kieran, 1989, 1992; Linchevski and Livneh, 1999) have pointed out that the roots of the problem lie in students' lack of awareness of the structure of arithmetic expressions. This does not allow the students to understand the properties of operations which can be consistently used in arithmetic contexts and which can be subsequently generalized to deal with symbolic algebra. Students often fail to judge the equality/ inequality of expressions like 345-237+489 with 489+345-237 or 237-345+489 without computation (Chaiklin and Lesgold, 1984) and are inconsistent while evaluating arithmetic expressions. They sometime solve an expression 50-10+10+10 as 50-30 and at other time would solve the expression 27-5+3 correctly as 22+3 (Linchevski and Livneh, 1999). Further they do not see a way of computing the expression 217+175-217+175+67 other than solving step-by-step from left to right and would even be tempted to cancel the '175's (ibid). These are errors due to misperception of structure of the expression and over generalization of rules of order of operations and the same are transferred while working on symbolic algebra. The rules of transformation are for the first time formally defined in algebra but do not make sense to them due to lack of a referent for the letter and validation of the rules, making the students feel that the rules of symbol manipulation are arbitrary. Thus, the reason for arbitrariness or meaninglessness which the students experience during their exposure to algebra is not due to applying or emphasizing rules of transformation, but due to the lack of emphasis on structure of expressions, making appropriate links with properties of operations and explanations for the rules, like distributivity, associativity (Kirshner, 2001). This cannot be simply solved by practicing manipulation of algebraic expressions but through specialized activities focusing on articulating and justifying the usage of rules in the classroom (ibid).

1.3 Approaches to teaching of algebra

Researchers' concern with students' poor understanding of properties of operations and structure of expressions and their resulting failure to deal with algebraic symbolism and its meaning and purpose led to various reconceptualizations of algebra. Many efforts have been made through research studies to convey to the students the essence of algebra and make sense of the symbols and operations on them. This includes introducing algebra through meaningful contexts like pattern generalization, using concrete materials, embedding algebra in problem solving situations, using technology supported approaches like spreadsheets, LOGO, CAS. The various approaches to algebra emphasize different aspects of algebra and may have certain limitations. Some of these approaches focus on creating meaning for the symbols, especially the letter, and the purpose of algebra, leaving the syntactic transformations to be handled by software. However, it has been realized that some basic understanding of symbols and syntax is required to make sense of the rich problem solving contexts or even judge if technology assisted solutions are correct or to use technology profitably in solving problems (Kieran, 2004). Further, it has been argued that when students work with syntactic transformations, they create meaning for the symbols by using them and acting on them. Therefore, separating the contexts in which meaning of the symbols are created from the syntactic aspects of algebraic symbols is not very helpful and both the competencies are required, which is the emphasis in this study.

Studies have also introduced algebra through the route of generalized arithmetic, which focuses on the structural aspects of the number system (Wagner and Kieran, 1989) and encodes the general rules of operations in arithmetic (Kaput, 1995). The "early algebra" studies by Kaput (1998), Carraher et al. (2000, 2001, 2003), Brizuela et al. (2000) and Carpenter, Franke and Levi (2003) are also efforts in the same direction, demonstrating in the process young children's capabilities to understand symbols, to create them, to work with them and explain their reasoning and solution process. The generalized arithmetic approach is not limited to generalizing regularities in operations and patterns which is a major focus in the "early algebra" studies. It also encompasses a deeper understanding of the structure of expressions which is another line of work used in the studies with students in middle school. These studies try to enhance students' understanding of symbolic expressions and syntax of algebra, which is also the approach adopted by the present study. These researchers have attempted to exploit the arithmetic-algebraic connection, by focusing on the similarities in the two domains in different ways: (i) correct parsing followed by order of operations and exploration of properties of operations (e.g. Thompson and Thompson, 1987), (ii) procedural/ computational similarity (e.g. Liebenberg et al., 1998, 1999a, 1999b; Malara and Iaderosa, 1999; Livneh and Linchevski, 2003) or (iii) representational/ notational similarity (e.g. Booth, 1984; Malara and Iaderosa, 1999). Except for the study by Thompson and Thompson (1987) which actually trained students to perceive the structure of expressions and appreciate the constraint of certain transformations but in a limited situation, the other studies focused largely on computational features and their generalizations to make the transition to algebra. This always did not lead to the desired effect on the students and they still failed to see the equivalences in the transformation rules in arithmetic and algebra and continued to work on algebraic expressions similar to computational arithmetic without abstracting properties and constraints of operations. The present research study builds on these insights from the literature and proposes a way to deal with the arithmetic-algebra connection and tackle the errors due to faulty perception of structure of expressions which have been found to be hurdles in understanding symbolic algebra.

2.0 *Defining the research study*

The research study being reported here is a design experiment on grade 6 students from two schools in Mumbai. It tried to systematically investigate the arithmetic-algebra connection and explored the introduction of algebra as generalized arithmetic by enhancing and connecting students' prior knowledge of arithmetic to algebra and exploiting the structure of arithmetic expressions to learn algebra. In the process it aimed to identify precisely the arithmetic concepts and tasks which would help in making the transition to algebra. Its objectives were to strengthen both *procedural knowledge*, that is, the calculus of algebra – knowledge of rules, conventions and procedures for working on expressions, and *structure sense* – ability to think of an expression as having a value, to identify the components of an expression (surface structure) and to see the relationships of the components in an expression among themselves and with the value of the whole expression (systemic structure) (Kieran, 1989; Hoch and Dreyfus, 2004).

The teaching-learning sequence was not restricted to generalizing properties of operations from arithmetic by emphasizing the structure of the expressions. It was complemented by using tasks which took a more comprehensive view towards generalization – exploring and finding relations among numbers/ quantities, sequence of operations and shapes in patterns, representing and generalizing them and justifying and proving some of the patterns. These tasks provided opportunities to translate the informal processes or arithmetic structures into formal arithmetic or algebraic sentences, which is essential for an algebraic way of thinking. Thus, students learnt the syntax and rules of transforming expressions, with numbers serving as referents for the letter; together with the use of expressions as tools for generalizing, proving and justifying in problem situations. For a complete sense of algebra, one would need to build an understanding of both the syntactic (based on structure of expressions/ equations and rules which define the nature of possible transformations) and the semantic (based on meaning of the letter/ expression/ equation as derived from symbolic statements and problem situations) aspects of algebra. For example, one not only needs to understand the constraints on the possible transformations of the expression 12+3×5-18 but also appreciate the change in value when the expression is slightly changed, say, 3+12×5-18 whose explanation will require a semantic understanding. This kind of knowledge would also help while representing a situation using arithmetic or algebraic expression (e.g. distinguishing a representation $x+3\times2-5-x+4-x$ from $(x+3)\times2-5-x+4-x$).

Students were first engaged in reasoning based on syntactic transformations of expressions like evaluating expressions, identifying correctness of an evaluation procedure for an expression and justifying it, comparing and identifying equality of expressions (e.g. 23-14×34+65 and 23-14×65+34) and its implications for evaluating/ simplifying expressions. These tasks did not require students to generate the symbolic expressions but only to reason about equivalence or non-equivalence of symbolic expressions in various computational and non-computational situations based on rules of syntactic transformations. Hence these tasks are included in the category of reasoning *about* expressions. The purpose of engaging students in activities which required reflection on rules of transformations was to begin the separation of the meaning of the expression from the value of the expression in the context of arithmetic itself, where this is not essential but lays the ground for further algebra learning (see Arzarello et al., 2001). Disparate looking expressions could have the same value with different information/ relation contained in them and similar looking expressions could have different values. Moreover, the familiar arithmetic symbol system was used in the teaching approach as a 'template'[10] for the development of the new algebraic symbolism. It enabled the numbers to be gradually replaced by letters, initially understanding algebraic expressions as only computational processes (inventive-semiotic stage of Goldin and Kaput, 1996); before interpreting them based on the structure of arithmetic expressions (period of structural development of Goldin and Kaput, 1996). It is only after this that algebraic expressions and symbols can be considered independently as objects with certain properties which can represent other entities and can be acted upon (autonomous stage of Goldin and Kaput, 1996 and object mediated phase of Sfard, 2000).

[10] The word 'template' is derived from Sfard's (2000) distinction between 'template-driven' phase and 'object-mediated' phase in the development of new signifiers/ symbols.

The tasks based on syntactic transformations exploiting the structure of the expressions could only help students to move from the 'inventive-semiotic' phase to the phase of 'structural development' but not to the 'autonomous' or 'object mediated' stage. To lead the students to this stage, they were engaged in a set of tasks which developed a culture of generalization, justification and proving, where algebra was treated as a tool for representing general relationships and concluding through manipulations on them. These tasks required the knowledge of rules, conventions and procedures for working on them and have been categorized as reasoning *with* expressions. However, it is important to note that the transition to the 'autonomous' or 'object-mediated' stage through reasoning *with* expressions is not the only way, this being considered most appropriate for this study. In fact, reasoning *about* expressions can itself lead to this advanced stage (e.g. complex operations on algebraic expressions, thinking of expressions as functions and exploring changes and transformations in functions).

The study also intended to observe and characterize the changes in students' understanding of algebra in the context of the teaching sequence which was to develop as a result of repeated attempts to make it more coherent. The study did not aim to compare the efficacy of the instructional approach being discussed with other approaches. It aimed instead, at an internal understanding of its effectiveness by exploring the changes in students' understanding and thinking processes as they developed new concepts and tools through interaction with the instructional sequence, and the possibilities it gave rise to in terms of student responses and the use of various concepts and procedures in different tasks.

3.0 *Designing the study*

The study was a design experiment (Cobb et al., 2003; Shavelson et al., 2003), the teaching-learning sequence evolving over five trials between 2003 and 2005. Design experiments are carried out in educational settings based on prior research and theory seeking "to trace the evolution of learning in complex, messy classrooms and schools, test and build theories of teaching and learning, and produce instructional tools that survive the challenge of everyday experience" (Shavelson et al., 2003, pp. 25). The first two trials (PST-I and PST-II) were preliminary and more exploratory in nature and the last three trials formed the main study (MST-I, MST-II, MST-III) which aimed at making the teaching learning sequence coherent. The teaching learning sequence co-evolved with the developing understanding of the research team about the phenomena under study as well as with the growing understanding of the students as evidenced from their performance and reasoning on various tasks.

3.1 Research questions

The study aimed to address the following research questions:

- What kind of arithmetic understanding would help in learning symbolic algebra?
    - How should the teaching of arithmetic expressions be restructured to prepare for a transformational capability with algebraic expressions?
    - How effective is such a teaching learning sequence in understanding beginning syntactic algebra?
    - Which tasks of the ones identified are more effective in making the shift possible from arithmetic to symbolic algebra?
- Does understanding the syntax and symbols of algebra support students in understanding the purpose of algebra and in the application of algebra for generalizing and justifying?
- What meanings do students attach to letters, expressions and syntactic rules of transformations in this learning approach?

- How do procedural understanding and structure sense of expressions mutually support one another?

3.2 Sample

The study was conducted in the research institute with grade 6 students (11-12 year olds) during the vacation period of the school in Summer (April-May) before the beginning of the school year and mid year (October-November). Each trail lasted for 11-15 days, with each session of approximately one and a half hour. Grade 6 is the first level when algebra is introduced to the students. The students came from nearby English and vernacular (Marathi) medium schools which catered to students from low and middle socio-economic backgrounds. Five schools were involved in the study at various stages of the programme but only two schools participated throughout the study. The choice of the schools was based on convenience; the first reason being their proximity to the centre and the second, due to a need for long term collaboration and support from the school to carry out the study. Students from these schools volunteered to attend the programme by filling in an application form distributed in the schools before the vacations. The final group of students attending the programme was randomly selected from the applicants. In the last two trials (MST-II and MST-III) the same students who attended MST-I were invited to attend the programme. 31 students (15 English medium and 16 Marathi medium students) who participated in all the trials of the main study were chosen for the final data analysis. The students had just appeared for their grade 5 year end exams when they came to attend MST-I and they completed grade 6 during MST-III. The pre-test performance at the beginning of the first main trial showed that the Marathi medium students were better than the English medium students in their knowledge of arithmetic. The English medium students did not undergo any algebra teaching in their school but the Marathi medium students were exposed to preliminary symbolic algebra in the school. The teaching was carried out with multiple groups (two to three) in each trial to see how the different groups responded to the same teaching sequence. The English group had between 20-30 students and the Marathi group had between 30-40 students in each trial.

3.3 Data collection and analysis

The data was collected through pre and post tests, interviews with a subset of students after the second and the third trial of the main study (MST-II and MST-III) which were video recorded and later transcribed, video recording of the classroom proceedings, students' daily work and teacher's log of the daily classroom processes. The post tests were long, containing approximately 25 questions and took around 2 hours to complete. The interviews (14 students after MST-II and 17 after MST-III) were held 8 weeks after the end of MST-II and 4 months after the end of MST-III. The tasks used in the interview were similar to the post tests and were restricted to only arithmetic expressions after MST-II, whereas it included both arithmetic and algebraic expressions and context activities after MST-III.

The data was analyzed both quantitatively and qualitatively with a focus on the nature of responses, the type and number of errors and the students' reasoning as inferred from their responses to tasks or from their explanations given in the interview. The analysis was carried out to ascertain the extent of students' understanding of concepts, rules and procedures in different tasks:

- Understanding of procedures – Evaluation/ simplification of arithmetic and algebraic expressions

Students were asked to evaluate arithmetic expressions which were of two kinds: simple expressions like $3+4\times5$ and $13-6+4$ or complex expressions like $-28+49+8+20-49$ or $7\times18-6\times11+4\times18$ finding easy ways of evaluation. These exercises laid down the rules for operating on expressions and understanding the constraints on operations and dealt with the application of certain procedures on expressions, both arithmetic and algebraic to lead to numerical answers or simpler expressions.

- Rules for transforming expressions with brackets

Brackets were an important concept and were understood both as a precedence operation as well as connected to equality of expressions using bracket opening rules, for example, 23-(9+5)=23-9-5. This flexible understanding of brackets is important for algebra as the first meaning as precedence operation is used for purposes of representation and the second meaning associated with equality is needed to simplify algebraic expressions.

- Understanding of structure – tasks based on '=' sign (comparing simple expressions and filling the blank), identifying expressions equal to a given expression from a list without computation, generating equal expressions

A task that was used to develop the understanding of '=' sign was filling in the blank by computation so that the expressions are equal, e.g. 23+5=__-2. Many of these tasks deemphasized computations and instead focused students' attention on the structure of expressions, identifying relations among expressions and within an expression (e.g. 234+345=233__) in the process using students' intuitive understanding of operations and simple transformations like increasing and decreasing number/ terms, changing numbers and signs. Another set of tasks dealt with identifying and generating expressions equal to a given expression. For example, given the expression 23+34×15+42, which of these are equal: 34+23×15+42 or 15×34+42+23. Later these tasks also were extended to include algebraic expressions.

- Context based tasks – letter number line, calendar patterns, think-of-a-number game, pattern generalization

The tasks used in this section dealt with observing and expressing generalities using algebraic expressions and subsequently with justifying/ proving them which required manipulating the expressions. The letter-number line was a generalized representation of the number line with the use of a letter, which was further used to carry out two tasks: journey on the letter-number line and distance between two points on the letter-number line. Calendar patterns required the students to represent the simple patterns between the numbers in a calendar using the letter and then explore various patterns in the arrangement of the numbers and justify them. Think-of-a-number game required the students to follow a set of instructions on a number and explain and justify the pattern in the answer with respect to the starting number. Pattern generalization involved the students in representing a general rule for the growing pattern in a sequence of shapes.

Through an analysis of these tasks, students' understanding of '=' sign, order of operations, transforming expressions, meaning of letter and expression, their ideas about representing a situation using the letter and manipulating the expression to arrive at a conclusion were explored. The effort was to examine students' use of the concepts and rules that they had learnt during the program and the extent to which their learning facilitated performance on various tasks. The analysis gave a sense of the nature of the concepts required to make the transition from arithmetic to algebra and allowed one to gauge the effectiveness of the teaching approach in enabling students to make the transition from arithmetic to algebra and in understanding the purpose of algebra.

4.0 *The teaching learning sequence*

The study intended to explore the arithmetic-algebra connection building on the structure sense of expressions but the exact nature of the tasks, procedures and concepts, which would enable the transition from arithmetic to algebra, were to evolve through the teaching. The teaching sequence evolved over five trials between 2003 and 2005 with multiple groups of students. In the paragraphs below, is described the gist of the understanding arrived through the engagement with the process.

4.1 The framework

The following general principles guided the development of the instructional approach.

- Using students' understanding and intuitions/ anticipations in the context of arithmetic to guide their learning of algebra
- Developing students' understanding of algebra by using and extending their experiences with symbols in arithmetic in specific ways
- Reasoning as a basis for learning

Students' knowledge of arithmetic was used as a foundation on which algebraic formalisms could be built. In this study, students' understanding of syntactic rules and conventions was developed and consolidated using their anticipations with respect to operations on numbers, thus tackling the pedagogical problem of teaching the syntax of algebra. By the end of primary school, students have had sufficient experience with numbers and basic operations, and are likely to have attained a level of familiarity and concreteness, which can be fruitfully employed to learn formal symbols and actions on them. Some of their expectations/ anticipations are correct (like, addition of two numbers can be done in any order) and some are wrong (like, subtraction of two numbers can be done in any order) which need to be brought to their notice and which they may be unable to correct by themselves. It was important for this teaching-learning approach to be aware of students' expectations and identify the situations which invoke these expectations, so that they can be gainfully employed to understand the meaning of operations, properties of operations and constrains on transformations. It is in this context that students were engaged first in comparing simple expressions (e.g. 234+436 and 235+437 or 428-129 and 429-128) which required them to make explicit their expectations regarding the operations of '+' and '–'; and then identifying equality of expressions, like 34+13×25+49 with 13+34×25+49 or 25×13+49+34 without computation. Discussions about possibilities and constraints of transformations (that is, about commutativity, distributivity and associativity) are critical in these situations.

The approach not only attributed meaning to the symbols by working on various tasks but also used them in communicating understanding. New ways of interpreting the familiar symbols were created in the context of arithmetic expressions that could subsequently be transferred to algebraic expressions. The students were made to focus away from computations and instead asked to attend to the information or description of relation that is contained in the expressions (e.g. 4+3 is not just 7 but also a relation 'three more than 4' or 'sum of 4 and 3'). Further, the numbers were attached with the signs preceding it to denote a signed number (like -2, +3), which could also represent a change (increase and decrease) in a state. This enabled students to move from an interpretation of expression as encoding a sequence of binary operations to focusing on the units in the expression as contributing to the value of the expression by increasing or decreasing it by certain amounts. This proved to be a very important concept while judging equality of expressions from a list to a given expression as stated in the previous paragraph. Students' expectations and understanding of symbols were tied together by engaging the students to discuss and reason *about* and *with* them.

The connection between arithmetic and algebra was established by building the content which had the following characteristics:

- Exploiting structure sense of expressions
- Use of structural concepts (Terms and '=')
- Explicating connections between arithmetic and algebra

The arithmetic algebra divide was bridged by exploiting the structure inherent in arithmetic expressions to connect arithmetic with algebra using the familiar symbols, thereby giving the letter a referent of number, and also by explicitly giving visual and conceptual support to the students to perceive the structure of an expression correctly. The visual cues allow the perception of the surface structure which is important to analyze the components/ units of the expression or equation. Understanding of systemic structure is required to act on the interpretation of the surface structure. In particular, understanding the '=' sign, equality of expressions and properties of operations are important aspects of structure sense. The reason for emphasizing the structure of expressions in the teaching approach was to link procedures with a sense of structure, so that instead of being two separate skills one following the other, they complement each other to form an integrated knowledge structure. Knowledge of structure of expressions provides scope for flexibly exploring procedures and strategies for computing expressions rather than applying the conventional rules for evaluation, which are rigid. This is an important characteristic of the approach taken in the study and which distinguishes it from earlier efforts (e.g. Livneh and Linchevski, 2003; Liebenberg et al., 1999a) of using arithmetic for teaching algebra.

The above was made possible by providing the students with a set of concepts, namely 'term' (e.g. terms in $12-5\times3$ are $+12$ and -5×3) and 'equality', which allowed them to correctly identify the units of the expressions and further understand the contribution of each part of the expression to the value of the whole expression. This approach is described as the 'terms approach'. The terms could be simple term (e.g. $+12$) or complex term (e.g. product term: -

5×3 or bracket term: -(4+6)) Also, these concepts helped in reformulating the rules for order of operations and bracket opening in structural terms, thus integrating the procedures more closely with structure of the expression. The precedence rules of evaluating expressions were replaced by the structural counterpart of flexibly combining terms. One could combine only simple terms and the product term had to be converted to a simple term before combining with the simple term: $4+5\times2 = \boxed{+4}\ \boxed{+5\times2} = \boxed{+4}\ \boxed{+10} = \boxed{+14}$. Else, two product terms could be combined if they had a common factor using the distributive property. It can be easily appreciated from the above that the value of the expression 5×2+4 will be the same as 4+5×2 but the value of the expression 5+4×2 will be different. Reordering the terms kept the value of the expressions invariant and thus terms could be combined in any order. In this way, the familiar processes of addition, subtraction and multiplication were converted into 'objects' (operations on signed numbers), not necessarily requiring computation at each step and could be combined flexibly by attending to the relationships between the terms in an expression. Thus, students were moved from 'computing with numbers' to 'computing with expressions' using properties of operations. Bracket was another important concept which was given a dual treatment: as precedence operation and a dynamic use connected with bracket opening rules and equality of expressions.

The terms approach not only created meaning for the operations but also afforded a more direct approach to tackling the structural errors (like, computing sequentially from left to right in the presence of a multiplication sign as in the above example 'LR' or detaching the negative sign, 24-6+4=24-10) and other inconsistencies in evaluating expressions which have been widely cited in the literature (Chaiklin and Lesgold, 1984; Linchevski and Herscovics, 1996; Linchevski and Livneh, 1999; Kieran, 1989). Further, this paved the way for learning manipulation of algebraic expressions which requires the flexibility in perceiving the information and interpreting the relationships embedded in an expression, so as to be able to operate on them. Essentially, the manipulation of algebraic expression follows the same rules of transformation as in arithmetic. All these reconceptualizations with respect to arithmetic allowed students to reason *about* expressions by engaging in discussions with respect to syntactic transformations and ideas of equality and invariance of value, without computation. Thus, on the one hand arithmetic operations were being reified, and on the other, understanding of algebraic manipulation was being developed on this understanding of arithmetic.

Contexts for algebra: The students were later introduced to the use of expressions in the contexts of generalizing, explaining and justifying (reasoning *with* expressions). The main ideas that students needed to grasp in this part are (i) the importance of representing situations for general cases, (ii) knowing that justification/ proof needs a general argument/ explanation (verbal or symbolic) not specific to particular cases, (iii) appreciating the purpose of transforming an expression, (iv) transforming the representation using valid rules and (v) interpreting the result. Students, in this study, were first engaged in simple representation tasks similar to the CSMS (Kuchemann, 1981) test items so that they could learn that representations could be made when all quantities were not given, with the letter/s denoting one or more of the unknown quantities in the situation. Continuing with the spirit of a generalized arithmetic approach that was adopted, students worked on tasks, like the letter-number line, think-of-a-number game, calendar patterns and generalization of growing patterns among shapes.

4.2 The development process

As the study evolved, some of the initial assumptions were modified to enhance the effectiveness of the sequence. The thesis discusses these modifications and the rationale for them. For example, the teaching-learning approach began with the assumption that to teach algebra one only needs to worry about building the structure sense for expressions. The first trial itself (PST-I) led to the modification of this assumption and more efforts were directed at consolidating the procedures of evaluating and transforming expressions and bracket opening rules in the second trial. In the first trial ‘term’ and ‘equality’ were found to be two concepts which had the potential to connect arithmetic and algebra. The second trial (PST-II) involved a two group experimental design to explore the extent of effect of arithmetic knowledge (procedure and structure) on algebra learning. The concepts of term and equality were used in this trial only for structure tasks, again separating procedure and structure of expressions resulting in a separation of arithmetic and algebra and a limited understanding of algebra. Discussion of the two pilot trials and a preliminary discussion of the results of these two trials can be found in Subramaniam and Banerjee (2004). In an effort to make the arithmetic algebra connection stronger in the third trial (MST-I), the concept of terms was used for both procedure and structure tasks. Terms were given visual salience by putting them in the boxes (e.g. the terms of 19 – 7 + 4 are $\boxed{+19}$ $\boxed{-7}$ $\boxed{+4}$). The rules for manipulating expressions in arithmetic and algebra were formulated differently, and on hindsight, these rules were not flexible enough and did not exploit the potential of the concept of terms fully. In the context of arithmetic expressions, ‘terms’ were used only to analyze the expressions before deciding the rule to be applied to evaluate it. In the context of algebraic expressions, ‘terms’ were used to identify like terms before adding or subtracting them by imagining them to be sum or difference of ‘singletons’ ($3\times x+4\times x = \underline{x+x+x}+\underline{x+x+x+x}$, Linchevski and Herscovics, 1996). The complementarity of procedure and structure could not be established and the connection between arithmetic and algebra did not get abstracted by the students. Simultaneously, contexts (like letter-number line, area and perimeter) were created to give meaning to the letters, using it to represent general relations so that students accept the non-closure of algebraic expressions. Students’ poor knowledge of transforming algebraic expressions was a hindrance in using that knowledge in these tasks and they could not make sense of the use and purpose of algebra in the contexts.

The fourth trial (MST-II) was devoted to making the teaching-learning sequence coherent and radicalizing the structural treatment by making terms and equality as the key concepts which bound the whole sequence. The rules were made flexible and uniform across the domains and were structurally reformulated. Integer operations were also subsumed in the ‘terms approach’. This was the first time that the precedence rules were completely done away with and was replaced by the idea of combining terms (which is nothing but adding integers) which has been briefly described in the previous section. New tasks like evaluating expressions using easy ways which required students to flexibly combine terms to minimize the steps for computing (e.g. -28+49+8+20-49 or 7×18-6×11+4×18), and generating equal expressions for a given expression (e.g. 25-3×5+18) were created which utilized the complementary nature of procedure and structure sense. The connections between procedure and structure sense and between algebra and arithmetic were established more securely. Efforts were also made to enable students to make sense of these algebraic symbols in contexts (like letter-number line and calendar patterns) and use them as a tool for solving problems. The last fifth trial (MST-III) was used for consolidating the teaching-learning process. It emphasized verbalization and articulation of various procedures and rules of evaluating/ simplifying expressions, rules of opening brackets and use of the concepts and rules learnt up till now in different tasks requiring reasoning. Students were also encouraged to explain and articulate their understanding of patterns in numbers and figures and generalize them verbally in the contexts created to embed algebra before moving to symbolic representations. This led to the opening of another dimension of the arithmetic algebra connection and needs further work.

5. Findings and conclusion

The study evolved from the indications made in various studies about the importance of building structure sense for arithmetic and algebraic expressions and the need to move away from computations to be able to connect the two domains. The analysis of the data revealed that the *radicalized* structural treatment of arithmetic (as is seen by the end of MST-II) with a deeper understanding of expressions and constraints and possibilities of transforming them enabled the transition to algebra by allowing flexibility in computing expressions. Identifying relationships between and within expressions and finding conditions for keeping the value of an expression invariant were the key ideas here. The effects of this approach are further elaborated below.

5.1 Procedural tasks

The students improved their overall performance in the procedural and structural tasks and understanding of rules. The students gained in flexibility while evaluating simple expressions (e.g. 3+4×5 or 13-5+7) and the more complex expressions (e.g. -28+49+8+20-49) finding easy ways of computing them, indicating their appreciation of the structure of the expressions and the ability to take advantage of it. There was a reduction in structural errors (due to faulty parsing, like 'LR' and detachment) but they did resurface in more complex situations, suggesting the lack of automaticity among students in the simpler contexts. Integer operation was another weak point resulting in low performance of the students in some items. The interviews and classroom discussions indicate that the students could avoid structural errors in the simple situations and were aware of uniqueness of the value of the expression even though one could use multiple ways of evaluating them. These achievements of the students were significant in the light of the results reported in the literature (cf. Liebenberg et al., 1999a; Malara at al., 1999). The flexibility in manipulating arithmetic expressions together with correct perception of structure of expressions paved the way for the manipulation of algebraic expressions.

By the last trial, most students were comfortable with simplifying algebraic expressions (e.g. $3\times x+4+4\times x-5$), applying the same rules as in arithmetic. Interviews with the students with respect to algebraic expressions after MST-III revealed their awareness of equivalence of all the steps in the process of simplification. For example, the expressions $3\times x+4+4\times x-5$ and $7\times x-1$ are equivalent and so are the steps in between. Although most students were able to evaluate algebraic expressions for a given value of the letter even when they made sign and calculation errors; a few students, however, failed to substitute the letter by a number till the last trial. The students interviewed did not show any such difficulty.

The appreciation of the similarity between manipulating arithmetic and algebraic expressions was a difficult task and developed only in subsequent trials when attempts were made to focus away from computation in the context of arithmetic. Consistency in perceiving the structure of expressions and understanding the properties of operations that can be used in the context of arithmetic is an important step to move to algebra. The coherence in the teaching-learning sequence which was developed by MST-II (discussed in section 4.2) could be a factor influencing the change as is seen by the end of the last trial. The students successfully generalized their understanding of rules of simplification from the context of evaluation of arithmetic expressions to simplification of algebraic expressions, displaying the connection between the two domains in their understanding.

5.2 Rules of transformation of expressions with brackets

In the course of the program, more students learnt to use bracket opening rules to evaluate expressions but for some students, this was accompanied by a lack of appreciation of the meaning of the bracket as enclosing parts which have to be given precedence in operation. Both the written test and interviews revealed that the two notions of bracket were absorbed by some of the students as procedures and not as 'procepts' which did not allow them to anticipate the effect of removing and putting the brackets. They failed to simultaneously understand that the bracketed (sub-)expression could be substituted by either a number or another equal expression, which is an indication of an evolved 'proceptual' understanding, which is useful for generating representations for problem solving. Students made more errors when the bracket was preceded by a negative sign rather than multiplication sign. Additional suggestions about ways of dealing with the brackets which emerged as the structural approach evolved are described in the thesis.

5.3 Structural tasks

These tasks revealed students' deeper understanding of expressions. Students' responses revealed a fair degree of understanding of constraints and possibilities of transformations, properties of operations and anticipation of the result of those operations. They understood that terms can be rearranged to keep the value same or they can be changed in ways that the net result does not change, rearranging the signs or numbers changes the value, a positive term increases the value of the expression and a negative term decreases it. Research literature quoted earlier, both exploratory and classroom interventions, indicate the difficulty students in general have in understanding these ideas.

Students' understanding of the '=' sign, that it signifies the equality in value of the expressions on both sides of the '=' sign, which is an important structural notion connecting arithmetic and algebra was elaborated through many tasks. Although students at times made errors in equalizing expressions by filling the blank (e.g. 23+4=__-3), they could judge both arithmetic and algebraic expressions for their equality/ inequality with respect to a given expression without computation and also generate expressions equal to a given one, focusing on the relationships between the terms and the transformations that were applied to it. In particular, classroom discussions of how a given expression could be transformed while keeping its value invariant led to significant revelations about students' understanding. Further, these tasks served as better diagnostic and learning tools with respect to the understanding of equality than the more traditional task of filling in the blank.

Interviews also revealed students' ability to identify equal expressions from a list of complex expressions and to compare them with the original given expression identifying the greater/ smaller expression in a pair. This was accomplished through a meaningful, rather than a mechanical, short-cut procedure, use of the concept of 'terms'. Comparison of such complex expressions was unfamiliar to them and their flexible use of terms in the task was an important finding in the interview[11]. They performed well in the written test in both arithmetic and algebraic expressions, although there was a decrease in their performance in arithmetic expressions with product terms, where a few of them consistently failed to use the correct parsing/ unitization to identify the equal expression in the post test. A few students also faced difficulty in judging equality of expressions when it involved brackets, a problem which was noticed in the evaluation tasks as well. However, the interviews and the classroom discussions showed that they had strategies in place to deal with these tasks and to rectify their errors and they were clear about equality in value as an essential criterion for two expressions to be equal. The students further pointed out that two equivalent algebraic expressions (e.g. $3\times x+4+4\times x-5$ and $-5+4\times x+4+3\times x$) will have equal value for all numerical values of the letter. Two ways of justifying it were seen: by replacing the letter by the number in both the expressions to arrive at two arithmetic expressions which they knew would have equal values or directly inferring that particular cases would hold true since the general case is true.

[11] Students had been exposed to tasks which involved comparing simple two termed expressions like 68-29 and 67-28, results of which are discussed in Naik, Banerjee and Subramaniam (2005).

5.4 Context tasks

Although the tasks discussed earlier had created in the students a predisposition for symbolic representations and thinking with an expression, fewer students could use these resources adequately for the tasks of reasoning *with* expressions or use this to appreciate the 'purpose of algebra'. The issue is not simply one of transferring the abilities from the syntactic world to the context situations where algebra is to be used as a tool or of giving meaning to the letter by embedding them in contexts. Two elements that play an important part in these tasks are (i) the culture of generalizing, proving and verifying, with which the students had very little experience and (ii) students' belief about the effectiveness of using algebra in these tasks.

In the initial trials, students either did not understand the goal of the task and therefore randomly manipulated the representation they had created, or knew the goal, wrote the correct answer in the end but could not manipulate the expression correctly to arrive at that answer. In the last trial, however with a change in the approach to deal with this issue which encouraged verbalization, some students engaged in algebraic thinking and used narrative arguments, often displaying a quasi-variable approach (Fujii, 2003), to convince others about the generality of a result or to draw conclusions. One must note however, that this did not necessarily require algebraic representation. A few also successfully used algebraic representations, could anticipate the goal and accordingly manipulate it to prove the result. Still, a few continued to repeatedly verify the conjecture/ proposition for specific instances, not realizing the limitation of the approach. This pattern of responses led to the understanding that students' abilities to manipulate algebraic expressions and their knowledge of transformation rules is put into use only after they understand the purpose of the task, the need for algebraic representation and can anticipate the goal. Otherwise, the manipulation of algebraic expressions in the contexts is random or the use of algebra is completely ignored. Possessing the syntactic knowledge of algebraic expressions predisposes students to think in terms of expressions within the contexts but does not guarantee success. Thus, besides the 'push' from arithmetic which lays the ground for initial understanding of algebraic symbols and expressions and reasoning about expressions (phase of structural development), one needs the 'pull' from a culture of generalization and the need for general justifications, not restricted to specific instances, to move to the autonomous stage.

5.5 Meaning of the letter and the expression

The emphasis in the teaching approach was on seeing an expression in flexible ways: as a statement expressing relationship and a value. One of the major hurdles in making sense of algebraic symbolism is understanding the meaning of the letter and the duality of the various symbols (see Wagner et al., 1999). From the analysis of the tasks in this study, it was found that, excepting a few, most students seemed to understand the meaning of the letter as a number and the dual meaning of the expression as something to be evaluated as well as expressing a relationship. Students could verbalize the meaning of simple expressions like 5+4 or x-3 (four more than five or three less than x) as well as see a statement like x-3+5=x+2 (in the context of a task on the letter-number line) as expressing a relation between x-3 and x+2 (x-3 is five less than x+2) and the fact that subtracting three and adding five to x leads to x+2. Instances of perceiving expressions in this dual manner were also seen in the tasks described above, especially in the structure tasks.

5.6 Procedure-structure connection

Students' responses in the tasks on reasoning *about* expressions in the context of syntactic transformations revealed the inter-linkages between procedure and structure of expressions. Their scores in procedural tasks and structural tasks are highly positively correlated. There is some indication to the fact that one needs a minimum competence in procedures to internalize and abstract those properties for perceiving structure and answer questions consistently related to them. A preliminary analysis of the data over three trials (PST-II, MST-I and MST-II) had revealed that the structural understanding of expressions developed as a result of consistent application of the rules and procedures over many situations sharing the structural features and that structure oriented approach to teaching helped in strengthening both procedural and structural understanding (Banerjee and Subramaniam, 2005). But, it is the qualitative data analysis, as discussed in the previous sections, which show the complementary use of these two senses and which allows students to work efficiently in both, predominantly procedural and predominantly structural tasks.

6. Conclusions

The study pointed out the purpose, strengths and the limitations of the various tasks used at different points of the study. It thereby elaborated on the specific supports, in the form of vocabulary, concepts, rules and procedures required for making the transition from arithmetic to algebra, without which it is difficult for students to see the connection between arithmetic and algebra. Further, a teaching guideline is proposed on the basis of this study for making a smoother transition from arithmetic to algebra.

The approach which was adopted and evolved during the study has the potential to substantially bridge the gap between arithmetic and algebra. The specific features of the approach which facilitate this connection are:

(xi) building on students' understanding of arithmetic operations and intuitions

(xii) moving away from computation and emphasizing structure of the expressions

(xiii) fostering an understanding of expressions in terms of information it contains, relationship embedded in it and the value it stands for

(xiv) identifying concepts of terms and equality, which are structural and can help in consistently understanding rules of transformation of expressions

(xv) reformulating the procedures of manipulating expressions in structural terms and using the same rules, terminology, notations and conventions in solving tasks in arithmetic and algebra

(xvi) deepening the understanding of structure of expressions by focusing on invariance of value of expressions, thereby elaborating the understanding of equality and equivalence of expressions

(xvii) choosing tasks so that procedures get connected with structure sense

(xviii) explicit attention to the number as a referent for the letter

(xix) emphasizing the process-product duality or flexible 'proceptual' understanding through tasks

(xx) developing the ability to communicate and reason with symbols

These are important aspects of the arithmetic-algebra transition and have been points of concern in many of the exploratory studies quoted in the introduction of this synopsis and elaborated in the thesis.

The approach succeeded in many ways in dealing with the syntactic and the semantic aspects of arithmetic and algebraic expressions. Although students' understanding of rules of transformations and operation sense was visible in the context of syntactic transformations and reasoning *about* expressions, it was not fully used while reasoning *with* expressions. Students could display algebraic thinking by the end of the last trial and convincingly explain their solutions but the transfer to the symbolic mode was not easy, even when they could understand the process of the representation and manipulation to draw conclusions. The unsatisfactory development of the teaching approach with regard to this aspect of algebra, largely guided by the assumption that knowledge of algebraic symbols and manipulation would directly lead to their use in contexts, was probably responsible for many of the effects seen in students' responses. Symbolic proofs/ justifications need to be preceded by developing understanding of the need for algebra and engaging students in verbalizing the process of solution, a point which was realized only in the last trial. It is hypothesized that reasoning *about* expressions may help in reasoning *with* expressions by enabling the students to think in terms of expressions.

The study tried to explore and show the potential of the approach in making the teaching and learning of the two domains, arithmetic and algebra, more coherent and connected. It was not designed to experimentally establish the efficacy of this approach with respect to the traditional or any other approach. One direction in which the study can be extended is to include problem solving by framing and solving equations with in the scope of the approach and also include rational numbers in the arithmetic expressions and as referents for the letter. Another challenge is to evolve the approach to incorporate non-linear algebraic expressions, multiple variables in expressions and operations on linear and non-linear expressions.