INVESTIGATING AND SUPPORTING TEACHERS' KNOWLEDGE OF AND RESPONSES TO STUDENTS' MATHEMATICAL THINKING

A Thesis

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by

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DECLARATION

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions.

The work was done under the guidance of Professor K. Subramaniam, at the Homi Bhabha Centre for Science Education, Tata Institute of Fundamental Research, Mumbai.

Shikha

Shikha Takker

In my capacity as supervisor of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.

K. Subramaniam

Thesis Supervisor

Date: 08 Jan 2021

Place: Mumbai, India

Dedicated to…

… living through
the struggles of being and becoming,
a theoretician and a practitioner,
a researcher and a teacher,
a speaker and a doer,
a critique and a listener.

… breaking boundaries
and seeking support
when in need,
and learning to offer
the love and support
to the others around.

… shared learnings
through dialogues,
discussions and the long walks
For the patience, respect and care
towards multiple perspectives,
and an openness to discover
new ideas in unison
with the others.

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LIST OF PUBLICATIONS[1]

Peer–reviewed Journals

1. *Takker, S. & Subramaniam, K. (2019). Knowledge demands in teaching decimal numbers. *Journal of Mathematics Teacher Education, 22*(3), pp. 257–280. http://link–springer–com–443.webvpn.fjmu.edu.cn/content/pdf/10.1007/s10857–017–9393–z.pdf [Included in Chapter 5]
2. *Takker, S., & Subramaniam, K. (2018). Teacher Knowledge and Learning In–situ: A Case Study of the Long Division Algorithm. *Australian Journal of Teacher Education, 43*(3). Retrieved from http://ro.ecu.edu.au/ajte/vol43/iss3/1 [Included in Chapter 7]
3. Takker, S. & Khunyakari, R. (2016). Re–imagining Classrooms as Spaces for Learning & Professional Development. *Voices of Teachers and Teacher Educators*, *4*(2), 47–53. National Council for Teacher Education, Ministry of Human Resource Development, Government of India, New Delhi.

Conference Proceedings

4. *Takker, S. & Subramaniam, K. (accepted). Contingencies as moments of collaboration: A report on investigating and supporting mathematics teachers' knowledge. *Fourteenth International Congress on Mathematics Education,* Shanghai: China. [Included in Chapter 7]
5. Marcone, R., Takker, S., Milani, R., D'Souza, R., Linardi, P. R. & Shah, P. (accepted). Discussion Group on Data gathering and interpretation: Challenging the role of theoretical frameworks. *Fourteenth International Congress on Mathematics Education,* Shanghai: China
6. *Takker, S. (2018). Knowledge demands placed on a mathematics teacher learning to teach responsively. In S. Ladage & S. Narvekar (Eds.), *Proceedings of epiSTEME7: Seventh International Conference to Review Research on Science, Technology, and Mathematics Education*, pp. 323–331, India: Cinnamon Teal. [Included in Chapter 5]
7. Takker, S. (2017). Challenges in dealing with social justice issues in mathematics classroom. In A. Chronaki (Eds.) *Proceedings of Ninth International Conference of Mathematics Education and Society*. Greece: Volos.
8. Takker, S. (2017). Developing noticing in middle school mathematics teachers. Paper presented at *Eighth Comparative Education Society International*

[1] The publications related to the thesis work are listed here.

Conference: Criticality, Empathy and Welfare in Contemporary Educational Discourses, November 16–18, Jammu University.

9. *Takker, S. & Subramaniam, K. (2016). Changing teacher knowledge–in–practice: A case of decimal fractions. *Proceedings of Thirteenth International Congress on Mathematics Education,* Hamburg: Germany. [Included in Chapter 5]
10. Takker, S. (2016). Evolution of a professional learning space for mathematics teachers and researchers. *Seventh Annual International Comparative Education Society of India: Comparative Educational Destinies: Visions, Dilemmas and Challenges* hosted by S.V. University, Tirupati, Nov 19–21.
11. *Takker, S. (2015). Confluence of Research and Teaching: Case Study of a Mathematics Teacher. In K. Krainer & N. Vondrova (Eds.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education.* pp. 3269–3275, Prague: Czech Republic. [Included in Chapters 7 and 8]
12. *Takker, S., Kanhere, A., Naik, S. & Subramaniam, K. (2013). From Relational Reasoning to Generalisation through tasks on number sentences. In Nagarjuna, G., Jamakhandi, A. & Sam, E. (eds.), *Proceedings of epiSTEME 5: International Conference to Review Research in Science, Technology and Mathematics Education*. pp. 336–342. HBCSE, India: Cinnamonteal. [Included in Chapter 3]
13. *Takker, S. & Subramaniam, K. (2012). Teacher's Knowledge of and Responses to Students' Thinking. In *Proceedings of 12th International Congress on Mathematics Education.* pp.4906–4915. Seoul: South Korea. [Included in Chapter 3]

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14. Takker, S. (2020). Developing *knowticing* among mathematics teachers, *Teacher Plus*, 18(5), 29–31, Azim Premji University.
15. Takker, S. (2019). Understanding learners' thinking through an analysis of errors. *At Right Angles*, Azim Premji University.

Book Chapters

16. Takker, S. (2011). Using Classroom–based Tasks as Contexts for Reflection and Situating Teacher Learning. In Premlata Sharma & C.G. Venkatesha Murthy (ed.) *Readings in Teacher Education: Preparation and Professional Development* (pp. 239–258). Regional Institute of Education (NCERT), Manasagangothri, Mysore, India.

* The results reported in these publications are a part of the thesis.

ABSTRACT

The thesis is an attempt to characterise teachers' knowledge of students' mathematical thinking, as it gets manifested in their practice. The current research in mathematics teacher education focuses on (a) the assessment of teacher knowledge through the use of standard instruments, and (b) supporting teachers through tasks that deepen their professional knowledge of the subject matter. Some researchers have argued that such a discourse does not capture the dynamicity of teachers' knowledge manifested in the classroom. The thesis is an attempt to respond to such a critique by presenting a way of systematically investigating teaching practice, in order to capture the dynamic aspects of teacher knowledge manifested in the act of teaching.

The thesis reports an ethnographic case study of the practice of four experienced elementary school mathematics teachers. Data was collected through observations, interviews, and formal and informal interactions with the participating teachers for two consecutive academic sessions. Evidences from teachers' classroom practices suggest that: (a) the knowledge of the teacher is not uniquely possessed by the individual but is a joint province of teachers and students in a classroom, and (b) the tools used to investigate the dynamic aspects of teacher's knowledge need to be reimagined, for instance, students' responses might help in unpacking some aspects of such knowledge.

The analysis revealed that teachers became more responsive to students' *anticipated* and *actual* ways of (mathematical) thinking from the first to the second year of the study. As teachers became more responsive to students' ideas, they experienced mathematical challenges in handling classroom situations. The thesis presents the *knowledge demands* underlying the teaching of a specific topic, decimal numbers. These knowledge demands, arising from *contingent* classroom situations, were analysed to unpack the aspects of topic–specific knowledge required for teaching mathematics. Teachers were supported in handling these knowledge demands through *in–situ* support in the classroom, and *ex–situ* support through teacher–researcher meetings.

Through the nature of support, demanded by and offered to the teachers, the study witnessed the evolution of a community of learning involving teachers and researchers. The centrality of practice, both in investigating teachers' knowledge and in developing a process of supporting them, has implications for mathematics teacher education, research on mathematics teachers, and for bridging the gap between research and practice in education.

Chapter 1

THE RESEARCH PROBLEM: WHY, WHAT AND HOW?

> *Any in–service programme, whether it attempts to seed new ideas, challenge existing notions and assumptions or simply provide content knowledge, needs to acknowledge and respect this professional identity and knowledge of the teacher and work with and from it.* (NCFTE, 2010, p.67)

1.1 Abstract

In this chapter, I attempt to sketch the role of teachers and researchers in the backdrop of the changing landscape of Indian education. Section 1.2 below discusses the rationale and motivation of the study, which draws from my school teaching and research experience. Reforms in the national curriculum offered a context to study teachers' perspectives on their classroom practice. After a brief description of the origins of the study, in Section 1.3, I discuss the process of arriving at the research problem and operationalise the key terms used. Section 1.4 describes the chapter wise organisation of the thesis. This chapter concludes by defining the scope and briefly describing the limitations of the study.

1.2 Origin of the Research Problem

The study investigated ways in which school mathematics teachers' knowledge of students' thinking manifests in their practice. It charts my exploration of how teacher knowledge can be studied from a standpoint which situates the knowledge in the practice of teaching. The early ideas of research emerged during my school teaching, where students' questions were opportunities for unpacking mathematics differently, and it was difficult to find resources or other support structures to discuss the struggles arising from handling the uncertainties of teaching. While being a mathematics teacher, university education gave me an opportunity to investigate how other mathematics teachers interpreted and handled students' emergent ideas in their classrooms. My Masters' dissertation (Takker, 2007) aimed to understand the struggles faced by teachers in realising the reforms proposed by the National Curriculum Framework (2005). The data collection involved interactions with teachers from a variety of schools, including government, aided, elite–alternative, and private to understand the struggles arising from classroom teaching. The findings of the study assured me that teachers struggle to engage with the proposed reforms. The encounters with the knowledge manifested in teachers' practice, or as it is called in this thesis "knowledge *in–situ*", at different levels of schooling and with a variety of teachers, served as an initial motivation to investigate the problem of teachers' knowledge more systematically and deeply.

It is around that time that the changes in the curriculum were seen vis–a–vis their implementation. The National Curriculum Framework 2005 (henceforth, NCF 2005), called for shifting the goals of education from transmission of knowledge and facts to knowledge construction, and proposed that education should be a tool for transformation of society. The National Council of Education Research and Training (NCERT), the apex body responsible for the development of the national curriculum, was creating revised textbooks and offering teacher training on how to implement the reforms proposed in the new curriculum framework.

NCF 2005 acknowledged the stress of school learning experienced by students and parents and suggested some guiding principles for designing a curriculum which provides equal opportunities to all students for learning without fear (pointed put by the document *Learning Without Burden* MHRD, 1993) or differences in their social status. These principles are "(a) connecting knowledge to life outside school, (b) ensuring that learning shifts from rote methods, (c) enriching the curriculum so that it goes beyond the textbooks, (d) making examinations more flexible and integrating them with classroom life, and (e) nurturing an over riding identity informed by caring concerns within the democratic polity of country" (NCERT, 2005, pp. viii). India has been dealing with the problem of bringing several out–of–school children into the school. So, parts of NCF 2005 such as, education for all children, needed to be enforced by other systemic structures. The landscape of education changed in the last decade with the Right of Children to Free and Compulsory Education (RTE) Act in April 2010. Five years after the curriculum document was in place, RTE (2009) increased the challenges of implementing reforms in classrooms. In the context of implementation of RTE, Ramanujam (2012) wrote,

> If one was asked to isolate and point to one single challenge as the most important among the plethora of problems, … it would have to be that of *creating a pool of good mathematics teachers in the required numbers.* At the elementary stage, the numbers exist, but not with the required understanding of mathematics or attitude towards mathematics or comprehension of how children learn (or fail to learn) mathematics. (p.9, emphasis in original)

Thus, the challenge of preparing teachers with the knowledge required for teaching does not just include improving teachers' knowledge of the content but also challenging their existing attitude about mathematics teaching and learning.

In the following sub–sections, I will detail two kinds of issues related to changed classroom practices under the reform context. First, the expectations made of teachers and the demands posed on them due to the curricular reform. This sub–section will critique the existing modes used to support teachers and flag some alternate ways of imagining teacher education or of working with teachers. Second, the subsection on meeting of research and teaching in the context of practice will problematise the contribution of education research and researchers to the practice of teaching. I will make some remarks on how research can be more responsive to the needs of the agents (primarily teachers and students) and engage with the realities of classrooms. Both these issues, that is, the role of teachers in a reform context and the gap between research and teaching are used to place practice at the centre of the educational discourse.

1.2.1 Teachers in a reform context

NCF 2005 expects the teacher to engage every child in the classroom and enable their participation in the learning of different school subjects. The curriculum document demands that the teachers internalize the principles envisioned in the document and enact them in the classroom. The National Focus Group Report on Teacher Education (NCERT, 2006*a*) proposed that the teacher education programmes must prepare teachers to be an active member in curriculum renewal and have good knowledge base. The document noted that the existing pre– and in–service teacher education programmes were insufficient in preparing teachers as professionals having a strong knowledge base. These reform propositions were communicated through the top–down professional development workshops organized for in–service teachers using a cascade model and through structural changes in the pre–service teacher education curriculum (Kumar, 2018). It was recognized that the content, substance, learning experiences and modalities of training teachers in pre–service teacher education had remained unchanged for decades, and had not caught up with the changed expectations of the curriculum, and society in general (NCERT, 2006*a*). Similarly, the in–service teacher education programmes were critiqued for using the lecture method by the focus group report on teacher education (NCERT, 2006*a*), for not encouraging active learning among school teachers. The workshop mode helped in communicating the propositions of the new curriculum, but the ways in which it informed teachers' practice was not studied. Further, the irony was that ideas such as, activity or inquiry based teaching, handling multigrade and large classrooms, using materials for teaching, etc., were communicated through the lecture mode (NCERT, 2006*a*). Teacher education has been recognized as the weakest link in the education system (Banerjee, 2012) and in the implementation of NCF 2005 (Batra, 2005). Additionally, there has been a need for good models for teacher professional development and for developing the knowledge of the cadre of teacher educators and administrators, which can support teachers in the process of reform (Banerjee, 2012).

One of the ways in of implementing the propositions of NCF 2005 was through the design of new textbooks for each school subject. The revision of the school textbooks was an intense exercise, involving teachers, teacher educators and researchers. Some of these textbooks, particularly at the primary grades, were significantly different from the old textbooks, in their writing style, appeal to the learner, concept introduction, and the anticipated learning trajectory. These textbooks also had some brief notes for the teacher, wherever deemed necessary by the textbook writers, on how to expand or extend an important idea. While the textbook renewal exercise was underway, NCF 2005 had suggested moving away from using textbooks as the only source of knowledge and bringing in local resources to bridge the gap between outside knowledge of the learners and the formal school knowledge. However, whether teachers understand how to use these textbooks or select appropriate resources from the outside, remains questionable.

For the teaching and learning of mathematics, NCF 2005 emphasised students' construction of knowledge while learning in a classroom (NCERT, 2005). The aim of teaching mathematics was shifted from a focus on procedures to the processes involved in doing mathematics. These processes included problem solving, approximations or looking for intelligent solutions, systematic reasoning, mathematical communication, and making connections (NCERT, 2006*b*). The changes in the textbook, particularly at the primary level, align with these new goals of teaching mathematics.

The proposal of NCF 2005 to create learner centered classrooms and make connections with outside school knowledge, poses demands on teachers. The teachers are expected to make sense of the propositions of NCF 2005 and use the new textbooks and local resources to enact this vision. Teachers who have not encountered such reformed pedagogies through their schooling or teacher education programme struggle to implement such propositions in practice (Rampal & Subramanian, 2012). Batra (2005) argued that the NCF 2005 evaded the question of, "How do you enable critical thinking and meaning making among children (the aim of the NCF) with a teacher who has not been through such a process herself?" (p.4350). The ways in which teachers make sense of reforms (often communicated to them through changes in textbooks) is varied. First, it was noted that teachers attributed a variety of meanings to the terms and ideas proposed in NCF 2005 (Takker, 2011). The study reported that teachers showed familiarity with the vocabulary of NCF 2005, perhaps acquired from participation in professional development workshops. However, teachers implemented these reforms based on what they thought was important for students to learn. It has been noted that in an attempt to accommodate the reforms without modifying the larger structure of thinking and understanding mathematics, teachers might combine aspects of the old and the new curriculum, without critically challenging the existing practices. Teachers who are unwilling to accept the reforms completely, but have an obligation to follow them and teach accordingly, tend to create a blend of open–ended activities with traditional procedural practice (Ebby, 2005). This recalls the case study of Ms. Oublier (Cohen, 1990), a teacher who believed that she had revolutionised her teaching following the educational reforms, but her practices were observed to be largely traditional. Such practices have been identified as "hybrid practices" (Brodie, 2011) or "instructional hybrids" (Cuban, 2007) in the literature.

The teachers' struggles with the reform agenda raises the question of how they can be supported. Batra (2005) suggested the restructuring of existing teacher education programmes, creating structural supports within teacher education institutes, and connecting institutional spaces to prepare and strengthen teachers' agency in implementing the curriculum. Recognising the symbiotic relation between the school education reforms and teacher education, the National Curriculum Framework for Teacher Education (NCFTE, 2010) recognised the teacher as a crucial mediating agent and charted a terrain for the professionalisation of teaching in India. It emphasised that teacher training should be designed to develop a stronger knowledge base, so as to prepare teachers to deal with the challenges arising in the classrooms. In a broader sense, NCFTE recommends a stronger preparation and support system for teachers through planning of continuous teacher professional development, the preparation of a cadre of teacher educators Who can prepare teachers for this arduous task, and encouraging research *by* and *with* teachers to inform their practice. These issues on the nature of knowledge that teachers need in order to teach effectively and the teachers' role in educational research are discussed in the following sub–sections.

1.2.2 Mathematical knowledge for teaching

The claim that teachers need specialised knowledge in order to teach school subjects effectively, has had a widespread influence on education research as well as on the design of interventions in teacher development internationally (Edwards, Gilroy & Hartley, 2005). Sustained efforts have been made by researchers to develop characterisations of specialised teacher knowledge that remain close to the actual work of teaching (Petrou & Goulding, 2011). The design of teacher education curricula or professional development interventions is founded on a conception that individual teacher's knowledge of mathematics teaching impacts their practice.There is a recognised need to identify both the form and the content of teacher knowledge that is most likely to translate into changed classroom practice.

Existing frameworks of teacher knowledge attempt to identify its components, especially those components that are missing from typical trajectories laid out by formal teacher preparation programmes (Ball, Thames & Phelps, 2008). Teacher knowledge is characterised by focusing on the teacher and the knowledge that the teacher brings to the classroom (elaborated in Chapter 2). Such frameworks have been criticised for at least two reasons. First, the existing frameworks view teacher knowledge as static. Hodgen (2011) argues that, "teacher knowledge is *embedded* in the practices of teaching and any attempt to describe this knowledge abstractly is likely to fail to capture its dynamic nature" (p. 29, emphasis in original). Second, the notion that the teacher acts as an individual in the process of teaching and learning, and therefore that teacher knowledge is uniquely the province of a teacher, needs to be problematised. Thus, there is a need to go beyond the individualistic assumptions about teacher knowledge and engage with the dynamic system in which teachers' work is located (Rowland & Ruthven, 2011).

These criticisms have several implications for the design of professional development interventions. Brodie (2011) argues that there is a need for textured descriptions of the difficulties faced by teachers when implementing the reformed curriculum. Further, Cobb and Jackson (2015) suggest that a study of teachers' existing practices can be used to identify aspects which can be leveraged to design support for learning. Taking these two arguments together, the thesis presents an approach of engaging with the work scenarios of teachers to develop an understanding of the challenges faced by them *in–situ* and design appropriate support structures. This approach, which takes the realities of teachers' work into cognisance and engages deeply with the practice of teaching, has the potential for the formation of learning communities involving teachers and researchers (Takker, 2015).

1.2.3 Research on and with teachers

In this section, I discuss the salience of the role of research in teacher education. Education research on investigating students' and teachers' knowledge and learning, and on teaching methodologies can be used to inform teaching practice. Let us discuss different ways in which research has informed or can inform practice first in the Indian and then in the International context.

In a review of literature on research in mathematics education in India, Banerjee (2012) found that a majority of studies on students focused on identifying the causes of fear of mathematics, listing errors in specific problems, finding reasons for mathematics anxiety and comparing different teaching methods. Such studies, the author argued, are limited in their methodologies, as they use psychometric designs of data collection and analysis, which do not help in understanding the complexities involved. Post 2000, mathematics education research in India has focused on understanding students' thinking and proposing learning trajectories for the teaching of specific topics (Menon, 2015; Banerjee & Subramaniam, 2012; Naik & Subramaniam, 2008). Research on students' thinking and learning in specific topics might be a useful resource for teachers in planning and conducting their lessons. Therefore means of disseminating such research to teachers needs attention. Very few parts of the research on learning trajectories have found their way into the development of textbooks and in planning the content for teacher professional development (TPD) workshops. Use of research findings for TPD is sporadic and there are no systems in place which encourage the use of research findings as a resource for teachers. On the contrary, most of the state led workshops for practicing teachers have a prescriptive character with the aim of telling teachers how to implement a text, use a teaching method or apply an assessment technique (Kumar, Dewan & Subramaniam, 2012). Other teacher professional development programmes where teachers are given an opportunity to deepen their content knowledge are organised in pockets by a few research and teacher education organisations. Further, it is important to note that communicating important ideas through workshops is not the most effective way of supporting teachers or envisioning changes in their practice. Putnam and Borko (2000) suggest that while workshop can be a means of developing certain practices and enhancing teachers' knowledge, the enactment of such practices needs to be supported through an engagement with teachers in their classrooms. Some independent groups, such as Eklavya, Quest, Jodogyaan, Navnirmiti continue to organise small scale trainings for mathematics teachers, in order to enhance teachers' content and pedagogical content knowledge of mathematics. In her research study, Kumar (2018) explored mathematics teachers' beliefs and knowledge by supporting teachers through a professional learning community comprising teachers, teacher educator and researcher. The teachers participated in workshops and collaborated with the researcher in the classroom. The research proposed situatedness, challenge, and community as three central aspects of the professional development to help teachers revise and reflect on their knowledge and beliefs and renegotiate their identities.

Initiatives such as these are significant in the Indian context for two reasons. First, they call attention to the features of a professional development programme which situate it in the context of classroom practice. Second, they offer an exemplar for approaches which connect research more closely with practice rather than offering mere accounts of practice.

Although some of these efforts on supporting teachers, make an attempt to use the insights gained from research to plan their interactions with teachers, a more systematic attempt at imagining a sustained relation between research and teaching calls attention. Further, the role of teachers in such research endeavours remains a question. In other words, how do teachers and researchers view themselves and the other in such research?

In the Indian context, teachers are unaware of the research that is happening on classrooms or with students; and no parts of the research practice encourage researchers to make explicit connections with practice. Further, while working through the bureaucracy of the school system, it often becomes difficult for researchers to conduct research that can be directly used by or done in collaboration with teachers. The question then is what would be role of the researcher and a teacher in a collaborative research endeavour. In order to think about ways in which teachers can engage with the research, we need to ask the following questions.

1. What kind of research would teachers find useful?
2. What could be the role of teachers in a research which focuses on students' or teachers' views, perceptions, knowledge, etc.?
3. How can teachers contribute to making research more grounded in practice?
4. What are the ways in which teachers and researchers can work together to improve students' learning?

A discussion on some of these issues, can benefit from the remarks made by the participating teachers during the course of this research study (refer Excerpt 1.1).

Excerpt 1.1: Teachers' perspectives on their role in research

TP	Researchers come and go, they are not actually interested in our (teachers') problems.
TV	You can tell me what you want to see and tell people through your thesis. I will teach using that method. I want to help you.
TN	See the researchers can spend time to analyze everything, but we have to teach, teach in the class, fast, complete the syllabus. We don't have so much time to think. This is my job.
TJ	Researcher can report one incident or performance, like results of a test. In teaching, every day matters.
TR	See I have a lot of work, so you tell me what you need. I can give that to you and then both of us are done.
Legends: T – Teacher (followed by the initial letter of the pseudonym of the teacher)	

The comments reflect the perceptions of teachers about their roles in a research study on teaching. The perceived image of the teachers in research is quite similar to the image of the teacher in the enactment of a curriculum framework (as noted by Batra, 2005). In research endeavours, the role of the teachers is either that of an implementer of a teaching methodology or as subjects whose teaching or knowledge will be assessed by the researchers (much like the inspectorial system). In the Indian context, where connections between research, school education and teaching are significant part of the contemporary discourse, it would be relevant to ask – what kind of image or role of teachers is imagined in the research on teacher education and development and also perhaps, how can teachers' knowledge contribute to the changing research questions. In this context, the accountability of researchers studying teaching or learning in classrooms, also needs to be figured. This points to the need for thinking about the connections between research and teaching in ways which can mutually support the development in the two fields and impact practice.

As the research on classrooms and teaching is growing in the Indian context, it is important to note that a comprehensive review of the literature in mathematics education research internationally points to the need for connecting teaching and research. Cai, Morris, Hohensee, Hwang, Robison and Hiebert (2017) present a vision of future of mathematics education research by drawing researchers' attention to the question of how the impact of research on teaching can be improved. Cai et al. (2017) argue that teachers need to play a more purposeful role in research. Charting the role of the researcher and the teacher in such a collaborative endeavour, the authors present a fictional account of a teacher–researcher partnership which exemplifies their interdependence. This partnership is aimed to generate the professional knowledge base required for improving teaching. Such a knowledge base includes identifying, developing, and revising artifacts tagged with the specific learning objectives. A salient component of this impact on practice includes creating better learning opportunities for students' learning mathematics. Since the knowledge base would be created in collaboration with teachers, its accessibility and dissemination would be taken care of. The vision of partnership proposed by the authors demands rethinking methodologies which enable such knowledge generation through building and sustaining partnerships by breaking the 'isolation' that teachers and researchers endemically suffer from. This isolation is considered as a roadblock to building a knowledge base, which has the potential to amplify the impact of research on practice (Cai et al., 2017). Such a partnership demands that teachers and researchers break out of their traditional boundaries and work together to impact practice.

Summary: In the Indian context, as in other parts of the world, the national curriculum reforms advocate a more student–centered approach to teaching and a shift from mathematics as learning procedures to engaging with the processes of doing mathematics (NCERT, 2006*b*). Teachers are expected to take into cognisance students' ideas and build on them. Teachers, who have not experienced such approaches during their schooling experience or in their teacher preparation, struggle to notice and build on the students' ideas emerging during classroom interactions. In the context of education reform, it therefore becomes critical to understand what additional knowledge demands are made of a teacher who is struggling to make a transition towards more responsive teaching. Teacher education in India demands an invested approach, which does not just support teachers in implementing the reforms but helps in engaging them in the process of reform. An overlap of goals between education researchers and teachers suggest the possibility of development of communities which can work together to generate artifacts of practice, engage with the struggles arising from practice, and create possibilities for learning of students, teachers and researchers.

1.3 Defining the Research Problem

Following from the reformed curriculum, the mathematics position paper (NCERT, 2006*b*) suggests that teachers use multiple approaches, focus on the processes of learning mathematics, and draw on students' knowledge for introducing different topics while teaching. All these recommendations are expected to help learners participate in the mathematics that they are learning, by shifting the emphasis from rote memorisation to doing and understanding mathematics. Teachers become the central mediating agent in transforming these reformed propositions into practice.

Such expectations place demands on teachers' knowledge and challenges their existing beliefs, knowledge and attitudes. It is evident that teachers who have not experienced such attitudes and/or pedagogies through their schooling or teacher education experiences, would need support to engage with such reform propositions. So, the question was – what kind of knowledge base do teachers need in order to engage with the vision of NCF 2005 in practice? While the literature on mathematics teacher knowledge (reviewed in Chapter 2) in specific topics is helpful, the need to identify aspects of knowledge salient in the practice of teaching, were deemed important. The thesis identifies the knowledge needed to teach mathematics through a systematic study of teaching practice. The knowledge required for teaching is defined for the topic of decimal fractions, which was selected in consultation with the participating teachers. The specific research questions addressed through the research study are:

1. How does teachers' knowledge about students' thinking manifest in their practice?
2. How can responsive teaching be identified and characterised? What is the relation between teacher knowledge and responsive teaching?
3. What is the nature of knowledge demands placed on the teachers' knowledge during teaching?
4. How can teachers' knowledge of students' thinking be supported? How do teachers engage with the support provided by the researcher?

The research questions have been explained further in Section 4.2 of Chapter 4.

1.3.1 Constructs used in the thesis

The research on teachers and teaching reported in this thesis attempts to (a) examine how teachers engage with the reform context, (b) report their struggles in the process, (c) explore and design the nature of support that teachers might need in this period of transition, and (d) concludes with how teachers' engagement with the reform propositions of NCF 2005 in practice can feed back into the future curriculum design, revision and development.

The central construct investigated through this research is teacher knowledge as it gets manifested in practice. It is called as knowledge in practice or knowledge *in–situ*. In other words, the study focuses on the knowledge of the teachers as it gets manifested in their practice. This way of studying or defining knowledge allows for

(a) treating knowledge as a dynamic entity (as in *knowing*) which emerges and becomes identifiable during the act of teaching as opposed to knowledge that is ossified, static, and internalised as abstracted concepts (reductionist and measurable though paper–pencil tests).

(b) appreciating and acknowledging that knowledge gets triggered and constructed through classroom interactions between the teacher and students, and at times between students.

(c) examining knowledge that is experienced by the teachers in the act of or through a reflection on teaching.

A characterisation of knowledge situated in practice or knowledge *in–situ* seems suitable to focus on specific aspects of specialised teacher knowledge which gets manifested in teaching. From among the many types that constitute teachers' specialised knowledge (such as that of curriculum materials, mathematical content, pedagogies useful for teaching mathematics, etc.), knowledge about students' mathematical thinking and ways of learning is the focus of this research.

Knowledge about students' mathematical thinking is operationalised as knowledge of ways in which students' deal with a problem (or topic), that is, recognising their intuitive ways of sense making (using prior knowledge of facts, tools such as explanations and representations known) and building on this by challenging or supporting this knowledge.

Teacher knowledge is connected with the individual (yet collective) teacher's beliefs, resources, and attitudes not just towards mathematics learning of their students but also of their own. Thus, a professional development initiative which aims to challenge teachers' existing knowledge, beliefs or experiences, needs to offer the experiences of learning mathematics differently and support teachers in the process of engaging learners in learning mathematics differently.

1.3.2 The methodological stance

As stated earlier, in the Indian context, the research on mathematics education has varied from understanding factors leading to learners' fear towards mathematics to designing learning trajectories in specific mathematical topics. While there are a few institutions which work to organise professional development of teachers, research on teachers and teaching is scarce. What has been learnt from effective professional development initiatives in India and from across the world is that the agency of the teacher is central, and determines teachers' participation and sustainability of these initiatives in practice. One of the questions that were asked in this research was – how can a research design acknowledge the agency of teachers and encourage their participation in the process of exploring and supporting teachers' knowledge.

In the reported research, teachers were familiarised with the research objectives and the course of action was discussed. Deviations in their role and respecting their choices were maintained through the course of study. For instance, each teacher's decisions about allowing recording of their lessons, the changes in the worksheet designed by the researcher for students, the need and frequency of teacher meetings, etc., were respected. Similarly, if the teacher wanted to discuss some of students' work in the meetings, the plan for the meeting was flexibly modified to accommodate this change. The video data of classroom observations was shared with the respective teachers. An interpretativist orientation and the use of a qualitative research design for the research study enabled appreciating the differences (between the plan of research and the actual field work), and probing of meanings and perspectives underlying teachers' actions. For instance, in beginning of the study, it was difficult to initiate conversations with the teachers about teaching. On probing, it was found that teachers were apprehensive about whether and how will research impact practice. However, the nature of interactions between the teachers and researcher changed considerably during the course of the study from not talking about their teaching decisions to reflecting on them for planning further lessons, seeking support, and so on.

While teachers were encouraged to articulate their opinions consistently, they were challenged and supported through listening to their students and colleagues, research literature in topic specific domains, discussions on the intent of the new curriculum, and so on. Even though teachers were challenged, at all times, they were respected for the experience and the opinion they brought to the discussion. The invested approach of the research, with an interest in teachers' existing practices and notions, seemed to have helped in breaking the traditional boundaries of the role of the researcher and in helping teachers in articulating their struggles.When teachers were challenged to examine their views about students' mathematical ways of thinking, sensitivity not just to the mathematical ideas that students bring to the classroom but also to students' social contexts was discussed. An instance of this kind has been reported in Chapter 4. It is important to mention here that although the literature, the current study, and the observations on working with mathematics teachers reported in the thesis focus on the "mathematical" aspects, the research acknowledges an engagement with the "affective" aspects of teachers' work.

1.4 Organisation of the Thesis

The research problem was identified through reflection on my school teaching, several hours of classroom observation of other mathematics teachers, reading of the literature and through interactions with mathematics education researchers and teachers. The research reported through the chapters of this thesis broadly follows the chronology of the research process. However, like any other research, there were several occasions of moving back and forth, whenever the research situation posed challenges. For instance, while field visits were ongoing during the literature review stage, an attempt was made to examine students' work in real classrooms to identify the misconceptions reported in the literature. Similarly, during data collection, if a specific representation was used to teach a mathematical idea, resources such as literature and other texts were examined to understand the efficacy of multiple representations along with their limitations.

The structure of the thesis captures the phases of the research study.

In (this) Chapter 1, I have tried to locate the rationale and motivation of pursuing the research study using my early experience of being an elementary school mathematics teacher. This includes linking the personal narrative on the struggles faced during teaching to the wider perspectives on the challenges faced by the other mathematics teachers in the Indian context. The landscape of Indian education is used to raise some questions about the role of the teacher in a reform context, in teacher education initiatives, and in research on classrooms. Further, an attempt is made to problematise the relation between research and teaching to justify the possibility of reimagining this connect, which is then supported with the contemporary vision of mathematics education research internationally.

Chapter 2 summarises the literature on the important constructs used in thesis, that is, teacher knowledge, knowledge of students' thinking, and connecting knowledge and practice to enable teacher learning. An attempt is made to historically trace the changing notions of these constructs in order to call attention to the need to "study" rather than "assess" teaching if the purpose is to identify and support the knowledge that mathematics teachers need to teach in classrooms. The chapter discusses the learning from a reflection on the existing frameworks or approaches and draws attention to some of the challenges faced by these frameworks in the contemporary context.

After setting the theoretical background for the study, Chapter 3 reports the pilot studies planned to operationalise the research constructs through field experience. It encompasses two experiences – engagement as a researcher and as a teacher–researcher. The first pilot was a case study of a teacher in a mathematics classroom, to engage with the aspects of knowledge that come into play during the act of teaching. Through classroom observations, interactions and some tasks, the teacher's and students' perspectives on classroom teaching were studied. In the second pilot study, the researcher taught middle school students as part of a summer camp. The aim was to examine students' engagement with the learning trajectory which was designed based on an awareness of the literature on students' thinking in the specific sub–topic and a study of other resources. The alignment and engagement with the process of teaching was maintained through the two roles of being a researcher and a teacher in two different contexts. The chapter concludes with the findings of these pilot studies followed by a reflection on what was learnt from these studies and how it helped in refining the constructs used in the main study.

In Chapter 4, I describe and justify the use of case study methodology to pursue this research. I examine the choice of an exploratory and interventionist case study, done in an ethnographic style. The chapter elaborates on the tools used for data collection in the three phases of the main study, that is, exploring teacher's knowledge in classroom, organising teacher–researcher meetings in school, and supporting teachers in their classrooms. The chapter discusses the decisions made during the course of the study, and why they deviated from the original plan. In the process, the issues of reflexivity, ecological validity and positionality are foregrounded.

The case study of two (of the four) participating teachers is reported in Chapter 5. The chapter uses the data collected from classroom observations and interactions with teachers and students in Phases 1 and 3 of the main study. The chapter proposes and exemplifies the construct of "knowledge demands" as a way of analysing knowledge needed by mathematics teachers during teaching. Although the scope of knowledge demands identified through the study of teaching practice is specific to the topic of decimal numbers and also to the knowledge that gets triggered in particular classroom situations, it is argued that a reflection on these demands has the potential to create a map of teachers' knowledge required for teaching mathematics. As a backdrop, the chapter also reports the evidences of change in teachers' practices in the two years of the main study.

Chapters 6 and 7 present an analysis of the support offered to the teachers in two different modes. Chapter 6 presents an analysis of the data from teacher–researcher meetings, including their design and conduct. It provides the details of how tasks for these meetings were designed along with teachers' changing participation and engagement with these tasks. The analysis of teachers' engagement with tasks is used to identify ways in which teachers' knowledge can be developed. Explicit connections between the content discussed during these meetings and the challenges faced by teachers in their practice contributed to the development of a community of teachers and researchers. Chapter 7 analyses the nature of classroom–based support for each teacher as they engaged with the challenges in using reformed classroom practices. An attempt is made to unpack the reasons for the changes noted in the teachers' beliefs, knowledge and practice. A reflection on one case study led to defining the analytical construct of "contingent moments" in teacher–researcher collaboration, which created opportunities for change in practice. In the second half of this chapter, an analysis of such moments for the other participating teachers is presented. The chapter concludes with a reflection on the process of converting such contingent moments into learning opportunities for teachers and researchers, when working collaboratively.

Chapter 8 proposes the construct of mathematical responsiveness, abstracted from the nature of support offered to the teachers through the study. The construct is defined using the existing literature on teacher listening, and refined through evidences from the data used in Chapters 5, 6 and 7. The second part of the conclusions chapter is a reflection on how mathematical sensitivity or responsive teaching can be developed in the context of practice. The last part of the chapter details the implications of this study for research in mathematics education, and for pre– and in–service teacher education programmes. This is followed by a discussion on how this research contributed to the design and development of tasks for teacher learning, and the emergent themes which can be potentially explored further.

1.5 Scope of the Study

The purpose of the study was to develop an in–depth understanding of teachers' knowledge about students' thinking through a study of their practice. Since the intent was to investigate teaching practice, the focus was on a few classrooms. Adopting a case study methodology, the sample of the study was limited to four teachers and nine classrooms in which they taught in two consecutive years. Due to an intensive focus on the everyday practice, the teachers from within the same school were selected and pursued. Thus, the context of the study is limited to a school system located in the urban parts of Mumbai.

The ethnographic style of the research study led to making decisions about recordings with the participating teachers, negotiating the role of the researcher, and offering support to teachers in aspects which were unplanned. All these unpredictabilities constitute an important part of the study. Since the purpose of the study was to understand teaching and work with teachers collaboratively, no standardised tools were used to assess either teachers' or students' knowledge, or in analysing the data from classroom observations.

The data collected over two years included observations of teachers' activity inside and outside their classroom, interactions with different staff members of the school, records of students and teachers, etc. The data used in the thesis is limited to a few topics and settings from among the whole set. However, it is important to mention that the data (which has not been used or cited directly in the thesis) has helped the researcher in understanding the social space better and has a bearing on the analysis.

The qualitative standpoint of the study does not allow for making generalisations about the findings per se. However, the analysis presented achieves two purposes. First, an analysis of the textured descriptions of teachers' work helped in theorising the analytical constructs, which have the potential for generalisability and wider use. Some of the ways in which the proposed analytical constructs can inform decisions about the content of teacher education, design of teacher professional development programmes, and collaborative research on classroom teaching are indicated. Second, a nuanced understanding of the complexities entailed in teaching, allows us to ask specific questions to understand the phenomena better, for further research, and offers directions which are alternatives to the existing ways of investigating teachers' knowledge.

Chapter 2

CONNECTING TEACHER KNOWLEDGE, PRACTICE AND LEARNING

> *Research has a greater impact on practice when teachers play a purposeful role in the research process, whether in defining research problems, in identifying learning goals, subgoals and learning opportunities; or of course, in implementing learning opportunities in the classroom.* (Cai, Morris, Hohensee, Hwang, Robison & Hiebert, 2017, p.466)

2.1 Abstract

In the last few decades, there has been an increasing focus on refining the construct of mathematical knowledge required for teaching (Speer, King & Howell, 2015). The construct of teacher knowledge has been studied for mainly two, sometimes overlapping, purposes. First, to assess or investigate teachers' knowledge, using valid and reliable instruments. Second, to develop this knowledge through the design and implementation of tasks which enable teacher learning. This chapter is organised around these two purposes.

An interest in specialised knowledge of mathematics teachers emerged from Shulman's seminal talk in an AERA meeting in 1985, where he proposed that teacher knowledge is specialised. In an attempt to operationalise the categories of teacher knowledge, proposed by Shulman, research studies focused on designing instruments to measure teachers' specialised knowledge of the content. On the other hand, Shulman's proposal for the shift in emphasis from knowing the content to pedagogical content knowledge of the teacher, generated interest in the practice of teaching. A focus on teaching as an indicator of teacher's knowledge gained prominence in research. An analysis of teaching is complex as several interconnected constructs play

in–the–moment to determine teaching decisions. An attempt to understand the complexity of teachers' knowledge through a study of practice, in a way, contributed to the changing gaze from assessing teachers' knowledge to unpacking it. In the first part of this chapter, I will summarise the frameworks that have evolved from Shulman's conceptualisation of teacher knowledge, and discuss how a reflection on these helped in developing a theoretical background for the research study. Then I will review the approaches which connect teacher knowledge with teaching practice and learning. I summarise by making connections between teachers' knowledge, practice and learning.

2.2 Central Questions

The purpose of this chapter is to review different lenses and the tools used to investigate and support the knowledge required for mathematics teaching. The central questions are

(a) What constitutes mathematics knowledge for teaching (or teachers' knowledge) and how is the operationalisation of this construct changing?

(b) How does the knowledge about students' mathematical ways of thinking link with the other aspects of teachers' knowledge?

(c) What kind of conceptualisation of teacher learning emerges from the relation between teachers' knowledge and practice?

The chapter is organised around these three questions. Section 2.3 discusses the changing notion of the construct of mathematics knowledge for teaching and the tools used to assess it. A reflection on the contemporary research studies suggests that the tools used to assess or investigate teacher's knowledge address specific aspects of knowledge. The question then is whether teachers' knowledge can be studied as a whole, along with its dynamicity as it gets manifested in the act of teaching. Section 2.4 presents a summary of the literature on how teachers' knowledge about students' mathematical thinking has been operationalised in the existing frameworks. An attempt is made to see the connections between teachers' knowledge about students and the other aspects of teachers' knowledge identified in the literature. In Section 2.5, I discuss how different perspectives on the connection between teacher knowledge and practice have been used to support teacher learning. Existing programmes of professional development are used as cases to distinguish different perspectives on the connection between teacher knowledge, practice and learning. The concluding Section 2.6 discusses how the current study responds to the existing gaps identified in the literature.

2.3 Investigating Mathematics Teacher Knowledge

What constitutes mathematics knowledge for teaching has been a central question in the last few decades. The changing definition of this construct has influenced the choice of methodological tools and the purposes for which these tools have been used. In this section, I will review the relevant literature on how teacher knowledge has been studied and what kind of knowledge was elicited through the use of different tools.

2.3.1 Testing teacher knowledge

Teachers' knowledge has been associated with the knowledge of the content that is to be taught. Ball, Lubienski and Mewborn (2001) asserted that while there is a consensus on the idea that teachers need more content knowledge to teach mathematics effectively, the nature and extent of this knowledge remained debatable. The problem of what constitutes teachers' knowledge has been addressed using the policy and the research perspective (Ball, Lubienski & Mewborn, 2001). From a policy perspective, it is important to determine what teachers need to know, which has been addressed through the design of tests of content knowledge. A set of mathematical topics are selected to formulate test items to assess teachers' knowledge of the subject. The approach is quite similar to the tests of teachers' content knowledge used for teachers' recruitment and promotion in India. The research based approach to the problem of unpacking teachers' knowledge include – first, characterising teachers' knowledge by examining the content and skills learnt through the university courses and second, by focusing on the nature of teachers' pedagogical content knowledge. In an attempt to validate empirically the impact of teachers' content knowledge on students' learning, there were studies on connecting the practice of teaching with the knowledge of the teacher. The relation between what teachers do in their classroom (the process of teaching) and what students learn through instruction (the product) was described as process–product research. These studies used large sample sizes, protocols to observe and measure classroom processes, and assess students' learning of what was being taught (Hiebert & Grouws, 2007) in order to claim reliable results. This research foregrounded the relation between teaching and learning by operationalising observable indicators, such as, the kind of teaching method used, classroom management, participation, etc. One of the major findings of such studies was about the effectiveness of some teaching methods over others in aiding students' learning.

Hiebert and Grouws (2007) summarised the challenges faced by the process–product research. First, it was critiqued for suggesting some teaching methods as more effective, without engaging with the questions of how these methods enable better learning, or whether there is a direct correlation between these ways of teaching and student learning. Second, such research paid little attention to the classroom environment within which teaching took place, thus not acknowledging that there are several features that might support or inhibit practice and therefore influence student learning. It did not acknowledge that teaching happens in an integrated system where such features interact to determine student learning. For example, even though group work is considered effective for any mathematics classroom, cultural variations or norms have the potential to determine the nature and extent of its impact on students' learning. Attributing teaching to some independent and interchangeable variables rather than interacting features, was limiting. Hiebert and Grouws (2007) state that, "the most problematic aspect of interpreting this work is characterising the nature of dependent measures. Many studies used standardised achievement tests to assess students' learning. Although these tests often are composites of a range of items, they include a heavy skill component, require relatively quick responses, and are restricted to closed–ended, multiple choice formats." (p.381). Third, but a related critique is that, there were some noted methodological challenges encountered by the process–

product research. These included isolating the variables under study (from other variables that impact teaching or learning), the validity of the tools used to collect data, reducing complex constructs into measurable parts, the role of mediating variables, the ecological validity of the tools used, etc.

The challenges in the process–product research emanated from an understanding that the constructs such as teaching and learning are complex and interdependent. In more recent studies on connecting teaching with learning, such a critique has been addressed. For example, in the TIMSS video study, the strong correlation between teaching methods and students' learning was explored in detail. The finding was refined to a strong causal connection between higher order questions (pursued in Japanese classrooms) and students' performance in international (PISA) testing (Stigler & Hiebert, 2009). Further research asked questions about the cultural features of such classrooms and teaching which supported students' learning and performance.

Conceptions of (a) measuring teachers on content tests, and (b) identifying a direct co–relation between teaching and student learning are still used as indicators to assess teacher knowledge, for recruitment and career advancement purposes. The question, however is, whether knowing more content and/or using specific teaching methods ensures effective teaching.

2.3.2 Teacher knowledge is specialised

Lee Shulman in his lecture at the American Educational Research Association (AERA) in 1985 challenged the view of equating content knowledge with effective teaching. Shulman (1986) criticised the separate tests on content and pedagogical knowledge used to assess teacher knowledge, arguing that such testing left a "blind spot" on the knowledge that teachers actually use to do their work. For instance, tests on content knowledge did not assess the questions that teachers ask when teaching a specific topic or the nature of explanation that they are likely to offer to support students' learning. Shulman proposed a redefinition of "content knowledge" from knowing the facts or rules to understanding the structure and ways of establishing validity within a discipline. He classified content knowledge into subject matter knowledge (SMK), pedagogical content knowledge (PCK), and curriculum knowledge (CK). Table 2.1 summarises the nature of knowledge captured by each of these categories and the ways in which such knowledge is used to make pedagogical judgments. Although all these knowledge forms appear in the context of teaching, he argued that separating them offers a framework for analysing and supporting novice teachers' specialised knowledge of mathematics.

Table 2.1: Categorisation of content knowledge (Shulman, 1986)

Category	Nature of Knowledge	Pedagogical Judgments
Subject Matter Knowledge	Knowing what, how and why of a procedure, kind of warrants that support or weaken a claim, centrality of a topic over another.	Justifying a procedure, the nature and structure of justification and the varied curricular emphasis on different topics.
Pedagogical Content Knowledge	Ways of representing and formulating the subject to make it comprehensible for students.	Selecting useful forms of representations, powerful analogies, illustrations, examples, and explanations.
Curriculum Knowledge	Knowing the difficulties in learning of specific topics and students' preconceptions around them.	Identifying students' misconceptions and offering explanations that help in reorganising students' thinking.

Shulman's categories of content knowledge helped in reshaping the focus of teacher education and research to how the knowledge is organised in the mind of a teacher (Petrou & Goulding, 2011). The research studies following from Shulman's work operationalised and refined the categories of teacher knowledge by designing and using instruments to measure specific parts of it. For instance, in their study on assessing primary school mathematics teachers' PCK on different topics, Chick, Baker, Pham and Cheng (2006) found that teachers lacked in (a) understanding students' misconceptions, and (b) in helping students overcome such conceptions. This study concluded that the enactment of PCK in classroom is complex and can be differently seen in instances where (a) pedagogical knowledge is clearly visible, such as, identifying cognitive demands of the task, (b) content knowledge is used in a pedagogical context, such as, making connections between topics, and (c) pedagogical knowledge is used in a content context, such as, drawing and maintaining students' focus on a method or a strategy. Such a characterisation is useful in sharpening the varied role of PCK in different teaching situations.

While Shulman's work has influenced the research in the field of mathematics teacher knowledge, it has been critiqued for (a) not sufficiently operationalising the categories of teacher knowledge, (b) missing a discussion on the nature of interactions between the three categories of teacher knowledge, and (c) for presenting teacher knowledge as a static entity (Petrou & Goulding, 2011). Several models of teacher knowledge (discussed below) have refined Shulman's categories by accommodating these critiques.

2.3.3 Teacher knowledge is context–bound

One of the critiques of Shulman's framework on teacher knowledge was deeming it as a static entity. In response to this critique, Fennema and Franke (1992) placed the context at the centre of their framework on teacher knowledge. They argued that the dynamicity of teachers' knowledge can be captured by studying its interactions with teachers' beliefs. They defined teacher knowledge as composed of knowledge of mathematics, pedagogy, and learners' ways of thinking within mathematics. While these constitute teacher knowledge, how it gets enacted is largely determined by teacher beliefs in relation to each of these kinds of knowledge, that is, beliefs about content, pedagogy and students' learning (refer Figure 2.1).

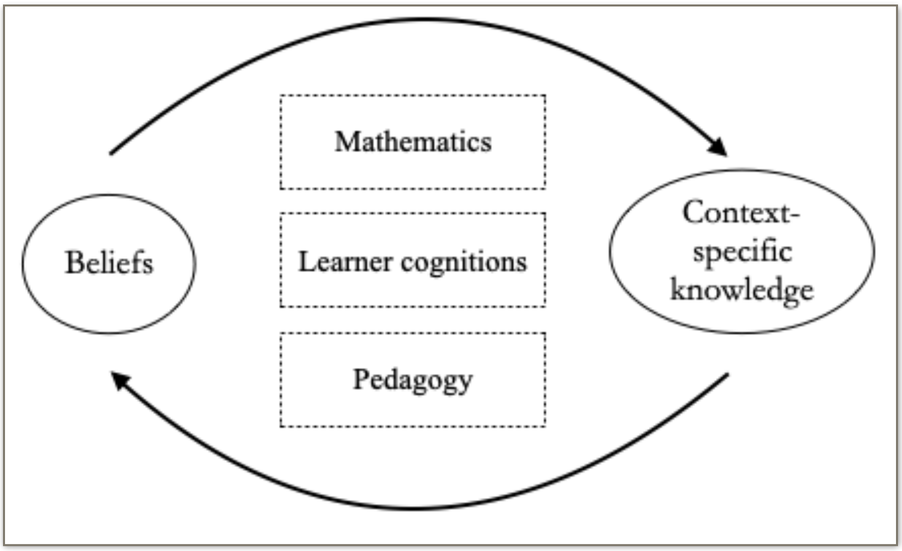


Figure 2.1: Belief-Knowledge Interaction
(adapted from Fennema & Franke, 1992)

Contrasting Shulman's framework with Fennema and Franke's proposal, it is noted that there is a clear focus on the subject matter knowledge or the knowledge of mathematics required for teaching. This category includes the knowledge of procedures and why procedures work in mathematics. Shulman's notion of PCK aligns better with the category of "knowledge of learner cognitions" rather than "pedagogical knowledge" in Fennema and Franke's conceptualisation. While PCK in mathematics refers to the pedagogy specific to the teaching of mathematics, "pedagogical knowledge" refers to the general principles of teaching such as classroom routines or management, use of effective techniques of managing content, and creating motivation among learners. The category "knowledge of learner cognitions" includes making a teacher cognisant in selecting appropriate representations and anticipating how students would deal with them.

Fennema and Franke (1992) acknowledged that none of these categories (mentioned in rectangles in Figure 2.1) exist in isolation from each other, all of them contribute to an understanding of teachers' knowledge.

> Knowledge is developed in a specific context and often develops through interactions with the subject matter and the students in the classroom. In their model, all aspects of teacher knowledge and beliefs are related to each other, and all must be considered to understand mathematics teaching. They suggest that no one domain of teacher knowledge has a singular role in 'effective' mathematics teaching (Petrou & Goulding, 2011, p. 14).

Further work on this framework involved designing a professional development programme, called Cognitively Guided Instruction (CGI), for mathematics teachers. CGI investigated the impact of providing structured knowledge about students' mathematical thinking on teachers' knowledge (of content, pedagogy, students' cognition) and beliefs in practice. The key foci of this programme were – studying the impact of using students' work as an artefact for teacher learning, and identifying different levels of learning or change among teachers. Both these aspects will be discussed in detail in Section 2.4.

Fennema and Franke's model of teachers' knowledge contributed to the understanding that teaching is interactive (a critique of the product–process research) and its enactment is a complex interplay of teacher knowledge and beliefs. While the model proposed studying teacher knowledge in context and interactivity of knowledge and beliefs, Petrou and Goulding (2011) argue that the methodological tools that can be used to measure the interaction between different categories of teacher knowledge in the context of classroom along with their changing roles and impact on learning remains a challenge.

2.3.4 Teacher knowledge and classroom practice

Fennema and her colleagues initiated the idea of studying practice in its complexity to understand teachers' knowledge. A more detailed analysis of practice was offered by Ball, Hill and Bass (2005) who invested in the problem of identifying the knowledge that underlies the work of teaching. Although several studies in different disciplines following from Shulman's work have demonstrated a co–relation between teachers' knowledge and students' achievement, there is a lack of measures which can be rigorously used to identify mathematical aspects of teacher knowledge (Hill, Schilling & Ball, 2004). In order to develop measures of effective mathematics teaching, a group at the University of Michigan, studied "high–quality instruction" in three mathematics classrooms. An analysis of teacher's work in these classrooms lead them to propose the construct of mathematics knowledge for teaching (MKT), which was classified into two broad categories, borrowed from Shulman's framework, namely, subject matter knowledge (SMK) and pedagogical content knowledge (PCK). The sub categories within each of these knowledge types are summarised in Table 2.2.

The categories of teacher knowledge account for a spectrum of mathematical knowledge from that held by educated adults (such as CCK) to the specialised knowledge of the teacher (such as SCK, KCS, KCT). This spectrum was used to map the general and specific aspects of teacher knowledge. While the organisation of teachers' knowledge uses content knowledge as a means to distinguish between knowledge types, the authors also proposed "tasks of teaching" as an important construct to understand teachers' work. The tasks of teaching included selection of representations and examples, assessment of students' understanding, evaluating the correctness of curriculum materials, etc.

Table 2.2: Classification of MKT (Ball, Thames & Phelps, 2008)

Category	Sub–categories	Operationalisation
Subject Matter Knowledge	Common Content Knowledge (CCK)	Identify incorrect responses, inaccurate definitions, use terms and notations correctly.
	Specialised Content Knowledge (SCK)	Determine the validity of a mathematical argument, select appropriate representations.
Pedagogical Content Knowledge	Knowledge of Content and Students (KCS)	Predict what students will find interesting, motivating, easy or hard; hear and interpret students' emergent and incomplete thinking.
	Knowledge of Content and Teaching (KCT)	Sequence content for instruction, selection of examples and representations, decision on the presentation of the content in class.
	Knowledge of Curriculum	Sequence and organisation of curriculum.

Further work in the same direction led to two kinds of research endeavour. It was realised that the tools used to assess teachers' knowledge indicated little about how this knowledge impacts instruction. So, a systematic study of classroom teaching was also used to design a rubric to assess the "mathematical quality of instruction (MQI)". Table 2.3 captures the different aspects assessed by the MQI rubric. MQI included an interaction of dimensions which characterise rigour and richness of mathematics in a lesson. The dimensions include the presence or absence of mathematical errors, mathematical explanation and justification, mathematical representation, and related observables (Hill, Blunk, Charalambos, Lewis, Phelps, Sleep & Ball, 2008). Hill et al. (2008) measured the nature and extent of relation between MKT and MQI by scoring teachers' lessons using the MQI rubric and correlating these scores with teachers' performance in the paper–pencil tests on MKT.

The researchers (ibid) found a strong correlation between what teachers know, how they know it, and how they use it while teaching. It was found that each MKT category did not have the same extent of co–relation with the teacher knowledge. For instance, mathematical errors (refer Table 2.3) including language errors were found to be more strongly related to teacher knowledge as compared to the density of accurate mathematical language. Several researchers have used the MQI rubric to assess teachers' instruction. For instance, Garet, Heppen, Walters, Parkinson, Smith, Song, Garrett and Yang (2016) used the MQI rubric to assess teachers' instructional practice before and after a professional development intervention. Such studies attempt to answer questions such as what is the nature of relation between teachers' knowledge and their instructional practice. However, they do not offer insights into the dynamic aspects of teachers' knowledge manifested in practice, such as, teachers' in–the–moment decisions.

Table 2.3: Categories of Mathematical Quality of Instruction (Hill et al., 2008)

Category	Description
Mathematical Errors	Nature of computational, linguistic, representation or other mathematical errors, like mathematical language, in instruction.
Inappropriate response to students	Degree to which a teacher misinterprets or fails to respond to a student's utterance.
Connecting classroom practice to mathematics	Degree to which classroom mathematical practices or activities are connected to key mathematical ideas or procedures.
Richness of mathematics	The use of multiple representations, linkages between them, mathematical explanation and justification. Explicating mathematical practices such as proof and reasoning.
Responding to students appropriately	Degree to which a teacher interprets students' mathematical utterances and address students' misunderstandings.
Mathematical language	The density of accurate language use to convey mathematical ideas.

A second kind of work which emanated from the MKT framework, was the development or adaptation of tools used to measure teacher's mathematical knowledge. Ma (2010), in her work on understanding mathematics teachers' knowledge, used some interview items developed from the tasks of teaching to create a map of knowledge that teachers need to have in order to teach conceptually. The interviews of teachers revealed that a deep and thorough knowledge of the subject matter includes identifying key ideas in the teaching of specific topics and connecting these ideas with the structure of mathematics. The elicitation of teacher knowledge through such tasks of teaching revealed aspects of knowledge which are significant for teachers to know, learn and develop through their education.

Some researchers (Garet et al., 2016; Carillo, Climent, Contreras & Munoz–Catland, 2013) have used the categories of teacher knowledge (mentioned in Table 2.2) to assess or analyse classroom teaching. Carillo et al. (2013) found that these categories of teacher knowledge do not comprehensively represent the construct of teacher knowledge that gets enacted in the classroom. The use of these categories to analyse the excerpts from actual teaching reveals that they are instances of an intersection of sub–domains that constitute MKT. This critique gave rise to frameworks which tried to characterise the knowledge that a teacher needs to handle everyday and specific tasks of teaching more effectively.

2.3.5 Knowledge of a mathematics teacher

Carrillo et al. (2013) proposed a revised focus on the knowledge of the mathematics teacher (MTSK), instead of specialised knowledge of teaching, which makes this knowledge different from the knowledge of other mathematics professionals and teachers of other disciplines. In their model, they do not acknowledge common content knowledge (CCK in Ball et al.'s framework) and broaden the scope of horizon content knowledge (HCK). Carrillo et al. emphasised the knowledge that teachers use, that is, knowledge of how and why in mathematics, and students' mathematical thinking. Broadly they classified teacher knowledge into the categories of mathematical and pedagogical knowledge, with their sub–categories (refer Table 2.4).

Table 2.4: Classification of teacher knowledge (Carrillo et al., 2013)

Category	Sub–category	Description
Mathematical Knowledge	Knowledge of topics	Knowledge of mathematical concepts and procedures along with their theoretical foundations.
	Knowledge of the structure of mathematics	Knowledge of key ideas, structure of the discipline, properties and connections, and ways of working within mathematics.
	Knowledge about mathematics	Knowledge of ways of knowing and creating mathematics, mathematical communication and reasoning, testing, how to define and use definitions, correspondences and equivalence, arguing, generalising and exploring.
Pedagogical Content Knowledge	Knowledge of feature of learning mathematics	Knowledge about how mathematics is learnt, difficulties faced by leaners in doing mathematics, different psychological frameworks on mathematics learning.
	Knowledge of mathematics teaching	Knowledge of resources, including representations and examples, and their appropriateness for the content.
	Knowledge of mathematics learning standards	Knowledge of progression in curriculum, learning and assessment standards.

The revised focus of MTSK implies studying the teaching practice for its own sake, and avoiding prescriptions about good teaching. The researchers argue that the perspective of studying practice for its own sake allows us to develop an understanding of the complex work of teaching. Further, it requires an explication of researchers' own positioning on the nature of knowledge that teachers need. Some more work is needed to operationalise the aspects of mathematics teachers' knowledge proposed in this model.

2.3.6 Studying mathematical discourse

Adler and Rhonda (2015) proposed a different perspective, although with the similar aim of understanding teaching, as Carrillo et al. They recognised that teaching is at the core of evaluating teachers' knowledge of mathematics. Teaching, they suggest, is an act of mediation towards the scientific concepts, generality and objectification in mathematics. In order to understand the nature and quality of this mediation, studying the mathematical discourse of the classroom, becomes salient. Mathematical discourse in instruction (MDI) can be used to define the instructional elements of mathematics teaching. Using a socio–cultural perspective on learning, mathematical discourse is studied in primary and secondary rural mathematics classrooms in South Africa. MDI has four interacting components to be mapped through a study of classroom instruction. These include exemplification, explanatory talk, learner participation and the object of learning. The relation between these constitutive elements of MDI is summarised in Table 2.5. Methodologically, the analytical framework is applied by dividing a mathematics lesson into episodes, which become the unit of analysis, and identifying examples sets related to a task, the explanatory talk that accompanies them and how these get accumulated as the lesson progresses. In this way, a description of what is mathematically made available to the learner is created and the object of learning is identified within and across episodes.

The framework offers a way to distinguish between different mediational means (explanation and exemplification) as well as how they interact with each other in classroom discourse (relation between examples and explanations). It provides a novel and promising way of describing practice and has the potential to be used for organising teacher development as well. While MDI has emerged in a particular context, its generative potential, through use in different contexts, needs exploration.

Table 2.5: Analytical framework for Mathematical Discourse in Instruction
(adapted from Adler & Rhonda, 2014, p.12)

Objects of Learning	Mediation through	Level 1	Level 2	Level 3
Exemplification	Examples	Experience separation and contrast	Experience separation, contrast or fusion	Experience fusion and generalisation
	Tasks	Carry out known procedures and operations	Apply and select operations and procedures, compare and contrast	Can make multiple connections and use multiple concepts using proofs and reasons
Explanation	Talk/ Naming	Use of colloquial language including ambiguous referents such as this, that	Use of mathematical language to name objects or reading string of symbols	Use of appropriate names and math objects and procedures
	Legitimating criteria	Use of visual cues or metaphors to relate to some characteristics of real objects	Use of specific, single case real life application and established short cuts and conventions	Use of equivalent representations, definitions, established generalisations, partial use of principles, structures and properties*
Learner Participation	–	Learners answer using yes–no or complete teacher's sentences	Learners answer what and how questions in phrases or sentences	Learners answer why questions, contribute ideas to discussion, teacher revoices, confirms and asks questions

* There is a fourth level where the understanding of all these is full (as opposed to partial).

2.3.7 Teacher knowledge–in–play

In their work on studying teacher knowledge in context, Fennema et al. (1992) stressed the need for research to look closely at the act of teaching. The weak boundaries between SMK and PCK, emanating from Ball et al.'s research, were becoming evident through studies of classroom teaching. Responding to these ideas, Rowland, Huckstep and Thwaites (2005) suggested categorising teaching situations to propose a well organised framework for teacher knowledge. This group of teacher educators video–taped mathematics lessons taught by pre–service elementary school teachers in British classrooms, and used the analysis from these lessons to propose an analytical framework for teacher knowledge. The Knowledge Quartet (KQ) framework, thus offered, had four dimensions namely foundation, transformation, connection and contingency. The meaning of these four dimensions is summarised in

Table 2.6. The codes within each of these dimensions were used to provide descriptions of teachers' work, while teaching in a classroom.

Table 2.6: Dimensions of Knowledge Quartet (Rowland, Huckstep & Thwaites, 2005)

Dimension	Description
Foundation	Knowledge, beliefs and understanding gained from the course work to prepare for the teaching.
Transformation	Knowledge used to plan and enact lessons.
Connections	Knowledge needed make decisions about connecting discrete parts of mathematical content.
Contingency	Knowledge required to deal with unpredictable moments arising while teaching in classroom.

The KQ framework moves away from the categories of teacher knowledge *per se*, to characterise teaching practices which invoke teachers' knowledge. This framework allows for an integration of categories of teacher knowledge identified by earlier research, for instance, connections between SMK and PCK, or between teacher knowledge and beliefs.

The KQ framework has been extensively used to analyse and give feedback to pre–service teachers by their teacher educators in different countries (refer Petrou, 2010). For instance, Goulding, Rowland and Barber (2002) used the KQ framework to examine subject matter knowledge of primary school teachers in England and Wales, and the connection between this knowledge with specific tasks of teaching, such as planning. One of the limitations of the framework is its lack of attention to the curriculum resources and their use.

2.4 Teacher Knowledge About Students' Mathematics

Existing frameworks on mathematics teacher knowledge propose an overlap between teachers' knowledge of students and the mathematical content. The integration of this knowledge is an important part of knowledge required for teaching mathematics. In the Section 2.4.1, I will discuss how an integration of knowledge of students and mathematics is proposed in the contemporary frameworks on mathematics teachers' knowledge. An attempt is made to highlight what was learnt from these frameworks and the challenges faced in using these frameworks to capture teacher knowledge in play. In Section 2.4.2, I will operationalise the topic–specific knowledge as an important sub–set of teachers' knowledge, using the literature on students' mathematical thinking in decimals. The purpose of this section is to create a map of the topic specific knowledge which teachers can be prepared with, for better anticipation and response to students' utterances while teaching in the classroom.

2.4.1 Knowledge of students and mathematics

In Section 2.3, I discussed frameworks which have tried to conceptualise 'teacher knowledge' by categorising it into sub parts. These frameworks have been criticised for not accounting for the teachers' in–the–moment decisions while teaching in the classroom. Teaching decisions, such as choice of appropriate representations, pressing some learner meanings, responding to a students' alternate conceptions, etc. are a part of teachers' routine and require a rich knowledge base. As discussed earlier, Shulman (1986) proposed the construct of pedagogical content knowledge (PCK) to direct attention to the amalgamation of content and pedagogical knowledge that teachers need to be able to teach students effectively. A part of PCK includes the knowledge of conceptions and preconceptions that students of different ages and backgrounds bring to the learning of specific topics. Teachers need to be aware of the common student preconceptions and the strategies which are helpful in reorganising students' knowledge. Research based knowledge on students' prior conceptions and of instructional conditions necessary to transform students' ideas, is useful in supporting this kind of a knowledge base (Shulman, 1986). Ball and her colleagues refined the construct of PCK to include – knowledge of content and students, knowledge of content and teaching, and knowledge of curriculum. Knowledge of content and students (KCS) includes the knowledge of common student errors and decisions concerning the errors that students are more likely to make, prediction about whether the students will find a task motivating and interesting, and the teacher's ability to hear and interpret students' emerging or incomplete thinking (Ball, Thames & Phelps, 2008). Ball, Hill and Bass (2005) assert that teachers need insight and understanding of the content in order to identify students' errors and select appropriate representations for dealing with them. Such knowledge lies at the interface of a mathematical idea or procedure and ways in which students think about it. Much of this understanding has developed from the research literature on students' thinking, learning trajectories, and cognitive research such as cognitively guided instruction. Empirical research following from Ball's framework has illuminated specific aspects of teacher knowledge in different topics for instance, teachers' PCK in subtraction (Chick, Baker, Pham & Cheng, 2006), arithmetic problem solving (Carpenter, Fennema, Peterson & Carey, 1988), primary mathematics (Baker & Chick, 2006), etc.

Carrillo, Climent, Contreras and Muñoz–Catalán (2013) expand the notion of KCS to include the knowledge of ways in which learning theories can be utilised for teaching in classroom. The knowledge of different ways in which students' think and learn is useful to develop this knowledge. Additionally, teachers need to understand how different theories or models of learning can contribute to the process of describing mathematical learning. The knowledge is not limited to knowing the theories or models of students' learning, but also how these can be used to orchestrate or plan learning experiences. Knowledge of features of learning mathematics (KFLM) includes identifying how mathematics is learnt and the features of mathematical learning.

Llinares (2013) notes teachers' ability to identify the mathematical elements of the students' talk as a skill and calls it professional noticing. This skill of noticing, understanding and inferring from students' productions, allows the teachers to plan learning trajectories and make informed instructional decisions. Sherrin, Jacobs and Phillips (2011) argue that research on adaptive teaching, decomposing practice and learning from reflection on teaching rest on the the idea of noticing, originally coined by Mason (2002) as researching one's own practice.

In the frameworks mentioned above, content specific knowledge related to students' conceptions is identified as significant for teachers to teach effectively. Rowland, Thwaites and Jared (2015) investigated when such knowledge is triggered or activated while the teachers are teaching in a classroom. In other words, what is the form and content of this knowledge in play. They define "contingency" to refer to those situations, which cannot be predicted by teachers while planning a lesson, and come to them as a surprise during teaching. Such surprises might arise from an unanticipated student remark or answer (Rowland & Zazkis, 2013). Handling such situations in the classroom, also described as "knowing to act in the moment" (Mason & Spence, 2000), is an important part of teaching. Although these surprises are unanticipated, teachers can be prepared to handle such situations more effectively if they are aware of the common student conceptions and misconceptions, and topics that students might find difficult. This information is likely to feed back into teacher's anticipation of the obstacles faced while teaching specific content.

The existing frameworks on teacher knowledge illuminate our understanding of how knowledge about students and content is intricately linked, particularly in the contexts of teaching specific topics. Further, it has been suggested from the existing literature on students' thinking, that a knowledge of students' conceptions might be a useful resource for strengthening teachers' knowledge base and in supporting their classroom instruction. However, the use of such frameworks for analysing teaching practice has raised some challenges. The first challenge is the difficulty in using these frameworks to capture the dynamic aspects of teacher knowledge in practice (Hodgen, 2011). In classroom teaching situations, it becomes difficult to identify different subdomains of knowledge. Carrillo et al. (2013) have pointed the difficulty, for instance, in demarcating between the subdomains of specialised content knowledge (SCK), horizon content knowledge (HCK) and knowledge of content and students (KCS) from Ball et al.'s framework. To give an example, the property of commutativity of addition and multiplication works for natural numbers. However, there is a difference in the two cases, that is, the meaning in a multiplicative situation may not necessarily be commutative (for instance, 3 groups of 4 objects each is not the same as 4 groups of 3 objects). Apart from this difference it has been noted that students apply the commutative property learnt during natural numbers to the multiplication of matrices. Carrillo et al. argue that the knowledge of this difference is a part of HCK, but since it is linked to the learning of students, it is also KCS. The second challenge is how teachers use the knowledge gained through their pre– and in–service courses in specific instances of teaching. In designing professional development experiences for teachers, it is hard to determine how such knowledge can be developed, particularly in ways that facilitate its impact on practice. Translating the knowledge learnt into practice has been identified as an endemic problem in teacher education (Kumar, 2008). Kazemi and Franke (2004) argue that an awareness of students' conceptions might not translate into informed decision making while the teachers are teaching. Even (2008) proposes that the integration of knowledge about students' conceptions or ways of thinking and using this knowledge to inform practice, gives rise to a new object, called "knowtice". She suggests that developing knowticing in teachers requires special attention in teacher education programmes.

Teachers' knowledge and noticing of students' thinking influences and gets influenced by how they listen to (and interpret) students. Since such listening depends on how knowledge is constructed in specific classrooms, an important part of teaching is listening and responding to unanticipated student ideas. Doerr (2006) argues that expertise in teaching is not uniform, and cannot be achieved through the learning of a fixed set of constructs, rather it is knowledge that develops across varying dimensions and in varied contexts for particular purposes. For a teacher to be responsive to students' ideas, they have to be able to hear the mathematics underlying students' responses, carefully scaffold a response which is within the reach of the student, and support students in reorganising their existing ideas or in learning new ideas. Researchers who have studied responsive teaching have stressed the importance of listening to ''children's mathematics'' (Empson & Jacobs, 2008). Teaching responsively has two aspects – first, an aspect of listening to what students are saying and understanding what they are thinking and second, responding pedagogically to students' thinking in order to facilitate learning. Researchers, who have attempted to characterise responsive teaching, have recognised both aspects and also that they entail special knowledge demands. Potari and Jaworski (2002) stress both cognitive and affective sensitivity to students, as well as the skill in managing mathematical challenge and learning. Davis (1997) distinguished interpretive listening, where the teacher is attending to students' ideas from evaluative listening, where teacher's listening is filtered by her prior expectations of how students ought to respond. A further category of hermeneutic listening pointed to the teacher's readiness to engage with students' ideas in changing them and also possibly changing the teacher's own understanding. Empson and Jacobs (2008) draw a similar distinction between directive, observational and responsive listening. Building on this literature, Doerr (2006) identifies three dimensions of teacher knowledge as important for responsive teaching. These are "(a) an understanding of the multiple ways in which students' thinking might develop, (b) ways of listening to that development, and (c) ways of responding with pedagogical strategies that support that development" (p. 256).

In contrast to the metaphor of a "map" of teacher knowledge described earlier, approaches on responsive listening emphasise the dynamic aspects of classroom interactions; they lay stress on the teachers' ability to anticipate paths that learners may take as they navigate the construction of new knowledge from what they have known previously. I argue that the literature on responsive listening helps in illuminating how the components of KCS, KCT, and SCK interact dynamically in the context of classroom interactions. Interpretive listening is strengthened by knowledge of how students interact with content, for example, by knowledge of common student errors and difficulties, which is a part of KCS. SCK deals with making the features of mathematical content visible to students through the choice and use of effective representations, justifications, etc. The demands entailed in responding to students in pedagogically appropriate ways are a part of KCT. Knowledge of the affordances of representations, and of ways of deploying them have been identified as a part of this component (Ball, Thames & Phelps, 2008). All these knowledge components are associated with how the teacher makes decisions in the ebb and flow of classroom interaction. Hence, the elaboration of knowledge of students and content required for teaching calls for a closer study of the demands implicated in such decision making, which is the focus of this thesis. Researchers have attempted to analyse these demands by focusing on topic–specific tasks of teaching; for instance, teaching integers with representations (Kumar 2018; Mitchell, Charalambous & Hill, 2014), preparing teachers to use different problem types in arithmetic (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989), making sense of students' responses to subtraction with regrouping (Ball & Bass, 2000), etc. In the thesis, the topic of decimal numbers was selected by the teachers who participated in the study as a focus for the interactions between the teachers and researchers.

2.4.2 Topic specific knowledge for teaching decimal fractions

Descriptions of topic–specific knowledge needed by teachers draw not only on studies of teaching practice, but also on prior research on student errors and difficulties. In this section, I will discuss some empirical findings on students' errors and difficulties related to the topic of decimal numbers and the underlying reasons for these as suggested by research studies. I illustrate how these research results, as well as suggestions on how to design instruction to deal with student difficulties, imply a set of knowledge demands on teachers, some of which have been identified explicitly by researchers. The topic of decimal numbers is the focus of the study as it was selected by participating teachers for discussion and support.

In an early study, Resnick, Nesher, Leonard, Magone, Omanson and Peled (1989) found that, when comparing decimal numbers of varying lengths and digits, students tended to judge the decimal number with more digits after the decimal point as larger. For instance, students may judge 4.63 as greater than 4.8 since 463 is greater than 48 or 63 is greater than 8. This has been described as the whole number rule, where students ignore the fact that the decimal portion of the number is a fractional part of the whole. Students who focused on the digits also faced difficulty in comprehending how a decimal number 2.593 would change if a zero is inserted at any of the four decimal places. In a later work, Steinle and Stacey (2004) classified students' responses on a variety of decimal magnitude tasks. Some students believed that the longer decimal number, with more digits, was larger (similar to Resnick et al., 1989), while other students believed that the shorter decimal was larger. Students provided different reasons for their choice, even for the same behaviour. For instance, students who thought that the "longer is larger" could be either guided by the length of the string (and say 2.78>2.9 as 78>9) or ignore the decimal point (and compare 278 with 29) or ignore the size of the part (considering 6.03 and 6.3 as the same). Such errors were often accompanied by errors in decimal number operations. For example, Grossnickle (1943) found that students made mistakes in placing and shifting a decimal point, annexed or omitted zeroes in the quotient, and so on when performing division with decimal numbers.

The analysis of students' errors when working with decimal numbers reveals their roots in the students' prior knowledge of whole numbers and the difficulty that they face in making the transition to rational numbers (Resnick et al., 1989). Steinle (2004) reported that an over–generalisation from whole number thinking in school mathematics is a root cause of student misconceptions in the comparison of decimal numbers of varying lengths and values. It is difficult for students to extend their understanding of place value notation and combine it with the understanding of fractions to make sense of rational numbers. Behr and Post (1992) argue that decimals are an important extension of both the base ten place value system and of rational numbers and can therefore be interpreted using either or both of these perspectives. The place value perspective logically extends the decimal understanding to the base ten numeration system by including tenths (one–tenth of one whole); hundredths (one–tenth of one–tenth) and so forth. The fraction perspective makes decimals a special case of the area based part–whole interpretation where a whole is divided into parts which are powers of ten, commonly 10, 100 or 1000. Behr and Post argue that the understanding from these two perspectives interacts as students try to make sense of decimals and their operations.

Specific suggestions to deal with this difficulty while teaching decimals include presenting numbers in fraction, natural number, and decimal forms to show the invariance among these representations (Vamvakoussi & Vosniadou, 2007). Desmet, Grégoire and Mussolin (2010) suggest that students should work with several examples of decimal fractions, where varying digit values and length will help in creating a conflict between their understanding of whole numbers and rational numbers. Brousseau, Brousseau and Warfield (2007) propose a curriculum for rational numbers, where decimals are used to approximate the measurement of continuous quantities, differentiating them from natural numbers where discrete and imprecise measurements are permitted.

Research on student difficulties and related literature on teaching decimal numbers suggests that students need to restructure their knowledge of conceptions, rules and symbols learnt for whole numbers (Irwin, 1996) and fractions. What are the knowledge demands on teachers suggested by this research? Firstly, teachers need to be secure in their own understanding of the magnitude of decimal numbers. Research suggests that this may often not be the case as teachers face difficulties similar to students in judging the magnitude of decimal numbers and understanding their density (Muir & Livy, 2012; Widjaja, Stacey & Steinle, 2008). Francisco and Maher (2011) have noted a lack of opportunities for teachers to learn about students' mathematical thinking and reasoning. Muir and Livy (2012) found that pre–service teachers were unaware that they had developed flawed understanding of decimal numbers unless their content knowledge was challenged. This dimension of teachers' knowledge is part of CCK in the MKT framework. Further, Tirosh and Graeber (1989) found that practicing teachers face difficulty in justifying the procedure for multiplying a decimal with ten (also noted by Chick, 2003). This is an example of knowledge that is specialised for teaching beyond CCK, and hence may be classified as SCK. Another example of SCK is knowing why annexing a zero does not change the decimal number. The research on student errors and difficulties shows that such errors have systematic misinterpretations underlying them that the teacher needs to be aware of. This forms a part of KCS. Some of this knowledge overlaps with KCT since student responses are related to teaching decisions.

It has been noted that inadequate attention to the analogy drawn between whole numbers and decimals might perpetuate student misconceptions instead of addressing them. For instance, consider the rule of annexing a zero to make the length of two decimal numbers equal. Swan (1990) suggests that such an emphasis on this rule by the teachers without reference to the place value might provide correct answers but does not support conceptual understanding. We may think of such knowledge as a part of both KCS and KCT in the MKT framework. Jackson, Gibbons and Dunlap (2014) reported that teachers attributed students' difficulty in decimals to students' traits, or deficits in their family or community and dealt with it by lowering the cognitive demands of the task when noticing students' facing difficulty. The authors added that although teachers attributed students' difficulty in learning decimals to the lack of instructional opportunities, they did not respond to students in ways that would enable participation in rigorous mathematical activity.

2.5 Connecting Teacher Knowledge and Practice to Learning

A large body of literature has identified the need for developing "practice–based tasks" to enhance teachers' knowledge of mathematics teaching. What is common in different practice based approaches to professional development (PD) is an invitation for teachers to participate in professional learning communities, and reflect on their teaching practice using the knowledge gained from the literature and field experience. Several PD programmes use students' work as an artefact to situate teachers' learning in their practice. Professional learning communities where student work is detailed, analysed and used to make teaching decisions has the potential to support teachers in making informed teaching decisions. Bannister (2018) suggests that teacher development through participation in such communities has the potential for "humanising mathematics teaching and learning" by impacting classroom learning directly.

The ways in which students' work gets used in the existing PD programmes is influenced by the theoretical stance on teacher knowledge and learning. I borrow the framework proposed by Cochran–Smith and Lytle (1999) on the relation between knowledge, practice, and learning to illuminate the differences between various approaches to PD of teachers and locate the approach undertaken by the reported study.

Table 2.7: Teacher Knowledge, Practice and Learning (Cochran-Smith & Lytle, 1999)

Knowledge–Practice relation	**Source of knowledge generation**	**Teacher learning**
Knowledge *for* practice	Theory or formal knowledge generated by university professors and researchers for use by teachers.	Knowledge generated by experts is passed on to teachers for use in classroom.
Knowledge *in* practice	Practical knowledge embedded in teachers' work and generated from reflection on teaching practice of expert teachers.	Teachers get opportunities to probe knowledge embedded in the work of expert teachers and deepen their knowledge through interactions in a community.
Knowledge *of* practice	Treating classrooms as sites of enquiry and using the knowledge developed by others (in the field) to interrogate practice.	Teachers learn by generating knowledge in their local contexts of practice being a part of inquiry communities and theorise their work.

Cochran–Smith and Lytle (1999) proposed a distinction between three conceptions of teacher learning by unpacking the image of teachers and assumptions about what constitutes valuable teacher knowledge (summarised in Table 2.7). In the first conception, knowledge *for* practice, it is assumed that formal or theoretical knowledge is produced by university researchers for teachers to improve their practice. The knowledge of the subject–matter or research–based strategies is communicated to the teachers. Teachers are expected to learn new skills, knowledge and techniques based on the existing standards. Teachers learn through participation in professional development workshops where such knowledge is transacted and are then expected to use it in practice. In this conception, teachers work individually to implement reforms by using certified procedures acquired through teacher preparation and professional development programmes. The second conception, knowledge *in* practice, assumes that knowledge is embedded in teachers' work and is essentially practical in nature. This "knowledge–in–action", according to Schön (1983), is tactic and implicit. It is made explicit through deliberative reflection on the experience of teaching. This stance is rooted in the constructivist image of knowledge generation and includes how (a) experienced teachers make judgments while teaching, (b) conceptualise or describe classroom dilemmas, (c) attend to different aspects of classroom life and, (d) think about and improve their craft (Cochran–Smith & Lytle, 1999). The epistemology of practical knowledge is accessed from embodied stories of teachers' personal actions (Clandinin & Connelly, 1998). Teaching is considered at par with the other professions such as making music and doing surgery. Therefore, novices learn by studying the work of experts and by replicating such learning environments in their classroom. In recent conceptualisations, the term "craft knowledge" has been used to amalgamate reflection on practice with the knowledge of teaching. Reflection on teachers' own actions, observation of others' practice, coaching to reflect on teaching are some of the ways of learning the practical knowledge. The third conception, knowledge *of* practice, breaks the divide between the formal and practical knowledge. It assumes that the knowledge needed for teaching is generated from systematic inquiries about teaching, learning, subject–matter and curriculum. Such knowledge is collectively constructed by teachers in partnership with those interested in classroom inquiry and learners' sense making. Teachers learn through participation in inquiry communities by generating the local theories of practice and connecting them with the existing theoretical perspectives. Such knowledge construction is considered 'transformative', where teachers' agency inside and outside of classroom is acknowledged.

A summary of the differences between the three perspectives is as follows.

> …the image of practice in the first conception, knowledge–for–practice, emphasises how teachers use the knowledge base to solve problems, represent content, and make decisions about the daily work of the classroom. The image of practice in the second, knowledge–in–practice, emphasises how teachers invent knowledge in the midst of action, making wise choices and creating rich learning opportunities for their students. Although different in important ways, both of these refer primarily to what teachers do within the boundaries of their roles as classroom managers, orchestrators, and planners. On the other hand, this third conception of teacher learning, knowledge–of–practice, emphasises that teachers have a transformed and expanded view of what "practice" means. Teachers' roles as co–constructors of knowledge and creators of curriculum are informed by their stance as theorisers, activists and school leaders…We are not suggesting that an expanded view of practice results from adding teachers' activity outside the classroom to what they do inside but, rather, that what goes on inside the classroom is profoundly altered and ultimately transformed when teachers' frameworks for practice foreground the intellectual, social, and cultural contexts of teaching. (Cochran–Smith & Lytle, 1999, p. 276)

2.5.1 Approaches to professional development

Cognitively Guided Instruction (CGI) is a professional development programme organised to provide teachers with the research based knowledge on students' strategies when solving arithmetic word problems, and then study their instructional decisions while teaching in classroom (Carpenter, Fennema, Franke, Levi & Empson, 2000). The project started with an investigation of students' strategies when solving addition and subtraction word problems (Carpenter, Hiebert & Moser, 1983). These word problems were classified into types, namely, combine, compare, equalise, and separate. The problem types were classified on the basis of the nature of thinking invoked and the strategies used to solve them. For instance, combine problems invite addition of two different sets, and compare problems involve a relation between the two sets with the start and relation known but the result unknown. The knowledge generated from the research on students' strategies was provided to teachers. Teachers participated in work group meetings, where they were familiarised with different problem types and students' strategies. The study was extended to understand how teachers used this knowledge in their classroom and its influence on students' learning (Carpenter, Fennema, Peterson, Chiang & Loef, 1989). The project followed a few teachers longitudinally to understand their "generative growth" through participation in structured CGI workshops (Franke & Kazemi, 2001). The findings from this study indicated that experienced teachers possessed some intuitive knowledge about students' thinking, but this knowledge often remained fragmented and was not utilised for decision making in classroom. The CGI framework has been used to analyse teachers' participation in PD workshops centered on students' work, their use of arithmetic problem types in their classroom, and the changes in teachers' beliefs and practice (Franke & Kazemi, 2001; Empson & Jacobs, 2008).

In the CGI programme, the knowledge of students' strategies provided to the teachers, was generated by the researchers and ways in which teachers used this knowledge was recorded. In the professional development workshops, teachers were provided with the knowledge of problem types in arithmetic and of ways of students' thinking around these problem types. Teachers used these problem types during teaching, which was documented. Thus, the programme is an instance of generating and using knowledge *for* practice.

In contrast, the lesson study approach to professional development is initiated by the teachers to plan and reflect on their teaching in work groups. Lesson study originated in Japan and gained attention from the international community after the TIMSS video study. A group of teachers interested in teaching a particular topic meet and plan a lesson, which is called a "research lesson". The research lesson is taught by one of the participating teachers, and observed by other teachers and some other observers (teachers from the same school or veteran teachers from elsewhere in the country). In the debriefing session, organised after the research lesson, the observers discuss the lesson with the teacher and suggest modifications. These suggestions are used by other participating teachers when teaching this lesson in their class. Apart from a focus on lesson sequencing, relevance of questions, individual and group work time; attention is drawn to the ways in which students respond to the teacher's questions and how a teacher builds on students' existing knowledge. With participation in different lesson study experiences, teachers learn to connect classroom practice to the broad curriculum goals, experience and discover novel practices, explore conflicting ideas between reform suggestions and their implementation, and improve their knowledge base (Lewis, 2000; Doig & Groves, 2011). The lesson study approach has been adapted in several countries to improve teachers' subject knowledge (Ono & Ferreira, 2010) and planning (Cerbin & Kopp, 2006), promote changes in teachers' knowledge and beliefs (Lewis, Perry & Hurd, 2009), encourage reflection, direct teachers' attention to students' thinking and learning, and build communities of practice.

Lesson study offers an approach for sustained professional development experience through participation in teacher communities. Teachers participate in collaborative activities before and after teaching the lesson. Experienced teachers document their practices and reflections from teaching a research lesson. Fraivillig, Murphy and Fuson (1999) argue that documentation of instructional practices of classrooms where teachers provide students with an opportunity to "explore mathematical objects and to synthesise their own mathematical meanings" is productive in generating descriptions of effective mathematics teaching. In a lesson study, a group of teachers create the knowledge required to improve their practice. The approach seems to be guided by the theoretical proposition, that knowledge is generated in the field, making it as example of knowledge *in* practice.

Both PD initiatives, CGI and lesson study, offer a perspective on supporting teachers' generative learning through reflection on students' work and classroom teaching. CGI is guided by the framework of generating knowledge *for* practice by providing research–based knowledge to teachers and recommending problems which can be used during classroom teaching. Lesson Study, on the other hand, is guided by a perspective on generating knowledge *in* practice by promoting teacher communities and building on knowledge situated in teaching.

In an attempt to reduce the gap between research and practice, the reported research combines aspects of knowledge *for* practice and knowledge *in* practice to offer an exemplar of generating knowledge *of* practice. This was done by using the research literature on topic–specific knowledge in ways that support or challenge teachers' experiences of teaching, leading to the organic evolution of the teacher–researcher community with the shared goal of students' learning. Ways in which teachers used this knowledge to inform their practice and the knowledge demands placed on them were analysed (reported in Chapter 5). Teacher reflections as they engaged with the research literature on student conceptions and students' actual responses, were supported through teacher–researcher meetings (discussed in Chapter 6).Teachers were also supported in using this knowledge to inform their practice through the *in–situ* support provided by the researcher (discussed in Chapter 7).

2.5.2 Bridging the research–practice divide

Challenging the recent work on learning trajectories and mathematics teachers' development, Cai et al. (2017) argue that the research is yet to identify the "grain size that is compatible with teachers' classroom practice" in order to seriously address the divide between research and practice. Additionally, research on mathematics teachers can be made more impactful by redefining teachers' roles in the research endeavour. A survey of research on teacher learning though collaboration (Robutti, Cusi, Clark–Wilson' Jaworski, Chapman, Esteley' Goos, Isoda & Joubert, 2016) reports the difficulty in explicitly relating teacher learning to their collaboration in the project.

Another critique, offered by Bannister (2018), is that despite an extant literature on teacher learning communities, little is known about the nature of teachers' learning and how they learn. One of the suggestions is to situate collaborations with teachers in their working days along with the aim of supporting their students' learning and improving their working conditions.

More recent work in organising professional learning experiences focuses on evidence based decision making where teachers are encouraged to learn in and from practice.

Approaches where knowledge is generated by teachers and researchers (or other professionals) collaboratively and used to improve classroom practice needs to be experimented. Brodie (2016) asserts that, in developing nations, the focus of professional learning communities can be to “support deliberate, collective learning, drawing on local data and the knowledge base” (p.157) available in the field. In this case, the knowledge that researcher or facilitator brings to a professional learning space becomes as significant as that of the participating teachers.

Kazemi and Franke (2004) note that bringing together teachers to “look at students’ work” does not necessarily contribute to teachers’ learning. Additionally, teachers who have never engaged in an analysis of the mathematical aspects of students’ work, through their schooling or teaching, might find it extremely difficult to develop a language to participate in such discussions. The analysis of mathematics underlying students’ work requires that teachers are aware of the potential ways in which students’ think about a topic, possible connections they might make between their prior knowledge and new knowledge, and examine or imagine different ways of dealing with such student utterances in the classroom. As noted from the literature on teacher knowledge of content and students, research on students’ alternate conceptions is a useful resource in developing an awareness of student difficulties or connections in a particular topic. Again, an awareness of the literature on students’ difficulties may not necessarily influence teachers’ practice (Even, 2008). Franke and Kazemi (2001) noted that often teachers struggle to make connections between the development of their students’ mathematical thinking and decisions to be made while teaching. The modality of communicating with teachers is an important factor in understanding the reach of professional development experience to the classrooms.

In India, teachers are familiarised with the propositions of the reformed curriculum through workshops in a cascade mode. While workshops are useful in communicating the key ideas of reform, they are known to make little impact on the teacher’s existing practice (Kumar, 2018). The diversity of the Indian classrooms, has also raised questions about what is the nature, focus, and content of the knowledge that teachers from different locales might need to teach mathematics effectively. Acknowledging the complexity of working with the teachers and supporting them in the contexts of practice while using the literature in the field, becomes a challenging task.

The reported research proposes constructs which served as a suitable “grain size” for research to impact practice. Teachers’ local knowledge base was supported with the relevant research–based knowledge to achieve the purpose of investigating and developing teacher knowledge. Tasks for supporting teacher knowledge were designed by combining aspects of teaching that were observed in practice and the existing literature in the field of students’ and teacher knowledge. The knowledge *of* practice perspective acknowledges teachers’ agency in selecting aspects of the professional development that they find most suitable depending on their classroom. Further, teachers were supported on the use of these aspects in their classroom to promote student learning.

2.6 Reflection on Frameworks of Teacher Knowledge and Learning

In this section, I attempt to address the question of what was learnt from a reflection on the existing frameworks of teacher knowledge and learning. These reflections influenced the design and conduct of the reported study.

The past few decades have seen an emerging interest in understanding teachers’ knowledge of mathematics and how it informs teacher learning (Zaslavsky & Peled, 2007). Hill, Ball and Schilling (2008) highlight the significance of distinguishing teacher’s knowledge of mathematics from the knowledge of mathematics for teaching. Mathematical knowledge for teaching includes thinking of suitable pedagogies using which mathematics can be communicated to students at different cognitive levels. Shulman (1986) called the specialised form of teacher knowledge as Pedagogical Content Knowledge (PCK), complementary to and distinct from content knowledge and general pedagogical knowledge of a teacher. PCK is “an understanding of what makes the learning of specific topics easy or difficult; the conceptions and pre–conceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons” (Even & Tirosh, 2008). Hill, Ball and Schilling (2008) proposed a comprehensive model for mapping teachers' knowledge that includes PCK which suggests that (a) teachers' knowledge about students' mathematics is an important aspect in the scheme of teachers' knowledge, and (b) there are different ways in which this knowledge interacts with the other inter–related aspects of teachers' knowledge base. Teachers' knowledge about students would be incomplete without conscious reflection on aspects significant to teaching and learning in classroom. In fact, Schön (1983) calls teacher a reflective practitioner, a professional capable of knowledge–in–action, reflection–in–action and reflection–on–action. The two strands of knowledge and reflection can be tied meaningfully to facilitate teacher learning.

Let us recall the organisational framework offered by Cochran–Smith and Lytle (1999) which relates knowledge and practice to understand teacher learning. They classify the relation of knowledge and practice as knowledge *for* practice, knowledge *in* practice, and knowledge *of* practice. Knowledge *for* practice is the knowledge generated by the professionals and shared with teachers through professional development workshops. This knowledge is then used by teachers in classrooms. Knowledge *in* practice is the craft knowledge of teaching that competent teachers gain from their experience and practice. Teachers deliberate on their teaching and learn from it. Knowledge *of* practice is generated by teachers when they work in communities and relate insights from their practice with the larger social, cultural and political contexts of inquiry. In the present study, an attempt is made to support teachers' learning by developing their knowledge *of* practice. There is a need to encourage teachers to focus on students' mathematical thinking and develop a critical perspective towards their teaching in the light of their experiences and wisdom that exists in the field.

The existing frameworks of teacher knowledge and learning illuminate our understanding about studying and supporting teacher knowledge. A refection on these frameworks opened up several questions for inquiry. Some of them include – how do textured descriptions of the work of teaching inform exploring and supporting teachers' knowledge; how can local knowledge base be elicited, challenged and supported in the context of practice; in what ways can the practice of teaching be interwoven with the existing research on teacher knowledge; and so on. The issues identified from the literature review, particularly those which influenced the design and conduct of the reported research are discussed below.

2.6.1 Accessing teacher knowledge through a study of practice

The existing frameworks suggest using the work of teaching to investigate teachers' knowledge of mathematics teaching. A practice–based perspective has also influenced the design of tasks for teacher learning. Artefacts from teachers' work have been used to support teachers' knowledge of the subject matter through reflection on practice. Lampert (2010) synthesises the use of practice–based approaches to teacher learning to make three insightful claims. First, teaching is relational work requiring intellectual and social collaboration. This collaboration implies that teachers learn about the subject matter in relation to their students. The proposition implies that teacher knowledge can be developed through an understanding of ways in which students interact with the subject matter and also that this kind of learning can take place together with the students. Second is the acknowledgement that teachers work through complex issues arising in the context of practice while making connections between the dynamic understanding of groups of students and the subject matter. Actual teaching deviates from any prescriptions as it is dynamically constructed in a classroom. Third, the word practice can be used in multiple ways. It can be used as opposite to theory, or having a unique set of professional standards like medicine and law, or as an act of continual performance. Teachers need to be encultured into specific practices that constitute effective teaching while at the same time providing adequate exposure to the issues encountered while teaching. This makes teaching both an object of thought and action. One of the important questions raised by Lampert's review of the practice–based approaches is whether learning to teach is an individual activity or an enculturation into professional practice which would rather be learnt through collective enquiry.

2.6.2 Teaching is non–individualistic

Recent literature on teacher learning suggests that teacher learning happens through enculturation into reformed practices. Valentine and Bolyard (2019) analysed the moments of "shift" in mathematical learning and teaching of pre–service teachers and found that teachers' learning was – (a) triggered by their relation with the others, (b) spread across different time frames, and (c) through their material relations with mathematics (working on and constructing mathematics). How does an understanding of teachers' learning collectively affect research on investigating teacher knowledge?

The review of literature on teachers' knowledge shows that the interest has shifted from measuring teachers' knowledge to understanding it through an examination of teachers' work. Recognising that teaching is a complex act, an analysis of teaching is done by breaking it into parts, for instance by identifying particular types of knowledge enacted in a moment of teaching. Ball (2017) asserts that, such an analysis of teaching, that is, breaking it into parts, is very different from the actual enactment of teaching, which is complex. While the existing frameworks on investigating teacher knowledge (discussed in Section 2.3) are useful in analysing some aspects of teacher knowledge, their use as assessment tools is highly individualistic, and merely indicates teacher's recall to the questions posed.

We know, from the literature (Lampert, 2010; Cai et al., 2017; Valentine & Bolyard, 2019) that the work of teaching is complex, and the triggers of teaching decisions lie in several connections that are made in the act of teaching. The frameworks on teacher knowledge that have been discussed so far (Rowland, Huckstep & Thwaites, 2005; Ball, Thames & Phelps, 2008; Carillo et al., 2013; Adler & Rhonda, 2015) foregrounded specific aspects of teaching by locating them in the tasks of teachers' everyday work. These frameworks have dealt with the complexity of teacher knowledge by offering different lenses to understand and analyse teacher knowledge. Not only have these frameworks, shown that the teachers' knowledge is specialised, but they have also identified the characteristics which help in preparing teachers for this specialised activity. A focus on studying the enactment of knowledge in teaching led the researchers to design tools to qualitatively understand teaching. Despite a focus on the nuanced aspects of teaching with the locus on the work of teaching, the existing frameworks face the same challenge. These frameworks largely situate the knowledge of a teacher in his/her mind, that is, focusing on a teacher as an individual. Petrou and Goulding (2011) note that, even though some of these cognitive frameworks acknowledge the role of context in teacher's knowledge, they emphasise the knowledge that an individual teacher brings to a classroom. This leads to a deficit view of the knowledge held by a teacher, which can be fixed using interventions. Through an analysis of teachers' knowledge in two different countries, South Africa and Nigeria, Brodie and Sanni (2014) conclude that researching the knowledge-practice relation in the context of teachers' work has the potential of producing non-deficit understandings.

Some questions become salient here – what does it mean to understand teachers' knowledge in their context, what are the theoretical and methodological standpoints that can be utilised to unpack teacher knowledge situated in their practice, and what would the standpoint on studying teacher knowledge in dynamic practice reveal about teacher knowledge, which is missing in the existing frameworks?

2.6.3 Teaching is context bound and dynamic

Through a case study of a mathematics teacher and teacher educator, Alexandra, Hodgen (2011) demonstrates how different kinds of knowledge gets elicited from an examination of different tasks of teaching. The study used evidences from investigating a teacher's knowledge in two different teaching situations. When asked about the area representation for the division of fractions during an interview, Alexandra seemed to not possess this piece of knowledge. However, during classroom observations, it was noted that, while teaching multiplication and division of fractions, Alexandra used this piece of knowledge to help students visualise the operations on fractions. The study concluded that the teacher's knowledge was stronger in the context of teaching than in an out–of–context experience. The other interpretation that needs checking is whether the interview situation triggered Alexandra in thinking about this strategy in classroom. The question to be asked then is – how is a teaching situation different from an interview situation, which supported Alexandra's knowledge of representations.

Hodgen's research raises questions about the validity of the tools that are used to measure teacher knowledge. First, it makes us question the nature of knowledge that gets elicited by the use of specific tools. It seems that the tools used to assess teacher knowledge might uncover only a part of knowledge and more *in context* descriptions might be needed to understand and unpack teachers' knowledge. Second, while a researcher would expect that an analysis of teacher's interview around a problem and her teaching of the same problem can be used as two data points to triangulate judgments about teacher's knowledge, it is clearly evident that two data sets unveiled different aspects of teacher knowledge. This implies that using one method or another is not sufficient to make conclusive judgments about whether or not teachers possess a piece of knowledge. Also, such methods cannot be used to make statements about the lack of knowledge in teachers, as the triggers might lie in teaching situations which demand elicitation of specific kinds of knowledge. Hodgen suggests addressing such problems by adopting a situated perspective as opposed to an individual one. A situated perspective acknowledges that "classroom knowledge is not a straight forward conceptualisation or application of a more abstract and general a priori mathematical knowledge" (p.36). Therefore, knowledge needs to be studied in the complexity of its practice. Secondly, a situated perspective allows for a discussion of the evolution of a teacher in learning to talk *about* and *within* mathematics and not just learn new knowledge. As Hodgen (2011, p.39) suggests, "the testing of individual teachers is likely to focus on de–contextualised mathematics knowledge which, as in the case of Alexandra above, may be very different from their classroom knowledge."

2.6.4 Abstract versus particular descriptions of practice

I have established that an investigation of teachers' knowledge through a systematic study of teaching practice is complex. A focus on the knowledge that the individual teacher possesses, as typically revealed in tests of teacher knowledge, is limiting and does not help much in unpacking the complexity of knowledge construction in a classroom (Petrou & Goulding, 2011). On the other hand, a framework, which captures teachers' knowledge in play and an inter–animation of ideas during whole class teaching, is difficult to conceptualise. I have discussed how one of the influential approaches to building a ‘‘practice based theory’’ of mathematical knowledge for teaching (the MKT framework) examined teaching practice by identifying the ‘‘recurrent’’ tasks of teaching such as presenting mathematical ideas, providing justifications, evaluating and managing explanations, using appropriate representations and examples, and making connections (Ball, Thames & Phelps, 2008). Inferences are then made about the ‘‘mathematical knowledge, skills and sensibilities’’ that are required to manage these tasks, leading to a characterisation of teacher knowledge and its major components, which include common content knowledge (CCK), specialised content knowledge (SCK), knowledge of content and students (KCS), and knowledge of content and teaching (KCT). It is important to note that the construct of recurrent tasks of teaching abstracts away from the specificities of particular instructional enactments in classrooms. While it allows for a generalisation from observations of individual lessons to a common body of teacher knowledge, some researchers have called for paying closer attention to the dynamicity of the knowledge manifested in teachers' practice (Adler 1998; Hodgen 2011).

2.7 Focus of the Thesis

The research reported in the thesis was aimed at understanding teacher knowledge from the standpoint of practice. Teacher knowledge situated in the specific context of practice is studied and links to research–based knowledge are made to combine aspects of knowledge *in* and *for* practice to generate knowledge *of* practice with the aim of supporting teachers. Further, an enculturation into alternate practice(s) which were then appropriated by teachers for use in classroom through teacher–researcher discussions uses assumptions of a situated framework (also noted by Franke & Kazemi, 2001). Through the course of the study, any judgments about the strict presence or absence of teacher knowledge are avoided, and the knowledge was studied using different modes of data collection, an insight gained from Hodgen's (2011) case study. Additionally, it is shown through data analysis that the knowledge that gets manifested in classroom (during teaching) is not the sole prerogative of the teacher but is often a result of discussions between students and teachers. Although the analysis of knowledge demands posed on the teachers, begins with a description of particular aspects of teacher knowledge that were triggered in the act of teaching, it is extended by a reflection on the anticipated teacher decisions and alternate trajectories. This allows for a comprehensive analysis of the knowledge–in–play. The thesis focuses on the demands placed on the teachers' knowledge due to contingent classroom situations for the topic of decimal numbers. The focus is on the dynamic nature of knowledge demands that arise *in–situ*, in the course of a teacher listening and responding to students in contingent classroom moments while teaching decimal numbers.

Chapter 3

EARLY BEGINNINGS TO THE LAUNCH OF THE RESEARCH STUDY

> *Practical actions are proposed which can afford access into the lived experience of others, by asking oneself what someone would need to be attending to, and how, in order to say what they say and do what they do. This pedagogic action can function as a research tool for analysis of what subjects say and do.* (Mason, 2017, p.1)

3.1 Abstract

Research on teacher knowledge about students' thinking can help teachers in making informed decisions about teaching and learning. Little is known about the nature of knowledge that teachers possess about their students and ways in which it shapes their teaching. This chapter reports two pilot studies which helped in refining the research questions and methodology of the main study. The first pilot study was a case analysis of an experienced school mathematics teacher with the aim of understanding how teachers' knowledge manifests in their practice, and ways in which this knowledge can be accessed and studied systematically through observations of classroom teaching and interactions with the teacher. Classroom–based tasks were designed to unpack mathematical aspects of teachers' knowledge about students' ways of thinking in a specific topic.

For practicing teachers, routine interactions with students are an important source of knowledge about their ways of thinking and responding. The research literature on students' thinking in specific topics is another important source (although seldom used in the existing teacher professional development programmes). In the second pilot study, both these sources of knowledge were integrated to design and test a teaching module on a specific topic. The module was designed based on the

knowledge of students' thinking gathered from the research literature and was modified through routine interactions with students. The researcher taught as well as observed a co–researcher's teaching of the module and conducted students' interviews to understand their ways of thinking and integrated these in modifying the lesson plans for the other sessions.

The two pilot studies provided insights into the process of investigating teachers' knowledge through a study of their practice and offered a perspective on the nature of knowledge that might be needed to support teachers.

3.2 Central Questions

The existing literature on investigating teacher knowledge shows the use of a wide variety of structured instruments to make claims about their knowledge. An increasing focus on close–to–practice items has led to more qualitative insights about teachers' knowledge and indicates ways in which such knowledge is closely connected to the act of teaching. Some of such instruments include (a) paper–pencil tests on content, pedagogical knowledge or pedagogical content knowledge, (b) rubric to be filled based on classroom observations of a teacher's teaching, and (c) interview of teachers based on questions which they are likely to encounter in their classrooms. While these instruments help us in unpacking aspects of teachers' knowledge, the knowledge that gets triggered and becomes explicit in the act of teaching needs to be brought in focus, to make claims about the knowledge of the teacher. An interest in understanding teachers' knowledge from the standpoint of practice raised the question – how can teachers' mathematical knowledge be accessed from a study of their practice. The specific questions addressed in this chapter are

(a) What is the nature of teachers' knowledge about students' thinking that can be captured from an investigation of their practice?

(b) What are the sources through which teachers' knowledge of students' thinking may be supported and what are the forms of such support in practice?

Clearly, the first question is about studying teachers' knowledge and the second is about exploring ways in which this knowledge can be developed. The first pilot study reported in this chapter included studying the practice of an experienced middle school mathematics teacher in order to understand her knowledge about students' thinking. Claims are made about teachers' knowledge using evidences from classroom observations, tasks centred around classroom teaching, and interactions or reflections on teaching. The insights gained from the first pilot study about the design of tasks for students and the topic–specific literature was used to design a module for the second pilot study. In the second pilot study, the researcher and a colleague taught module on algebraic thinking to two batches of students. The module was designed based on knowledge about students' ways of thinking gathered from the literature and modified from the experience of teaching. The findings of this study indicate that teachers' knowledge of students' conceptions gained from the research literature, and an analysis of and reflection on students' actual responses helps in making informed teaching decisions which translate into increased students' participation and engagement in learning.

3.3 Pilot Study 1: Observing Teaching of Proportions

This pilot study aimed to investigate teacher knowledge through a study of teaching practice without the use of standardised instruments. The purpose was to explore how a study of practice can be used to understand the knowledge that the teacher brings to the classroom.

3.3.1 Objectives

The study aimed to investigate the (a) nature of teacher's knowledge about students' mathematical thinking and learning, (b) relation between teacher's knowledge and her responses to students' mathematical thinking, and (c) teaching practices which reflect knowledge about students' thinking.

3.3.2 Background and context

Knowledge about students' thinking can be an important resource for teacher preparation and professional development. In Chapter 2, I discussed that knowledge of students' mathematical thinking includes knowing about students' (alternate) conceptions, their conceptual difficulties, potential learning trajectories, and developing sensitivity to what students think and do in a mathematics classroom. The sources of teachers' knowledge about students' thinking could be teachers' shared experiences, their own and peer reflection on students' conceptual difficulties and insights drawn from the research literature in the field. The knowledge of mathematics along with knowledge about students' learning mathematics guides teachers in planning and taking in–the–moment decisions in classroom. Knowing about students' mathematical thinking supports opportunities for asking questions linked to students' ideas, eliciting multiple strategies, drawing connections across strategies, and so on (Franke, Kazemi & Battey, 2007).

Unfortunately, knowledge of content and students' thinking are dealt with separately in the teacher preparation and teacher education programmes in India. The psychology courses deal with the components of students' thinking and learning. The concept–related discussions are confined to the methods courses such as Pedagogy of Mathematics. It is believed that the experience of teaching would help teachers to integrate the two knowledge pieces together and blend them in their teaching. Discussions on concept–specific students' thinking and learning need to find a place in teacher education in the Indian context.

Another issue at hand is the scarcity of interventions where teachers are engaged with research on students' topic–specific thinking and analyse its potential for teaching. Cognitively Guided Instruction (CGI) (Carpenter, Fennema, Peterson, Chiang & Loef, 1989) has been an intensive attempt where teachers were provided with research–based knowledge about student trajectories in whole number concepts through use of semantic problem type framework. But CGI misses the analysis of teachers' knowledge about students gained from their diverse experiences and building on it through research–based materials. Another consideration is that programmes like CGI tell us nothing about whether teachers who are not involved in such professional development process possess such knowledge and if so what shape it takes (Hill, Ball & Schilling, 2008). Research works try to relate teacher knowledge with students' thinking but "missing are the analysis that take into account the complexity of actual mathematics instruction that needs to consider various (and sometimes conflicting) factors, facets and circumstances" (Even & Tirosh, 2008).

Despite the extensive work done in the field of developing teachers' knowledge, there are difficulties in identifying its nature and extent. Teachers know the most about their students and their ways of thinking and learning. They make conjectures about students' learning, listen and respond to them in the classroom and share intellectual and affective moments with them. All this helps in formulating teacher knowledge which remains largely unexplored and unchallenged. Ball, Hill and Bass (2005) question whether this is due to the nature of methods that are used or the nature of (teacher) knowledge that remains tacit and unarticulated.

In the South African context, Brodie (2014) reports how the use of learner errors in a professional learning community created opportunities for shifts in teachers from identifying to engaging with errors. Exemplars or models through which students' thinking can be utilised as a tool for teacher learning in various teacher education programmes, particularly in the Indian context are yet to be explored. In this study (both the pilot and the main), an attempt is made to characterise the complexity of teacher's knowledge about students' mathematics and explore ways in which it can be used to enhance teacher learning.

3.3.3 Methodology

Exploratory Case Study was considered as an appropriate methodological design to probe deeper into teacher's knowledge, thinking and decisions while teaching. Classroom observations were used to capture the dynamics of classroom teaching and learning. Task–based interviews were conducted with the teacher and the students to understand their perspectives on the concept being taught. Task–based interviews involve a subject and an interviewer interacting in relation to one or more tasks (questions, activities, problems). They are generally used in psychological studies to make inferences about mathematical thinking, learning, and problem solving (Goldin, 2000). They are used to focus subjects' attention on the process of solving the mathematical tasks rather than on the final answers.

3.3.3.1 Participants and settings

A case study of a mathematics teacher, TJ, teaching in a Grade 7 classroom is reported. TJ taught in an English–medium private school in Mumbai which followed an Indian Certificate of Secondary Education (ICSE) curriculum. Unlike most schools in India, students in this school address teachers by their name indicating equality of respect. Initially in this study, four mathematics teachers from this school were followed in their classrooms. However, practical limitations and attempt to focus on one teacher and her classroom teaching was found to be suitable for meeting the objectives of the study. TJ was a sensitive teacher who allowed students to talk and ask questions in her classroom. Also, she was more like other teachers in her beliefs about what should be taught in mathematics and how it can be taught. She thus presented a "typical case" of a middle school mathematics teacher. TJ was followed for a period of three months (21 sessions of 30–90 minutes each). Data was collected through classroom observations, task–based interviews with students and the teacher throughout the duration of the study. TJ taught in two Grade 7 classrooms, with 34 students in each class. The topic selected for analysis from TJ's teaching was "proportions". She taught this topic to both the classes. The teaching of other topics, such as, geometry, ratios and mathematics projects were observed by the researcher to gain familiarity with the setting, students and the teaching.

3.3.3.2 Data collection and analysis

The study was carried out in two phases. Before the field work, Phase 1 of the study was planned. Phase 2 of the study emerged from the interactions with the teacher and students during the course of field work.

The first phase aimed to capture teacher's understanding of students' mathematical thinking through classroom observations and task–based interviews with students and the teacher. Classroom observations were video and audio–recorded, and were supplemented with field notes by the researcher. All the interviews with the teacher and students were audio–recorded and transcribed. The teacher was interviewed prior to and after every lesson while teaching the topic of Proportions. The nature of questions posed to the teacher were related to objectives of teaching the lesson, considerations for lesson planning, connections with the previous lesson(s), etc. This was followed by the classroom observation of her teaching. Instances where she deviated from the plan, or of her responding to the students' questions or responses in class, her questions posed to students, etc., were focused in the post–lesson interactions. (Any) Five students from the class were interviewed after each lesson. Questions were posed to understand how they solved the problems given in the class, what kind of questions, strategies or thoughts they had around these problems. The teacher was interviewed after each lesson to elicit more about her thinking behind the instances highlighted in classroom observations. TJ was more comfortable in discussing specific cases of students' questions and responses (on proportion problems) than questions about why a particular topic was to be taught.

Phase 2 of the study included the design and implementation of a task. Six problems on proportional thinking were created or modified using the existing literature (Lamon, 1999). The problems were discussed with the teacher for their suitability with students. The teacher was then requested to anticipate students' responses to these proportion problems. 11 students, selected by the teacher as representative of the range of ability in her class, were asked to solve these problems and justify their solutions. Their verbal and written explanations were taken up for discussion with the teacher. Thus, the data sources included observations, interviews, discussions with the students and the teacher. Written documents like teacher's lesson plans, assessment records, students' notebooks and test papers, background of the teacher and students, etc., were also collected.

The detailed notes of the teacher's teaching in the classroom and her responses to students' mathematical thinking from Phase 1 enabled the creation of a teaching profile including teaching routines and expectations from students. Patterns in teacher's responses to students' errors, alternate solutions, justifications, etc., while teaching, were identified. Pre– and post–lesson interviews helped in triangulating teacher's responses and offered instances for reflection and discussion. The questions posed by the teacher, students and their responses were analysed. Teacher's responses to specific students' strategies were gathered from Phase 2. The data from this phase was organized in categories (refer Table 3.1).

Table 3.1 Analysis of a proportion problem

Proportion Problem	**Teacher's Anticipation**		**Students' Responses**		**Teacher's reflection**
	Strategy	**Error**	**Strategy**	**Error**	
The cost of 10 pens is Rs. 42. What will be the cost of 15 and 20 such pens?	Cross–multiplication	Cancellation errors	Halving cost of 10 and adding to the cost of 10 for 15 pens. Then, doubling the cost of 10 pens to find the cost of 20 pens (S1,2,3,4,5)	None	The methods are good but they are commonsensical. I don't know how far will these methods help them. See this person (pointing to S10) has done it using algebra. They [students] need to work systematically like this.
	Using algebra (beginning with the unknown as x)	Writing the ratios incorrectly like $\frac{10}{42} : \frac{x}{15}$	Cross–multiplication with the unknown (S2)		
	Unitary Method (find the cost of 1 pen and multiply it with the number of pens needed)	Calculation errors	Finding the cost of 5 pens (as it is a common factor of 10,15,20) and using it to find the cost of 15 and 20 pens (S6,11)	None	
	R: Don't you think students might use halving or doubling to solve this problem? T: I don't know if they know that much. If I would have been at their place I would not have used this method. There is a direct method of working the proportion method, so why go for some long or complicated method. They might use unitary method but not doubling and all.		Unitary method to find the cost of 15 pens, doubling cost of 10 to find for 20 pens (S7,8,9)	None	
			Algebra Method (unknown as x and finding its value) (S10)	None	
			Unitary Method (finding the cost of 1 pen and then multiplying it with 15 and 20) (S11)	None	
Legends	T – Teacher, R – Researcher, S1, 2, 3, …12 – Students				

3.3.4 Findings of the study

It was found that the teacher's notion of what constitutes formal mathematics aligned with her goals of teaching mathematics. In clarifying the difference, the teacher stated that "formal mathematics is about algorithms and routes to problem solving which are precise, while commonsensical or out–of–school mathematics includes using strategies like halving and doubling" (TJ, PL1). Therefore, the goal of school mathematics teaching is to make "students learn these algorithms for better performance in standard examinations". In light of this goal, "once the students have been taught the algorithms, they are expected to use them while solving problems". The teacher was found carefully selecting students to answer the questions posed by her based on their attention in class. The decisions on which student should respond depended on her personal knowledge of the student (as quiet, shy, participative, hyperactive, etc.). Teacher's knowledge about students' mathematical capability was also guided by these personal qualities attributed to students. The teacher considered it as her responsibility to respond to any questions asked by the students (as opposed to letting students think about or respond to each other's questions). Students' questions were not revoiced or discussed in the whole class. The same was true for the strategies and errors made by students (refer Excerpt 3.1). TJ did not consider that students' knowledge and thinking needed articulation and sharing in classroom. The teacher also believed that students cannot solve a problem "correctly unless they are taught". Thus, students' intuitive knowledge for problem solving was not explored.

Excerpt 3.1: Finding square root

TJ	How to find the square root of 2025 [which is the product of 25 and 81]. To remove a square we put a square root on the other side. Use factorisation method
BSt4	There is a easy method
TJ	I know
BSt4	Can I show you the method?
TJ	No [Teacher shows factorisation on board]
BSt4	J (calling the teacher) you can directly do it
TJ	Wait [Teacher completes factorisation and leaves the class as the time gets over]

	After the lesson was over, the researcher interviewed B St4 and asked him about his way of working.
BSt4	We made two thousand twenty five [2025] from twenty five and eighty one [25 and 81]. Twenty five is five into five and this [81] nine into nine. So, five times nine is forty five, then why [do] factorisation?
	In the post–lesson interview, BSt4's explanation was shared with the teacher and her comments noted.
TJ	These are common–sense answers. They [students] are in school to learn algorithms. These answers will not help them in board examinations.
Legends: TJ – Teacher Jasmine, B St – Boy student.	

The teacher considered student's strategies as "common–sensical" and distinguished these ways from the 'algorithm' used for finding the square root though factorisation. Looking closely, one can see that the student is finding those factors which are repeated, and therefore does not understand the need to find the prime factors, which would then be combined to give the square root. The teacher's difficulty in noticing the mathematics underlying student's response and connecting it with the algorithm is noted here. It was also found that the teacher's knowledge about students' understanding was justified through criteria like "attentiveness, listening to the teacher, and his/her personal interest in mathematics", as is evident in the teacher's remarks to the students' responses to a proportion problem (refer Excerpt 3.2).

Excerpt 3.2: Post-lesson discussion on students' responses

TJ	He is intelligent as he goes by what is being taught. He pays attention and listens to me in the classroom [S10]
TJ	I don't know how he is going to cope further [in board exams] because he is not listening to all the topics so the basics is not being dealt with, now he is managing to get a B (grade) because see he has solved most of the questions using logic but these things don't work later, he is using his common sense to find answers, that's very good but I don't think how long will he be able to do this... [S6]

Note that the discussions centred around students' work served as a context for probing teacher's knowledge of students' mathematical thinking deeply. It seemed that the teacher had a broad sense of students' understanding gathered from listening to their responses to the questions posed, students' responses in the written assignments or tests and work in their notebooks. Her response to the students' incorrect written work showed that she marked incorrect responses but they were not accompanied with descriptive feedback. If the teacher noticed that several students were making a similar mistake in solving problems, she would repeat that specific idea in a later lesson.

An important finding was that although the teacher was fluent in solving mathematical problems, that is, had sound content knowledge and was concerned about students in general, her idea of what students need to learn in mathematics was restricted to reproducing the algorithms. This finding makes us think that the teacher's knowledge of students is not a by–product of her sound content knowledge. The teacher in this case understood knowledge about students' thinking as being able to predict their performance. Although the teacher provided space for students to talk in the classroom, attempts where students' mathematical responses were noticed, revoiced, or used for discussion were missing.

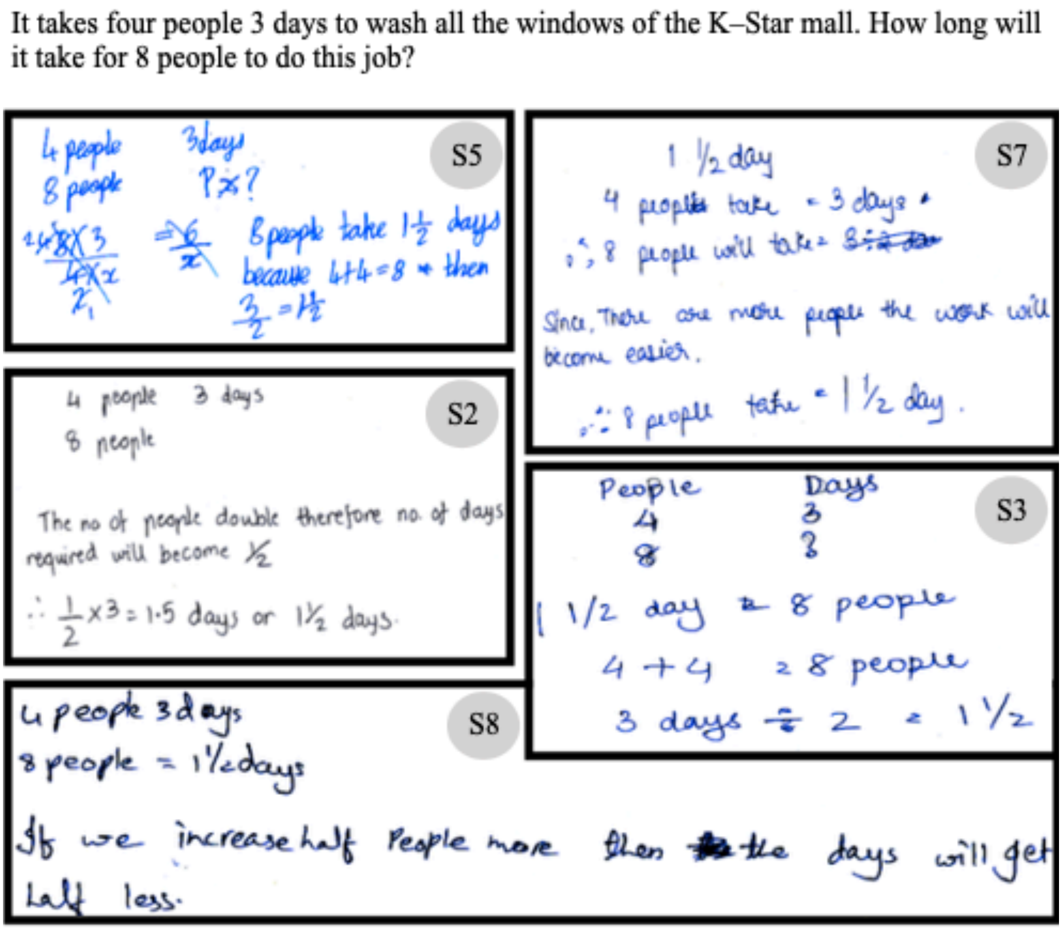


Figure 3.1: Students' Responses to a Proportion Problem

It was interesting to notice the enthusiasm with which students approached the proportion word problems posed by the researcher in Phase 2 of the study. Students were also keen to share different ways in which they solved the problem. The set of proportion problems (given in Phase 2) elicited different student strategies and their justifications. Some of the strategies used were halving and doubling, estimation, using the common factor, unitary method, proportion method, generalising the relation or solving algebraically, comparison with half, and sometimes a combination of two or more strategies (see Table 3.1). An example of the use of different strategies can be found in Figure 3.1. The problem addressed the idea of inverse proportion, which was not taught by the teacher. TJ anticipated that none of her students will be able to solve this problem. Contrary to her expectation, it was found that all the students attempted and solved the problem correctly using different strategies.

Analysis of the data from the Phase 2 of the study (a glimpse shown in Table 3.1) showed a clear mismatch between the problem solving strategies or errors anticipated by the teacher and those used by the students. On reflection, the teacher rationalised this gap by classifying students' strategies as logical but distant from algorithms and therefore "unacceptable in school". However, some other unanticipated evidences from students' strategies like the potential of a students' response, their conceptual understanding, ability to relate mathematical concepts, solve problems by reading the context despite not being taught (as in case of the problem above) conflicted with the teacher's knowledge about students' thinking. A case in point was the discussion around a student S11's response, whom the teacher classified as weak, inattentive, and generally possessing no understanding of ratios and proportions (refer Figure 3.2).

During reflection, TJ's initial response was that the student left the solution incomplete, which indicates the student's lack of understanding. The researcher suggested that they together try to solve the problem using the student's method. As the teacher began solving the problem using the student's method, she was surprised by the complexity of her solution. The teacher remarked, "I never thought she could think like this". Instances like this where the teacher and the researcher discussed the mathematics underlying individual student responses and used it to identify the

Which vehicle has faster average speed - a truck that travels 126 miles in $1\frac{1}{2}$ hours or a car that travels 135 miles in $1\frac{3}{4}$ hours?

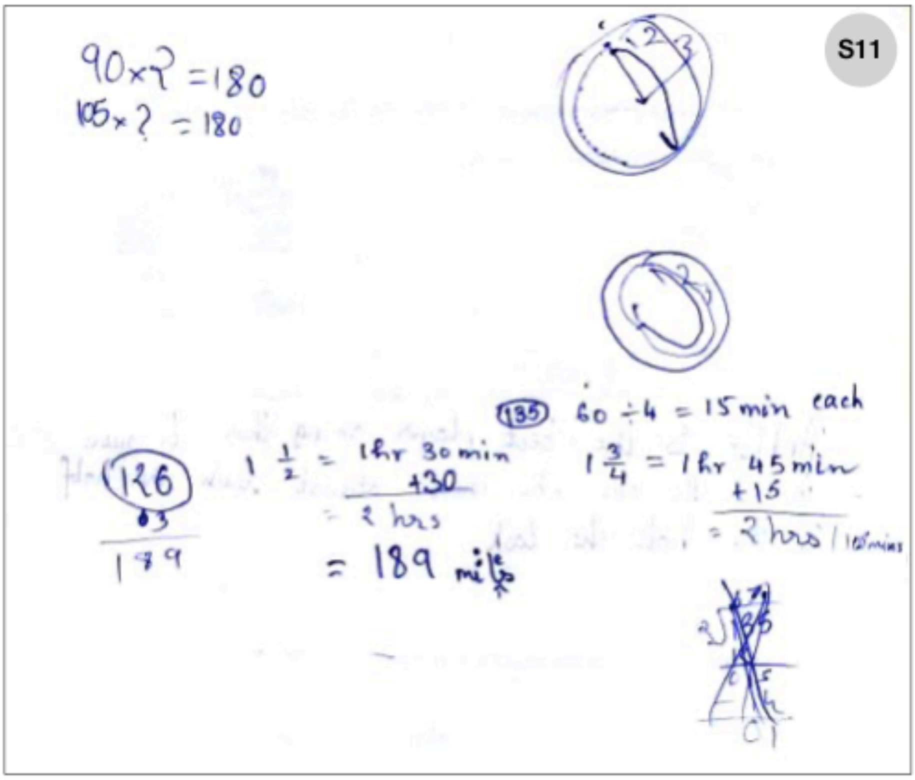


Figure 3.2: Student's (S11) Response to a Proportion Problem

similarities and differences between these responses, helped the teacher in revisiting her assumptions about students' capabilities. Unpacking the complexity in students' responses by using, examining or comparing their strategies seems to have helped the teacher in developing mathematical appreciation and sensitivity to students' ways of solving problems. It is difficult to hypothesise whether or how developing an appreciation of the students' mathematical ways of problem solving would have affected the teacher in structuring (selecting, modifying or creating) problems and engaging students differently while teaching in the classroom.

3.3.5 Conclusions and discussion

The pilot study was an attempt to discover the nature of teacher knowledge about students' mathematical thinking and its manifestation in classroom. The in–depth analysis of teaching of a topic by a teacher in two classrooms was insightful in characterising teaching–learning practices in terms of goals of the teacher, sources of knowledge about students, and decisions while teaching. Different ways in which students approached the proportion problems enriched the reflections with the teacher and served as an authentic context to challenge the teacher's existing notions about students' thinking. The study suggests the need to engage teachers in the process of articulating their knowledge and problematising their assumptions. The affordances arising from developing knowledge of students' thinking and seeing it in play while teaching in the classroom would be an interesting extension to the work. Evidences from the teacher's engagement in a classroom–based task, which involved thinking about students before and after the lesson, anticipation of and reflection on students' thinking and learning, provides strong support to the use of such tasks as potential sources of teacher learning and gaining knowledge in practice. Further, the potential of consistent efforts with the teacher to unpack students' mathematical thinking and sustain reflection on teaching could be explored further. Further, the study called attention to a more participatory role of the researcher where together with the teacher, an inquiry into students' ways of thinking could be investigated and ways of scaffolding it could be tried.

To summarise, the following insights were gained from the first pilot study which gave way to the second pilot study.

1. Teacher's knowledge about students' mathematical ways of thinking is individuated and is often based on attributes such as (lack of) students' attentiveness and listening in the classroom.
2. When directed to think about students' mathematics, the teacher underestimated the students' capability which in turn might determine their choice of tasks (such as not solving inverse proportion problems or word problems that are not "taught").
3. While the teacher was broadly aware of students' mistakes, she found it difficult to articulate the thinking underlying students' responses. The possibility of using the knowledge about students' thinking to inform teaching practice was therefore not considered.

4. Teachers can be initiated into appreciating students' mathematical ways of thinking by challenging their anticipation through the use of actual students' work from their own classrooms and a reflection on them.

So, for the second pilot study the starting point was an awareness of students' ways of solving problems and planning teaching tasks around them to identify the gains in students' learning.

3.4 Pilot Study 2: Teaching Early Algebra

The current literature on algebra education calls for considering early algebra as a beginning to algebra teaching and learning. In this study, tasks on number sentences were used as a context to explore the development of algebraic thinking in Grade 6 and 7 students. The tasks were developed based on the research literature and modified based on interactions with individual and groups of students. The strategies used by the students to solve these tasks and the justification or explanation given to support their response, are discussed. The findings of the study suggest that students move from purely computational strategies to relational reasoning and later generalised thinking as justifications. The use of *box* as a representation for number sentences supported students' thinking about structures and the movement from relational to generalised understanding. The study offers an instance of how early algebraic thinking develops in students in a classroom environment guided by students' thinking, conflict generation, and learning by consensual meanings.

3.4.1 Objectives

The study attempted to explore students' algebraic reasoning when exposed to early algebraic ideas through contexts like number sentences, pattern generalisation, proof and justification, etc. The study was designed on the premise that students make sense of new experiences based on their intuitive knowledge and if tasks are designed in a fairly open ended way, there is a possibility of variability in students' learning. Also, the design of tasks is informed by topic–specific research literature, resources such as textbooks, and the knowledge about students' gained from direct interactions.

3.4.2 Background and context

Algebra is one of the most difficult topic areas in elementary school mathematics, with the use of letters for unknown numbers and variables presenting a major hurdle to students. The shift from working with numbers to working with letter symbols requires well designed instruction that facilitates this transition (Banerjee, 2008; Subramaniam, 2004). There are other identified challenges in the learning of algebra such as understanding of equality, making generalisations, operating with letters, and flexibly dealing with procepts. Here, a small subset of the study is reported which focuses on the use of number sentence tasks to investigate students' thinking and build on their algebraic thinking.

In a typical mathematics curriculum in India, students are first exposed to algebra in Grade 6. Algebra begins with a discussion on the arithmetic properties (like closure, commutativity, associativity, distributive property and identity) with the use of variables. The idea of a variable is strengthened through pattern generalisation tasks which are extended to forming and solving simple linear equations. In Grade 7, solving algebraic equations becomes a major theme. Methods of solving linear equations (trial and error, balancing and transposing) are followed by framing and solving equations from word problems. In Grade 8 students enter the world of quadratic equations and polynomials. There is an emphasis on doing algebra successfully in middle school mathematics. However, the perspective on developing the tools for thinking algebraically has not yet entered the current mathematics curriculum. By algebraic thinking, I mean the act of deliberate generalisation and expression of generality (Lins & Kaput, 2004), analysing relationship between quantities, noticing structure, studying change, generalising, problem solving, modeling, justifying, proving and predicting (Kieran, 2004).

Algebra research in 1980s and 1990s focused on formulating stages for algebra learning and identifying student difficulties and its sources (Lins & Kaput, 2004). The later research conceptualised early algebra and teaching approaches to try it in the classroom with younger students. Early algebra means building background contexts for problems to be solved using intuition or previous knowledge (Carraher, Martinez

& Schliemann, 2008), with the objective of exposing students to a generalised mode of thinking while they are dealing with arithmetic. The significance of relational understanding and focus on the structures is an important part of early algebra. In the context of number sentences, relational (or structural) understanding means students attending to the structure of the sentence to decide what numbers make the number sentence true, instead of carrying out the calculations in order to determine the values of the missing number (Fuji & Stephens, 2001). Therefore, students who are able to use relational thinking to solve open number sentence problems consider the expressions on both sides of the 'equal to' sign while students with computational thinking view numbers on each side as representing separate calculations (McNeil, Grandau, Knuth, Alibali, Stephens, Hattikudur & Krill, 2006). One of the ways in which development of structural thinking can afford processes of abstraction and generalisation (Mulligan, Vale & Stephens, 2009) is exemplified in this chapter.

3.4.3 Methodology

The data was collected from a teaching camp organised for Grade 6 and 7 students from three English medium schools in the vicinity of the research institute. 68 students (37 boys and 31 girls) participated in the camp. The students were in the beginning of their academic year. The number of students in the two batches were: 33 students (majorly Grade 6) in the morning batch and 35 students (majorly Grade 7) in the evening batch. The teaching camp continued for a period of 9 working days with a two–hour session every day for each of the two batches. Two researchers (myself and a co-researcher) were the teachers for the camp. Data sources included classroom observations, teacher logs, and students' written and oral responses. The objectives of teaching were informed by the research literature on early algebra and student difficulties in learning algebra. Different contexts were used through the summer camp. For this chapter, I will elaborate on students' responses to the tasks on number sentences. After viewing the videos of all the lessons on number sentences, episodes demonstrating a change in the students' ways of dealing with number sentences were noted. These episodes were transcribed. Both oral and written student responses were analysed in the context of classroom discussion.

3.4.3.1 Task Design and Implementation

Teaching of algebra depends on how students are introduced to express qualitative relationships focusing on general mathematical relations (Fuji & Stephens, 2001). An important consideration in designing tasks was that students' engagement in tasks should provide some evidence of their reasoning and abstract thinking capabilities. Earlier research informed us that one of the useful routes is working on algebraic expressions through the broadening of arithmetic ideas, which can create opportunities for student learning (for details refer Banerjee, 2008). Since it was the first time that these students are exposed to algebraic thinking (or algebra), I was keen on using number sentences as a beginning context. I was curious to find out the affordances of the number sentences task as students' reasoning progressed.

The beginning tasks on number sentences (refer Figure 3.3) were designed to understand students' identification of the relations between numbers and make their thinking explicit. As the tasks progressed, students' movement from procedural ways to reasoning structurally, was observed. Thus, the later tasks were designed to support students' movement from relational to generalised thinking.

Objective: Making students explicate/ verbalise their (relational) thinking

Beginning tasks on number sentences	Later tasks on number sentences
$76 + 47 = \_\_\_ + 48$ $876 + 547 = \_\_\_ + 878$	$876 + 547 = \square + 878$ $a + b = a - 1 + \_\_\_$ $a + b = a + \square + b - \square$
$57 + 41 = 56 - \_\_\_$ $457 - 341 = \_\_\_ - 342$	$457 - 341 = 456 - \square$ $a - b = a - 1 - \_\_\_$ $a - b = a - \square - b + \square$

Figure 3.3: Tasks Used in Four Teaching Sessions

The beginning tasks on completing number sentences were guided by the notion of equality and relations in numbers. The tasks began with examples like 76 + 47 = ___ + 48 and soon shifted to using larger numbers in order to direct students' attention to the structure of number sentences. The initial student responses to these tasks were largely computational. A majority of the students added the two numbers on the same side of equal to and subtracted the number on the other side from the sum. As the students started identifying and talking about relations in the numbers on either side of the equal to sign, they were introduced to the need for expressing any number in the form of relations they identified. The notation of "box" emerged as a placeholder for an unknown number in this process. The reason for using a box instead of a letter as an unknown aligns with the indications from the research literature that variables are difficult for students to decipher as numbers. The box was introduced as a place–holder representing "a place for any number", or precisely as students said, "any number can go inside it". It was found that the box representation gave freedom to the students to talk about generalisations and facilitated mathematically rich discussions around the given equations. Apart from filling the missing value in addition and subtraction number sentences, there were also tasks on true–false sentences and creating such sentences individually and in groups.

3.4.4 Findings of the study

Before presenting the transition in students' thinking from computational to relational to generalised thinking, there is a brief description of the classroom culture and pedagogic moves which supported us in knowing about students' thinking and therefore take (unplanned) decisions while teaching in classroom.

Typically, each teaching session began by asking students to respond to a set of problems either in a worksheet or on the chalk board. Students could choose to work either individually, with partners or in groups. After they finished spending some time on the problem, they would explain their method to the whole class, during which other students and the teacher posed questions if they were not convinced. After one strategy had been discussed and agreed upon, students who proposed a different strategy came up and explained their strategy. The blackboard was used to record different strategies proposed by students. There was a discussion on the effective strategy and what makes some strategies more effective than others. The evolution of a classroom culture where students would refer to each others' strategies by citing their names, pose questions when in doubt, or feel free to comment on each others' strategy respectfully was witnessed.

Students were introduced to the idea of number sentences and were encouraged to explicate the reasons for the truth of a number sentence. Students' explanations served as a way for teacher(s) to know about their prior knowledge and the connections they make, their approach to problem solving, etc. There were also discussions on the significance of (thinking and) asking why to find the reasons for responses. The accepted reasoning was consensually defined as trying to explicate what we are thinking when we solve a problem and why we think the strategy we choose works.

3.4.4.1 From Computational to Relational (or Structural) Thinking

In the beginning, almost all students had a computational approach towards addition number sentences. Students carried out the calculations for the pair of numbers on one side of the "equal to" sign and subtracted the number on the other side to fill the missing blank (refer Figure 3.4). These responses reveal students' exposure to solving linear equations.

S23 (using computations)	S49 (blank as variable)	S18 (using variable x)
$53 + 38 = 54 + \_\_\_$ $\begin{array}{r} 53 \\ +\ \underline{38} \\ \underline{91} \end{array} \quad \begin{array}{r} 91 \\ -\ \underline{54} \\ \underline{37} \end{array}$	$48 + 39 = 40 + \_\_\_$ $87 = 40 + \_\_\_$ $87 - 40 = \_\_\_$ $47 = \_\_\_$	$53 + 38 = 54 +$ $53 + 38 = 54 + x$ $53 + 38 - 54 = x$ $91 - 54 = x$ $37 = x$

Figure 3.4: Students' Responses to Number Sentences (Session 1)

While doing this procedure, all the students were convinced with the rule that "sign changes when we move from one to the other side of equal to". The conversation that follows (see Excerpt 3.3) is representative of (several) students' responses in interviews or during classroom teaching.

The proposition of a sign change was treated as a given rule. Neither did the students raise a question on why this is true, nor did they know the reason for it. It was difficult to make them think about the need to know why this is true and holds for any equation.

Excerpt 3.3: Number sentences - procedural understanding

	Number sentence $48 + 39 = 40+$____
S	Forty eight plus thirty nine is eighty seven. Forty is subtracted from eighty seven.
T	Okay, how?
S	When it goes to the other side, it will be eighty seven minus forty. So answer is forty seven.
T	How does plus become minus when it goes to the other side?
S	It is a rule.
T	But why does it work?
Sts	It is a rule only. It works.
T	Are you all convinced about it?
Sts	Yes
Legends used: T – teacher, S – student, Sts – students	

Another approach that exemplified the use of procedures was replacing the blank with a specific letter. Many Grade 7 students substituted the blank with an *x* stating "let the blank be *x*" followed by which they solved the equation to find the value of *x*. They continued to think that any unknown should be replaced by the letter *x* and then all the numbers should be taken on the other side of *x* and computed (for example see S18, Figure 3.4). This shows prior exposure to algebra, particularly solving linear equations in one variable. Students using this procedure had the same idea about sign change as others, using the rule without knowing the reason. Also, while going through students' work, it was found that a majority of students did not face the commonly reported difficulties in literature like interpreting "equal to" as "something to do signal" or as "closure of expression". Not making such errors might have been due to the students being older and the instruction that they have had. It was noticed that some of the students (particularly in Grade 7) exhibited knowledge about solving linear equations with one variable.

The computations done by students assured that students got the correct answer but as Fuji and Stephens (2001) suggest, the goal is to focus students' attention to the underlying mathematical structure exemplified by the sentence. Students figured out the uniqueness of number sentences being posed to them in the next session. Explicit attention was drawn to the relation between the given numbers in number sentences (refer Excerpt 3.4).

Excerpt 3.4: Towards relational understanding

T	There is something similar in all the number sentences, right?
Sts	Yes.
T	There is something common. What is it?
	Different responses from students.
Sts	All of them have plus, dash (blank), some numbers, same way to reach answer.
S7	Teacher, in each sum of the three numbers… two numbers are very close.

The students were convinced of the similarity stated by S7 and as the discussion went on, another student S2 expressed that “actually equal to is like a balance. If we take away something from one side we have to give it back. So we take it away from the other side also or add it to the same side”. This was a crucial juncture and students readily accepted this idea. Despite this discussion, it was found that many students were using computations to solve number sentences. On probing, it was discovered that students felt that computation was a secure way to get a correct answer. However, the new discourse in the classroom was about effective strategies, relation between numbers, equal to as a balance, etc. It was interesting to note that the students who were using procedural approaches realised that the efficient strategy was to compare numbers on either side of equal to and so their justifications changed in the later sessions. In the second session, it was found that, a number of students started using both the methods to solve a number problem, where they treated one way to solve and the other to verify their answer (refer Figure 3.5).

S36 (relational then procedural)	**S4 (procedural then relational)**
$79 + 46 = \underline{\quad} + 48$ $48 - 46 = 2$ $\quad$ $79 + 46 = 125$ $79 - \underline{77} = 2$ $\quad$ $125 - 48 = \underline{77}$	$62 + 19 = \underline{\quad} + 20$ $\begin{array}{r} 62 \\ +\ \underline{19} \\ \underline{81} \end{array}$ $\quad$ $\begin{array}{r} 81 \\ -\ \underline{20} \\ \underline{61} \end{array}$ $\quad$ or $\quad$ $20 - 1 = 19$ $62 - 1 = 61$

Figure 3.5: Students’ Use of Relations and Computations (Session 2)

The evidence from comparing students' responses from Session 1 and 2 showed that almost all the students used computations to solve number sentences. A change in students' strategies and reasoning from procedural to beginning relational thinking in the later sessions was witnessed.

3.4.4.2 Nature of Relational Thinking in Students' Reasoning

Students continued to use their methods (computational and/or relational) for fill in the blank problems, true–false number sentences, and for creating and solving their own sentences. However, students were using different representations to express relations in numbers. These included explanations with words, using diagrams, using numbers and computations, writing more than one reason, etc. (refer Figure 3.6). Students stated that these solutions (using relational thinking) made their responses quicker. They were found to be gaining confidence in the use of relations between numbers. Often they would also look for similarities in different representations to justify their strategy.

S31	S33	S64
-1 +1 45 + 22 = 23 + 44	27 + 32 = ____ +28 27 + 1 = 28 32 − 1 = 31 Therefore, ____ = 31	62 + 19 = ____ +20 Answer is 61. If we subtract a number from one of the numbers and add the same number to the other, answer will be the same.

Figure 3.6: Responses on Number Sentences (Session 3)

The idea of "equal to" as a balance was also getting strengthened. There were other related ideas which were emerging. Some students started using the diagrammatic representation of the balance to show commutative property (refer Figure 3.7).

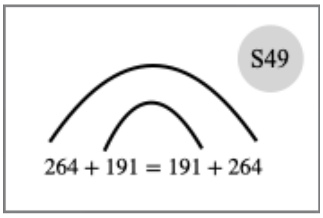


Figure 3.7: Representing Balance in Number Sentences

Students started using this explanation to support other claims for instance, S40 wrote that "20 = 20 because equal to is a balance and on each side equal weight should be there". Also, the discussion on the sign change was revised and students now could make sense of the changing sign with the explanation of balancing. This was evident in utterances such as, "if we take (away) 2 from left side, we take it (away) from this (right) side for balance". The justification for sign change was extended from number sentences (with the relation between two numbers) to any two expressions on either side of equal to (S64, Figure 3.6).

3.4.4.3 From Relational to Generalised Thinking

After students attained a level of comfort in working with number sentences, the trajectory took a different turn. A student in the beginning of the fourth session said that "this (pattern) works for all the numbers... I take any number add one to the first number and subtract one from the second number, I get the same answer". At this point, there was a discussion on whether it is possible to express this relation as a generalised mathematical statement. This was accompanied with the introduction of a new representation called "box". Broadly, the process of generalisation happened in different levels (refer Figure 3.8).

Level 1	$a + b = a + 1 + b - 1$
Level 2	$a + b = a + 5 + b - 5$ (5, 6 or 10)
Level 3	$a + b = a + 100 + b - 100$
Level 4	$a + b = a + \square + b - \square = a - \square + b + \square$

Figure 3.8: Levels of Generalisation

The sequence of number sentences made students generalise with the *box* as representing any number. When asked about the conditions under which the above number sentence will be true, students became more specific. The conditions stated by them were "*a* and *b* hold the same value on either side of equal to and the box refers to the same number in a number sentence". This was extended to saying that "the sign of the numbers inside box should also be the same and it can be a fraction, decimal or integer". The box thus signified the representation for any number. They extended their understanding of *a* and *b* as any whole numbers to *a* and *b* being numbers other than whole numbers. It was interesting to see the enthusiasm with which students pursued the idea of generalisation with *box* as a generalised number and proving that "the sum of the two numbers remains the same if any number or box is added to the first number and the same is subtracted from the second number" (Level 4).

Thus, it was found that the strategies used by students while justifying number sentences involved complex interweaving of computational–structural understanding, articulation of relational thinking and the movement to generalisation. There were noticeable shifts in students' reasoning from computational thinking to developing relational understanding to the need for generalised statements and their proofs. The later sessions focussed on the ideas of justification and proof of generalised statements.

3.4.5 Conclusions and discussion

Achieving generalisation is a cornerstone in learning algebra at the school level. The analysis of students' work on number sentences and the trajectory in their reasoning verified the potential of these tasks for triggering relational reasoning. The trajectory of working on number sentences witnessed a movement from procedural (or computational) to relational to generalised thinking, supporting the view that understanding of structures is a key to generalisation (Mulligan, Vale & Stephens, 2009). Generalised reasoning reinforced students' idea of equations as a *balance* where they were found demonstrating compensation of quantities symbolically. Along with the role that the classroom culture and students' prior knowledge played in the development of this trajectory, it was also identified how intermediate resources such as the use of a *box* supported the trajectory towards generalised thinking. The use of box provided liberty to students to put anything inside it – fractions, negative numbers, etc. The box represents a partial symbolisation of the concept of variable and students found it easier to relate its use both as an unknown and a variable.

Number sentences offer a powerful context and can be integrated with the existing Indian curriculum. Students' use of number sentences brings forth the algebraic nature of such arithmetic tasks. However, the movement from computational to generalised thinking in a flexible mode entails a significant role of the teacher, including identifying the appropriate prompts, and planning for the unexpected student responses. Understanding the teacher's role in the trajectory of students' thinking is one of the crucial components of teaching.

3.5 Learning From Pilot Studies

The findings from the two pilot studies helped in forming hypotheses about teacher knowledge. First, experienced teachers have an intuition about students' ways of thinking and responding, but these ways are largely classified as correct and incorrect and not explored for their nuances. Second, teachers attribute students' difficulties to non–mathematical aspects, and engagement with the mathematics underlying students' thinking needs some pressing and direction from the researcher. Third, even when the teacher talks about students' mathematical ideas, they underestimate students' ideas. The anticipation–reflection task has the potential to challenge teachers' knowledge and beliefs about students' capabilities. Fourth, knowledge of research literature supports a teacher in making decisions about what to teach, adopting a more open–ended approach to problem solving, and handling unanticipated moments. Lastly, making sense of students' responses in–the–moment is complex and is guided by several considerations such as, goals of teaching, setting classroom culture, careful listening and guiding, etc.

Chapter 4

RESEARCH DESIGN AND OVERVIEW OF THE STUDY

> *Small scale studies have an advantage for the theory practice relationship since it is easier to integrate teachers into research. Also, research results from such studies can be written in the form of "stories" which give an authentic view of practice and give principals, administrators and policy makers an insight into the complexity of change in the teaching profession. Such studies also provide useful contrasts to tables of percentages that can give the impression that teacher education and teachers' growth is as easy as calculating numbers and counting means. In addition, such stories are also a good starting point for working with teachers, in particular because they can compare their situation with those of the case.* (Krainer's commentary in Adler, Ball, Krainer, Lin & Novotna, 2005)

4.1 Abstract

A study of teaching practice opens itself to several methodological questions. What is an appropriate methodology to study teachers' knowledge situated in their practice? How can such a methodology be designed and modified based on the contextual demands of the study? What are the affordances and constraints of this methodology in light of the research questions? How do the roles of the researcher and participants get negotiated through participation in the study? In this chapter, I shall try to address these questions from the methodological perspective. I will discuss the details of the data collection procedures, decisions made in different phases of the study, and how the data was analysed. The process of negotiating with the limits of a case study as a methodology, are discussed. The chapter is a report of how the methodology evolved

during the course of the study and a post–hoc reflection on the decisions made and their affordances.

4.2 Research Questions

The research study aimed to investigate teachers' knowledge of students' mathematical thinking as it gets manifested in their practice. Further, attempts were made to develop teachers' knowledge of students' thinking through the design and development of practice–based tasks. The following research questions were addressed through the study.

1. How does teachers' knowledge about students' thinking manifest in their practice and what are the ways in which such knowledge can be studied or identified?

2. How can responsive teaching be identified and characterised? What is the relation between teacher knowledge and responsive teaching?

3. What is the nature of demands placed on teachers' knowledge during teaching? How can such knowledge demands be understood in light of the existing frameworks on teacher knowledge?

4. How can teachers' knowledge of students' thinking be supported? What do teachers learn from the support provided by the researcher and how does it manifest in their practice?

Since teacher knowledge is dynamic and fluid, any claims about the strict presence or absence of a part of such knowledge is avoided. Students' thinking is defined as mathematical ways in which students process an idea – it could be their ways of problem solving, making sense of representations, forming explanations, facing conceptual difficulties, their common conceptions, etc. Practice refers to the act of, as well as reflections on, teaching, that is, what teachers do in their classroom and ways in which they think or reason about it.

4.3 Theoretical and Methodological Stance

In this chapter, I discuss how research questions were refined and operationalised. The choice of the research methodology, suitable for this study, was guided by the literature on investigating teacher knowledge, an awareness of the developing field of teacher education in India, and researcher's own experience as a teacher in the past. These considerations also influenced some of the practical decisions made in the field during the process of data collection. In this section, I attempt to discuss these influences and what was learnt from a reflection on them.

The literature that has influenced the design of this study is taken from the field of mathematics teacher education, practice–based teacher education, and theories of situated learning. A review of the existing frameworks used to investigate teacher knowledge reveals that the individualist assumption of a teacher is frequently a common denominator (Petrou & Golding, 2011). The assumption manifests itself in investigating teacher knowledge as a sole prerogative of the teacher, without any contact with the educational systems within which it is located. Such an assumption underlies several research studies which use questionnaires or structured interviews as tools to measure teacher's knowledge of the subject matter. Petrou and Golding (2011) observe that although the existing frameworks of teacher knowledge acknowledge the importance of context in understanding teacher's work, the research emphasises the knowledge that an individual teacher brings to the classroom. This kind of an emphasis, the authors (ibid) argue, can lead to a deficit view of an individual teacher's knowledge, demanding some fixing through teacher development programs. Extending the critique, Hodgen (2011, p. 29) notes that teacher knowledge is embedded in their practice and an abstract description of such knowledge would fail to capture its dynamic nature. I recall Hodgen's case study of a teacher's knowledge of multiplication and division of fractions (described in Section 2.6.3 of Chapter 2) which revealed that the knowledge of the teacher was qualitatively different in the two situations of lesson planning and the structured mathematics interview. While this teacher did not draw upon the knowledge of different models of teaching fractions in the interview setting, it became prominent in the lesson planning experience. The author (ibid) concluded that the teacher knew more than what she could articulate in the interview situation.

The individualistic assumption noted in the research on mathematics teacher knowledge resonates with the cognitive theories on learning, which focus on the mind of an individual. The cognitive theories posit the knowledge in the mind of an individual and define learning as an acquisition of knowledge and skills which can be transferred to other situations. Situated theorists (Greeno, 1998; Putnam & Borko, 2000), on the other hand, challenge the assumptions of the cognitivists to propose that knowledge is social and contextual. They argue that the cognitive core or the mind of an individual cannot be separated from the context in which it operates or the activity of which it is a part. Also, the activity within which the mind operates is fundamentally influenced by the physical and the social context, such as interactions with the individuals, materials and representation systems in this environment. Situated theorists suggest that measuring teacher knowledge is complex, and it requires tools which can be used to unpack this knowledge in the context of its emergence. Several mathematics education researchers align with the situated perspective to argue that the knowledge of the teacher is not located in her mind, but gets realised through the practice of teaching (for instance, Hegarty, 2000; Mason & Spence, 2000).

I believe that such a critique raises two questions for deciding a methodology for investigating teacher knowledge. First, whether the tools used to measure teacher knowledge are valid, in adequately measuring what they purport to measure. Second, what kind of knowledge gets unpacked through a study of knowledge situated in the context of practice. A further question would be whether and how the knowledge triggered through this kind of a theoretical standpoint is similar to or different from, the kind(s) of knowledge identified by the use of existing frameworks on teacher knowledge.

In Chapter 1 (Section 1.2.3), it was discussed that the research on mathematics education in India is scarce. The limited research on comparing teaching methodologies such as traditional teaching with activity based teaching, has led to predictable results (Kumar, Dewan & Subramaniam, 2012). Such research studies use psychometric, survey and experimental methods as modes of data collection. These studies have been critiqued for the lack rigour in data collection procedures, and their over–reliance on anecdotal or impressionistic understandings (Banerjee, 2012; Kumar, Dewan & Subramaniam, 2012), rather than actual data from the field. Situating the research on mathematics teacher education in the dearth of research in mathematics education in India; these authors have suggested the need to use case and ethnographic studies in order to develop a deeper understanding of teachers' work, teachers' knowledge and its development.

The developing field of mathematics teacher education research in India poses challenges to early researchers. Theoretically, it becomes difficult yet compelling to situate their research work in the international literature, which may not necessarily correspond with the realities of classroom teaching in India. Practically, a decision to undertake a systematic study of classrooms, lends itself to challenges such as, completion of field work within the duration of research given the field challenges in securing permissions from the local authorities, getting teachers on board with the research objectives and in justifying the significance of such research.

The literature on teacher knowledge and the status of research in the Indian context opened up several methodological questions. The primary question was to think of a methodology which will make the dynamic aspects of teachers' knowledge visible. This standpoint (of understanding the dynamic aspects of teaching) was supported by the theory of "landscapes of learning" (Wenger–Trayner, Fenton–O'Creevy, Hutchinson, Kubiak, Wenger–Trayner, 2014) which defines teacher knowledge as "knowing in practice, and a reflective analysis of teaching" (Lampert, 2001) where the focus is on the teacher's in–the–moment decisions and the complex considerations that underlie such decisions. These perspectives helped in understanding that the knowledge of an individual teacher changes when it enters the classroom due to an inter–animation of ideas. Thus, the knowledge of a teacher is like a changing landscape which becomes accessible as some connections get triggered in the contexts of their practice. Such triggering can happen, for instance, when a teacher attempts to provide a simpler explanation to a student, offers an alternative representation, or notices an underlying mathematical idea in a students' question or remark. Ball and Bass (2000) suggest that such an understanding of the teacher's knowledge can be developed by systematically studying the "work of teaching". Ball, Hill and Bass (2005) define the work of teaching as,

> …the predictable and recurrent tasks of teaching, tasks that teachers face that are deeply intertwined with mathematics and mathematical reasoning – figuring out where a student has gone wrong (error analysis), explaining the basis for an algorithm in words that children can understand and showing why it works (principled knowledge of algorithms and mathematical reasoning), and using mathematics representations. Important to note is that each of these common tasks of teaching involves mathematical reasoning as much as it does pedagogical thinking. (p.21)

Such perspectives reinforced the attention or the methodological intent of this study to shift from assessment to explorative investigation aimed at understanding mathematics teachers' knowledge. Further, the research reported here shows that such an intent links well with the professional development initiative to support teachers' knowledge in the contexts of their practice. A perspective on knowledge *in* practice (discussed in Section 2.5 of Chapter 2) helped in unpacking knowledge that teachers have and need to become more specialised in their work.

4.4 Case Study as a Research Design

The research adopted a case study methodology to engage with the teachers' knowledge of students' mathematical thinking. The case study features "descriptions that are complex, holistic, and involving a myriad of not highly isolated variables; data that are likely to be gathered at least partly by personalistic observation; and a writing style that is informal, perhaps narrative, possibly with verbatim quotation, illustration, and even allusion and metaphor" (Stake, 1978, p.7). In this study, each teacher's teaching constituted a case. It was bounded by all the interactions within and about teaching. Through a case study approach, the aim was to make sense of teachers' practices by closely examining them in the context of their work.

An attempt was made to capture all teaching practices that were observable. Each case (teacher's teaching) was studied for its particularity and complexity by examining the teacher's individual and combined activity in situations such as teaching in a classroom, discussing with colleagues, or with parents, interacting with an individual child or a group of children, and so on. Following a case study methodology in school settings, the teachers were observed in their classrooms during teaching, as well as in other situations while inside the school setting. In order to develop an understanding of the work of teaching, the "emic" perspective on the teacher knowledge was prioritized. Aligning with this perspective, an attempt was made to understand, probe and clarify the meanings that the teachers attributed to their classroom practices and the related issues through discussions on the researcher's observations and teachers' interpretations on their practice. An ethnographic approach to observing teachers in different settings within the school and an "interpretivist orientation" (Harrison, Birks, Franklin & Mills, 2017) to understand their perspectives on their practices, helped in grounding the study in the context of practice. An interest in understanding teachers' perspectives by being closer to their natural settings (Creswell, 2013) helped in building a rapport with the participants, develop a vocabulary to discuss about teaching, and create a space for discussion on problems arising in the teaching.

Yin (2014) argues that precision or accurate reporting and a rigour in the process are central to case studies. Since the attempt is to develop an understanding of a case in its real settings, case study methodology opens itself to a variety of data collection methods (Merriam, 2009; Harrison et al., 2017). In this case study, teaching was followed for two academic sessions and data was collected through various modes. Teaching was understood by observing the practices of teachers, seeking teachers' opinion about these practices, interacting with students to understand their views about teaching, probing teachers' knowledge about students and their mathematical thinking, determining the considerations that guided their decision making, and supporting thinking aloud about particular events that arose while teaching. Triangulation of teaching practices was done by observing the repeated use of a practice at different occasions, discussions about these practices, and creating a scaffolded recall with teachers during reflective interactions. In order to engage teachers in discussions about their teaching, the researcher's role became more interactive and participatory in the course of the study. Stake (cited in Harrison, Birks, Franklin & Mills, 2017) points out that the participatory role of the researcher is significant in examining the integrated system within which the case unfolds.

While in the first phase of the study, the researcher examined and analysed each case, in the second and third phases of the study, the case acquired a shared character. In other words, teaching became a shared artefact for reflection among the teachers and researcher. In fact, evidence supports that discussions around teaching have potentially contributed to teacher learning and reflections on their practice. This *hermeneutic* character of the case is noteworthy. Figure 4.1 describes the changing nature of the case, and how it was treated during different stages of research.

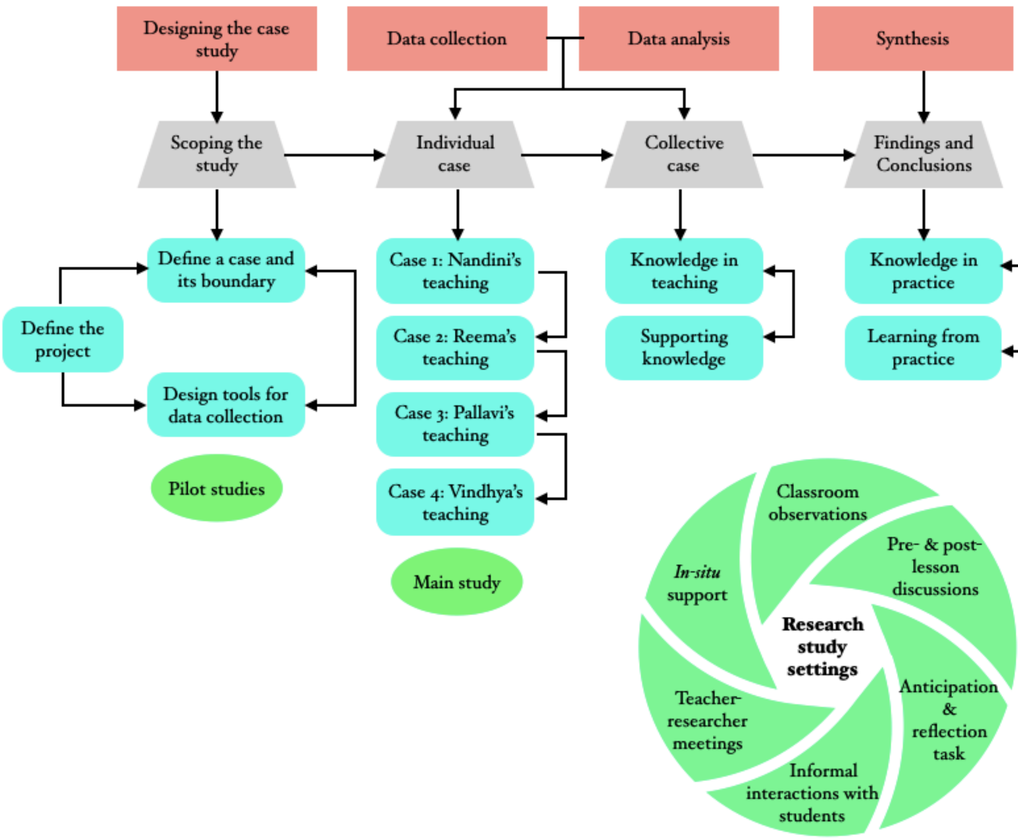


Figure 4.1: Case Study Methodology

After identifying the problem statement, that is, to study teacher knowledge from the standpoint of practice, tools used to understand practice needed to be identified. Since the focus was on understanding practice, a teacher's teaching was defined as a case. The scope of the case was bounded by observing classroom teaching, and understanding students' and teacher's perspective on the content that was taught. The two pilot studies of analysing a teacher's practice and the researcher as teacher respectively (discussed in Chapter 3) helped in clarifying the focus on teachers' questions, explanations and responses, and students' questions and explanations. The pilot studies helped in refining the tools such as the anticipation and reflection task. For the main study, a case study of four teachers' teaching was followed. The data collection and analysis involved studying the data of each teacher's teaching separately and identifying patterns in practice. The purpose of the analysis was to unpack the knowledge underlying teaching as well as identifying ways of supporting it in practice. This required studying all the cases collectively and identifying similarities and differences in the teachers' practice. For instance, understanding how different teachers explain a rule and then supporting them in developing conceptual explanations. Apart from classroom observations, an abstraction of knowledge manifested in teaching was identified through tools such as, pre– and post–lesson discussions with the teachers, anticipation and reflection task, teachers' engagement with the *in–situ* support offered by the researcher and so on. The findings of the study attempt to characterise responsive teaching from an analysis of teaching and how teachers can be supported in learning from practice. An analysis of exploring and supporting teacher knowledge in practice emerged from the individual and collective study of the cases in different settings (classrooms, school, teacher–researcher meetings).

It is almost a recurring question for case studies about whether and to what extent they can be generalised. Questions such as – how is the uniqueness of this case relevant for other contexts need to be foregrounded. While no two teachers teach in the same manner, creating a discourse around teaching, the methods used, the contextual tools developed and the nature of discussions have a wider applicability in understanding diversity of teaching in the Indian classrooms and in the world.

4.5 Participants and Settings

The intent of the study was to understand teachers' knowledge of students' mathematical thinking, *in–situ*. Therefore, it was important to study teaching in naturalistic settings, that is, in teachers' classrooms. In this section, I will outline the process of selection and profile of the participating teachers (participants). A description of the school culture, teachers' routines, and expectations from teachers will help in setting the background for understanding the teachers' work.

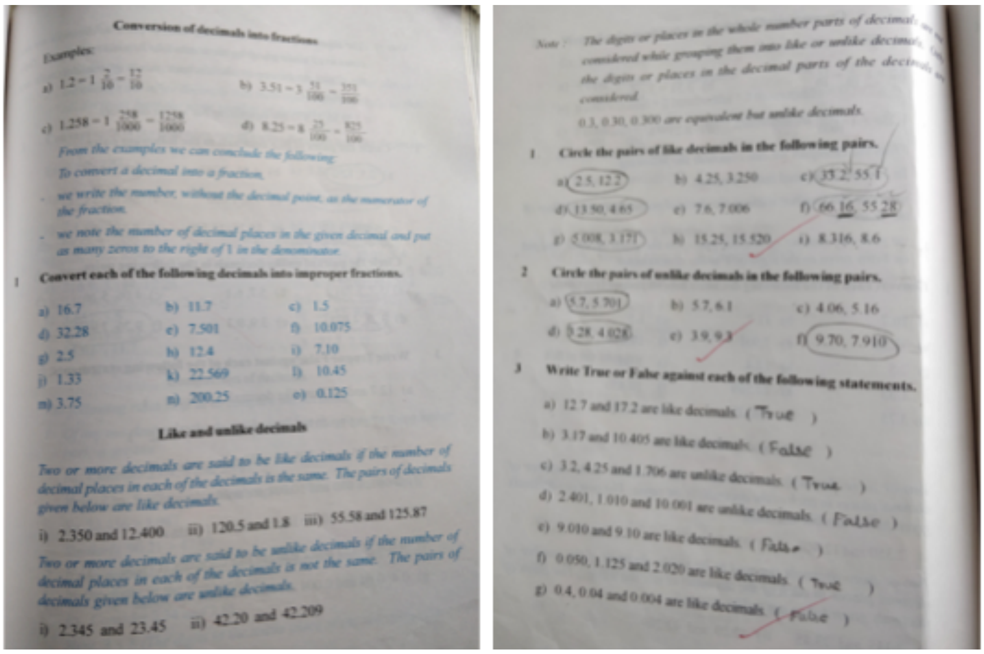
4.5.1 Sample selection

The researcher visited the office of the governing council of a group of schools located in the vicinity of the research institute. The council gave permission to conduct the study in one of its six schools. These six schools are located a few kilometres away from the office of the governing council. The researcher visited each of these schools and interacted with the principal and the vice–principal of the school. One school was selected (from the six) based on its representativeness of students from different socio–economic backgrounds. A meeting was organised between the researcher, principal and all the mathematics teachers. The meeting with all the seven mathematics teachers, working in the selected school, revealed that a few teachers might get transferred to a different geographical location (a common practice with these schools), in the next few months. The mathematics teachers who would not be transferred immediately became the participants of the study. So, a combination of purposive and convenience sampling was used to select the four mathematics teachers who participated in the study. Meetings between these teachers, the school principal and the researchers were held to explain the purpose of the research, the roles of teachers and researchers, and the logistics of data collection. Other clarifications regarding the research study were provided to teachers, individually and as a group. There was an initial period of negotiation where the teachers were skeptical about their participation in the study. They perceived it as an additional burden and were reluctant in getting their classroom teaching recorded (details in Section 4.8.1). Over a period of time, teachers became willing and voluntarily increased their participation in the study. They initiated pre- and post-lesson interactions with the researcher, began discussing students' thinking with peer teachers, and consented for the recording of their lessons. Their consent for data collection was secured. The details of the school and participants can be found in the following sections.

4.5.2 About the school

The study was carried out in a school located in Mumbai, a city in the state of Maharashtra, in India. Mumbai is one of the five metropolises in India. It has people from mixed socio–economic and cultural backgrounds. Mumbai is an archipelago of seven islands. The islands coalesced into a single landmass in 1784. The island has been ruled by dynasties in the past, was used as an overseas port, has witnessed the uprising of several political movements and worker unions. The city has emerged as an economic and entertainment capital of the country.

The school is located in the urban parts of Mumbai. It was founded over four decades ago with an explicit goal of serving students from different religions, castes and creeds. During the period of the study, the school catered to 833 students and had 47 permanent staff members. Additionally, there were contractual staff, including teachers. The school was selected as the site for the study since it caters to students from mixed socio–economic backgrounds. The occupation of students' parents varied from being a scientist, a schoolteacher, a cleaner or a watchman. The students and teachers came from varied linguistic backgrounds. The medium of instruction in the school was English. The Indian languages that were heard in the school premises included Gujarati, Hindi, Kannada, Malayalam, Marathi, Tamil, Telugu and Bengali.



Conversion of decimals into fractions

Examples:

a) $1.2 = 1\frac{2}{10} = \frac{12}{10}$ b) $3.51 = 3\frac{51}{100} = \frac{351}{100}$

c) $1.258 = 1\frac{258}{1000} = \frac{1258}{1000}$ d) $8.25 = 8\frac{25}{100} = \frac{825}{100}$

From the examples we can conclude the following:

To convert a decimal into a fraction,

- *we write the number, without the decimal point, as the numerator of the fraction.*
- *we note the number of decimal places in the given decimal and put as many zeros to the right of 1 in the denominator.*

1 **Convert each of the following decimals into improper fractions.**

a) 16.7	b) 11.7	c) 1.5
d) 32.28	e) 7.501	f) 10.075
g) 2.5	h) 12.4	i) 7.10
j) 1.33	k) 22.569	l) 10.45
m) 3.75	n) 200.25	o) 0.125

Like and unlike decimals

Two or more decimals are said to be like decimals if the number of decimal places in each of the decimals is the same. The pairs of decimals given below are like decimals.

i) 2.350 and 12.400 ii) 120.5 and 1.8 iii) 55.58 and 125.87

Two or more decimals are said to be unlike decimals if the number of decimal places in each of the decimals is not the same. The pairs of decimals given below are unlike decimals.

i) 2.345 and 23.45 ii) 42.20 and 42.209

Note: The digits or places in the whole number parts of decimals are not considered while grouping them into like or unlike decimals. Only the digits or places in the decimal parts of the decimals are considered.

0.3, 0.30, 0.300 are equivalent but unlike decimals.

1 **Circle the pairs of like decimals in the following pairs.**

a) 2.5, 12.2	b) 4.25, 3.250	c) 35.2, 55.1
d) 13.50, 4.65	e) 7.6, 7.006	f) 66.16, 55.28
g) 5.008, 3.171	h) 15.25, 15.520	i) 8.316, 8.6

2 **Circle the pairs of unlike decimals in the following pairs.**

a) 5.7, 5.701	b) 5.7, 6.1	c) 4.06, 5.16
d) 5.28, 4.028	e) 3.9, 9.3	f) 9.70, 7.910

3 **Write True or False against each of the following statements.**

a) 12.7 and 17.2 are like decimals. (True)

b) 3.17 and 10.405 are like decimals. (False)

c) 3.2, 4.25 and 1.706 are unlike decimals. (True)

d) 2.401, 1.010 and 10.001 are unlike decimals. (False)

e) 9.010 and 9.10 are like decimals. (False)

f) 0.050, 1.125 and 2.020 are like decimals. (True)

g) 0.4, 0.04 and 0.004 are like decimals. (False)

Figure 4.2: *Ladders* for Grade 5 (2005, pp.43–44)

Figure 4.3: Photographs of the School

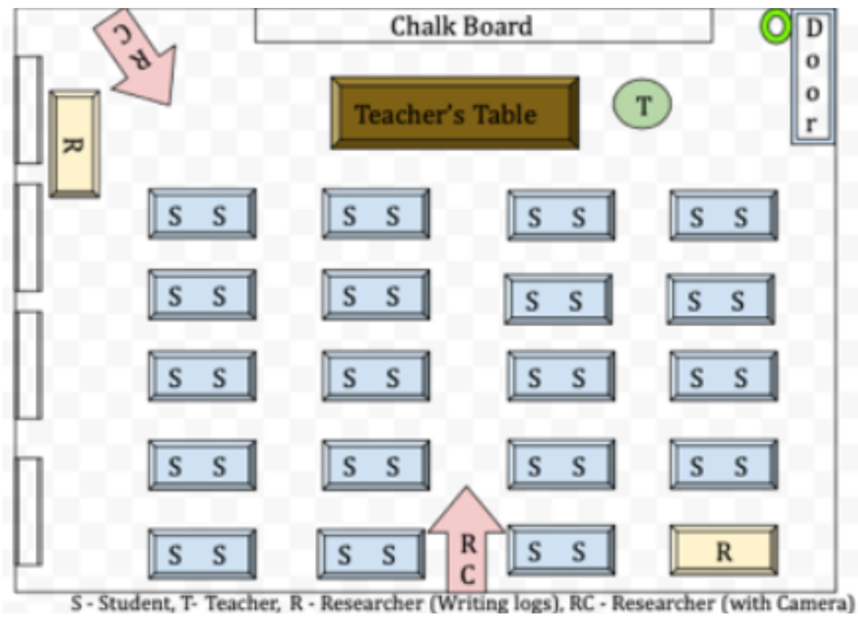


Figure 4.4: Layout of the Classroom

The school followed the curriculum prescribed by the Central Board of Secondary Education (CBSE) for Grades 1 to 10 and used the textbooks prescribed by National Council for Educational Research and Training (NCERT), the apex national body that frames the National Curriculum Framework and designs textbooks in India. Until a few years ago, the teachers from Grades 1 to 5 in the school were following a book series called “Ladders”. The subject teachers from different schools spread across the country, but functioning under the same governing society, prepared this book series. Ladders had several problems for practice for each chapter in a grade–wise manner. A sample page from Ladders for Grade 5 can be seen in Figure 4.2. Pallavi (one of the participants of this study) was a co–author of the Ladders book designed for Grades 3 and 4. She mentioned that the book was based on the old NCERT curriculum for mathematics, and therefore had several problems for practice. However, during the study (and a few years prior, as reported by the participants) the Ladders books were no longer in use and all the teachers followed the NCERT textbooks in the sequencing of content, examples, problems, giving homework and other teaching purposes.

The working hours of the school were from 0715 hours to 1345 hours for students, and till 1430 hours for the teachers. A typical school day began with the assembly, which had a prayer in Sanskrit, followed by news headlines read aloud by a student in English, then announcements by the vice principal or some thoughts by the principal, and conclusion with the national anthem. With the beats of a drum, the students would march to their classrooms. There would be five minutes for each class teacher to mark students’ attendance for the section assigned to them and then the teacher would move to the classroom where s/he is supposed to teach the first lesson. The school had a lunch break from 1145 to 1215 hours. Some visuals of the school that capture the ethos are shown in Figure 4.3.

The school had well lit classrooms, one staff room, and a separate male and female toilet on each floor of a three–storied building. Each classroom had a green board, four windows on one side, space for about 35–40 students to sit on desks in pairs, and a teacher’s table sometimes with a chair. Students sat in rows facing the green board and the teacher’s table (refer Figure 4.4 depicting the layout of a typical classroom).

4.5.3 Participants of the study

Four experienced school mathematics teachers participated in the research study. Each teacher had more than 10 years of experience in teaching mathematics at the school level, at the beginning of the study. All the four teachers were female and belonged to the same school but had an experience of teaching in other schools as well. Two of these teachers, Pallavi and Reema (pseudonyms), taught mathematics and environmental science to the primary school students (Grades 1–5, age 6–10 years). The other two teachers, Nandini and Vindhya (pseudonyms), were teaching mathematics and physics to middle grade students (Grades 6–10, age 11–15 years). Both these teachers had taught mathematics to Grade 11 and 12 students (of age 16–17 years) for a few years, early in their career. A subject teacher teaching two or more subjects is common in the Indian school system. The details of the educational qualifications and the classes taught by the teachers during the research study can be found in Table 4.1. The study was carried out in the span of 2 years, 2011–13; or two academic sessions, 2011–12 and 2012–13. All the four teachers participated in both the years of the study.

Table 4.1: Professional details of participants

Pseudonym	Educational Qualifications	Languages Known	Teaching Experience	Grades Taught	Subjects Taught
Pallavi	Bachelor of Science (B.Sc.), Bachelor of Education (B.Ed.)	English, Hindi, Tamil	22 years	I to V (Age 6–10 years)	Mathematics, Environment Sciences
Reema	Bachelor of Science (B.Sc.), Bachelor of Education (B.Ed.)	English, Hindi, Marathi	20 years	I to V (Age 6–10 years)	Mathematics, Environment Sciences
Nandini	Master of Science (M.Sc. Physics), Master in Education (M.Ed. Systems Management), Bachelor of Education (B.Ed.)	English, Hindi, Malyalam	10 years 8 months	VI to X (Age 11 –15 years)	Mathematics, Physics
Vindhya	Masters of Science (M.Sc. Mathematics), Bachelor of Education (B.Ed.)	English, Hindi, Telugu	25 years	VI to X (Age 11 –15 years)	Mathematics, Physics

Pallavi has been teaching primary grades for 22 years. In most of the interactions with the researcher, Pallavi was articulate about her opinions. She often brought to notice her concerns about students, their backgrounds and mathematics learning. While teaching in the classroom, she appeared confident. She had contributed to the design of the textbook for primary mathematics, Ladders. She expressed how she preferred the Ladders book, since it had clear explanations and several problems for practice. She did not prefer the NCERT textbooks (that were currently being used), as they are more like ‘language books’ and do not have enough problems for practice. She was often articulate about her beliefs about mathematics, its teaching and learning and students’ capability, during various interactions. In the classroom, she clearly stated the algorithms or rules and often provided some mnemonics to facilitate their recall. She expected students to remember these rules and use them for solving problems. Most of her teaching time was spent explaining the procedures, and helping students in copying the correct answers from the board.

Reema taught the same grades as Pallavi. She had been a primary school teacher for 20 years. Reema was awarded with the best teacher award in 2009 by the governing council of the school. She was usually sensitive to students’ physical health and their social backgrounds. She was found comforting students, when they felt physically unwell or had financial difficulties at home. She opened up with the researcher only after a few months of data collection, but would rarely speak in front of her (particularly senior) colleagues. She was usually given the responsibility of extra–curricular activities in the school. In the first year of the study, she was also the remedial teacher for primary mathematics. Remedial teaching meant that she was expected to teach those students who either failed in the mathematics exam or were identified as ‘weak’ mathematics learners by the school. The remedial teaching happened after school hours, that is, from 1345 to 1430 hours. Reema admired the contextual approach of the new NCERT mathematics textbooks. She used the contexts suggested in the textbook for introducing a topic. However, she would soon move to procedures and algorithms. One of her common teaching practice was to state the procedure and ask individual students to repeat it in order to gather their attention.

Nandini had approximately 10 years of teaching experience. Being a physics and mathematics teacher, she often made connections between these subjects. She had earlier participated in a science workshop, organised by the Homi Bhabha Centre for Science Education, on constructivist teaching. She believed that the teacher professional development workshops assumed an idealised classroom where students are interested, disciplined and self motivated. She added that workshops, which focused on telling teachers about innovative pedagogies did not engage with the realities of school teachers, particularly, that they have several non–teaching responsibilities. She believed that teachers needed to evaluate whether the inputs given during workshops were at all implementable. Nandini was put in–charge of the examinations in the second year of the study. Her responsibilities included monitoring the design, moderation, implementation and assessment of examinations. She was entrusted with the timely results of science and mathematics exams for the middle school. In her classroom, she was observed to be using a mix of conceptual and procedural explanations. She paid individual attention to students, who had difficulty in understanding a procedure, during her class time.

Vindhya was considered to be a senior, respected and experienced teacher in the school. Pallavi often invoked Vindhya's authority to legitimise her teaching practices such as, correcting students' mistakes immediately when they appeared, repeating the procedures until students' have memorised them, and so on. Vindhya had been teaching mathematics in the schools under this governing body in different geographical locations. Her duty was transferred from her home town to this state, which was a cause of concern to her. Usually, she would take minimal non–teaching responsibilities, owing to her poor health. Vindhya committed herself to completing the syllabus, the complete textbook, with the old and the new curriculum. She was often found disciplining the students in her classrooms by explicitly stating the norms, such as no talking among the students when the teacher speaks, writing the questions with a black pen and answers with the blue pen, recording all the steps of the procedure systematically, and raising a hand when asking a question from the teacher. Some of the students, not from the classrooms taught by her, would approach her if they had difficulty in solving a math competitive exam question.

In the beginning of this study, when the researcher followed these teachers in the staff room, the four participating teachers rarely spoke to each other. Some brief interactions included reminding each other of the circulars, asking for some data needed by the school such as fee collected from the students, and passing on the attendance registers from the previous years.

4.5.4 A teacher's routine

Each teacher of the school had multiple responsibilities. Either a teacher was a class teacher or a subject teacher. A class teacher would have the responsibility of collecting fee from students, record class attendance everyday, stay with the students during breaks, etc. The class teacher was expected to be aware of each student's performance in academic and non–academic activities. A subject teacher had the responsibility of teaching specific subjects, in which they had a professional degree. Every class teacher was a subject teacher, but not vice versa.

Table 4.2: Lesson Planner (reproduced for representation purpose)

Date	Class	Content	Signature
15–08–12	V B	Exercise 7.2, Ques 1–5	Pallavi
15–08–12	V C	Ch 7, Introduction, Exercise 7.1	Pallavi
15–08–12	VI A	Exercise 5.6, Ques 10–15	Vindhya
15–08–12	VI B	Exercise 5.6, Ques 10–15	Vindhya

The first task of the day for every teacher, after entering the school premises, was to fill a lesson plan in a specified format (refer Table 4.2) for the entire day. The lesson plan was a brief record of the content to be taught by the teacher on a particular day. This plan was used by the school administrators to keep a track of what is being taught, and to select classrooms for inspection. Everyday a new sheet of lesson plan was placed in the vice principal's office, where each teacher would go and sign the plan for the day. If a teacher forgot to fill this plan, she was marked absent for the day.

After this routine, all the class teachers would take the attendance register (and fee receipts, if needed) from the reception and proceed to their class. If the time permitted, the teachers took the attendance to record the number of students who were present and absent. Else, she would gather the students and accompany them to the assembly ground (school playground). The class teachers made sure that the students stand according to the ascending order of heights (a routine which students became accustomed to, soon after joining the school) and had the responsibility of maintaining discipline through the assembly time. Usually, the teachers kept walking beside students during the period of assembly and accompanied them to the classroom post–assembly. A subject teacher, on the other hand, had "ground duties", after filling the planner. These duties included ensuring the closing of the school gates in time, checking incoming students' uniforms, trimmed nails and neatly tied hair; along with helping class teachers in maintaining discipline among students during the assembly. The class teachers accompanied their students back to the classroom and the subject teachers punished the students who did not adhere to the school norms, such as, not wearing a tie or having long nails. The punishment usually was in the form of collecting scrap from the playground, taking a few rounds of the playground, raising their hands up and standing in the sun, etc. The assembly happened in, what was called as the, "zero period" of the school time–table. Before the beginning of the first period, the class teacher was required to complete the attendance and some administrative responsibilities such as fee collection, and depending on the time they engaged with students' concerns such as fights or name–calling.

The teachers proceeded to the classroom, where they are expected to teach the subject, in the first period. If the teacher was free during this time, she would usually sit in the staff room and correct students' notebooks (written work) or exam papers, create a test or exam paper, make a record of the number of students present or absent in her class, type circulars to be given to the students, write notes for parents, etc. Sometimes, meetings were arranged among colleagues or with the administrative staff to plan a school event such as an annual day, a parent teacher meeting, or an inter–school competition. In the lunch breaks, the class teacher was expected to sit in the class to ensure that the students have completed their lunch. At the end of the school day, the teacher had to ensure that all the students have left the school premises by either waiting in the classroom until all students had left or standing near the school gate to hand over students to their guardians.

4.5.5 Teaching schedule

Each participating teacher taught for about 5–7 periods everyday from the 8 periods, listed in the school timetable. Each period was 40 minutes long. Each teacher was assigned four sections (each grade has two or more sections depending on the number of students) for teaching in a year. Except Vindhya, all three teachers were also class teachers in the first year of the study. In the second year, all the teachers had the responsibility of being the class teacher. In the school culture, a class teacher is considered to be responsible for all the academic and non–academic aspects of one batch of students for an academic session. The academic work of the class teacher included collating the results received from each subject teacher, for the students of a batch (called *section*). Also the class teacher discussed the overall development of the student in the parent–teacher meetings organised by the school a few times in a year. The non–teaching aspects of their work included taking attendance everyday, dealing with classroom management issues, collecting fee from students, organising school picnics, preparing students for annual day, etc. Thus, being class teachers, the participating teachers had all these responsibilities. Apart from being a class teacher, each of the participating teachers had some other academic or non–academic responsibility, for each academic session. For instance, in Year 2, Nandini was the examination in–charge for Grades 6–10. This responsibility involved monitoring the entire process of conducting the examinations in an academic year, from the creation of the question papers for mathematics and sciences to declaring the results. Similarly, in Year 1, Reema had the responsibility of preparing students for dance performances or plays for occasions like the school annual day and sports day.

Inside the classrooms, almost all the time, students were given individual work and the teachers addressed the entire class. Some teachers had extra responsibilities, for instance, Reema conducted remedial teaching for some students of the classes (not only for the classes she taught) who were found failing in mathematics examination or the tests conducted by the school. The remedial teaching happened for an hour, twice in a week. The teachers usually gave individual attention to the students during this time, and tried to teach some basic ideas to help them pass the examination. If the student passed in the exam that followed after the remedial teaching, the student was exempted from this stay–back. The researcher observed some of these academic and non–academic activities with the participating teachers' permission in each year.

4.6 Phases of Research: Exploration and Intervention

The research study was carried out over two academic sessions 2011–12 and 2012–13. In the first academic year (Y1), the researcher spent 8 months in the school. During this period, the researcher observed lessons, interviewed participating teachers in formal and informal settings, shadowed the teachers in academic and non–academic tasks, interviewed students, visited other classrooms, interacted with the school administrators and collected documents, such as circulars, exam papers, students' notebooks, etc. In the second year of the study (Y2), the researcher spent 6 months in the school. Similar to the first year, the researcher observed lessons, interviewed students and teachers, and collected documents. Additionally, meetings between teachers and researchers were organised after the school hours. These meetings happened in the school almost once in a week in the second year. Table 4.3 depicts the distribution of time during the two years of the study. The time of the study has been classified into phases, based on the objectives. The three phases of the study, elaborated below, were temporal in nature and varied in terms of the role of the researcher. Phases II and III happened concurrently.

Table 4.3: Phases of the Research Study

Phases of the Study	Time Period	Modes of Data Collection
Phase I	Sep 2012 – April 2013	Classroom observations Pre and post–lesson interviews with teachers Long interviews with teachers Anticipation task with students and teachers Interview with students
Phase II	July 2013 – Dec 2013	Teacher–researcher Meetings
Phase III	July 2013 – Dec 2013	Classroom observations Pre and post–lesson interviews with teachers Anticipation task with students and teachers Interview with students

4.6.1 The first phase

The aim of the first phase was to understand teacher's knowledge of students' mathematical thinking, through a study of teaching practice. In this phase, teachers were observed while teaching in their classrooms. The details of the lessons observed, for each teacher, can be found in Table 4.4.

Table 4.4: Classroom Data from Year 1

Teacher	Class	Topics observed	Duration of data (minutes)
Pallavi	V	Multiplication, Division, Area and Boundary, Decimals, Smart Charts	1555
Reema	V	Subtraction, Multiplication, Division, Area and Boundary, Fractions, Decimals, Mapping, Boxes and Sketches, Smart Charts	2405
Nandini	VI	Fractions, Mensuration, Ratio and Proportion, Decimals, Algebra, Practical Geometry	1990
Vindhya	VI	Mensuration, Ratio and Proportion, Decimals, Algebra, Data Handling, Symmetry	1625

Each math lesson was observed by two researchers – the principal researcher and an observer. The observer assisted the principal researcher in the video recording of lessons and by writing notes. The position of the researchers while doing observations can be found in the classroom layout, represented in Figure 4.4. The observer was briefed about the purpose of the study, and was familiarised with the literature on mathematics teacher knowledge and classroom observations. During the data collection, the two researchers compared their notes, discussed what they found significant in a lesson, read and commented on each others' notes. Although no observation protocol was followed, the researchers made notes of the interactions among teacher and students, and attempted to record the movement, gestures, and actions; observable during the lesson. The researchers reflected on whether their written record captured how a concept was dealt in the class, through the use of some key questions (refer Section A2.3 of Appendix). These questions or pointers emerged over time from the post–lesson discussions among the researchers.

Apart from the researchers' notes, data was collected using audio recorders. Three audio recorders were placed in a classroom – one near the teacher (on the teacher's table) and two of them close to the students who were sitting away from the researcher. Since the teachers were uncomfortable with the placement of a video recorder in their classroom, for the first 3 months of the study, no video recorder was used for lesson observations. After each lesson, the researchers discussed their observations, and filled what was missed during the observation using the audio records and through discussions with each other. The principal researcher made a note of the significant events during each classroom observation, such as, an unanticipated student's question, shift made by the teacher from one representation to another, the choice of context, etc. These events were used to pose questions to the teacher in the post–lesson discussion. The classroom observations of each participating teacher were transcribed using the researchers' notes and audio (or video, wherever available) records of the lesson.

Before and after every lesson observation, the principal researcher tried to talk to the teacher, whenever the teacher's time permitted such a discussion. The pre–lesson discussions happened prior to the lesson observation and focused on understanding teacher's plan and the considerations which guided the plan (refer to the questions in Table 4.5).

Table 4.5: Pointers for Pre-Lesson Discussion

	Pre–lesson discussion
1	What do you plan to teach today? *Based on response to this question, ask specific content related questions, such as, if the teacher tells a specific sub–topic to be taught, ask how will it be taught, will s/he use a single or different representations, how will it be linked with students' prior knowledge of ___ (sub–topic), etc.*
2	Is this topic linked to what was taught yesterday or any time before? How is it related?
3	What kind of resources do you intend to use during the lesson? *Go through the material that the teacher has referred to. Make a copy of it. It could be a lesson plan, pages from a textbook, material from Internet, etc.*
4	What is the progression of the lesson? *Ask this in steps, that is, what all will you teach today. Use phrases such as after this, before that, etc.*
5	What would you want students to learn by the end of this lesson?
6	How do you expect the students to respond to this topic/ question/ idea or representation?

After the lesson observation, post–lesson discussion was done with the teacher. Although initially triggered by the researcher, gradually, teachers started initiating these discussions with the researcher. These post–lesson discussions focused on eliciting reasons for the decisions made by the teacher, such as ignoring or detailing a particular student error, postponing a discussion, shifting from one representation to another, and so on (refer Table 4.6).

Table 4.6: Pointers for Post-Lesson Discussion

	Post–lesson discussion
1	What did you think about today's lesson?
2	How was it similar to or different from what was planned?
3	What was the basis of some of the teaching decisions? *Identify some decisions made by the teacher, such as, expanding on a particular students' response, elaborating on a specific representation, deviation from the plan, etc. Ask questions on these decisions.*
4	What did you think about a specific student's question? *Identify a student's question that was linked to the topic and probe teacher's thinking about the importance of that question.*
5	Why did you think students' responded in this way?
6	Does this change the way you would plan the next lesson? If so, how?
	Note down anything else that the teacher found interesting, routine, or important.

The pre– and the post–lesson discussions with the teachers varied from 5–30 minutes, depending on the teacher's availability.

In the first phase of the study, the researcher also had two long interviews with each participating teacher (refer Section A2.4 in Appendix for the complete interview schedule). The objective of these in–depth semi–structured interviews was to understand the meanings that teachers attributed to their practice (or actions) and the significance attached to them. An in–depth interviewing addresses the context of the participants, which adds substance to the meanings attributed by them. Although such an interview is not free from presuppositions, the researcher (or interviewer) is not directive, since the purpose is to understand the participant's construction of reality in their own terms (Jones, 1985). The design of the semi–structured interview schedule was informed by the context of teachers' work. For instance, consider the question on the number of problems covered by a teacher in a lesson (refer 1.13 in Table 4.7). This question was informed by the classroom observation that teachers intended to cover some number of problems in a lesson, with less wait time for the students. The criteria of the number of questions covered in a lesson was also used to compare the quality of teaching, among teachers and by school administrators.

Table 4.7: Some questions from Interview Schedule 1

	Interview Schedule 1
1.13	It has been observed that some teachers cover 2–3 problems in a lesson, while others do 5–10 problems. One an average you seem to cover ___ problems. How do you decide how many problems can be covered in a lesson? Does this change from one lesson to another? What determines this change?
1.16	How important is repetition and drilling in your math lessons? Can you tell me some exercises that you use to help students memorise an idea?
2.2	Do you think teachers are different from private tutors, parents or elder siblings who are interested in teaching mathematics to students? If so, what makes teachers different from others?
2.5	Would you think that the concepts in mathematics are related to each other? Can you give a few examples to support what you think? Do you think a teacher needs to be aware of these connections?
3.2	Why do you think students need to know the algorithm for (any one of them) multiplication of two digit number by a two digit number or unitary method?
3.3	Sometimes students do not understand what is being taught. What are some of the ways through which you know that an individual student is not understanding? How do you deal with such a situation in the classroom?
3.4	Do you think there is any difference between the way boys and girls learn or perform in mathematics? Why?
	In column 1, the first number refers to the part of the first interview (1, 2 or 3) and the number after the dot indicates the question number in the interview schedule.

The interview schedule was validated by two mathematics education researchers, two school mathematics teachers, and two education researchers. The first interview aimed to understand the teachers' views about teaching practices observed in their and other teachers' classrooms. It included a discussion on the teaching strategies used by them and a reflection on the impromptu decisions made in the classroom. This interview was divided into three parts, namely,

Part 1: Explanations of teacher's own practices.

Part 2: Views about knowledge required for teaching mathematics.

Part 3: Awareness of students' mathematical thinking

Each part had about 8–10 questions. The questions were nested, that is, the response to one question informed the subsequent question. The interview was planned to happen in a conversational style. Some sample questions from each part of the interview can be found in Table 4.7. Although each interview was planned for an hour, the time spent with each teacher was different (refer Table 4.8).

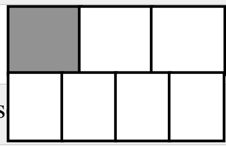
Table 4.8: Data from Long Interviews in Year 1

Teacher	Interview 1	Interview 2
Pallavi	1 hour 10 minutes	1 hour 30 minutes
Reema	1 hour 30 minutes	1 hour 20 minutes
Nandini	1 hour 15 minutes	1 hour 20 minutes
Vindhya	1 hour	30 minutes

The second interview was a task–based interview (Goldin, 2000), that is, the interview with the teacher was centred around a task that was given to the students. The interview included (a) an anticipation phase: anticipation of students' ways of thinking around specific problems, (b) a testing phase: testing the anticipation by noticing how students approach these problems, and then (c) the reflection phase: reflecting on one's own knowledge about students by studying students' responses. In the first part of this interview, the anticipation phase, teachers were requested to anticipate students' responses to a set of problems (refer Table 4.9 for an example).

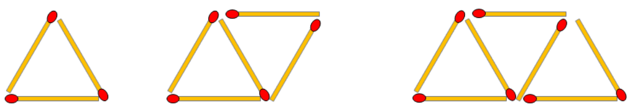
Table 4.9: Interview II: The Anticipation Phase

S.No.	Sample Question
	What is the fraction of the shaded part? (a) $\frac{1}{6}$ (b) $\frac{1}{7}$ (c) $\frac{3}{4}$ (d) Can not say
1	(Questions to be asked before showing the given options for the problem.) What are the different correct and incorrect responses that students would give for this problem?
2	What would be the different ways in which students might solve this problem?
3	Do you tell students different ways of solving a problem?
4	Do you think students can solve some problems without 'knowing' the algorithm or without being taught?

S.No.	Sample Question
5	Can you tell me one common mistake that the students will make while solving this problem?
6	(Now show the four options to the problem.) Which among these options would students select? Why? 
7	What can you say about the student's thinking if s/he responds by s
8	How would you address with such a response?

These problems were then posed to the students in the form of a worksheet, designed by the researcher. For example, consider the problem on finding the shaded part of the region (elaborated in Table 4.9). Here the teacher was asked to solve the problem, predict how different students will deal with this problem, anticipate common students' errors (until this point the four options were not shown to the teacher), and then discuss students' thinking underlying each of the four options given for the problem.

Table 4.10: Anticipation and reflection task in Year 1: Worksheet

S.No.	Sample Question
5.1	Shahni has 17 chocolates. She wants to distribute them equally among her four friends. Which of the following should she use to find chocolates that each friend will get? (a) $17 \div \frac{1}{4}$ (b) 17×4 (c) $17 \div 4$ (d) $17 \times \frac{1}{4}$
5.5	How many halves are there in 57?
5.9	Fill in the missing blanks to make the mathematical sentence true. 48 + 97 = ____ + 99 67 – 58 = 64 – ____
6.1.1	The cost of 10 pens is Rs. 42. Find the cost of 15 and 20 pens.
6.1.3	Farida says that "Sum of an even and a odd number is always odd". Do you think she is right? Can you prove this?
6.1.5	Which vehicle has faster average speed: a truck that travels 100 kms in 1½ hours or a car that travels 140 kms in 1¾ hours?
6.2.5	Kiran and Saheb are making bridges with matchsticks. They are playing in turns. See how they are playing and be a part of it.  (a) How many sticks will be used in the 5th and 100th design? (b) Which design would require 57 sticks? Show your working.

The problems given in the worksheet covered the topics that were observed in the first phase of teaching. These topics included large numbers, fractions, decimals, ratio and proportion and mensuration (refer Table 4.10 for some selected problems). The worksheet problems were designed using the existing research literature on students' thinking in specific topics (refer Sections A1.1 to A1.6 of the Appendix for the worksheets used in Year 1) .

In the second part of the task, the testing phase, students were invited to solve the worksheet problems and give reasons for their response. Since the worksheet was administered in teachers' classrooms, in some cases, teachers volunteered to be present. Alternatively, they could go through the students' work just before the third phase of the interview. While solving the worksheet problems, students were told that the worksheet is not a test and that they can talk to the researchers, the teacher or with each other, for clarifications or discussions. Before taking these worksheets to the teachers (for reflection), the researcher collated different kinds of responses for each problem. The purpose was to alert teachers to the diversity of students' responses. In the reflection phase of the second interview, teachers were shown these students' responses, they were encouraged to use students' ways to solve problems and then comment about them. Then teachers were asked to reflect on their anticipation of students' responses and in the process compare it with the students' actual responses. Teachers were probed for the reasons for predicting students' responses and then about the difference between their prediction and actual student responses.

The aim of the anticipation reflection task was to understand teacher's knowledge about students' mathematics and challenge it through the use of actual students' work. The purpose was also to expand teachers' noticing of (the diversity in) students' responses and help them revisit their beliefs about students' capability.

4.6.2 The second phase

Before the beginning of the next phase, the researchers examined the data that was collected in the the first phase from the four classrooms, taught by the participating teachers. Since it was the end of the academic year 2012–13, a meeting was organised between the teachers and the researcher to discuss the observations from the first phase. Much like the observation from Phase 1, during this discussion, the principal researcher had one to one talk with each teacher. Teachers did not interact with each other, even though the similarities in the students' responses across classrooms were discussed.

In one of the meetings with the teachers, the researcher made a proposal to form a group of teachers and researchers, which would meet once in a while, to discuss classroom teaching. While the teachers were skeptical about the time at which these meetings would be organised, they agreed to the idea of having these meetings in the school. It was decided to hold these meetings in the mathematics laboratory of the school, once every week, and after school hours. The data from these meetings between teachers and researchers constituted Phase 2 of the study. The aim of the these meetings was to invite teachers to talk about their classroom practice and engage with the problems or difficulties arising in teaching of a specific topic. Further, the attempt was to create a safe space for teachers and provide them with a community, with whom they could share their problems of teaching. Since the classroom data from Phase 1 was vast (refer Table 4.3), it was decided to focus on one topic to have detailed discussions, during these meetings. Although the researcher was considering a different topic: early algebra, fractions or multiplicative thinking (since these were the focus of the pilot studies), the teachers decided to discuss decimal numbers with the researchers. Their rationale was that decimals were introduced at Grades 5 and 6, and students made several errors while solving decimal problems. In order to plan for these meetings, the classroom observation data on decimals from the first phase was taken and viewed by a team of researchers, and detailed notes were made about events such as students' errors, questions, teacher's use of representations, nature of explanations, and so on. The researcher identified those situations from classroom observations where the teacher faced difficulty in responding to the students, or missed an important student utterance (question, response, suggestion), a connection or representation, etc. It was noted that such classroom situations arose from difficulties in (a) understanding a student's question, (b) identifying appropriate representations, (c) providing justifications, (d) providing support to some students identified as weak in mathematics, or (e) using contexts and models suitable for the topic of study. These considerations and a review of literature on students' thinking in decimal learning and tasks used to support teacher learning, were used to plan the teacher–researcher meetings.

In Phase 2, teachers and researchers met in the school for after–school meetings. The meetings were organised, almost once every week, and the day was selected based on teachers' convenience. There were 20 such meetings which lasted for 40–90 minutes. The participants in these meetings were the four mathematics teachers, the principal researcher, researcher's supervisor, and at least two co–researchers. These meetings were audio and video recorded, with prior permission of the teachers. Apart from the digital records, written records for each meeting were prepared by a teacher or a researcher, and summarised before the next meeting. Initially researchers planned these meetings by designing classroom–based tasks, that is, tasks which were designed based on the episodes selected from the classroom observations of the first phase. The details of all the tasks, their design considerations, implementation, and reflection can be found in Chapter 6.

4.6.3 The third phase

The third phase was concurrent with the second phase and similar to the first phase of the study, in terms of modes and nature of data collection (refer Table 4.3). There were two major differences between the first and the third phase. First, in Phase 3, teachers were more active in initiating discussions with the researcher before or after the lesson, and also at times when they were planning a lesson. Second, each teacher independently proposed a video recording of their lessons. The decision of video recording was guided by an interest in seeing their own classroom teaching after the lesson. Even though, the focus was on the decimal lessons, the researcher decided to observe the teaching of other topics as well. Summary of the lessons observed in the Phase 3 can be found in Table 4.11. Like the first phase, in Phase 3, data was collected from classroom observations, pre– and post–lesson interviews and anticipation–reflection task.

Table 4.11: Classroom Data from Year 2

Teacher	Class	Topics taught	Number of lessons	Data (in hours)
Pallavi	V	Decimals, Area and its boundary, Patterns, Division	24	17
Reema	V	Decimals, Area and its boundary, Multiplication, Division	33	23
Nandini	VI	Decimals, Mensuration, Practical Geometry	24	17
Vindhya	VI	Decimals, Mensuration, Practical Geometry	23	16.5

As stated earlier, in Phase 1 teachers avoided the pre– and post–lesson discussions with the researcher. In contrast, in the third phase teachers proactively interacted with the researcher in planning and reflecting on their lessons. On a few occasions the teachers called the researcher to discuss their lesson plan. Interestingly, these discussions continued even after Phase 3 (or researcher's field work for the study) was over. The content of these discussions had also changed over time. While in the beginning the researcher posed some questions (listed in Table 4.5 and 4.6), teachers began discussing some important, perplexing events from their classroom in the discussions initiated by them. Instances of such discussions include, how does an alternative algorithm given in the textbook work, gaps in the textbook content, designing worksheet, reflection on decision to spend more time on teaching place value, etc (details in Chapter 7).

Teachers started seeking support from the researcher in planning their lessons, following up with some weaker students, validating an exam paper, designing worksheets on the topics which were missed in the textbook, etc. Their planning of lessons slowly moved beyond stating the exercise number to explicating the topics and tasks for a particular day. Unlike the time bound planned interviews, this phase witnessed punctuated interactions with the teachers, which happened as the teachers' time permitted and when they wanted to discuss something. When such interactions were planned, they were audio recorded, a researcher made notes, and the principal researcher recalled the discussion and made notes after the discussion. However, in impromptu situations, the researcher's notes were validated by the second observer. In this year, the worksheet designed for students, for the anticipation task focused on the topic of decimal numbers, and teachers validated this worksheet based on their anticipation of students' responses (refer Sections A1.7, A1.8 and A.19 of Appendix for the worksheets used in Year 2). A few problems from the worksheet can be found in Table 4.12.

Table 4.12: Anticipation and Reflection Task in Year 2: Worksheet

S.No.	Sample Question
1	Circle the smaller number. Give reasons for your choice. (a) 4.63 or 4.8 (b) 0.7 or 0 (c) 0.6 or 0.37 (d) 8.24 or 8.245 (e) 0.25 or 0.100
5 (iv)	Circle the numbers that are the same as $\frac{8}{100}$ (a) 0.80 (b) 0.800 (c) 0.08 (d) .08
8	Reena thinks that the following statements are true. Do you agree with her? Give reasons. (a) The sum of 3.7 and 8.6 is more than 1. (c) The product of 16.5×0.2 is more than 16.5.
11	Show these decimal numbers using a diagram. (a) 0.5 (b) 0.67 (c) 1.35

Notably, teachers' anticipation was considerably different from that of the first phase. Unlike the first year, teachers did not attribute students' lack of capability to their inattentiveness in reading the question or to "not being taught" the algorithm. Instead, they predicted how based on their prior knowledge students would solve some problems. Nandini's precision in prediction and her reflection on individual students' ways of thinking is a case in point (refer Section 6.4.2 of Chapter 6 for details).

The anticipation of and reflection on students' responses or teacher's moves was used by teachers to organise the discussions during pre– and post–lessons. In one of the pre–lesson interactions, Reema articulated the purpose of that lesson based on her anticipation about how students' might deal with this idea. At the end of this discussion, Reema had developed a worksheet along with the researcher. In her next lesson, she used this discussion and the worksheet to build students' understanding. (This episode has been elaborated in Section 7.5.3 of Chapter 7).

A new development in Phase 3 was the changing relationship between the researcher and individual teachers. It was noted that each teacher needed a different kind of support while teaching in their classroom. For instance, Pallavi invited the researcher to be a co–teacher during teaching of one of the lessons about which she felt less confident. Reema started seeking support in planning and reflecting routinely before and after her lessons, identifying gaps in the textbook in use and making worksheets to fill this gap. Nandini used the reflection time with the researcher for thinking aloud about her decisions and making plans for the upcoming sessions. Vindhya discussed her pedagogy and specific student responses usually after teaching a few lessons.

Apart from the changing dynamics between the researcher and the teachers, changes were noted in teachers' teaching. The teaching was becoming more responsive to students' thinking. New demands were posed on the teachers due to decisions such as, probing students' mistakes, encouraging multiple ways of problem solving, consistency in the use of contexts and representations, etc. The knowledge demands posed on the teachers as they became more responsive to students' thinking have been analysed in Chapter 5.

A summary of the three phases of the research can be found in Table 4.13.

Table 4.13: Summary of Research Phases

Phases of the Study	**1 (Year 1)**	**2 (Year 2)**	**3 (Year 2)**
Objectives	Explore teacher's knowledge of students' mathematical thinking manifested in their classroom.	Enhance teachers' knowledge of students' mathematical thinking through *ex–situ* support.	Examine how teachers use the knowledge of students' mathematical thinking in their classroom.
Duration	8 months	6 months	6 months
Site of data collection	Classroom Staff room Other school premises	Mathematics laboratory in the school Researcher's institute	Classroom Staff room Other school premises
Modes of data collection	Classroom Observations, Pre– and Post–lesson interviews, Anticipation–reflection task, Long interviews.	Planning; observations, summaries, and reflections of teacher–researcher meetings.	Classroom Observations, Pre– and Post–lesson interviews, Anticipation–reflection task.
Tools for data collection	Audio records, Researcher notes, Documents, Video records (of some lessons).	Audio and video records, Researcher notes, Written summaries.	Audio and video records, Researcher notes, Documents.
Participant's role	Teach in their class. Discuss plan and reflections.	Participate in the meetings and later organise these meetings.	Teach in their class. Discuss plan and reflections.
Researcher's role	Non–participant observer.	Teacher educator.	Participant observer.

4.7 Data Reduction and Analysis

The time spent on the field in each of the two years was 8 and 6 months, respectively. All the data (except some informal interactions with the teachers) was audio and/or video recorded. The data collected during field work can be classified as classroom observations, interactions with the teachers, teacher–researcher meetings, and interactions with other officials. Table 4.14 summarises the number of entries of data collected during the study. The time spent on collecting documents or records, such as students' written work, attendance sheets, circulars, etc., is not mentioned in this table. Due to the vastness of data, it was important to reduce it for an in–depth analysis. However, care was taken to ensure that data reduction did not compromise on the diversity in teachers' responses to students' questions, selection of representations, etc., during the teaching of different topics. In order to account for this diversity, the analytical techniques used for the decimal lessons – teacher's practices, coding, responses to students, explanations, questions, etc., were also validated with other topics that were observed during each year. Further, even though the decision about observing decimal lessons was made in Phase 2, a few other topics after the decimal lessons were observed in Phase 3 to examine the extent of consistency in each teacher's practice.

Table 4.14: Field Data

	Number of instances	
	Year 1	**Year 2**
Classroom Observations	149	104
Interactions with teachers	72	34
Interactions with school officials	4	4
Teacher–Researcher Meetings	0	21
Interaction with students	8	10

4.7.1 Data from classroom observations

The first level of data reduction was done by separating the decimal lessons, from the classroom observations of all the topics taught in the two years. The number of decimal lessons, each of 40 minutes duration, taught by each teacher in the two years is mentioned in Table 4.15.

Table 4.15: Classroom Observation of Decimal Lessons

	Number of decimal lessons	
Teacher	**Year 1**	**Year 2**
Pallavi	9	10
Reema	12	13
Nandini	13	21
Vindhya	6	12

A coding scheme was developed through grounded ways of looking at the data of one teacher's teaching (see Table 4.16). The open coding of the lesson transcripts included, "breaking down, examining, comparing, conceptualising and categorising" (Strauss & Corbin, 1990). The broad categories that emerged from coding were questions, explanations, and responses of the teacher and students. The coding scheme was informed by the researcher's awareness of the instruments such as Learning Mathematics for Teaching (LMT, 2006), Knowledge Quartet (Rowland, Huckstep & Thwaites, 2005), literature such as Lampert's and Gutstein's teaching, and the observations of the (participating) teachers' classrooms. A detailed coding scheme was prepared using the first three decimal lessons taught by Nandini. To check for the consistency of the coding scheme, Nandini's other decimal lessons, and one decimal lesson selected randomly for each of the other 3 participating teachers, were coded. The coding scheme was then presented to a group of mathematics education researchers, who validated the codes, suggested changes, and coded one decimal lesson for each teacher. The suggestions made by the validators were mostly in terms of nesting some codes and separating others. The two researchers (principal researcher and her supervisor) independently coded all the decimal lessons taught by Nandini. The discrepancies in the coding were discussed and resolved. The coding scheme was refined and expanded based on the new practices that were observed in the second year of the study. For instance, the category of teacher's response which earlier included *restating* and *correcting* were refined to *restating* a student's utterance, *evaluating* by saying "good" or "yes", *correcting* a student's mistake immediately, and posing students' response for *public thinking*. New categories, such as *student's evaluate* or *student's adds*, were added as some of the student's utterances involved evaluating or building on each other's responses, in the second year (discussed in detail in Sections 5.5.1 and 5.6.1 of Chapter 5). After the coding scheme was finalised (refer Table 4.16 for the final codes), the principal researcher coded the decimal lessons taught by the four teachers.

Table 4.16: Coding Scheme for Classroom Observations

Code	Description	Example
TQ–work	Teacher question related to students' work.	Have you drawn a number line?
TQ–textbook	Teacher question taken from the textbook.	Convert the following fractions to decimals.
TQ–bin	Teacher question requiring a binary response.	Is the length 15 or 16 cm?
TQ–what	Teacher question about factual details.	What is the equivalent fraction for $\frac{5}{6}$?
TQ–how	Teacher question asking for procedure.	How do you covert this fraction into decimal?
TQ–why	Teacher question to seek reasons for a procedure or statement.	Why is $\frac{86}{10} = 8.6$?
TQ–elicit	Teacher question to probe or elicit a response.	What did we do in the last lesson?
TE–tell	Teacher tells the explanation.	$\frac{86}{10}$ is equal to 8.6.
TE–procedure	Teacher explains the procedure.	First you count the number of zeroes in the denominator. Then start from the right and count that many digits. Then put the point.
TE–justify	Teacher gives reasons for the procedure.	$\frac{86}{10}$ Here 86 is made up of 8 tens and 6 ones. 80 divided by 10 is 8 and 6 divided by 10 is the same as 6 times 0.1.
TE–scaffold	Teacher offers support to the student who is struggling.	Yes, now that you have found an equivalent fraction can you tell me how to convert it into decimal?
TR–listen	Teacher listens to student response.	*Teacher nodding*
TR–evaluate	Teacher passes a judgment on student response.	Yes, that is good.
TR–correct	Teacher corrects a student response.	Wrong, $\frac{86}{10}$ is not 86.10. It is 8.6.
TR–restate	Teacher repeats (or slightly reformulates) a student response.	Alright, you say that both the numerator and denominator are multiples of 2.

Code	Description	Example
TR–expand	Teacher expands on a student response.	What she is saying is that 2 is a common factor for 86 and 10.
TR–publicthink	Teacher poses a student statement for the whole class to think and respond.	Dev is saying that 2 is the common factor of 86 and 10. Do you agree or disagree? Why?
TR-argue	Teacher poses an argument in response to a student's utterance.	What about 3.06, is it the same as 3.60?
SQ–work	Student asks a work related question.	Do we have to make this table?
SQ–textbook	Student states a question from the textbook.	Convert the following fractions to decimals.
SQ–clarify	Student seeks clarification from the teacher.	Teacher, is it 15 cm?
SQ–what	Student asks a 'what' question.	What comes in between 15 and 16 cm?
SQ–how	Student inquires about the procedure.	How to find an equivalent decimal of 7.8?
SQ–why	Student seeks reasons for a statement.	Why is 36.0 equal to 36.00?
SE–one word	Student responds to a question in one word.	15 cm (a correct one–word response)
SE–repeat	Student repeats the teacher's statement.	2 is the common factor.
SE–recall	Student recalls a piece of information from prior knowledge.	The last digit of the number has to be divisible by 2.
SE–error	Student gives an erroneous response.	17.5 cm (an incorrect one word response)
SE-completes TE	Student completes the teacher's explanation.	T: Three point six can be written as… S: thirty six by ten.
SE–procedure	Student explains the procedure to solve a question.	There are 2 zeroes in the denominator so the point will come before 8. So $\frac{86}{100} = 0.86$.
SE–justify	Student presents an reason for the procedure.	2 cm is equal to 0.02 m as 1 cm is a hundredth part of a metre, and 2 cm means two 1 cm parts of 100.
SE-adds SE	Student adds to another student's explanation.	S1: Three tenths means three by ten and… S2: or three times one by ten.
SE–observe	Student makes an observation about the ideas being discussed.	Teacher, why is there no oneths?
SE–argue	Student argues with another student or the teacher about his/ her point of view.	No, teacher. When we multiply with 10, it becomes 10 times, not lesser.
SE-evaluate	Student evaluates another student's explanation.	Srishti is wrong. She forgot zero in between.

The coding of the decimal lessons, taught by the four participating teachers, constituted the first phase of analysis. The codes helped in concluding that teachers were becoming more responsive to students' ideas. This was particularly evident in the codes of teacher's responses to students and students' responses to each other. The codes helped in capturing the nature of changes in teachers' practice. As Strauss and Corbin (1990) point out, the process of coding allows for more questions and explores researcher's assumptions about the phenomena, leading to new discoveries. Through the process of coding, it was realised that the coding scheme did not capture the challenges posed on the teachers as they explored a different pedagogy. These conceptual challenges faced by teachers in the context of their practice seemed important to be studied and analysed. I call these conceptual challenges arising in the context of teaching as knowledge demands posed on the teachers while teaching in the classroom. The construct of knowledge demands can be potentially used to (a) understand the complexities of teacher's knowledge manifested in their practice, (b) study the teaching decisions made in–the–moment, (c) reflectively discuss the underlying reasons of mathematical choices, such as, the nature of representation used, and (d) identify the knowledge required to support teachers in their practice (discussed in detail in Chapter 5).

It was observed that the changed practice posed knowledge demands on teachers. For instance, a classroom where students ask questions and propose strategies is more challenging for the teacher than the classroom where the teacher tells the procedure and students are expected to follow it. In order to abstract these knowledge demands, it was decided to study the decimal lessons from the two years more carefully, that is, by focussing on detailed interactions between students and the teacher. So, for a second level analysis, each decimal lesson was divided into episodes which dealt with a sub–topic. For instance, the episode on the expanded form of a decimal number was separated from the episode on writing the number names of decimal numbers, which happened consecutively in the same lesson. The duration of these episodes varied depending on the time spent on each sub–topic within a lesson. While looking at the decimal lessons from the two years, it was found that teachers took more time in dealing with some sub–topics while teaching in the second year. The question then was to identify the variable that triggered a detailed response from the teacher in the second year when compared with the teaching of the same sub–topic in the first year. It was found that in the second year, teachers decided to respond to *contingencies* arising from classroom teaching as opposed to ignoring them as in the first year. Rowland and Zazkis (2013) define “contingencies” as teachers’ responses to classroom events that were unplanned or unanticipated and were triggered by a student’s remark or question. Researchers have defined such moments as knowledge that plays out (Rowland, Huckstep & Thwaites, 2003), knowing to act in–the–moment (Mason & Spence, 2000), thinking on the feet (Schön, 1983), finding right practice at the right moment (Lampert, 2001), and improvisational coaction (Martin & Towers, 2009). The definition of contingency by Rowland and Zazkis (2013) was found closest to this research study, particularly, because of the focus on exploring teacher’s knowledge about students’ mathematics. The definition of contingency was expanded for the purpose of the study and was revised based on the observations of practice. Contingencies include an unanticipated student’s question or observation, a connection made by the teacher while teaching, connecting a representation to another (which the teacher might be aware of but had not thought of connecting before that moment), and so on. Thus, contingencies were identified as moments which were unanticipated by the teacher but happened or emerged during the teaching. Those episodes from the two years of teaching of decimals were selected, where the teacher decided to ignore a contingent moment in the first year but decided to respond to it in the second year of teaching. These episodes are referred to as paired episodes, that is, episodes dealing with the same sub–topic but treated differently. Table 4.17 shows how paired episodes were identified through a comparison of lessons from the two years of Nandini’s teaching. The lessons from which the paired episodes were taken, were compared using the coding scheme. For example, lessons Y1DL1 (Year 1, Decimal Lesson 1) and Y2DL2 (Year 2, Decimal Lesson 2) were compared since these lessons focused on introducing decimal numbers. Similarly, lessons Y1DL5 (Year 1, Decimal Lesson 5) and Y2DL8 (Year 1, Decimal Lesson 8) focused on the conversion between fractions and decimals and were followed by introducing students to the conversion between measurement units.

Table 4.17: Decimal Lessons in Year 1 and 2: Nandini

	Year 1 (Y1)	Year 2 (Y2)
DL1	Introduction to decimals using fractions and whole numbers	Worksheet (to check students' prior knowledge)
DL2	Place value table, expanded form, decimals on number line	Introduction to decimals using Measurement activity
DL3	Shading decimal parts and fraction–decimal conversion	Introduction to the decimal place value, place value table
DL4	Locating a decimal on a number line, Place value table, number names	Place value, expanded form, number names
DL5	Fraction–decimal conversion, conversion between millimetre and centimetre	Number names, fraction and decimal conversions
DL6	Conversion between millimetre and centimetre, finding whole numbers in between which a decimal number lies	Place value table, number names to decimal numbers, placement of zero in a number, conversion of fraction or expanded form to decimal
DL7	Conversion between millimetre and centimetre, representing shaded parts as fractions, place value table to decimal number, conversion between decimals and fractions to their lowest term.	Fraction to decimal conversion, decimal to fraction in lowest form
DL8	From expanded form to decimal numbers, number names, locating decimals on a number line	Decimal to fraction conversion, measurement context, conversion between millimetre and centimetre
DL9	Conversion from decimals to fractions in lowest form, comparison of decimal numbers	Shading decimal parts and fraction–decimal conversion
DL10	Comparison of decimal numbers, conversion between centimetre and metre, metre and kilometre, rupees and paisa.	Finding decimal numbers between whole numbers using a ruler and a number line
DL11	Conversion between paisa and rupees, centimetre and metre, metre and kilometre	Naming decimal numbers marked on a number line
DL12	Addition and subtraction of decimals	Conversion between decimals and fractions, Placing different types of numbers in a place value table
DL13		Using place value to locate whole numbers between which a decimal number lies, reading and writing decimal numbers
DL14		Representing decimals on a number line, comparison of decimals
DL15		Conversion between rupees and paisa, centimetre and metre, millimetre and centimetre
DL16		Conversion between grams and kilograms

	Year 1 (Y1)	Year 2 (Y2)
DL17		Word problems on addition of decimal numbers
DL18		Word problems on addition of decimal numbers (including conversion of units)
DL19		Subtraction of decimal numbers
DL20		Word problems on subtraction of decimal numbers
DL21		Worksheet (as a post–test)

The paired episodes helped in (a) identifying micro–level changes in each teacher's practice, for instance, in identifying how the teacher dealt with the multiple solutions offered by students, and (b) abstracting the knowledge demands posed on the teacher as the teacher decided to be more responsive to their students' mathematical thinking. Chapter 5 elaborates the knowledge demands abstracted from the teaching of Nandini and Reema by studying paired episodes from the two years of their teaching. The case of Nandini and Reema's teaching was selected as their teaching offered "maximum variation" across 2 years. The lessons analysed in Chapter 5 are representative of the other lessons taught by the two teachers.

4.7.2 Data from interviews

The other data set included personal interviews with each teacher in Year 1. The interview data was referred to as corroborating evidence when analysing each case (in Chapters 5, 6 and 7). The interviews helped the researcher in getting a general sense of teachers' practice and created scope for a further interaction exclusively focusing on teachers' views about students' mathematics. Such an interaction was planned during the anticipation–reflection interview, which was centred around a task. The teacher's responses to these interviews were used to triangulate the observations of practice (The interview data has been used to characterise pedagogical aspects of teachers' practice in Chapters 5 and 6 and cited directly in Chapter 7).

4.7.3 Data from teacher–researcher meetings

The *ex–situ* support offered to the teachers was in the form of teacher–researcher meetings, organised in Phase 2 of the study. In these meetings, teachers engaged with classroom–based tasks were designed to elicit, challenge and support teachers' knowledge. The audio and video records, researcher's notes, and self reported summaries were compiled to create a transcript of each meeting. These transcripts also included the written work of all the participants. The tasks planned for the meetings included topic–specific discussions such as how whole number affects decimal learning, how to choose a representation for introducing hundredths, how to handle students' errors in the classroom, etc.

Tasks with a similar focus were identified and classified under a theme. For instance, the theme of attending to students' errors included all those tasks where a student error in learning of decimal numbers was analysed and teaching decisions around the students' responses were discussed. Tasks within each theme were analysed based on the design considerations and the nature of teachers' learning though participation in these discussions. An attempt is also made to unpack progress in teachers' noticing and learning across different meetings. The process of analysis of teacher meetings and findings from this analysis can be found in Chapter 6.

4.7.4 Data from *in–situ* support

In the second year, teachers were supported in the contexts of their practice. This *in–situ* support was located in several sites: classroom, staff room, school premises, walks, etc.; but included an actionable outcome. The nature of this support varied for teachers. However, all the teachers requested such support due to problems arising in their teaching. For instance, Pallavi found it difficult to understand the "partial quotients" method given in the textbook. She struggled to use this method with large numbers. Reema found that there were word problems on currency in the decimals chapter, given in the textbook, which required multiplication of a decimal number with multiples of 10. She found that multiplication with 10 and its multiples had not been covered before the currency context and could not be assumed as students' prior knowledge. Vindhya faced the student error "longer is larger" even after several explanations in the class. Students' immediate response to a comparison problem such as, which is greater, 9.10 or 9.100, was 9.100. Nandini was becoming perceptive to students' reasons but would often find it difficult to collate their responses and connect their strategies to the algorithm. As teachers became more sensitive to such situations in their classroom, they also realised that they needed support to respond to these situations. The teachers approached the researcher for the specific support to handle such demands arising from their classroom. Since the demand for such a support was often time bound, that is, had to be handled in a lesson on the same day, or the next day; these moments acted as "contingencies" for the researcher. Similar in its character to the way teachers made a decision on whether and how to respond to such moments in their classroom, the researcher had to make a decision on how to support teachers. The salience of such moments also arise from the possibilities of learning they created for the teacher and the researcher. An analysis of the mathematical learning of the teachers and the researcher through such collaboration is discussed in Chapter 7.

4.8 Dilemmas in Studying Practice

The researcher adhered to the ethical considerations such as, anonymity of the participants, permissions for data use, seeking informed consent, etc., during the course of data collection, while analysing data and when presenting findings. In this section, I discuss some of these considerations and the dilemmas they posed while working in the field. This will be followed by a description of how these dilemmas were handled during the course of the study.

4.8.1 Skepticism about videography

At the beginning of the research study, the researcher took permission for data recording from the governing body and the local school authorities. The permissions were followed by an interaction with the participating teachers, where the purpose of the study, the role of the researcher, and the nature of records needed for the study were explained. The teachers were briefed about the nature of data that will be collected from their classrooms and ways in which it will be used by the researcher. During the meeting, the teachers agreed to participate in the study, after seeking clarifications.

During the lesson observations almost all the participating teachers shared their discomfort with the placement of a video camera in their classroom. The stated reasons included distraction among the students, deviation in the teacher's attention, the misuse of the video data by the school authorities or any other stake holder, etc. The teachers were also worried that this data might be used to make decisions about their salaries or career promotion. The teachers' concerns about videography were understandable given how, in general, classroom observations are handled in the Indian context. The classroom observations from inspectors (coming from outside the school system) and senior school leaders has been known to affect teacher's working conditions and salaries. Further, such inspections are particularly judgmental and teachers rarely find them useful (refer Excerpt 4.1).

Excerpt 4.1: Use of lesson plan: Nandini (Y1)

Line No.	Speaker	Utterance
14	R	Okay so you planned this. Is there a lesson plan I could see?
15–19	N	Lesson plan is a one line thing. We submit it everyday when we come to school. There is a register in the vice–principal's office. We have to write there before going to our classes, which all classes we have and what we are going to teach. It has a column for date, day, class, and concept. We mention the exercise or page number that will be done in class.
20	R	But why is it in VPs office? I mean as teachers you would need it, isn't it?
21–22	N	They come to check sometimes, vice princpal sir or principal sir. They check whether what we have written in the plan is what we are teaching in class. Teachers do not carry it to class.
23	R	So do they observe your classes?
24		[N nods in approval]
25	R	After that what is done with those classroom observations?
26	N	This report by vice principal or principal of our classroom observations goes as our CR [Confidential Report]. It does not affect promotion or anything. That is based on a teachers exam. But it goes in the file. It has some impact.

Legends: R – Researcher, N – Nandini

As a researcher, I was aware that no video records of the data of classroom observations might affect the validity of the data. Video records are the closest tools to getting a live experience of the field. The dilemma was of losing the visual data as the teachers were scared of the data being used to judge their performance by the school authorities. Teachers' "informed refusal" (Cohen, Manon & Morrison, 2013) on videography was respected. The decision of not recording against their will was maintained throughout the course of the study. However, teachers agreed to the audio recording of the data in Phase 1. In order to avoid the loss of data, due to no video recording, two researchers stationed themselves in the classroom and observed it from different locations. Thus, part of the data from the first phase of the study, comes from transcripts prepared using audio (and not video) records and researchers' notes. Interestingly, teachers' discomfort with the video records did not stay for the entire Phase 1 of the study. Through pre– and post–lesson discussions, the teachers became cognizant of the interest of the researcher in the issues of classroom practice, and saw a distinction from the inspectorial observations that they were accustomed to. After about 3 months of data collection in Phase 1, teachers themselves proposed video recording of their lessons. Thus, the modes of data collection remained unchanged till the teachers agreed to video recording of their lessons for self–viewing and discussions with other teachers. The sensitive use of data collection methods by respecting the concerns of the participants helped in creating a negotiated space and the emergence of videography for teachers' own use apart from using it for research purposes.

Another way of maintaining teachers' trust on the use of data was by seeking their informed consent for the use of classroom data for specified purposes only. In this research, each individual participant and the researcher signed a consent form (see Section A2.5 of Appendix), with a copy kept with both. The teachers' trust that the data from their classrooms will not be mis–used was helpful in building a relation, beyond that of the researcher as an objective observer and the teacher as a subject of the study.

4.8.2 Reciprocity of the research endeavour

In the plan of the study the researcher's role was that of a non–participant observer. Interactions with the teacher and the students before or after the lesson were planned. The questions asked to different teachers were kept similar. One among the several reasons for keeping the researcher's behaviour and actions similar to all participants, was to reduce the *researcher's bias* and collecting data in a way that would allow for comparison of similarities and differences in the views of different teachers. However, the field work required the researcher to engage with other issues or concerns.

First, the effect of researcher's presence on the natural settings was a concern as teachers repeatedly requested comments on the observed lessons. The request implied an implicit hierarchy in the roles of the teacher (doer) and the researcher (observer). It was difficult to change the dynamics of this space from the researcher giving feedback to the teacher, to the teacher and researcher discussing the teaching. During the course of the study, a focus on the practice helped in blurring the boundaries of this role and creating this space for discussion about students' learning, pedagogies and content to be taught.

Excerpt 4.2: Reema's participation in TRM (Source: Researcher reflections)

> Reema had mentioned some of the experiences from her class in the post–lesson interactions that she would share during the meeting. But in the last few (teacher–researcher) meetings she has not shared these classroom experiences and has been talking very little. So after today's meeting I checked with her about why she is not speaking in the teacher–researcher meetings. She mentioned that her classroom experiences are different from the senior teacher present at the meeting. She found it difficult to challenge a senior high school teacher by citing her experience, assuming that it might be limiting.
>
> She was critical of herself and would not share her experience in these meetings. In this situation, I found myself convincing her about each individual's right in a group or community to share their experience while being respectful to the differences. It seemed that my role as a researcher changed several times in the course of such conversations with Reema and the other teachers.

Second, a hierarchy of roles among teachers influenced their participation in teacher–researcher meetings (refer Excerpt 4.2). Since the purpose of the teacher–researcher meetings was to discuss and share experiences, the teachers were encouraged to talk about their classroom experiences. Several times, the teachers would check before the meeting, whether what they plan to share is relevant or appropriate to be discussed during the meeting. While this was not a planned role of the researcher, helping teachers in explicating their classroom experiences to generate discussions around them was supported through several other conversations (such as that reported in Excerpt 4.2 from researcher's reflection notes).

Third, sustaining teachers' attention to the mathematical aspects of their teaching was difficult. Teachers had other legitimate concerns about the issues that they struggled with (for instance, refer Excerpt 4.3). These issues varied across teachers. In the first few months, the researcher was skeptical in attending to such issues. An engagement with the issues of teaching became a part of the observations as the researcher was immersed in the field. The nature of this engagement varied based on the issues that teachers' shared. One such case has been reported in Excerpt 4.3.

Fourth, there were instances when teachers wanted to read the researcher's observation notes from their and others' teaching. It was difficult to respond to such requests due to concerns like – (a) how would the teacher interpret the researcher's notes, (b) whether it is fair for teachers to see the notes of another teacher's teaching, and (c) what could be the purpose of seeing these notes. While a part of researcher's notes, particularly data on students' utterances was shared with the teachers, it was negotiated that the teacher and the researcher would discuss these notes, as the teacher's time permitted. The researcher denied the request of the school authorities to examine or copy these notes. However, discussions on teaching with teachers and school authorities were done. In retrospect, the decision of teachers taking notes during the teacher–researcher meetings, was perhaps influenced by the teachers' interest in the recorded notes and how they were used.

As Cohen, Manion and Morrison (2013) state, "researchers need to reflect attitudes of compassion, respect, gratitude and common sense without being too effusive. Subjects clearly have a right to expect that the researchers with whom they are interacting have some concern for the welfare of participants" (pp. 59–60). The reciprocity of the researcher in responding to the teachers' mathematical and non–mathematical concerns played a role in gaining teachers' trust in sharing their actual struggles, idea and opinions. The reciprocal relationship with the participants helped them in unveiling their concerns to the researcher. The changing role brought in a new perspective on the nature of learning among teachers. The support that each teacher needed to cross the threshold created by the years of experience, requires a commitment to participate in their struggles and share their successes in the classroom.

Excerpt 4.3: Reema's concern for Agrima (Source: Researcher reflections)

Initially, Reema has been very conservative in talking about her lessons (even in pre– and post–lesson discussions). She did not talk about her students. Her interactions with the researcher in Phase I were mostly about the new and the old textbooks. One of the students in her class, Agrima (pseudonym), was a single parent child. Her father was a cleaner at the nuclear facility and had passed away due to severe exposure to radiation. Her mother managed to keep them going by doing cleaning chores at other households. The day when Agrima opened up to Reema about the reasons for leaving this school as her mother could not afford the school expenses, Reema was very disturbed. She invited me to walk with her out of the school. While we walked outside the school premises for an hour, she expressed how she felt helpless. We brainstormed some ways in which she could help in supporting Agrima's education in the school. The discussion around this student and how Reema could help her, without compromising on her professional identify, made us both, more open to each other. It was after this incident, that Reema began discussing her students' work and her dilemmas while teaching before and after her lessons. She would also sometimes call the researcher in the evening to discuss the plan for her next day's lessons. As a researcher, I realised that it is important to engage genuinely with the issues that concern our participants. This conversation revealed to me that Reema, unlike her popular perception of scolding children for not studying well and getting angry with them, was concerned about their education, and in ways which she found difficult to articulate with anyone at school.

4.8.3 Boundary crossing and researcher's positioning

The researcher's positioning in the study, varied at different phases of the study. In the beginning, the interest of the researcher was in gathering information about teaching, and designing and implementing tasks, which would inform teachers' practices. Through interactions with the teachers, the researcher's involvement became more of a participant observer. In the later stages of the study, the researcher became a co–planner of lessons or worksheets with the teacher, or co–taught with the teachers in the classroom. The researcher's role became varied in terms of updating teachers about the research in the field of mathematics education, clarifying the propositions of the reformed curriculum for mathematics, examining some online resources to be used for teaching, helping in completion of some non–academic work, designing a question bank along with all the teachers for use in classes, etc. All these roles were unanticipated at the beginning of the research.

The reflexive positioning of the researcher helped in building rapport with the teachers and helped gain entry into the teachers' professional lives. The mutuality of the developing relation between the teacher and researcher led to the changing interactions between the teachers and researcher. For instance, in the beginning of the fieldwork, the pre– and the post–lesson interactions between the researcher and individual teachers involved reflection on classroom teaching and were usually initiated by the researcher. The teachers later adopted this practice and took the role of discussing classroom teaching, sometimes demanding the presence of the researcher. The initiation of discussions by individual teachers is pertinent also because it marks a major shift in their perception of the researcher's purpose in the school.

As noted earlier (refer Excerpt 1.1) the school teachers viewed the researcher(s) as an outsider to observe their lessons as an authority and as someone who will judge their teaching. An interest in the teachers' practices and challenges, and participation in the teachers' everyday activities opened up a hybrid space for the teacher and researcher to interact. Similarly, teachers took initiative of bringing artefacts from their practice to the meetings with the other teachers and researchers.

The interactions between the teachers and researcher constitute an important component of the study as it provided a space for teachers to talk about their teaching with an interested other.

4.8.4 Waiting for the right time!

It has been acknowledged that the nature of inquiry in qualitative research is dependent on "fluid situations and changing rather than static events, behaviours evolve over time, and are richly affected by the context – they are situated activities" (Cohen, Manion & Morrison, 2013, p.22). The researcher needs to patiently wait for such behaviour to unveil itself in the naturalistic settings, rather than creating triggers through explicit intervention. In the beginning, all the teachers resisted any interactions with the researcher outside the classroom. Several reasons could explain such a response. First is the genuine lack of time since the teachers were shifting from one classroom to another, after almost every lesson. Second, not having the vocabulary to talk about teaching, since such discussions had not happened before. Third, could be a lack of interest in talking about their teaching with an outsider. This was followed by a phase where teachers engaged in brief conversations with the researcher on request. The focus of these conversations was mostly to gather some information about their colleagues' work. For instance, almost every teacher expressed an interest in knowing what was happening in the other teachers' class and how much of the content was completed. The other areas of interest were – the number of mistakes made by the students of different classes, nature of exam questions if one of the participating teachers was assigned the responsibility to make the final question paper, whether the notebooks were corrected in time, etc.

The researcher's response was to encourage them to ask these questions to each other and direct their attention to the student utterances arising from their own classroom. Gradually, the quality of these conversations changed and teachers started discussing about the way content was dealt within the textbook, challenges faced in handling social justice situations depicted in the word problems, difficulties faced by individual students, and seeking justifications for why algorithms work. With time, teachers also became more comfortable talking to each other to identify shared struggles in teaching specific topics, and ways in which they handled specific issues in classroom. Adler (1998) remarks, becoming a mathematics teacher involves learning to talk both within and about mathematics teaching and learning, rather than simply learning new knowledge.

In short, the field work required seeking permissions, building a rapport with the participants and the related others, convincing them of the need for a space for discussion, working with the teachers' busy schedules and non–teaching responsibilities, navigating between different bureaucratic and other conflicting interests, organising meetings, etc., and amidst these creating a discourse for reflecting on the mathematics being taught in the classroom. While it is recognised that qualitative research is time–consuming, as the researcher is expected to wait for the phenomena to occur in the natural settings, some structural or systemic issues need attention in reducing the time spent on several activities during fieldwork in schools. A step in this direction could be partnership between school and research institutions.

4.8.5 Focus of the researcher

In order to understand teaching practice, the focus of discussions with the teachers were contextually situated. An often raised critique by the teachers of the workshops organised for their professional development has been the use of alienating classroom situations – with classrooms which are fully lit, have all the resources in place, no classroom management issues, students and teachers are committed to listening to each other, etc. Teachers find these situations alienating when compared with their work place, which are more complex and dynamic. The decision to unpack teacher knowledge from the practice perspective was guided by – the theoretical considerations that teaching needs to be understood as a social practice, and the practical considerations of engaging with the complexities of actual classroom teaching. It was decided that the study would be located in the school setting, while teachers will have opportunities to visit the research institute from where the researcher was pursuing her doctoral research. The location of the study in the school is common to other countries like South Africa, Germany, etc., but it was a difficult choice to make in the Indian context. The issues of situating a study in the classroom from the research perspective are mostly linked to the time spent and access to issues of research interest. These include waiting for the issues of research interest to emerge or happen in the naturalistic setting, ensuring acclimatisation of research participants to the tools used for data collection, filtering the naturally occurring noises in the school premises from the data collection through video and audio recorders, delays due to unanticipated events in classroom or school, etc. Some specific instances where such issues were experienced by the researcher were – the participating teacher being given a substitution period, observer asked to teach a lesson in the absence of the teacher, a teacher asked to teach two classes at the same time due to shortage of staff, a teacher struggling to find time to teach mathematics with other responsibilities like preparation for annual day, collection of school fee, passing administrative information, shortage of electricity, fluctuating attendance during heavy rains, etc. These observations also indicate the school context in which the study was located.

The different goals of a teacher and a researcher might conflict. Teachers are burdened with the routine responsibilities and discussions about teaching are a small subset of their work. A researcher’s primary focus is however on magnifying and analysing parts of teacher’s work while keeping the larger context intact. The conflicts in the role of teachers and researchers arise from a lack of space in the system and of a vision of the roles that teaching and research play in the process of knowledge generation. The focus of the research on teaching needs to acknowledge this structural limitation and challenges arising from such conflicting goals.

Chapter 5

KNOWLEDGE DEMANDS IN TEACHING DECIMALS

Where might we begin to identify the elements of practice that need to be included in an analysis of this teaching? How can these elements be related in a way that captures the complexity of the work? How can we analyse practice in a way that will improve our understanding of the problems involved in doing teaching, of the resources teachers can use to address those problems, and of the work entailed in using those resources. (Lampert, 2001, p.27)

5.1 Abstract

Existing frameworks of teachers' knowledge required to teach mathematics do not adequately capture the dynamic aspects of knowledge manifested in teaching practice. In this chapter, I examine the knowledge demands that arise in situ, in the course of two teachers' listening and responding to students' thinking, while teaching the topic of decimal fractions. It is described how contingent classroom situations pose challenges to the teachers, through an analysis of "paired episodes", that is, episodes of classroom teaching of the same sub–topic by the two teachers in two consecutive years, with significant differences in their responsiveness to students' thinking. The topic–specific knowledge demands posed on the teachers, as students make connections between their prior knowledge of whole numbers and fractions with their emerging understanding of decimal fractions, are explicated. It is argued that an abstraction of knowledge demands, from a close study of practice and reflection on them, can be used to unpack the complex knowledge required by teachers to teach responsively.

5.2 Knowledge Demands

Before stating the central questions for this chapter, let us recall how knowledge demands are defined and the relevance of this construct to analyse teaching practices. Knowledge demands refer to the mathematical challenges faced by the teachers while teaching a specific topic. These knowledge demands could arise from the difficulty in supporting or challenging students' ideas, bridging students' ways of thinking and the content, using multiple and relevant representations, appreciating the mathematical deviations made by students, designing variations in problems, and so on.

The two case studies, discussed in this chapter, are of teachers who were making a transition from more traditional to student–centred or responsive teaching. Particular knowledge demands become more visible, when teachers are in transition. Such demands are especially significant in moments when teachers deal with the "contingencies" that arise in the classroom (Rowland, Huckstep & Thwaites, 2005). Contingent moments arise from an unanticipated student question or observation, or sometimes through a connection that the teacher makes between the mathematical ideas in play. The contingencies place demands on teachers' knowledge, as a teacher needs to evaluate whether these moments can be converted into learning opportunities (Rowland & Zazkis, 2013).

Thus, the focus is on the dynamic nature of knowledge demands that arise in the course of teachers' listening and responding to students in contingent classroom moments, while teaching decimal numbers. The construct of knowledge demands helps in capturing the knowledge *in situ* and offers a framework to analyse contingent moments of teaching by inviting teachers, researchers and the mathematics education community to deliberate on the richer knowledge base that underlies the teaching of specific topics. An abstraction of these knowledge demands can provide suitable entry points for teacher development and support.

5.3 Central Questions

This chapter presents an analysis of how teachers dealt with the contingencies arising in the teaching of decimal numbers in two consecutive years, in a more traditional and a more responsive way, respectively. The knowledge demands posed on the teachers as they decided to respond to contingencies while teaching in the second year are examined.

The literature on knowledge demands underlying responsive teaching, discussed in (Section 2.4 of) Chapter 2, emphasises that the teacher must be able to anticipate pathways of student movement from previous knowledge to new knowledge (Doerr, 2006). In the context of decimal learning, this suggests that the teacher needs knowledge of the connections of decimal numbers with the base ten structure gained from whole number thinking, with the measure and part–whole sub–constructs of fractions, and with equivalent fractions. The points of connection are several, and connections can be made in multiple ways in the classroom. It is expected that a close study of classroom enactments will reveal the nature of knowledge demands made *in situ,* which underpin teachers' moves and actions. Thus, the questions asked are:

(a) What are the knowledge demands posed on the teacher, who is teaching the topic of decimal numbers, when she/he is less and more responsive to students' thinking?

(b) How does an analysis of such knowledge demands enrich our understanding of the specialised knowledge for teaching mathematics and what implications does it have for the acquisition of such knowledge?

5.4 Data Collection and Analysis

The case studies of teaching of Nandini and Reema, two of the four school mathematics teachers who participated in the study, are used in this chapter. To recall, Nandini had a master's degree in physics (with mathematics as a subsidiary subject) and had been teaching mathematics and physics to students from Grades 6–10 for over 10 years. Reema had a bachelor's degree in science and education, and had been teaching mathematics and environmental studies to students from Grades 1–5 for 20 years (refer Section 4.5.3 of Chapter 4 for details). The data used is from the lessons on decimal numbers taught by Reema and Nandini in Grades 5 and 6 respectively, for two consecutive years. The number of decimal lessons taught by Nandini and Reema in the two years are listed in Table 5.1. Transcripts of decimal lessons were prepared using researcher notes and audio or video records. A part of the audio data from the first year was transcribed and supplemented with visual details using the (two) researchers' notes. The remaining data from the first year and the data of classroom observations from the second year were transcribed using video records with additional details from researcher notes.

Table 5.1: Data from Reema and Nandini's classroom

	Year 1		Year 2	
	Number of Decimal Lessons	**Number of students**	**Number of Decimal Lessons**	**Number of students**
Reema	12	11 Girls, 22 Boys	17	10 Girls, 24 Boys
Nandini	12	15 Girls, 19 Boys	20	14 Girls, 20 Boys

Decimal lesson transcripts were analysed at two levels – coding of the lessons and the comparison of paired episodes from the two years of teaching. The decimal lessons were paired based on the similarity of sub–topics discussed, that is, had paired episodes (refer Section 4.7.1 of Chapter 4 for details). Table 5.2 captures the details of the paired lessons for the two cases. Three decimal lessons[2] were randomly selected from all the four teachers' teaching and coded to check for the consistency of the claims made in this chapter.

Table 5.2: Paired lessons for analysis

Paired Lesson	**Lesson Code**	**Paired episode**	**Duration** (Hr.:Min.:Sec.)
Nandini			
1	Y1DL1	Place value in whole and decimal numbers	00:40:00
	Y2DL2	Why are there no oneths?	00:39:48
2	Y1DL3	Position of zero in a decimal number	00:35:03
	Y2DL5	Place value of zero in relation to its position	00:29:07
3	Y1DL5	Conversion of length measurement units	00:33:33
	Y2DL7	One division after one	00:35:50

[2] These three lessons from Nandini and Reema's teaching were different from those reported in the thesis.

Paired Lesson	Lesson Code	Paired episode	Duration (Hr.:Min.:Sec.)
Reema			
4	Y1DL1	Length measurement to introduce decimals	00:30:07
	Y2DL2	Length of the frog context	01:10:27
5	Y1DL2	Place value of decimals	00:37:19
	Y2DL4	Place value of decimals	00:48:02
6	Y1DL3	Fractions and decimal numbers	00:42:47
	Y2DL6	Fractions, decimals and grid representation	01:10:00
Legends: Y – Year, DL – Decimal lesson. The number after each legend indicates the position of the lesson in the sequence.			

The analysis sections (5.5 and 5.6) are organised case–wise. Section 5.5 is the case analysis of Nandini's teaching. It begins with a brief about her teaching, followed by a discussion of the patterns that emerged in coding of the 3 paired lessons (Section 5.5.1) and the analysis of paired episodes from these lessons (in Sections 5.5.2 to 5.5.4). These episodes reveal how Nandini responded to the contingent classroom situations differently in the two years of her teaching. The episodes are located in the context of the lesson in which they appeared. The second level analysis also takes into account the teaching decisions made by Nandini, classroom practices which supported learning, and the explanations built for mathematical statements during classroom discussions. In a similar way, Reema's case study has been analysed in Section 5.6.

The case of Nandini and Reema's teaching was selected as their classroom teaching offered "maximum variation" in the 2 years. The changes were broadly in the choice of representations, listening to students' ideas and making attempts to understand them, anticipating and reflecting on students' thinking, and making decisions in the classroom based on students' ideas. The variation in teaching practice allowed us to get a sense of the difference in knowledge demands made on the teacher as their practice became more responsive.

5.5 Case Analysis of Nandini's Teaching

Nandini had never taught using the old NCERT textbooks but she seemed to be aware of them. In the first year of the study, she mentioned using the new textbooks extensively, except in the first lesson for every topic. In her words, she began teaching every topic with an "interesting context", for which she would "search on the internet", in order to familiarise students with the "application of learning the topic". While this was corroborated through observations, Nandini was not found to be using this application context after the introductory lesson anytime during the teaching of that or any other topic. Nandini expressed that although she understands that some students need more attention from her, she was pressed for time, to complete the syllabus, and therefore could not afford to provide the support needed. Significant differences were noted in Nandini's teaching in Y1 and Y2, elaborated in the following sections.

5.5.1 Teaching in Year 1 and 2

Broad changes in Nandini's teaching practice are characterised using a comparison of the paired lessons, that is, lessons which focused on similar sub–topics from the 2 years. Table 5.3 indicates the frequency of a few selected codes in three pairs of lessons across 2 years of teaching. The selected codes are related to students' and teachers' explanations, and teachers' response to students utterances. The three pairs of lessons focused on introduction to decimal numbers (Y1DL1 and Y2DL2), representing decimal numbers using a place value table, writing decimals in words, and expressing fractions and expanded forms as decimals (Y1DL4 and Y2DL5), and conversion of decimal and fractions, and between measurement units – centimetre and millimetre (Y1DL5 and Y2DL7).

Table 5.3: Frequency of select codes in Nandini's (paired) decimal lessons

	Code	Y1DL1	Y2DL2	Y1DL4	Y2DL5	Y1DL5	Y2DL7
1	TE (Teacher Explain) – tell	15	13	13	13	15	9
2	TE – procedure	8 + 1*	4	9	5	26	8
3	TE – justify	2	7	1	10	2	14
4	TR (Response) – evaluate	17	7	3	3	11	5
5	TR – restate	9	18	4	19	9	22
6	TR – expand	0	7	3	14	2	11

	Code	Y1DL1	Y2DL2	Y1DL4	Y2DL5	Y1DL5	Y2DL7
7	TR – argue	0	1	0	0	0	1
8	TR– public think	0	3	0	4	0	6
9	SE (Student Explain) – one word	42	26	31	74	38	54
10	SE – erorr	15	0	0	7	15	1
11	SE – procedure	0	3	0	3	3	7
12	SE – justify	0	5	0	14	0	19
13	SE – observe	1	12	0	10	1	13
14	SE – completes TE	1*	8	0	6	0	1
15	SE – adds SE	1	7	0	28	2	6
16	SE– argue	1	1	1	3	0	2
17	SE – evaluate	0	1	0	8	0	5
Legends		* indicates an incorrect response, T – teacher and S– student.					

A comparison of codes suggests that there was an increase in the use of justifications by Nandini (Row 3, Table 5.3) and the students (Row 12) from Y1 to Y2 and a decrease in procedural explanations provided by the teacher (Row 2). In the first year, Nandini usually began with an explanation of the procedure to solve a problem and the students were expected to follow the procedure to solve similar problems. For instance, for conversion from decimals to fractions, students were expected to memorise the rule that the number of digits after the decimal point must be the same as the number of zeroes following “1” in the denominator of the fraction. In the second year, in contrast, students were encouraged to provide reasons to justify the steps of the procedure, and once the procedures were established, the class used them as reasons for other statements. For instance, students were encouraged to use their knowledge of the meaning of equivalent fractions and of conversion from decimal number to fraction as justification for comparing decimal numbers. While these patterns revealed Nandini’s increased focus on justifications and explanations, there were new practices observed in the class in the second year. I noted Nandini’s use of revoicing, that is, restating a student’s statement (Row 5), posing it for public thinking in class (Row 8), a decrease in the overt judgments of students’ responses as right or wrong and efforts at probing their thinking (Row 4). A sustained dialogue (over several turns) with a student to probe the student's thinking and calling everyone's attention to mathematical statements requiring examination or emphasis were often seen in the lessons. Further, there was an increase in the quality of students' talk – providing justifications, asking questions, and adding to or evaluating each others' responses (Rows 15–17). Nandini encouraged students to share their observations about the mathematical objects being discussed (Row 13) and provided scope for students to articulate and justify the incorrect responses (Rows 10, 12). There was also an emergence of the usage of ''because'', ''if...then...'', and ''so'' constructions in the second year.

I now analyse the paired episodes from Nandini's teaching, to identify the knowledge demands placed on her, when teaching specific ideas in classroom. These episodes are taken from the lessons which have been compared in Table 5.3.

5.5.2 Place value in whole and decimal numbers

In each year, Nandini began the teaching of decimal numbers by connecting it with students' knowledge of the place values of whole numbers and extending this knowledge. In this section, the focus is on how the relation between place values of whole numbers and decimal numbers was dealt with in each year.

5.5.2.1 Year 1: Place values in whole numbers and decimal numbers

In the first lesson on decimals in Year 1 (Y1DL1), Nandini began by asking students to guess the length of a duster. After listening to the students' estimates of 12, 15, less than 10 cm, etc.; Nandini measured the length using a ruler as 17.5 cm. She introduced a decimal number as the number where a (decimal) point is used. She explained that, ''point five is a part of full 1 cm'' and the decimals are used ''when there is no full 1 cm''. The length estimation task was followed by a discussion on the cost of half, quarter and half of a quarter litre of milk. Nandini helped the students to write the cost for fractions of a litre of milk, on the board, using decimal notation: Rs[3]

[3] Rs is used to denote the Indian currency, namely, Rupees. 1 rupee is equal to 100 paise.

10.50, Rs 5.25. Next, Nandini drew a 5×2 rectangular grid to show the fractions $\frac{5}{10}$ and $\frac{2}{10}$, respectively. Students responded with "same as half" and "point five" for $\frac{5}{10}$ and with "two by ten", "point two" and "ten point two" for $\frac{2}{10}$. Nandini did not respond to these student utterances and shifted the discussion to the place value of digits in the whole number 256 as an introduction to "how to write a decimal number". She explained that the decimal place values are written like whole number place values "but with different words" (example "tens" and "tenths"). A student then asked, "what is oneths?" to which another student responded that it does not exist. The student's question was not taken up by Nandini for discussion and remained unanswered. Next, Nandini discussed place values in a decimal number 0.256. She explained that the places are counted from the "left to right side", and named the first place to the right of decimal point as tenths. She called it "one by ten, one by tenth [sic]" ("one by ten" refers to the fraction "$\frac{1}{10}$" or "one divided by ten"). Similarly, she named "one by hundredths and one by thousandths [sic]". Then, Nandini returned to the question of writing $\frac{2}{10}$ as a decimal number by stating the rule that a zero in the denominator means one digit after the decimal point. She wrote "$\frac{2}{10} = 0.2$ and $\frac{5}{10} = 0.5$" on the board. She did not explicitly connect the rule to the just concluded discussion about place values in a decimal number. In the remaining part of the lesson, Nandini asked the students to draw a 10×10 grid. The grid was used to show fractions with denominator hundred. In the post–lesson discussion with Nandini, she explained the rationale for her lesson as follows (refer Excerpt 5.1).

Excerpt 5.1: Post-lesson reflection: Nandini (Y1DL1)

Speaker	Utterance
Nandini	I generally try to begin [a topic] with an introduction. I mean, I tell them [students] the need for that concept, why they are learning it. Like decimals, it is like telling them the practical applicability of that concept. Today fractions and decimals was something I wanted to teach. Fractions like half, one by four, one half, quarter, and half of quarter, they know. So beginning with that creates curiosity.

In this lesson, Nandini used her knowledge of the connection of decimal numbers with measurement, fractions and whole numbers, to build on students' prior knowledge gained both in school and in informal contexts. Nandini recalled the part–whole meaning of fractions and used an area representation to introduce the decimal notation for fractions with denominator ten. She sought to extend students' knowledge of the place names in a whole number by introducing the place names to the "right side of the decimal point". She emphasised the difference between the names, that is, tens and tenths. However, the correspondence drawn between the whole number part and the fraction part of the decimal number conveyed an implicit "mirror metaphor" (MacDonald, 2008). In this case, placing of a mirror at the decimal point was implicit in the phrases used by Nandini "left to right side (of decimal point)" and the student's question about "oneths". The lack of attention to the student's question about oneths and another student's comment that, "oneths does not exist" suggests that Nandini did not notice the importance of the question for learning about decimal place values. Although it is possible that Nandini did not hear or attend to these student utterances, the overall differences in her approach to teaching in the 2 years suggest that these were a matter of her teaching decisions. I conjecture that an awareness of the underlying student thinking, that is, identifying its source helps the teacher in noticing the mathematical potential of a student's utterance and linking it with the concept. Nandini referred to the connections between the place value names of whole numbers and decimal numbers, and fractions as representations of decimals. However, she did not connect these two pieces of knowledge to build a justification for why a distinct place for oneths does not exist or why a digit after the decimal point corresponds with a zero (ten) in the denominator when written as a fraction. In this lesson, the use of linear (measurement context) and area (5×2 and 10×10 grid) representations to show decimal numbers was also noted. The connection between these two representations was missing.

5.5.2.2 Year 2: Why are there no oneths?

In Y2, as in Y1, Nandini began teaching decimal numbers using the length measurement context. In the first lesson, she asked students to measure any five objects from their surroundings and write the measure precisely. This lesson was not observed by the researcher but was described by Nandini. She reported that students expressed the measures as "half, half of half, point five, more than point five". Nandini diagnosed that students understood that a decimal number is made up of a whole number part and a fraction part. In Lesson 2 (Y2DL2), she said that the decimal point "separates the whole number side and the fraction side". She wrote 7.39 on the board and asked students to identify the place value of each digit starting from the left. Students identified the place value of 3 as "three by ten" and "tenths". At this juncture, a student asked "*Ma'am, oneths kyun nahi hota?*" [Teacher, why are there no oneths?]. Another student contested this statement by saying that oneths exist. Nandini listened to these two students and revoiced the question to the whole class. She said, "What she is asking, when you write a [whole] number we start with ones place. Ones place, tens place, hundreds place, thousands place, but here just after the decimal [point] we started with tenths place, one by ten. Why there is no oneths place after the decimal (point)?" Nandini's revoicing of the student's question about oneths indicates an awareness of the mirror metaphor or at least of the problem that it causes. Students provided different explanations while agreeing or disagreeing that oneths exist. In the process, Nandini sought clarifications, asked questions, and offered counter–arguments. The transcript of the first explanation constructed by two students is reproduced below (see Excerpt 5.2). Code switching between Hindi and English language was common in the classroom. The researcher has translated the utterances from Hindi to English.

Excerpt 5.2: Explanation for oneths - I (Y2DL2)

Line No.	Speaker	Utterance
49	GSt2	*Ma'am samjha. Iske andar jab karenge na to one ka part one hi rahega.* (Teacher, I understand. In this when we do [partitioning], a part of one will remain one.)
50	BSt1	No Ma'am. There is no oneths place in the decimal part.
56–57	BSt2	Ma'am, because one is a whole number and tenths means starting with ones, this whole number [one] has ten parts. And tenths here means three tenths, as three is in the tenths position [in 7.39]. So 3 parts of one whole.
58	GSt2	No. Three [times] one–tenth of a whole.
60–61	BSt2	Ma'am, ma'am, one there is a whole ma'am and then there is a tenths place because ones there is one whole and one part, and tenths means one whole has ten parts in decimal.
Legends used: G St – Girl student, B St – Boy student		

The students' justification used the relation between consecutive place values to state that one–tenth of ones is tenths. They also used their knowledge of partitioning to conclude that partitioning a whole into one part will leave the whole intact. Two other students provided a slightly different argument (see Excerpt 5.3).

Excerpt 5.3: Explanation for oneths - II (Y2DL2)

Line No.	Speaker	Utterance
64	GSt3	Ma'am, first we write tenths as one by ten, but we cannot write oneths as one by one.
65	T	Why not?
66	GSt3	Ma'am, because it is a whole.
67	BSt3	It is a whole number.
68–69	T	Whole number, okay okay. What she is saying is what you have studied in primary class.

The student (G St 3 in Excerpt 5.3) extended the meaning of tenths as one by ten ($\frac{1}{10}$) to infer that oneths means "one by one", which is a whole number and not a fraction. By convention, its position is to the left side of the decimal point. After listening to different students' explanations, Nandini offered an argument based on contradiction. She began by assuming that oneths exists as a distinct place value and then rejected it.

Excerpt 5.4: Explanation for oneths - III (Y2DL2)

Line No.	Speaker	Utterance
71–72	T	Tenths place. Now S [G St3] is saying if I write another number, let us have a number [i.e. digit] here say "2" in oneths place. Hmm?
73	GSt3	Oneths is ones.
74–76	T	Hmm. If it is oneths, then you should write it as two by one. 2 7.^39. It [points at 2] is oneths, how should you write it? Two by one
77	GSt4	How?
78	GSt3	Because it is oneths [emphasizes–ths].
79	T	So does it become a decimal or a whole number [part]?
80	GSt4	Whole number.

Line No.	Speaker	Utterance
81	T	Two by one is a whole number so it should be a part of the left side, not of right side. Right side is all parts, divided by ten, divided by hundred, like that. That's why you say tenths [emphasis on– "ths"]. Tenths place, hundredths place.

As noted in Excerpts 5.2 and 5.3, students extended the definition of tenth as a part obtained by partitioning the whole into ten equal parts, or equivalently as the fraction $\frac{1}{10}$, to define oneth. Since oneth was just one part, it was the whole. Nandini supported the students in recasting this argument by applying it to a particular instance (refer Excerpt 5.4). She supposed that there was a distinct place for oneths, and went on to show that this would be the same as ones, and hence placed on the left of the decimal point. In all the arguments, the conclusion was that, there is no distinct place for oneths, because oneth corresponds to a whole and not a fractional part of the whole. In her explanation, Nandini used the arguments offered by the students – definition of oneths derived from the definitions of tenths and hundredths, and decimal point as a separator for the whole and fractional part. A few students maintained that oneths exist, explained that oneths is the same as ones, and used the two synonymously throughout the lesson ("seven by one or seven times one"). Nandini tried to persuade these students that they should say "ones" and not "oneths" since "7 times 1 is different from 7 by 1" and "oneths would be parts, while ones is a whole". Following this discussion, students extended the place value names for the fractional part, by using "hundredths, thousandths, ten–thousandths, lakhths, ten–lakhths, and so on". (In the Indian number system, a "lakh" is one hundred thousand; "million" is not used.) The class discussed place values of different decimal numbers and placed the digits of each number in a place value table. A student asked Nandini whether there should be a position for the decimal point in the place value table. Nandini emphasised that the decimal point acts as a separator and does not have a place value in a decimal number. After the lesson, Nandini talked about her observations from the lesson (refer Excerpt 5.5).

Excerpt 5.5: Post-lesson reflection: Nandini (Y2DL2)

Speaker	Utterance
Nandini	It was different today. They [students] came up with thousandths and ten–thousandths themselves. In fact, a girl was asking that there can be lakh–ths and ten–lakhths also. They were making connections and extending it.... Apart from place value, I wanted to do comparison also. But then I was expecting them to be thorough with it. I found that they were not. But place value is extremely important. So I decided to take it up completely.

5.5.2.3 Knowledge demands

What can be inferred about the knowledge demands from the descriptions and excerpts of the episode presented? As in Y1, Nandini drew on the analogy between the place values of whole number and decimal numbers to introduce the decimal numeral notation, in Y2. However, in this year, she responded to the student's question about the existence of oneths by revoicing the question to the class and generating a discussion. She not only listened to the student but also listened in a manner that was responsive. The responsive nature of listening is seen throughout the discussion as it is carried to a conclusion. This entails firstly, an appreciation of the significance of the student's question, which is an understanding of how the analogy between whole number place values and decimal fraction place values can mislead. Secondly, she evoked a discussion that led from the student's question to a conclusion that is adequate in answering the question. To successfully manage the discussion in this manner, Nandini needed not only to identify and support threads that moved the discussion towards a conclusion, but also continuously evaluate if the conclusion is within reach, recognise the conclusion as it emerges, and bring the discussion to a closure. From the teacher's moves I infer that, *en route* to the conclusion, she listened to what the students were saying, continuously evaluating whether their statements (i) were accurate, (ii) based on knowledge shared by other students, or (iii) based on definitions that have been accepted, and (iv) led towards the desired conclusion and closure. At times she made such evaluation explicit as when she said, "what she is saying is what you have studied in primary class" (lines 68–69 above). Staples (2017) argues that the challenges of teaching get amplified as a teacher proactively tries to create a common ground in a classroom with students of different mathematical backgrounds and learning styles.

The first aspect of the teacher's response – appreciating the significance of the student's question – is a form of interpretive listening (Davis, 1997). Such listening is already supported by rich knowledge and anticipation of student thinking. In the second aspect of the teacher's response – leading the discussion to a conclusion – knowledge is brought into play more explicitly. In making evaluations of the students' responses and in formulating her own responses, the teacher is called to draw upon topic–specific knowledge that is thick in terms of key ideas (definition of oneths, decimal point as separating whole number and fractional parts, etc.) as well as connections to other topics that the students already know (equipartitioning, fraction notation, place value names, etc.). Some of these ideas allowed students to progress further than the teacher expected. For example, the students extended the definitions to smaller decimal fractions (hundredth, thousandth, etc.), a response appreciated by the teacher in her remarks to the researcher. The teacher is aware of these definitions and supports students in using these definitions in an emerging argument. In other words, the knowledge demand concerns knowing the potential ways in which this piece of knowledge could be used in supporting an argument or an explanation.

It is noted that the pieces of knowledge that are manifested in the discussion, such as particular definitions of tenths, are specific to the situation. More generally, the knowledge demands that are manifested in specific episodes are those that are related to the specific concept or question that is focused. Hence, particular enactments cover only a portion of the knowledge map related to the specific concept. Reflection on the episode might uncover further portions of this knowledge map, and other directions that the classroom discussion might have taken. To illustrate this, let us examine why students are led to ask the question about oneths.

The teacher's references to the left and right side of the decimal point introduced an implicit metaphorical mirror located at the decimal point. While this is of some use in identifying the place names on either side of the decimal point, it suggests the presence of oneths. The correspondence between place names in the fraction part and in the whole number part is partly a matter of convention, but the fact that no distinct place exists for "oneths" is due to the underlying relation between consecutive place values. The base ten structure binds the place values in a relation of (positive and negative) powers of ten around the basic unit or "ones". All place values to the left and to the right of the units place refer to the basic unit. To the left, there are multiples of the basic unit (in terms of positive powers of ten); to the right there are fractions of the basic unit (with negative powers of ten). Expressing the place value in terms of powers of ten shows that the unit's place is the point of "symmetry" $\{..., 10^3, 10^2, 10^1, 10^0, 10^{-1}, 10^{-2}, 10^{-3}, \ldots\}$. This manner of clarifying the multiplicative relation between place values relocates the mirror at the "ones" place. Thus, the teacher might have chosen to lead the discussion towards recognising that the point of symmetry and the location of the mirror is the "ones" place and not the decimal point.

A sense of the landing place of the discussion, that is, the statement or explanation that would bring the discussion to an adequate closure, and the knowledge entailed in managing a discussion towards such closure, is an important component of the knowledge demands made on the teacher. There are potentially different mathematical ideas that can be utilised to provide a justification for the non–existence of oneths, including, relocating the mirror metaphor to the units place as a reference point, a weakening of the mirror metaphor by focusing on place values, and relation with the fraction notation.

I believe that Nandini implicitly recognised the mirror metaphor. Reflection on the episode could make this metaphor explicit, leading to deeper understanding of the affordances and limitations of the metaphor, including the mathematical understanding that the proper location of the "mirror" is not the decimal point, but the ones place. In this sense, although the knowledge that emerged in the particular episode is partial, it contains the possibility of elaboration to acquire deeper knowledge and understanding, which in turn, can lead the teacher to be better prepared to anticipate and listen to the students' utterances. This has implications for the role of reflection on the details of classroom interaction for the strengthening of knowledge of content and students required for teaching, which I will return to later.

Responsive teaching involves understanding how students' existing knowledge interacts with new knowledge. For the topic of decimal numbers, this involves a deep understanding of how students' knowledge of whole numbers interacts with the learning of decimal numbers. A piece of this knowledge concerns understanding the affordances and limitations of the mirror metaphor. The classroom interaction involved constructing an argument that showed the limitations of the mirror metaphor. In the episodes discussed above, Nandini interprets the decimal place values using the fraction notation, which is helpful in supporting the students' argument. The specific episodes do not open up other issues identified in the literature about the interaction between students' knowledge of fractions and decimal numbers (for instance, $1.2 = \frac{1}{2}$; Steinle, 2004) since these did not emerge from student actions in this classroom.

In the next section on the placement of zero, I discuss another piece of knowledge that involves the interaction of whole number and decimal number knowledge. It also involves an understanding of how students' fraction knowledge interacts with the learning of decimal numbers, and misinterpretation arising from relating string length in a decimal fraction to size.

5.5.3 Position of zero

Besides place value names, Nandini dealt with another concept connecting decimal numbers with whole numbers, which was the effect of the position of zero in a decimal number. Although there were several occasions in the two years when discussion on the placing of zero emerged, in this section, some representative episodes are discussed.

5.5.3.1 Year 1: Position of zero in a decimal number

In the 12 lessons on the teaching of decimals in the first year, the relation between zero and place value emerged in five different lessons. It emerged for the first time in Lesson 4 (Y1DL4), as students worked on the task of writing "one hundred and two ones" as a numeral. Nandini asked students for the place value of each digit and wrote "102" on the board. A student stated that the answer should be 102.0 "because they [the textbook] have said write in decimals". Nandini accepted the response and changed the answer to 102.0, but gave a different reason from the student. Her response was, ‘‘Yes. So there are no tenths. I will write point zero’’. She explained that zero can be put at the place for the missing tenths. I will examine the discussion on the same task in Y2 later. In another episode in Lesson 5 (Y1DL5), while expressing 2 mm as centimetres, students stated that $\frac{2}{10}$ is ‘‘point two’’. Nandini revoiced the students’ response by saying, ‘‘zero point two centimetre’’, emphasising the zero and the unit. She went on to explain that, ‘‘2 mm is 2 divisions out of 10 divisions’’ and equals 0.2. A student, who probably noticed that Nandini said ‘‘zero point two’’ when all the students said ‘‘point two’’, asked, ‘‘Teacher if we don’t put zero, it’s wrong?’’ Nandini said ‘‘no’’ and moved on to the next problem.

In an episode from Lesson 8 (Y1DL8), students worked on tasks of writing the decimal number for a given expanded form, such as, $23 + \frac{2}{10} + \frac{6}{1000}$. Nandini explained that the missing place needed to be filled by zero, which triggered a discussion among some students (see Excerpt 5.6).

Excerpt 5.6: Position of zero (Y1DL8)

Line No.	Speaker	Utterance
44	T	After ten, hundred is not there [refers to $\frac{0}{100}$]. So it is equal to $23 + \frac{2}{10} + \frac{0}{100} + \frac{6}{1000}$ [Teacher waits for students to copy].
45	BSt1	One zero six.
46	BSt2	Aaahh two zero six.
47	GSt1	Teacher if we don’t put zero, then the value would change.
48	BSt3	Yeah, the value will change.
49	T	[to whole class] It is the third place.
50	GSt2	[to G St1] If we don’t put it [zero], it will be more, the value will be more.
51	T	23.206

The students observed that the number obtained by removing the zero would be greater than the original number (Line 50, Excerpt 5.6). The conjecture formulated by the students, which is valid only when a non–terminal zero is removed from the right side of the decimal point, indicates that they were making a connection between the value of a number and placing a zero. It is noted that the students' attempt at formulating a general rule from the specific examples did not obtain a response from Nandini. In Lesson 9 (Y1DL9), while dealing with the comparison of numbers, Nandini explained the procedure of expressing decimal numbers in their expanded form. For example, to compare 3.01 and 3.10, Nandini said, "For exam if it is a one mark question you can directly write the answer otherwise you write like this: 3 (and) 3 are same, 0 and 1, 1 is greater". Students solved other comparison problems by following the same procedure. At the beginning of Lesson 10 (Y1DL10), the researcher noticed a student's (Soh) written work and had a brief conversation with him, which is reproduced below (refer Excerpt 5.7).

Excerpt 5.7: Comparison of 3.3 and 3.300 (Y1DL10)

Line No.	Speaker	Utterance
14–17		Soh is sitting next to the researcher. His written work to the problem "3.3 or 3.300 which is greater?" is as follows: $3.3 = 3 + \frac{3}{10} + \frac{0}{100}$ $3.300 = 3 + \frac{3}{10} + \frac{0}{100} + \frac{0}{1000}$ So 3.3 is greater.
18	R	How did you get this?
19	Soh	It has more numbers so it will be lesser.
20	R	Which has more?
21	Soh	This one [points at 3.300].
22	R	This is lesser?
23	Soh	Yes, because it has more numbers [means digits].
24–26	Soh	[He looks for the answer at the back of the textbook and says] Something is wrong, I think. They both are the same. [He looks confused].
27	R	And why do you say both are same?
28	Soh	Answer is given [in the answers section of the textbook].
Legends: R – Researcher, Soh – Soham		

Although Soh followed the procedure taught by Nandini, his conclusion was different. He concluded that 3.3 is greater than 3.300, as 3.3 has fewer digits, that is, reverse of whole number thinking. Knowing that the textbook shows the correct answers, he accepted that 3.3 and 3.300 are the same without reasoning. In the next episode of the same lesson, the students had to compare 1.23 and 1.2. Nandini started solving it by placing a zero at the end of 1.2, that is, 1.20. A student asked Nandini, "how zero" to which she said, "Here 1.2 means 1.20. Zero is not written, that's all." Then she compared each digit of 1.20 and 1.23. Nandini was teaching students to make the length of the two decimals the same by annexing zeroes. The reason for using the zero annexure algorithm was stated as the convenience in digit–wise comparison of decimal numbers.

In these episodes, I notice that students were trying to make sense of the position of zero and the corresponding change in the value of a number. Nandini did not seem to anticipate or notice students' difficulties with the placement of zero in a range of decimal number notation and comparison tasks and therefore did not respond to them. Nandini's goal was to present students with clear procedures and prepare them for answering questions in exams. These goals require the teacher to know the procedures and to illustrate them with examples. They are different from the knowledge demands posed by the more responsive approach that Nandini adopted in the second year.

5.5.3.2 Year 2: Place value of zero with respect to its position

In Y2, there were several instances, which led to a discussion on the position of zero. I will discuss two episodes where Nandini attempted to diagnose and challenge students' thinking underlying their responses. In Lesson 3 (Y2DL3), Nandini asked the students to expand 578.92 to show the place values, and made a connection between the fraction representation and place value of each digit. Then, Nandini inverted the task and asked students to derive back the decimal number from the place values. For representing the place value of 9, students mentioned "$\frac{9}{10}$, 0.9, .9", to which Nandini asked if 0.9 and .9 "are the same". A majority of students agreed that they were equivalent and Nandini did not pose any questions to probe further. Students represented the place value of 2 as "zero point zero two (0.02), point zero two (.02), and zero point two (0.2)". Nandini decided to discuss which of these was equivalent to the fraction representation of two by hundred. She placed all of these responses on the board for whole class discussion and did not tell the students which was correct (refer Excerpt 5.8).

Excerpt 5.8: Place value of 2 in 578.92 (Y2DL3)

Line No.	Speaker	Utterance
321	T	If I write zero point two?
322	SSts	It is wrong.
323	SSts	It is 0.02.
324	T	What is 0.2?
325	BSt	One by ten and one by ten.
326	MSts	Two by ten.
327	T	Hmm. If the number is two by ten, how do you expand it?
328	Sts	Two by ten
329	T	Two by ten [writes "2/10"]. Okay. Is this [points to 2/100] two by ten?
330	MSts	No.
331	BSt	It is 0.02.
332	Sts	Two by hundred.
333	Ssts	0.02.
Legends: BSt – Boy student, MSts – Many students, SSts – Some students, T– Teacher.		

The class reached an agreement that 0.2 was not a correct representation for $\frac{2}{100}$ by linking it with its fraction equivalent $\frac{2}{10}$ and further decomposing $\frac{2}{10}$ as $2 \times \frac{1}{10}$.

In another episode in Lesson 5 (Y2DL5), the class discussed the problem of writing one hundred and two ones in numeral form (refer Excerpt 5.9). This was the same example as the one discussed in Y1, and described above.

Excerpt 5.9: Equivalence of 102 and 102.0 (Y2DL5)

Line No.	Speaker	Utterance
169–170	BSt3	Teacher *agar isko decimal me likhna hai to* [Teacher if we have write in decimal, then] one hundred and two point zero, one zero two point.
171	T	Point zero.

Line No.	Speaker	Utterance
172	BSt1	*Ya* [Or] zero zero zero zero.
173	T	Is it [points to 102.0] the same as hundred and two?
174	T	He [B St 3] is saying we should write one hundred and two point zero.
175	GSt1, 2	Yes, correct.
177	GSt1	Ma'am, like we can write 6 and 6.0000.
178	GSt2	But 60,000 point nahi [But not 60,000 point].
179	GSt3	Or 600.8
181	T	So is this [102.0] correct you are saying?
182	MSts	Yes.

Similar to the previous year, a student proposed 102.0 as the decimal representation for one hundred and two ones. Some students extended it by placing more zeroes to the right after the decimal point. Students gave additional examples (6 and 6.0000) to support the equivalence of 102.0 and 102.000..., and non–examples (sixty thousand point something and 600.8). The class seemed to agree about the equivalence of 102, 102.0, 102.000..., and so on. At this juncture, Nandini modified the question and posed it to students (see Excerpt 5.10).

Excerpt 5.10: Examples and non-examples of equivalent decimals (Y2DL5)

Line No.	Speaker	Utterance
189		[T now writes 1020.0 just below 102.000].
190	Sts	Wrong, wrong.
192	T	These two [1020.0 and 102.00] are not same?
193	Sts	No, no.
195	T	But the number of zeroes are same [ignoring the decimal point].
196	BSt	But it is wrong.
197	T	Why?
198	GSt	Because after the point that has three zeroes but this one after the point has two zeroes [for 102.000 and 1020.0].
199–200	T	[Repeating what the student said] After the point there are three zeroes here [pointing to 102.000] and [correcting the student] here there are only two zeroes [pointing to 1020.0].

Line No.	Speaker	Utterance
201–202		[G St nods in approval. Then teacher changes 1020.0 to 1020.00 inorder to correspond to the student's statement, and asked students to compare 102.000 and 1020.00].
203	Sts	Ma'am, wrong.
204	T	Now for both you have three zeroes.
205	GSt2	Ma'am, one is thousand and [other] one is hundred.
206	GSt3	Ma'am, first one is one hundred and two, but second one is one thousand and twenty.

Nandini presented a set of decimal numbers for comparison based on students' responses in the order – (a) 102.0 and 102.00, (b) 102.0, 102.00 and 102.000, (c) 1020.0 and 102.000, (d) 1020.0 and 102.00. The first two sets (a, b) were confirming examples, while set (c) was a non–example. The last example (d) was intended to test whether students were merely counting the zeroes to establish equivalence. Nandini's statement, "but the number of zeroes are same" was an anticipation of students' thinking, of counting the zeroes and ignoring the decimal point, when comparing equivalent decimals. At the end of the discussion, students' justifications changed to "one is one thousand and the [other] one is hundred", thus reasoning about numbers as a whole. The discussion continued for another set of examples – 102.000 and 1020.00, and 103.00 and 1030.0 (see Excerpt 5.11).

Excerpt 5.11: Explanation for equivalent decimals (Y2DL5)

Line No.	Speaker	Utterance
		[These numbers are on the board—102.000 and 1020.00 and 103.00 and 1030.0].
258	GSt4	Ma'am if we add zero behind the number, the number has the same value. If we add it ahead of a number, it has a lot of value. So it is one thousand and thirty and that is one hundred and three.
261	T	So you are saying if we add zero after the decimal?
262	GSt4	No, not after the decimal, after the number.
268	GSt4	If there are one hundred and three and you add a zero behind it so you have the same value but in that if you add a zero in front of another number, so it has changed the point.
270	T	So this zero changed this number [in 1030.0], but this zero [in 103.00] did not change the number.

Line No.	Speaker	Utterance
272	GSt3	Ma'am here [103.0] three has ones place and here three has tens place [1030.0].
274	T	So Si [G St3] says that here 3 has a different place value from the 3 here.
278	T	So she has compared the same place value numbers. Hmm
279	BSt	Same digits
287	T	So finally are the two numbers the same [points to 103.00 and 1030.0]?
290	T	One is thousand and thirty, and the other one is hundred and three. We will see more numbers like this

While discussing this set of numbers, students were making attempts to articulate a general statement, using the specific examples recorded on the board. Nandini's move of changing the problem led the students to make such attempts. Later in the same lesson, Nandini placed an initial zero and asked students if the numbers, 0103.0 and 103.00, were the same. In the post–lesson interview, Nandini articulated the reasons for going beyond the example of 102 and 102.0 (see Excerpt 5.12).

Excerpt 5.12: Post-lesson reflection: Nandini (Y2DL5)

Speaker	Utterance
Nandini	When I asked them to write for 102, one girl said 102.0. I know this is correct and she was also right. But I thought that might be a confusion others might have. I thought I will ask the class. I wanted to try. Children have this difficulty with zero and its place in a decimal number. I think it is important. What do you think?... When this girl said that 102 and 102.0, I thought I will [take the opportunity to] discuss this. I took the numbers 103.00 and 1030.0 because I knew that they will count the number of zeroes in the end. But they need to see the placing of zero, its position in the number. Si [GSt3] gave a nice answer. She told that number changes because of the changed position of zero. I liked her answer.

It is important to note that in Y2, Nandini was listening to students, interpreting their responses, and prompting them to arrive at a general statement about the placement of zero and to provide supporting explanations.

5.5.3.3 Knowledge demands

A common thread in the excerpts from Y1 and Y2 is the effect of the placement of zero on the decimal number. In Y2, Nandini addressed this issue when a student asked if the numeral form of one hundred and two ones should be 102 or 102.0. She revoiced the question and posed it back to the class. Even though the class responded confidently, that they were the same, Nandini decided to probe students' understanding further, as she mentioned in the post–lesson discussion with the researcher. The variations that she produced to probe the students' thinking indicate that she was mainly checking for whether the students were basing their judgement about equivalence on counting the number of zeroes. Thus, her emphasis was on having students compare the numbers in pairs such as 102.00 and 1020.0, and 103.00 and 1030.0. The examination of different cases of positioning of zero in a decimal number and comparing its relation with the original number is an important part of teacher knowledge. The sequencing of examples and providing variations is important for students to build an understanding of this relation. Through her choice of examples, Nandini produces variations in the length of the decimal numbers without changing the digits, which was useful in generating the conflict with whole number thinking (also noted by Desmet, Grégoire & Mussolin, 2010).

In unpacking the knowledge demands underlying dealing with the cases involving the placement of zero in a decimal number, I will distinguish between two issues that are relevant. First, the students need to have clarity about how the placement of zero in a whole number affects its value. Second, this knowledge needs to be extended to understanding how the placement of zero in the fractional part of the number (that is, to the right of the units place) affects the value of the number. The second issue is related to the interaction of students' whole number knowledge with their learning of decimal numbers. A closer examination of Nandini's responses to students reveals that she did not adequately address the second issue. In the variations that she discussed, none of the fractional place values had non–zero digits. Rather, the discussion focused only on the change in value to the whole number part of the decimal number. In Lines 205–206 (of Excerpt 5.10), the students assert that the first number is one hundred and two (102.000), while the second is one thousand and twenty (1020.00). This is based on whole number knowledge and possibly involves visually identifying the whole number parts in the pair of numbers. The discussion thus draws on students' prior knowledge of whole numbers, but does not extend it in a way that is useful to understand the effect of placing a zero on the fractional place values.

In Y1, the student Soh had thought that the values of 3.3 and 3.300 are different. An example of this kind with a non–zero digit in the fractional part of the decimal number does not appear in Nandini's variations in Y2. As Stacey (2005) has pointed out, Soh's error is related to the incorrect over–generalisation of whole number thinking. A direct correspondence between whole numbers and decimal numbers would lead to a student thinking that inserting a zero after all the digits of a number will increase the value of the original number by ten times. I found evidences of such thinking in students' written responses to other problems in a worksheet designed by the researcher, for instance "4.4 × 10 = 4.40", and "3.600 > 3.60 > 3.6". In contrast, there were evidences of reverse thinking as well, as in Soh's response to the comparison problem. In stating that, "3.300 is less than 3.3 because it has more numbers [digits]", the student identifies the inverse relation between place values of the whole number part and the fraction part. In other words, students might think that, adding more places to the whole number part increases the value of the number, when compared with the original number while adding up places to the fraction part decreases its value. Alternatively, the students might be guided by an implicit mirror metaphor thinking that the values increase (from tens to hundreds, etc.) while extending the places to the left, and the values decrease (from tenths to hundredths, etc.) when extending the number to the right of the decimal point. Although Nandini discussed the issue of the placement of zero in Y2, she did not fully address the kind of over–generalisations that students like Soh might make.

Note that the students' justifications are frequently phrased in relation to a specific example rather than as a generalisation. Nandini accepts such justifications, but pushes the students to say something more general. In Line 258 (of Excerpt 5.11), a girl student (GSt4) attempts such a generalisation when she says that a number has the same value "if we add zero behind the number" but not if we "add it ahead of the number". However, this again is about placing a zero in a whole number. Another student (GSt3) observes that the placement of zero has changed the value of the digit "3" from ones to tens (Line 272, Excerpt 5.11). Nandini revoices this and points out to the class that the student has compared the place values in the two numbers of the same digit (although she misspeaks saying "number" instead of "digit" and is corrected by a student). The formulation by GSt3 is different from previous formulations and is more powerful, since it can be generalised to apply to both the whole number and the fractional part of the decimal number. The general formulation would be that if placing a zero changes the place value of a non–zero digit, then and only then does the value of the number change. However, this general formulation does not appear in the discussion. In fact, in her remarks to the researcher, Nandini only notes that G St3 said that "the number changes because of the changed position of zero," a formulation that is not helpful in understanding the effect of placing a zero to the right of the decimal point.

A general statement such as the above was well within the reach of the discussion. Nandini did make several moves that were "responsive", including introducing variations in the examples, and probing students to formulate sharper statements. However, she appears to have been constrained in two ways. Firstly, she focused only on the possible student misconception that, judgements about equivalence of two decimal numbers with the same digits could be made by counting the zeroes. She did not consider other possible incorrect generalisations from whole number knowledge that students might have made. Secondly, she did not seem to be aware of the need to specifically address students' difficulties with placement of zero to the right of the decimal point, in decimal number with non–zero digits in decimal fraction places. Since the examples in Lines 189–206 (of Excerpt 5.10) did not contain examples of numbers with non–zero decimal fraction places, a difficulty such as the one faced by Soh (3.300 is less than 3.3) did not get addressed. While comparing 0.2 and 0.02 in the earlier example, Nandini did deal with a decimal number with a non–zero decimal fraction place. However, this discussion was related to the fraction notation corresponding to the two numbers and did not explicitly focus on how the placement of zero changes the value.

5.5.4 Conversion of measurement units

Conversion between units of length measurements is treated as an important application of learning decimal numbers. Until Grade 5, students are asked to convert the measures in larger units to the smaller units, since they are unfamiliar with the decimal representation. After learning decimal fractions, students are expected to do conversion from smaller units to higher units. This sub–topic was dealt by Nandini in both the years of teaching decimal numbers.

5.5.4.1 Year 1: Conversion of measurement units

In Y1, Nandini used a ruler to introduce decimal fractions in the first lesson (Y1DL1). She asked students to guess and measure the length of the duster. Since the length was between 17 and 18 centimetres, the need for a precise measure was used to introduce decimal numbers. After the first lesson, reference to the measurement context was made in the fifth lesson. In Lesson 5 (Y1DL5), the class was given the task, from the textbook, to convert measures in millimetres to centimetres. Nandini initiated the discussion by drawing students' attention to the divisions between 0 and 1 cm on the ruler. She cued students to count every division after 0 on the ruler and named the indicated units as "millimetre". Then, she defined 10 millimetre divisions to be (the same as) 1 centimetre. Nandini introduced the decimal representation for each millimetre length by converting it into centimetre, naming the divisions as 0.1, 0.2, 0.3, and so on. The pattern of 0.1, 0.2, 0.3... was extended to convert bigger lengths given in millimetres to centimetres. In the remaining part of the lesson, students were asked to convert the given lengths, such as 30 mm, 16 mm, 4 cm 2 mm, etc., to centimetres. Nandini did not use the ruler or the number line in subsequent lessons for teaching or discussion on any other sub–topic.

In the Grade 6 textbook, conversions from smaller to larger units of length measurement are presented as an application context, where students are expected to use the knowledge of conversion between fractions and decimals. (In contrast, the Grade 5 textbook uses the length measurement context, specifically the conversion between millimetre and centimetre, to create a need for introducing decimal numbers.) Nandini used the conversions between the length measurement units as an application context in Lesson 5, as suggested by the textbook. She identified the labels for the smallest measure of length marked on the ruler using both the units, millimetre and centimetre. However, neither the fraction representation of the relation between the units nor the addition of fractions, which were known to the students, were mentioned in this discussion. The use of the fraction equivalent of the measure, might have helped students in connecting the fractional and decimal representation, and justified the link between the two units. In other words, 1 millimetre is 0.1 centimetre because of the relation between the number of sub–units that constitute the bigger unit. Also, the relation between 1 millimetre, one–tenth of a centimetre and 0.1 centimetre might have been strengthened by discussing the relation between place values, which in turn, could have been represented using the fraction notation.

5.5.4.2 Year 2: What is one division after one centimetre

In the second year, Nandini organised a detailed discussion on finding the labels for the millimetre measures. In Lesson 7 (Y2DL7), Nandini taught how to convert fractions with denominators as powers of 10 (or expressible as powers of 10) to decimals. Then, Nandini drew a ruler on the board and explained the reason for scaling up the millimetre divisions for greater visibility. She invited a student to come to the board and measure the length of the duster. The student measured the length of the duster using the ruler drawn on the board, and said "one division after one (centimetre)". Nandini revoiced the student's utterance and asked the whole class to think about this measure. She encouraged students to represent the measure in different units. Different students' responses were recorded on the board. These included "11 millimetre, 1 centimetre 1 millimetre, 1.1 centimetre, and 1 and one–tenth centimetre". She confirmed whether the measure is 1.1 centimetre or 1.1 millimetre. One of the students' responded that the length is 11 mm, so it cannot be 1.1 mm. Nandini asked the students to justify each of these responses. Students justified their responses using the relation between millimetre and centimetre, representing the relation using whole numbers, fractions or decimals. Nandini then asked students whether all the responses, listed on the board, were correct. She invited students to show equivalence between these measures. Students offered the following justifications (refer Excerpt 5.13).

Excerpt 5.13: One division after one (Y2DL7)

Line No.	Speaker	Utterance
197	G St.1	It is one, it is one by ten.
201	G St.5	Ma'am because each millimeter has, no, each centimeter has ten millimeter.
217	G St.2	Ma'am these are same only.
219	B St.2	One by ten is point one. So one ten plus point one. One point one.
226	G St.2	Ma'am we can see it in expanded form also. It is 11 millimeter also.
227	G St.3	*Ma'am one centimeter ke baad bhi equal to dalo. Wo ten our one hoga.* [Ma'am put an equal to after 10, there will be 10 and 1.]
233	B St.1	Ma'am one centimeter is equal to ten millimeter. And one more millimeter.
242	G St.5	We can write one millimeter as one by ten centimeter.

Students used the ruler, the place value of 1 in different positions, and the relation of one–tenths as justification to reach this conclusion. The lesson concluded with the consensus that 1.1 centimetre is the same as 1 centimetre and 1 millimetre or 1 centimetre and one–tenth of a centimetre or 11 millimetre.

Later in the lesson, students were asked to convert 2 mm to centimetre. A student proposed decomposing 0.2 as "0.1 cm + 0.1 cm" and each being "one–tenth of a centimetre". To find the conversion, they added one–tenths twice and concluded 0.2 cm as the answer. Nandini took this opportunity to introduce the hundredths place value by extending the conversion context to identify the relation between meter and centimetre.

Nandini's decision to spend the whole lesson on the question of "one division after one" created space for students to explore the relations between different units, thus moving beyond the procedural understanding of conversion between units. Public thinking in the class around the student's question led to elicitation of different representations of the same measure and justifying the relation between them. The flexibility in naming a measure using different units helped students to see the different representations of the measure. Further, Nandini's insistence on seeking for reasons, made students justify the equivalence of the numerical representations using equal divisions on the ruler, decomposition of a number, and relation between place values.

5.5.4.3 Knowledge Demands

In both Y1 and Y2, Nandini dealt with the conversion of length measurement at least twice, first to introduce the decimal numbers and second to reinforce the conversions from smaller to the larger unit. Unlike the first year, where Nandini labelled the measures as 0.1, 0.2, and so on; in the second year, she asked students to find the length measure of “one division after one (centimetre)”. The purpose of revoicing was not limited to eliciting students’ responses or gaining their attention to the naming of the length measure, but was extended to offering justifications and examining the similarities (and differences) in the different responses, which emerged from classroom discussion and were recorded on the board. Initiating a discussion around a student’s question and orchestrating it, in a manner that allows other students to contribute and justify their solutions, indicates an engagement of students in serious mathematical activity. Nandini’s decision to respond to a contingent classroom situation, an unanticipated student’s question, by allowing others to contribute and then shape the discussion, is guided by a rich knowledge base. I conjecture that Nandini’s knowledge of equal partitioning, iteration of a sub–unit to form a bigger unit, and the use of linear representation to discuss relations between units; might have helped her in anticipating the mathematical potential of the student’s question. Also, note that supporting students’ justifications and challenging them to see the similarities between the apparently different representations, was challenging and engaging for students. The flexibility of navigation between different representations helped students in decomposing 2 mm and identifying why it is the same as 0.2 cm.

Another important decision made in these lessons was about using the length measurement context for strengthening the decimal place values. In Y1, Nandini used the linear representation (ruler and number line) for the tenths place value. It was not extended to the hundredths place value. This meant that the number line with whole numbers was used to show the relation between the length measurement units of millimetre and centimetre only. The limited use of number line constrained students’ imagination in representing a decimal number with hundredths on a number line. This was evidenced in students' response to a question in the worksheet, where they were asked to represent 0.06 on a number line. Several students reported that they could not do the task since they have learnt how to locate this number using a grid, and not a number line.

On the contrary, in Y2, Nandini extended the relation between these two measurement units, to introduce the hundredths place value, by asking students to convert between length measures in centimetre and meter, using the same meter strip. Nandini iterated on the ruler, which had centimetre and millimetre marks, to show how 100 times a centimetre makes a meter. Her revoicing of the student's question, about what is one division after one, at this different situation, revealed that Nandini wanted to draw students' attention to the number of equal divisions between the measures and in identifying the relation between the given units. She helped students to see the link between the measure meaning of fractions and the relation of tenths and hundredths. The use of consistent representation (linear) for tenths and hundredths supported students' reasoning in moving flexibly from one measurement unit to another.

In the teaching of decimal fractions, measurement is chosen as a context for understanding the relation between different units. The multiplicative relation between units of length (weight and capacity) is structurally the same as the relation between the place value of the consecutive digits of a number in base ten system. Apart from being a context for decimal representation, length measurement offers a linear representation, from a ruler to a number line. This relationship between the measurement context and the number line representation was used by Nandini, to introduce the conversion from the smaller to the bigger unit in the second year. The structural similarity of the relation between the units of the metric system and decimal representation is an important part of teacher's knowledge in this case.

Another important piece of knowledge is the affordance of the selected representation(s). Nandini used the number line representation to introduce the relation of tenths, while hundredths was introduced using a grid. In Y1, the choice of different representations for tenths and hundredths created a disconnect between the continuity of units among students. This was evident in their difficulty in using a number line representation to show a decimal number with hundredths place value. In the second year, Nandini seemed aware of this disconnect and choose to use a consistent representation, that is, a number line to introduce both tenths and hundredths. Later, she also used the area representation to show different place values. The pattern of Nandini's responses to students, in the second year, reflects an awareness of subtle aspects of the mathematical concept. Her selection of appropriate representations and their coherent use, offering reasons for equivalence of two representations and providing the tools (using representations, previously known ideas, etc.) in order to direct students' attention to the key mathematical ideas are in action here. The depth in Nandini's knowledge of the use of measurement as a context for learning decimal fractions supported her in unpacking the mathematical potential in students' responses. Like other episodes of Nandini's teaching in Y2, I observe that her decisions are not just guided by anticipating potential students' difficulties and their sources, but also a preparedness to deal with the contingencies, in this case by leveraging the knowledge of connections between whole numbers, fractions, and decimal numbers.

5.6 Case Analysis of Reema's Teaching

Reema had been teaching Grades 1–5 using the old and the new textbooks. In the first year of the study, she mentioned that the new textbooks have more real life problems (or contexts) but less practice exercises. She used the contexts from the new textbooks to introduce a topic, and then quickly shifted to working with numbers and figures, which she considered important for students to learn. Reema made several attempts to work informally with the researcher to understand the textbook content better and discuss pedagogies, which might be more suitable for her learners.

5.6.1 Teaching in Year 1 and 2

Like in Nandini's case study, the changes in Reema's teaching practice are analysed by comparing a set of paired episodes. Table 5.4 summarises the frequency of selected codes in three pairs of lessons across the 2 years. The selected codes focus on students' and teacher's explanations, teacher's questions and her responses to students' utterances. The first pair of decimal lessons (DL) focused on introduction to decimals using the measurement context (Y1DL1 and Y2DL2). In the second pair of lessons (Y1DL2 and Y2DL4), place value of decimal numbers was the goal. The relation between fractions and decimals was discussed in the third pair of lessons (Y1DL3 and Y2DL6). Although Reema's classroom witnessed discussions on core ideas (such as oneths, position of zero) similar to that of Nandini, these ideas have not been selected for analysis here. The reason is to identify a breadth of knowledge demands by focusing on different sub–topics in the teaching of decimal numbers.

Table 5.4: Frequency of select codes in Reema's (paired) decimal lessons

	Code	Y1DL1	Y2DL2	Y1DL2	Y2DL4	Y1DL3	Y2DL6
1	TQ (Teacher Question) – textbook	4	0	0	5	0	5
2	TQ – elicit	1	24	0	17	2	12
3	TQ – what	18	115	41	74	113	79
4	TQ – how	0	13	1	5	3	2
5	TQ – why	0	5	0	8	4	11
6	TE (Teacher Explain) – tell	19	6	41	9	41	2
7	TE – procedure	3	4	8	2	15	3
8	TE – justify	0	23	0	9	0	14
9	TR (Teacher Response) – evaluate	1	6	12	8	17	6
10	TR – restate	0	43	12	30	21	23
11	TR – expand	0	20	2	29	2	10
12	TR – argue	0	1	0	0	0	
13	TR – public think	0	23	0	6	0	4
14	SE (Student Explain) – one word	34	196	54	121	154	121
15	SE – error	4	5	12	3	25	1
16	SE – procedure	0	9	7	5	7	2
17	SE – justify	0	7	1	21	0	9
18	SE – observe	4	7	7	15	4	4

	Code	Y1DL1	Y2DL2	Y1DL2	Y2DL4	Y1DL3	Y2DL6
19	SE – completes TE	3	0	0	8	0	1
20	SE – adds SE	0	15	0	4	0	1
21	SE – argue	0	9	3	3	0	1
22	SE – evaluate	0	14	0	5	0	3

There is an emergence in the use of justifications by Reema (Row 8, Table 5.4) and her students (Row 17) from Y1 to Y2. There is a decrease in the procedural explanations offered by the teacher (Row 7) in a majority of lessons and students (Row 16, except in the first pair of lessons). In the first year, Reema stated some rules, such as, two zeros in the denominator means two digits after the decimal point. There is a considerable decrease in telling such rules to students in the second year (Row 6). Students have made some observations in both the years, although, there is a rise in these observations in the second year (Row 18). These observations include questions about the existence of oneths, differentiating between methods, about position of a decimal point in the place value table, etc. Two new codes emerged from the classroom observations of teaching in the second year (Row 20 and 22). In Y2, when the teacher invited a student to share her explanation, other students added to this explanation. Students supported each other in completing an explanation by sometimes building on it. In Y2, Reema also invited students to revoice another student's method and try to solve problems using it (Row 20). For instance, in Y2DL4, a student Misha, had suggested converting 2 rupees 50 paisa to paisa and dividing 250 with 50, to find how many matchboxes of 50 paisa each can be bought with this amount. Reema acknowledged the student by referring to this method as "Misha's method", asked another student to restate the method, and then recalled it when solving another question later in the lesson. The student–student interactions included adding to each other's explanation, as well as challenging it by expressing their disagreement (Row 21). These instances increased markedly in the second year. For instance, in Y2DL2, a pair of students volunteered to add 30.5 cm to itself using a number line drawn on the board. Some other students were suggesting the total length while seeking and offering reasons for their answers. Students also evaluated each other's and the teacher's work by expressing whether they agreed or disagreed with it (Row 22). Like Nandini's teaching, 'revoicing' emerged as a new teaching practice, in Reema's teaching (Rows 10–13). In particular, Reema encouraged students to examine each other's method by posing them for public thinking (Row 13), refraining from passing judgments about the correctness of the method, in Y2. While there was an increase in the number of 'why' questions posed by Reema (Row 5), the number of 'what' questions varied for Years 1 and 2 (Row 3). The 'what' questions varied depending on the sub-topic that was been discussed. A practice unique to Reema was offering a background for any context when using it for introducing a sub–topic. For instance, when introducing the money context, Reema had a detailed discussion about how an amount such as rupees 499 and 99 paisa is used by the shopkeepers to attract customers (Y1DL1). In the second year, the class discussed about how paisa is obsolete and how different denominations of paisa would not be returned by a shopkeeper as change (Y2DL2). While the practice of providing contextual background was consistent in the two years, Reema began summarising discussions in every lesson in the second year. The practice of pausing and reflecting on what was done, mostly done by Reema and sometimes by students, was used by the students to pose alternative methods, clarify their doubts, express if they had not understood something, etc. In Y2, Reema had sustained dialogues with individual and group of students. She made several attempts to unpack students' thinking and offered scaffolds by asking simpler questions. In the following section, episodes from the paired lessons from Reema's teaching are elaborated to abstract the knowledge demands arising from teaching.

5.6.2 Introduction to decimals: Measurement context

In the revised mathematics textbook of Grade 5, the chapter on decimals "Tenths and hundredths" includes several contexts. These include – length measurement (guess and measure length of different objects for instance ant, pencil, candle, ladyfinger, notes or currency, etc.), money and currency exchange, and temperature. The measurement context introduces millimetre as a unit, followed by the length of the frog context (refer Figure 5.1). In the following section, I will discuss how decimals were introduced in the first and second year of Reema's teaching.

5.6.2.1 Year 1: Length Measurement Context

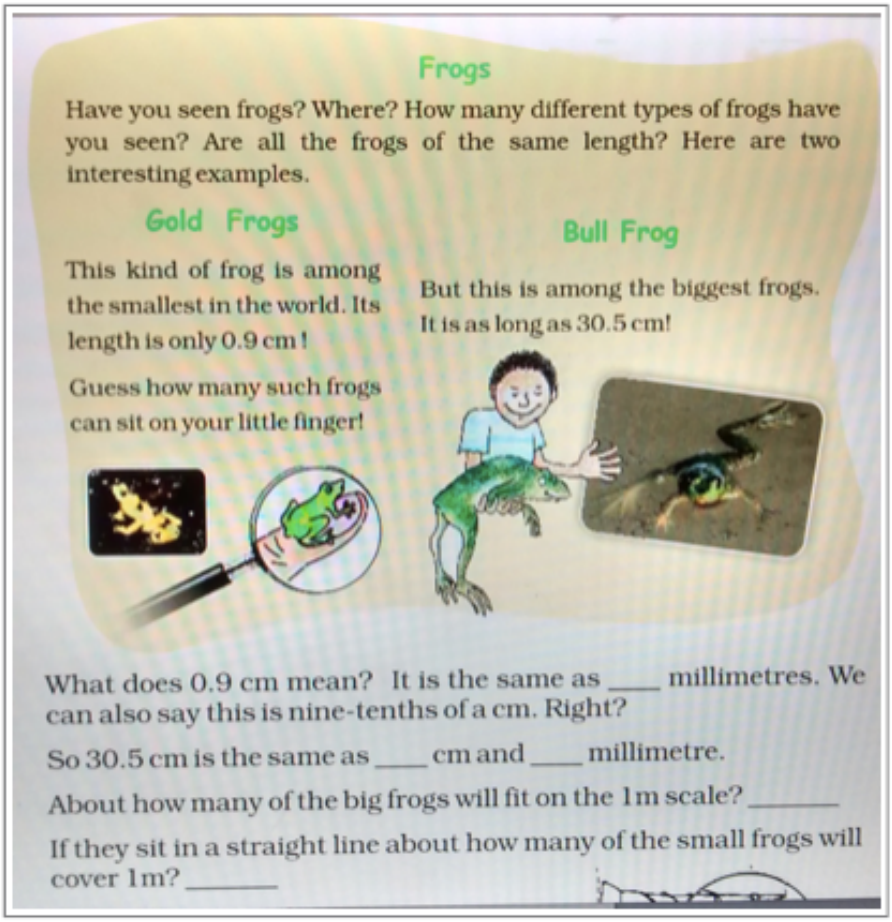
In the first year, Reema introduced decimals using the measurement context (Y1DL1). She asked students to look at their 15 centimetre (cm) long ruler to guess and measure the lengths of given objects. She pointed to the centimetre markings on the ruler and told students that each marking represented a measure – "1 cm, 2 cm, and so on". Further, between 1 and 2 cm, the first line (referring to the first division) is "point one". She referred to the other parts as "point 5, point 8 and point 9". She asked the students to draw – an ant of length less than 1 cm, a glass of length 11 cm with water up to 5 cm, and a pencil of length 7 cm. Reema told the students how to draw the given lengths. Consider Excerpt 5.14 (Line 70) for how Reema told the students to draw an estimated length of 7 centimetre.

Excerpt 5.14: Guess the length of ant (Y1DL1)

Line No.	Speaker	Utterance
70	Reema	(Draw a) Pencil of length 7 centimetre. Drawing, draw it horizontally, this pencil. Pencil of length 7cm, as I said. In your notebook, estimate how much is 1 centimetre. Now you know how much is 1 centimetre. (Length of) Ant is less than 1 centimetre. So you know how much is 1. Now such one one ant you draw and add for pencil.
77		A girl student's work: She measured the length of an ant using the marks on her first finger. Using this length, she drew 1 centimetre and doubled this length. She then used the length of 2 cm to make 4 cm and doubled this length to get 8 cm. Then, she measured 1 cm length from 8 cm, using the marked length of 1 cm on her finger and erased it to get 7 cm.

Records from classroom observations revealed that students used other ways to estimate the measure of the given lengths. For instance, a girl student (refer Line 77, Excerpt 5.14) used the length drawn by her earlier to make a length of 7 cm, that is, she drew 1 cm length (measure of an ant), then doubled this length to get 2 cm, doubled it twice to get 8 cm and then took away 1 cm from this length. In other words, she used her finger as a measure to draw the length of 7 cm. A less sophisticated strategy was used by a group of students who marked 1 cm on their eraser and then repeated it 7 times. A few other students measured one of the given lengths using the ruler and estimated the other lengths based on the measured length. While Reema suggested a method, students used different strategies to complete the given task. In another example to draw a length of 5 cm, Reema told the students that "5 is less than half of 11", thereby suggesting the students to use the 11 cm length to draw 5 cm. Students independently used this strategy to draw the given measure.

After the guess and measure problem, Reema asked the students to guess and cut a thread of length 10 cm. The task was to draw a circle of perimeter (since the students were aware of perimeter and not circumference, at this grade level) 10 cm. The lesson concluded with the students working on their drawings, and completing the table on finding the estimate and actual measures of the remaining objects, as homework. In the next lesson (Y1DL2), Reema introduced the "ths" in "tenths and hundredths" using the money context. Reema decided not to do the length of the frog task (refer Figure 5.1) as she felt it to be challenging for students who have just been introduced to decimals.



Frogs

Have you seen frogs? Where? How many different types of frogs have you seen? Are all the frogs of the same length? Here are two interesting examples.

Gold Frogs

This kind of frog is among the smallest in the world. Its length is only 0.9 cm !

Guess how many such frogs can sit on your little finger!

Bull Frog

But this is among the biggest frogs. It is as long as 30.5 cm!

What does 0.9 cm mean? It is the same as ____ millimetres. We can also say this is nine-tenths of a cm. Right?

So 30.5 cm is the same as ____ cm and ____ millimetre.

About how many of the big frogs will fit on the 1m scale? ______

If they sit in a straight line about how many of the small frogs will cover 1m? ______

Figure 5.1: Length of Frog Context (NCERT, 2007, p.135)

Reema used a variety of contexts such as money, currency, temperature; in different lessons and completed most part of what was given in the textbook. The discussion around each context was largely driven by Reema – she introduced the task, solved problems based on the task on the board and asked students to copy the solutions in their notebook. While teaching, her instructions were simple and explanations procedural, demanding students to follow and repeat.

In these lessons, I noticed that Reema took the “guess and measure” length problem, suggested by the textbook, to introduce decimal numbers. She used a ruler to direct students’ attention to the whole number measures and then introduced “point 1, point 5, and so on” as measures between wholes. The students used objects around them, as estimates of 1 cm, and iterated 1 cm to guess and draw the lengths – less than 1 cm, 11 cm, 5 cm and 7 cm. Reema did not discuss with individual students or the whole class about how they were estimating the given lengths. Although students used different strategies to estimate the given lengths, Reema suggested one method for drawing these lengths. The method suggested by Reema matched with the strategy used by one of the students.

It is notable that Reema simplified the task of ‘guessing and measuring’ by introducing the ruler in the beginning of this lesson. She seemed to be aware of the students’ familiarity with the ruler and its use to measure length. She used this prior knowledge to introduce the guess and measure problem. However, the lesson primarily focused on the revision of the whole number lengths, since the length measures given in the task did not have a fractional part of a centimetre. Further, this guess and measure problem was not extended to introduce decimal numbers in this or the next lesson. While the textbook suggested extending the ‘guess and measure’ task to the length of the frog context, in order to introduce fractional parts, Reema decided to omit this context. She believed that the length of frog context was challenging for students and perhaps struggled to support students while dealing with this problem (refer Excerpt 5.15).

Excerpt 5.15: Post-lesson reflection: Reema (Y1DL1)

Speaker	Utterance
Reema	I see that in the textbook they have given this frog context for measurement. But I think it is a difficult problem for children. It is complicated for them to understand this. I don't know how to take it. I did the first problem, this length guess and measure. I don't think I will do this (frog context). In the next class, I will do money to do practice on decimal point.

5.6.2.2 Year 2: Length of the frog and the meter strip

Like the first year, Reema introduced decimals using the problem on guessing and measuring the given lengths, in the second year. In the first lesson (Y2DL1), she asked students to recall the use of a ruler. A few students mentioned the units – centimetre and millimetre, and Reema took this opportunity to introduce the different divisions in a ruler. She asked students to estimate the length of a few objects that she had carried with herself. These included an envelope, a comb, a tin, and a marker pen. The choice of the objects seemed deliberate, as two of them measured in whole numbers and the other two had measures in centimetres and half centimetres. Reema used these measures to introduce "half or point 5" to students. The whole class discussion on estimated lengths was tabulated on the board and students were invited to measure the actual length of the objects and write it alongside the estimated measures. In the next lesson (Y2DL2), Reema introduced the meter strip as an extension of the ruler and used it to show the relation between meter, centimetre, and millimetre. She first asked students to estimate the length of this strip, and then posed a series of questions on the meter strip (refer Table 5.5).

Table 5.5: Metre strip task variations (Y2DL2)

Line No.	Utterance
68	Guess the length of this (strip).
119	It is divided into how many parts?
131	How much is half of this strip?
133	How many centimeters would be half of the strip?
140	How much is (the measure of) each part?
142	Five such parts (of 10 cm) would measure how much?
160	Seven parts (of 10 cm each measure)?

Line No.	Utterance
163	If from all the equal parts, I take one part, only one part, so what is the fraction?
176	What is the length of eight such parts?
178	What will be the length of two such parts on the strip?
181	In the whole strip, how many two two parts are there?
183	How many 5 parts are there?

After creating familiarity with the meter strip, Reema introduced the length of the frog context. She began by asking students to guess the length of the frogs, that they have seen. Students guessed different lengths, “5 cm, 10 cm, 1 cm, 15.5 cm”. Reema told the students that the “textbook writers have found that the length of the shortest and longest frogs is 0.9 cm and 35 cm”, respectively. She asked students to imagine that the frogs of the same length are sitting on a 1 metre long strip without leaving any gap, that is, very close to each other. She then posed the question, “how many frogs of length 1 cm can sit on the strip”. While the students were modelling, an iterative addition of 1 cm using their hand, Reema showed this action through jumps on the meter strip, drawn on the board. She restated and recorded students’ responses, such as, “100 frogs of 1 cm each”, on the board. Then, she changed the length of the frog to 2 cm and asked the same question. Students immediately responded “50” giving reasons using half. Then, she asked them about the longest frog, which is 30.5 cm long. A student, Ashwin, came to the board and located 30.5 cm on the strip. When he reached 30.5, Reema mentioned that it is half centimetre or 5 mm more than 30 cm. While thinking of the next frog, Ashwin pointed to 70 cm on the strip. Other students responded “71 cm, 75 cm”. Students gave reasons for their responses like “seventy plus point five plus point five”. When a majority of students seemed convinced with a response, Reema noted it on the board as “30.5 + 30.5 = 71 cm?”. Students noticed the response “71cm” on the board and then with some probing from Reema, revised their answers stating that “60 and 1 is 61 cm”. They added the length 30.5 for the third frog and concluded that 91.5 centimetre length is covered. Students used the strip to count up from 91.5 up to 100, and told Reema that 8.5 cm of space would remain. Reema took the opportunity to put the subtraction sentence “100 cm − 91.5

cm" on the board and verified the answer to be 8.5 cm. With support from the other students, Reema revoiced the whole discussion on the number of frogs of length 30.5 cm, decimal addition (adding 91.5 and 8.5), and subtraction (going backwards from 100 to 91.5). Followed by this discussion, Reema posed the question of "how many frogs of length 0.9 cm can sit on this (pointing to the metre) strip". Initial student guesses were – "definitely 100, 100 divided by 9", corrected by another student as "100 divided by point 9", another student said "99 frogs". Reema asked students to spend some time solving this problem. A few students volunteered to explain their methods to the class. Reema recorded these responses on the board and invited students to think aloud about them. The methods listed on the board included – (a) adding 0.9 repeatedly, (b) subtracting 0.9 repeatedly from 100, (c) multiplying 0.9 with 10 repeatedly, and (d) 100 divided by 0.9. After asking students to repeat each others' methods and evaluate them, Reema asked "how much is 0.9 cm less than 1 cm?". The students stated 1 mm. Reema pointed to the metre strip, pasted on the board, and marked 0.9 mm starting from 0. After leaving a gap of 1 mm, she pointed to 1 cm and the students said "0.9 again". Students extended the pattern and added 0.9 for every whole number, starting from 0. Together, the class kept a record of the remaining length from each 1 cm length. Thus, the assumed problem was visualised as frogs sitting on each whole number and the remaining length for each centimetre. Reema recorded this discussion on the table, reproduced as Excerpt 5.16.

Excerpt 5.16: Reproduction of blackboard work on 0.9 cm frog (Y2DL2)

Number of frogs	1 frog	2 frogs	3 frogs	5 frogs	10 frogs	100 frogs
Remaining length	−1 mm	−2 mm	−3 mm	−5 mm	−10 mm	−100 mm

Together, the class was convinced that the answer was more than 100 frogs. Reema then asked, "how much more", which initiated the students into counting the left over spaces. Now, the question was, how many frogs can sit in 100 mm or 10 cm of left over space. The students discovered that 11 frogs could occupy this space and concluded with 111 frogs as the answer.

Reema reflected on this lesson and recalled it in the 17th teacher–researcher meeting. She mentioned carefully sequencing the questions and how the problem generated interest among students (refer Excerpt 5.17).

Excerpt 5.17: Reflection on frog context (TRM17)

Line No.	Speaker	Utterance
252	Reema	On one metre scale how many frogs can be placed?
254	Reema	So, the length of the frog was 0.9 cm and they are placed end to end on a metre scale.
256	Reema	So how can we do this? How can you find? Some said, it is division, multiplication all that. Then we went practically like keeping the frog, in one centimetre, that is ten millimetre, we can place one frog, then second centimetre, second frog. Like that each time, point one, point one, point one was remaining. In that point one, point one ten times, again one more frog can be placed. So total how many frogs can be placed? Ten plus one that is eleven. So that way.

Reema also mentioned that she wanted them to identify that “ten times point 1 makes the space for 1 more frog to fit on the 1 meter strip”. In another meeting, she discussed how the use of a meter strip supported the “movement from tenths to hundredths” among students. She added that the students showed flexibility in the use of a number line to represent decimals of different lengths, unlike selecting a number line to represent tenths only, as in the first year.

5.6.2.3 Knowledge demands

In the first year, Reema introduced decimals using the ‘guess and measure’ problem and the ruler. The choice of measures limited the discussion to the revision of whole number measures. Since the guess and measure problem was not linked, in this or the later lessons to decimal fractions, it remained as a stand alone lesson on length measurement. Zooming into students’ ways of drawing the estimated measures, the observations revealed that, students created markings on another object (such as a finger, eraser, length on the desk) and used these markings to draw the given lengths. For instance, some students marked 1 cm on the eraser and estimated the length of 7 such erasers to mark water up to 7 cm in a glass, while a few others kept doubling the length of 1 cm to get 8 cm and then took away 1 cm to get 7 cm. Students’ strategies varied from being simpler, such as the repeatedly adding 1 cm, to more sophisticated, such as doubling the length and then subtracting a length to get the required measure. Listening to these strategies and seeing their underlying mathematical potential is an important part of teacher's work (Lampert, 2001). One part of examining the mathematical potential of students' strategies includes identifying their correctness and the level of sophistication. A consequent teacher move could be discussing these strategies and organising them in increasing levels of sophistication. Such a discussion involves pressing some strategies more than the others (similar to how Brodie, 2011 defines pressing some meanings) with the aim of helping students learn them. One of the teaching goals while supporting students in moving from less to more sophisticated strategies could be uncovering the mathematical connections made in the process. For instance, in this case, students created markings on an eraser, finger, or desk and used them as referents to estimate measures. In Line 77 of Excerpt 5.14, the student created a 1 centimetre mark on her finger (referent). Here one centimetre is a standard measure, and the referent is the marking made on the finger. This referent was used to estimate and draw other lengths. An identification of such referent for the estimation task seemed common to several student strategies. However, the selection (or creation) of such referents is not sufficient. In order to use a referent for the estimation task, the relation between the referent and the standard unit needs to be understood and/or articulated. The process of estimation requires an iteration of the standard unit measure using an appropriate referent. While students suggested different strategies to solve the estimation task, Reema legitimised one strategy by calling it as the method (refer Excerpt 5.14). Her emphasis on the use of this method by all students seems to be closely tied to the belief about use of one method (or algorithm). Aligned with the belief is the responsibility of the teacher to 'teach' this method to the students. These beliefs seemed to be guiding her decision of avoiding challenging problems, such as the length of frog context, attributing it to lack of students' capability and difficulty in dealing with it.

In the second year, Reema introduced the students to the meter strip. Her questions elicited students' prior knowledge about half and double ("how many 2 cm parts are there on the strip? how many 4 cm and 1 cm parts are there on the strip?"), part–whole meaning of fractions ("what fraction of strip is 50 cm, 20 cm, etc."), measures corresponding to fractions ("how much is three quarters of the strip?"), length measurement (objects such as envelope measured using the ruler and meter strip), and the relation between measurement units ("How many 20 cm make 1 meter?"). The variations in questions about (a) equal partitions of 1 meter long strip by changing the size of the part, and (b) defining the size of each part while changing the whole to be 10 cm, 30 cm, 45 cm, 50 cm, 75 cm, 1 m); helped students in developing familiarity with the use of the meter strip. This familiarity supported students in using the meter strip to justify the tasks of the type – how many frogs of n centimetres will sit on the meter strip?. Reema gave different values to n, such as 30.5 cm, 2 cm, 1 cm, 4 cm, 10 cm, 25 cm, 30 cm, and 0.9 cm. Initially, students used the meter strip to iterate a length and find the number of frogs. Gradually, they used the relation between the different lengths (2 cm and 1 cm, 2 cm and 4 cm, 10 cm and 30 cm, etc.) to find the number of frogs. At the same time, students' thinking was shifting from being additive ($n + n + n + \ldots$ till the sum is 1 meter or closer to it such that no more whole of n can be taken away from it, n is the length of each frog) to multiplicative (n times the number of frogs should be equal to 1 meter) and proportional (If n is 10 cm and 10 such frogs can sit on the 1 meter long strip then for $2n$ or 20 cm, half of 10 frogs will sit on the meter strip). Towards the end, students moved to using division, and stated the solution to be 100 divided by the length of the frog.

One of the challenges of introducing a tool or representation in teaching mathematics is to create familiarity with it in ways that students can use it for the task. In this case, a variety of questions on finding equal parts of a whole, the size of each part, relation between different parts and with the whole; helped students in developing this familiarity. Through a series of such tasks or an exercise (Watson & Mason, 2006), Reema modelled using the meter strip as a tool for justification and the students used the understanding developed through a series of tasks to formulate an explanation for the problem on length of frog.

Reema's teaching showed the use of linear measurement to invoke students' prior knowledge, introduce new knowledge of decimal fractions, make links between the context and the linear representation, support flexible movement among measurement units, and offer explanations using the representation. Unpacking the affordance of the context included using it to build connections between students' knowledge and mathematical content, creating a challenging task, and supporting students to solve a problem. These aspects form an important part of teachers' knowledge required for teaching mathematics.

Secondly, knowledge demands have to do with handling multiple student responses. It is important to note that in the second year Reema refrained from passing a judgment as correct or incorrect on the student responses, unlike the first year. In the second year, she was listening to and recording student responses on the board and held whole class discussions to judge the correctness of these responses. She encouraged students to seek justifications, acknowledged students' contributions and revoiced their responses for public thinking (see Rows 11 and 13 in Table 5.4). The justifications she offered or summarised were built on students' contributions. For instance, after hearing different students' strategies to the length of the frog context, Reema offered a more sophisticated solution and together with the class found the answer to the problem. Evidently, Reema heard students' methods carefully, encouraged other students to restate and use these methods, and used the knowledge from these strategies to propose the solution. However, Reema's struggles in consolidating different students' responses and in linking them explicitly with the proposed solution, were notable. The difficulty faced by a teacher in organising different students' responses and connecting them with more sophisticated explanation(s) needs acknowledgement. Students' independent problem solving helped them to participate in the discussion towards the solution guided by Reema.

Reema required more support in the second year due to the nature of demands posed on her by such classroom situations. As discussed, these situations included increased student talk, multiple student responses, affordances of a context and model, and consistency within and across representations. In the second year, Reema demanded more discussions with the researcher prior to the lesson mostly around the content from the textbook and post–lesson on ways of handling students' responses.

5.6.3 Place values in decimal numbers

In both the years, a real life context was used to introduce place values of digits in a decimal number. After introducing the real–life context, students were shown the place value chart and expected to place different decimal numbers in this chart.

5.6.3.1 Year 1: Money context and place value

In the first year, Reema began the second lesson (Y1DL2) using the money context. Although the context was borrowed from the textbook (refer Figure 5.2), the numbers were modified.

1. How many paise does a matchbox cost?
2. How many matchboxes can be got for Rs 2.50?
3. How many rupees does the soap cost?
4. Arun wanted to buy a soap. He has a five-rupee coin, 2 one-rupee coins and 4 half-rupee coins. Write in rupees what money will he get back.

Figure 5.2: Money Context (NCERT, 2007, pp.138–139)

Reema asked students to read the price of the items, shown in Figure 5.2, to which a girl student read the cost of the pen to be “six rupees point fifty paisa”. Reema’s response was “I want the correct method” to which a boy student repeated the same response as the girl before. Reema told the students that the digits of the number after the decimal point are read individually, that is, “six point five zero”. After telling this rule, Reema asked the students to read the price of the other items. Some of the students read the amounts in the same manner as before, such as, “eleven point seventy five”. A student asked Reema the rationale for reading the prices with the decimal point, as the shopkeeper would say it as “eleven rupees and fifty five paisa” (instead of eleven point fifty–five). Reema responded that the shopkeeper will give the bill in writing and then proceeded to defining the decimal point as “differentiating rupees and paisa”. She emphasised that, “the number before the decimal point is rupees and the number after the decimal point is paisa.” (Refer Section 6.5.2 of Chapter 6 for a discussion in a teacher–researcher meeting on this conception.) Reema discussed the relation between rupees and paisa and recalled students’ knowledge about fractions in order to express 1 paisa as rupees. After the students stated the relation between paisa and rupee, Reema recalled the relation between millimetre and centimetre. The movement from the currency to the measurement context could have been made to introduce both the tenths and hundredths place value. After the students stated the relation between millimetre and centimetre, Reema told them how to express a fraction as a decimal. While she stated that, “one by ten is written as zero point one”, she showed the complete division of 1 with 10 to get the quotient 0.1. She introduced the rule to check the number of zeroes with the number of digits after the decimal point, when converting from fraction to a decimal representation. She added that “one zero (in the denominator) means that we move the point to the left side, one digit.” She defined the default position of the decimal point to be on the extreme right of the number. After showing the decimal part to the right side and the whole number part to the left side of the decimal point, she defined the decimal point as separating the whole number and decimal part. She recalled the place value names of the whole number part and then stated the name for the place value names of the decimal part. A student asked, “where is the oneths?”, to which Reema responded that, “directly we start from tenths, one upon one is the same only”. She made the place value chart and told students how to place decimal numbers in the chart. Then, she invited students to write the given numbers in the place value chart. A student checked for the position of oneths in the place value table. The question about oneths recurred from a student who asked, “Teacher after the point is tenths, no? So for oneths?”. Reema restated her earlier response, “division by one is the same number”. (The students stayed with the question about the oneths as a place holder until the end of the lesson, as they stated this question to the researcher after the lesson.) Students were invited to write different decimal numbers in the place value table. Note the kind of errors students were making in writing the decimal numbers in the chart (refer Table 5.6a).

Table 5.6 (a): Error in placing decimal point in place value table

Number	Hundreds	Tens	Ones	Tenths	Hundredths	Thousandths
36.50	3	6	.	5	0	
14.4	1	4	.	4		
2.04			2	.	0	4

Reema modified the place value chart to create a separate position for the decimal point (refer Table 5.6 b). Now, the students wrote the numbers in the place value chart correctly, without looking at the place values.

Table 5.6 (b): Place value table with decimal point

Number	Hundreds	Tens	Ones	Decimal Point	Tenths	Hundredths	Thousandths
36.50		3	6	.	5	0	
14.4		1	4	.	4		
2.04			2	.	0	4	

Reema introduced the decimal place value using the money context, which was familiar to students. Using this context, she could introduce the "hundredths" place value. In order to introduce tenths and hundredths, she shifted to the relation between measurement units of centimetre, millimetre and meter. She realised that the measurement context could be used to introduce both the place values consistently. However, she left both these contexts (money and measurement), after introducing the place value names and moved to the place value chart. Since the relation between the consecutive place values was not elaborated, and the emphasis was on the place value names, the question about a place for oneths appeared a few times during the lesson. Although Reema provided a reason for the non–existence of oneths as a place value, it is not clear whether students were prepared to comprehend this reason of "one upon one is same only", an explanation offered by Reema. This way of understanding place values was not used to define the other place values, which could have possibly lead to the recurrence of the question. In between, Reema also explained the movement of the decimal point to the left when dividing by powers of ten, under the heading, conversion from fractions to decimals. However, I did not find students engaging with this conversion perhaps because they were still getting familiar with the decimal place values for which the anchoring explanation came from the place value names of whole numbers. In the end, when Reema asked students to write decimal numbers in the place value chart, some students re–wrote the number in the chart, without paying attention to the place value of the digits. Reema's move to create a place for the decimal point in the place value chart, seems to have reinforced students' technique of re–writing the number, instead of drawing their attention to the place value of the digits in the number.

5.6.3.2 Year 2: Measurement context and place value

In the fourth lesson on decimal numbers in the second year (Y2DL4), Reema took up the decimal place values. She had introduced tenths and hundredths in the previous lesson, using the length measurement context. Students had extended the place values to thousandths and beyond. In this lesson (Y2DL4), Reema aimed to draw students' attention to the relation between different place values. When the observer entered the room, a student Ashwin was arguing that the place value of tens is "1 multiplied by 10" and not "10 multiplied by 1". Reema was probing the difference between these two through an example. She wrote a number on the board "13.9" and asked him to state the difference. Ashwin revised his explanation and stated that they are the same, however, he continued that the relation is 10 times, when moving from ones to tens. When Reema asked about how to express ones in tens, students stated the relation of one–tenth, and with some probing extended the same relation to hundreds and tens, ones and tenths, tenths and hundredths, and so on. Then Reema asked, "how can we get tenths from hundredths?", and the students articulated the relation to be ten times. The class reached the conclusion that, "moving to the left of the place value table means increasing the place value by ten times and moving to the right means the relation is one–tenth". Unlike the previous year, the left and right were not defined with respect to the decimal point. At the end of this discussion, a place value table was drawn on the board (refer Table 5.7).

Table 5.7: Place Value Table (Y2DL4)

Thousands	Hundreds	Tens	Ones	Tenths	Hundredths	Thousandths
1×1000	1×100	1×10	1	$1 \times \frac{1}{10}$	$1 \times \frac{1}{100}$	$1 \times \frac{1}{1000}$

After introducing the place value relation, Reema invited students to write some numbers in the place value table. The students read aloud the decimal numbers with place values and then placed them in the table. For instance, a student read the number 35.6 as "thirty means three tens, five ones and six tenths" and then placed the number in the table. While students were invited to write the numbers in the place value table, other students evaluated or supported the peer who was invited to write on the board. One of the students, Ishita challenged the teacher by saying that the "place value table is wrong" as there should be a place for decimal point between ones and tens. Reema responded to the student by stating how decimal point does not hold a place value and is used to separate the whole number and the fractional part.

The students started to discuss the relation between different place values among themselves, for instance, the difference between thousands and thousandths, using the explanation discussed in the beginning of the lesson, of ten or one–tenth times from the ones place. Reema asked students to make the place value table in their notebooks and record the numbers written on the board.

In the next part of the lesson, Reema wrote down the prices of different items (refer Figure 5.2) on the board. She asked the students to read these numbers. Reema told the students that the decimal point "separates rupees and paisa". Unlike the first year, in this lesson she added that, if the price is read without the decimal point, the two units rupees and paisa are separated but the decimal point ensures that the decimal number as a whole is a measure in a single unit. Like the frog context, Reema asked several questions about the information given in the problem in order to make students familiar with the price of different things. A few examples of these questions include, "let us read the cost of all the things", "fifty paise means how many rupees", "how are the rupees and paisa separated". She then posed the question, "The cost of one matchbox is 50 paisa, how many matchboxes can Arun buy in two rupees and fifty paisa?". Students offered different ways to solve this problem. The familiarity with the money context helped the students in justifying their responses. Similar to the frog context, the relation between number of matchboxes and the amount was tabulated to find a general relation (refer Table 5.8). The additive relation between the increasing number of matchboxes (cost of 3 matchboxes is 50 p + 50 p + 50 p) was followed by the use of additive and multiplicative relation (cost of 3 matchboxes is the sum of the cost of 2 matchboxes and 1 matchbox and 2 is the double of 1).

Table 5.8: Cost of Matchboxes (Y2DL4)

Number of matchboxes	1	2	3	4	5	10
Cost	50p	100p	150p	200p	250p	500p

At the conclusion of the problem, Misha, a student came up with the explanation of finding the number of matchboxes by dividing the total amount with the cost of 1 matchbox (number of matchboxes is equal to 250 paise by 50 paise). When invited to show Misha's method on the board, Daksh, another student showed multiplication of 50 and 5 to get 250 and justified it by saying that the cost of 5 matchboxes is 5 times the cost of 1 matchbox. The students continued to solve other tasks related to this problem. While solving tasks such as these, students practiced conversions between rupees and paisa. In contrast to the first year, no rule about the relation between the number of zeroes and number of digits after the decimal point was noted in this lesson.

In this lesson, Reema began with the measurement context, which was introduced in the previous lesson. She extended the place value of tenths and hundredths to other place value names. Clearly, an extension of place value names, is stemming from the familiarity with the place value names of whole numbers. However, this familiarity with the whole number place values was not extended or repeated by the teacher and was just assumed. Instead, the relation between consecutive and non–consecutive place values was emphasised. Unlike the first year, Reema objected to creating a place for the decimal point in the place value chart, because she considered it mathematically incorrect. She justified it to the student and the overall emphasis on place value in the lesson supported this justification. The money context was used to reinforce the place value relation of hundredths. Reema made an important observation about how the decimal point indicates that the amount, as a whole, is a part of the bigger unit (in this case, rupees). In the absence of a decimal point, the amount can be read as two separate units (rupees and paise). Further, when posing the tasks around the money context, Reema encouraged students to think of different methods. She instructed the students to carefully see the methods that have been already shared by their classmates and are listed on the board, and suggest a method only if it is different from the stated methods. Also, notable is the students' movement from additive to multiplicative strategies when solving the same task.

5.6.3.3 Knowledge demands

The discussion on place values in decimal numbers using a context poses at least three different kinds of knowledge demands on the teacher. These are the choice of the context to introduce place value, the appropriate use of the context, and the extent of "pressing" of analogy between place value names in whole numbers and decimal numbers.

In the first year, Reema used the money context to introduce place value names. She realised that this context allowed her to extract the relation of hundredths only. The decision to shift to the measurement context in order to introduce tenths and hundredths indicates that Reema realised the limitation of the money context for the introduction of place value names. In contrast to the money context, the measurement context is more comprehensive since it can be extended to the place value thousandths (and more). The appropriateness of the measurement context over the money context for the introduction of place value is a crucial decision in order to direct students' attention to the relation between place values. A related knowledge demand is to think of the limit to which a context can be extended.

In the first year, it was noted that Reema (like the other teachers) mostly used a real–life context to introduce a sub–topic. The context was not extended through the lesson. At other times, a context might be a word problem given in the textbook. Within the same lesson, teachers made shifts to different contexts. While this teaching strategy might work in covering a larger set of contexts within a topic in a shorter span of time, it does not provide students ample time to engage with the richness of a context. The choice of an appropriate context demands that it has the potential to be used consistently or comprehensively, as in the case of place values. Another consideration, also discussed elsewhere, is the relation between the context and the representation. In other words, how appropriately does a context align with the representation, for instance, the alignment between linear representation and the measurement context. The third knowledge demand has to do with the relation between the place value names for whole numbers and decimal numbers. I conjecture that since the analogy was not "pressed" or repeated often by the teacher in the second year, students focused on the relation between place values and not on the naming convention so much to think about the oneths question.

5.6.4 Fractions, decimals and the grid representation

The relation between fractions and decimals is an important aspect of decimal teaching, mainly, for two reasons. First, like whole numbers, fractions are taught at earlier stages and therefore are used as prior knowledge for the teaching of decimals. Second, decimals are considered to be a more precise representation of a quantity over fractions. One of the sub–topics in the teaching of decimals is learning how to do conversions between fractions and decimals. In the first year of teaching decimals, Reema used the connections between fractions and decimals, twice. She used it to introduce decimal numbers, and for conversions between fractions and decimals and vice versa. In contrast, fraction representation was used as a justification for several other sub–topics, such as, decimal representation on a number line, expanded form, equivalent decimals, comparison of decimals, addition and subtraction, etc., in the second year.

5.6.4.1 Year 1: Fractions and decimals

In the third lesson (Y1DL3), Reema revisited the reading of decimal numbers. After a student read the number "nine point three zero", the next question was to read "40.05". A student asked the reason for reading the zero in the number (referring to 40.05), since zero does not have a value. Reema responded briefly to the question and postponed a detailed discussion on how the position of zero in a number changes its value (refer Excerpt 5.18).

Excerpt 5.18: Reading zero in a number (Y1DL3)

Line No.	Speaker	Utterance
14	B St	Teacher why do we say forty point zero five? Zero does not have a value. So, why do we say zero?
15	Reema	Zero has value before the number. That is after [the number] it doesn't, there is no value. We will come to that, okay? Wait for some more periods. … The point we are going to cover after two periods. Is it clear?

Reema then asked the students to convert fractions into decimals. She began with fractions $\frac{1}{10}$ and $\frac{1}{100}$. While some students answered the decimal equivalent correctly, a student said "$\frac{1}{100}$ is 1.1". Reema corrected him and stated the rule to convert decimals into fractions, that is, "Then, one more thing I taught you, if there are two zeroes in denominator, so point will move up to this, one and two. Here, there are no two digits, so you have to put a zero and point will come. So point zero one." Followed by this, Reema asked students to convert fractions such as "$\frac{5}{100}, \frac{31}{100}$" to decimals. Reema told the students to use the explanation (given above) in a reverse order when converting decimals to fractions. When she asked students to find the fraction equivalent of "3.6", different responses came up (refer Table 5.9).

Table 5.9: Students' responses: Decimal to fraction conversion (Y1DL3)

$3.6 = \frac{3}{6}$	$3.6 = \frac{6}{100}$	$3.6 = \frac{6}{10}$	$3.6 = \frac{3}{60}$	$3.6 = \frac{6}{30}$

Reema continued to ask this question to individual students until she found a student who gave the correct response. After the correct response, Reema explained the procedure for the movement of the decimal point, depending on the number of zeros in the denominator, in other words, powers of ten (refer Excerpt 5.19). Following this explanation, Reema asked students to convert a few other decimal numbers into fractions. While students made similar errors, such as those listed in Table 5.9, Reema continued to correct their responses.

Excerpt 5.19: Procedure of conversion from decimal to fraction (Y1DL3)

Line No.	Speaker	Utterance
84	Reema	Correct $\frac{36}{10}$. Now listen here. How it will be $\frac{36}{10}$? All listen. I just showed you $\frac{1}{10}$. This is one you know in the numerator. Yes and in the denominator there is ten, one zero. Ten is one and one zero. So point will move from right to left by how many digit? One digit. So point is always here and it will move here… So when we have to remove it, now this point one, when we have to remove the point, what you have to do? Just write one divided by, after point one digit, so here divided by ten. Is it clear? So here also we have three point six, so after points how many digits are there?

Reema told the students how to convert a fraction into its lowest form by finding the common factor of the numbers in the numerator and the denominator. The explanation was followed by a few questions on fraction to decimal conversion, such as, "$\frac{36}{2}, \frac{365}{100}, \frac{35}{5}, \frac{25}{5}$". The series of questions (a to h given below) asked by Reema was mostly the same for each of these parts.

(a) Is the numerator divisible by 2?,

(b) Is it divisible by 5?,

(c) Is it divisible by any other number?,

(d) Is the denominator divisible by that number?,

(e) What is the numerator divided by that factor?,

(f) What is the denominator divided by that factor,

(g) Can it be divided any further?, and

(h) So, what is the answer (lowest form)?

Reema asked the students whether they have understood the procedure of converting a decimal to fraction and then to its lowest term. Two students expressed that they did not understand the procedure. As a response, Reema took another fraction and asked these two students the same set of questions, as listed above (a to h). The last task in this lesson was to practice what was learnt and record it in the notebook. Reema wrote a few numbers on the board and asked students to note these down (refer Excerpt 5.20).

Excerpt 5.20: Board work - Decimal to fraction conversion (Y1DL3)

Line No.		Utterance
535–537	On the board	Change the following into fraction and simplify. (a) 2.4 = (b) 0.25 = (c) 1.6 = (later changed to 1.05) (d) 0.16 = (e) 5.5 = (f) 7.5 = (g) 10.00 =

She wrote the answers for each of these parts on the board, sometimes asking the same set of questions to the students and at other times, solving the problems by herself. Students were asked to copy these procedures in their notebook.

Reema taught the connection between fractions and decimals for the purpose of doing conversions. The rule for conversion was stated explicitly and students were expected to follow it. When the students expressed that they were unable to understand, the teacher repeated the procedure and asked students to pay close attention. Listening carefully to the teacher who is repeating a procedure, is a common pedagogical strategy used to respond to students who do not understand. Reema avoided a detailed discussion on the students' errors on conversion of 3.6 to fraction, and similar questions in the second half of the lesson. Instead, she waited for the student who would give the correct answer, without stating the reason for it, to close this episode. Her conclusion was repeating the rule for conversion from fractions to decimals. I do not know the reason for not taking up a discussion on students' errors, but classroom observations reveal that this was indeed a common teaching practice. The questions posed during the lesson included conversions between fraction and decimal. Equivalent fractions were discussed, but the phrase was not mentioned, to convert fractions into their lowest form. The variety in the questions was in terms of having tenths or hundredths place value. The last set of questions (Excerpt 5.20) indicates another variation that the whole number part was either zero or non–zero.

5.6.4.2 Year 2: Fractions and decimals: Linear to Area Representation

In the second year, Reema began discussing the relation between fractions and decimals, from the second lesson (Y2DL2). While the students were working with the frog context, Reema used students' intuitive knowledge about "$\frac{1}{2} = 0.5$" through the task "what is half of the metre strip?". She connected "$\frac{1}{2} \times 1$ metre" and "$\frac{50}{100}$ m" with "0.5 m". She extended this knowledge to introduce "$\frac{1}{10} = 0.1$". Students made sense of 0.9 mm in different ways, such as, "$\frac{1}{10}$ of 9", "1 mm less than 1 cm", "0.1 cm less than 1 cm", and "9 times 0.1 mm". In the third lesson (Y2DL3), Reema asked students to complete the length measurement task given in the textbook. The task involved measuring the lengths of four candles and expressing the measures in centimetre and millimetre. Later, students were asked to express these lengths using centimetre only. Reema asked students to explain how to write the length of the first candle, that is 2 cm 9 mm, using centimetre only. Reema then asked, "why point nine only? Why not zero point nine? Or why not point zero zero nine?". Two students tried to justify their response, 2.9 cm. They formulated an explanation using 1 cm length, stating that, "0.9 mm is close to 1 cm, so it is not full centimetre, less than 1 (cm)". Another student extended it by saying that "it (0.9) is one by ten centimetre less than one centimetre". They recalled that 9 times 0.1 gives 0.9 and justified the placement of the decimal point before 9 in 2.9. The class went on to discuss the measures of other candles found by the students, followed by conversions from centimetre and millimetre to centimetres only.

Reema brought up the conversions discussion briefly again in the fourth lesson (Y2DL4) when doing the money context. Students were asked how to write 50 paisa and 25 paisa in different ways. This was extended to writing the amounts in paisa as rupees. Students connected the fraction representation $\frac{25}{100}$, one–fourth of one whole, a quarter, with its decimal equivalent 0.25.

In the sixth lesson (Y2DL6), Reema held a detailed discussion on conversion from fractions to decimals using a grid. She introduced the students to 10 strips, which were further divided and extended to a 10×10 grid (refer Figure 5.3).

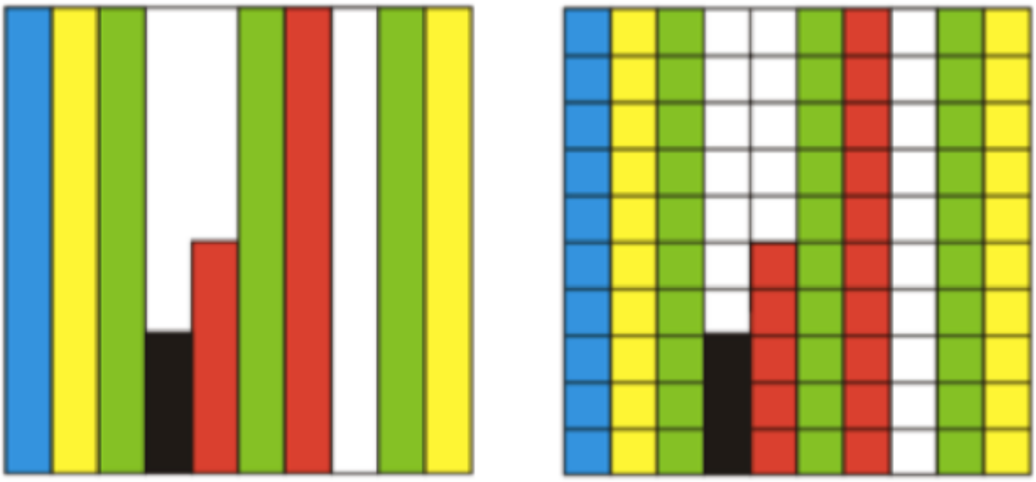


Figure 5.3: Fraction–Decimal Grid (NCERT, 2007, p.140)

Like the meter strip, Reema asked several questions to create familiarity with this representation. She asked, "what is half of this whole", "how can half be shown in different ways?", "what is half of each row?", "how can this half be represented in terms of the whole?", etc. The students mentioned these measures in decimals and fractions, and Reema recorded this discussion on the board. While seeing the measure of half of a row as 0.05 or $\frac{5}{100}$, Reema directed students' attention to how these two representations are connected. She asked students "zero point zero five, how did you get it?", to which a student Misha responded, "hundred has two zeroes". This was followed by a set of questions (adapted from the textbook), where Reema shaded strips in different colours and asked their fraction and decimal equivalent. The explanations emerging from these set of questions were of the kind, "one out of ten strips [is blue]", "one–tenth is blue", "zero point one of the whole square is blue". Similarly, 2 such strips meant "two times 0.1" or "0.2". Then, students were asked to find the fraction of area in red. The red colour covered 15 small squares of the 10×10 grid. The responses included, "1.5, one strip and half, $\frac{15}{100}$". The difference in responses was not pursued at the moment and was taken up later in the lesson. Thus, in first half of the lesson, Reema posed questions about the task of expressing the shaded regions and recorded this discussion carefully on the board (as shown in Excerpt 5.21). Since the shaded regions of each set was the same, students observed 0.1 is equivalent to 0.10.

Excerpt 5.21: Fraction and decimal equivalents (Y2DL6)

$\frac{1}{10} = 0.1$	$\frac{10}{100} = 0.10$
$\frac{2}{10} = 0.2$	$\frac{20}{100} = 0.20$
$\frac{3}{10} = 0.3$	$\frac{3}{100} = 0.30$
$\frac{15}{10} = 1.5$	$\frac{15}{100} = 0.15$

Students' prior knowledge of finding equivalent fractions was explicitly used as a justification for finding equivalent decimals. During the summary, while each part was being discussed, Reema noticed an anomaly in the last fraction. She asked the students that, even though 1.5 and 0.15 marked the same regions on the grid, "are they equal?". It is unclear whether Reema figured at this point, why despite showing the equivalent regions, the fraction or decimal equivalents did not match. When a student mentioned, "zero point one five is wrong", Reema took a moment and checked the shaded representations for 1.5 and 0.15 on the grid. After this pause, Reema asked the students, "why is zero point one five wrong?". While she helped students understand that 15 small squares from 100 small squares will be represented as $\frac{15}{100}$, a student said, "1.5 is wrong". When Reema asked for the reasons, a student tried to explain that "5 is half of 10", so 1.5 is "one and half out of 10". Reema posed the question about "how do we write half out of 10", to which a student proposed, "$\frac{1.5}{10}$". Reema revoiced the discussion on "1.5 divided by 10". Using equivalent fractions, the class discovered the equivalence of "$\frac{1.5}{10}$ and $\frac{15}{100}$". After revising the discussion on equivalent decimals, Reema posed the question whether "5 paisa is the same as 50 paisa". The students found ways to distinguish between these and the lesson concluded by showing the equivalence of 0.5, .5, 0.50, 0.500 using their fraction equivalents. Towards the end of the lesson, Reema removed the grid lines, and showed students a square. She asked students to shade 0.50 of it as yellow, 0.45 as red and so on. Students discussed among each other and completed this task.

In the second year, I see that Reema has been encouraging students to use the fraction knowledge in different lessons while teaching decimal numbers. She extended the measurement context from the previous lessons and used a measurement task to introduce the conversions between fractions and decimals. Reema used the context of measuring the candles to do conversions by strategically placing the shaded regions to helping students discover the equivalent decimals. Students' movement between different representations stands out, for instance, in interpreting measures such as, half of a meter, 0.9 as a decimal number, 1 and half of a whole, etc. The flexibility in seeing the decimal numbers was supported by the area representation, that is, the shaded region. Reema discovered the discrepancy in representing 1.5 along with the students. Unlike the first year, Reema was now open to accepting her doubts in front of the students, and in resolving them while teaching in the class and in discussion with the students.

5.6.4.3 Knowledge demands

Like the analogy between whole numbers and decimal numbers, the analogy between fractions and decimals has its affordances and constraints. The affordances lie in the use of fraction representation as a justification for locating a decimal on a number line, representing equal parts of the unit, finding equivalent decimals, comparing decimals, multiplying and dividing decimals by powers of ten, and so on. All these affordances were used in the second year of teaching. The constraints were visible in the students' responses, such as those listed in Table 5.9, where students mis–understand the whole number part as the numerator and the decimal number part as the denominator of a fraction. It is evident in students' responses such as, $\frac{3}{6} = 3.6$ and $\frac{8}{10} = 8.10$. Another noted mis–interpretation emerges from a mixing of the 'numerator as whole number and denominator as the fractional part' with the rule of placing the digits after the decimal point based on the number of zeroes in the denominator. This is evident in responses such as, $\frac{8}{10} = 8.0$, $\frac{50}{100} = 50.00$. The knowledge demand has to do with how the fraction knowledge is invoked while doing decimals. For instance, Reema used students' fraction knowledge to represent a measure in different ways. It is in this context, that she introduced the decimal number 0.1. The relation between one–tenth (indicates a fraction and used as a decimal place value name) and 0.1 (decimal representation), was used to interpret other measures such as 0.9, in the second year. The back and forth movement between the fractions and decimals, encouraged by Reema, helped students in using fraction equivalents as justifications for several sub–topics within the learning of decimal numbers.

The second knowledge demand has to do with an emergent practice in Reema's teaching, which is the use of variations, to strengthen a new context or representation. Her use of a variety of questions or an exercise helped in developing a familiarity with the representation and in beginning to use it as a justification for several related tasks. This careful moderation of variations is an important teaching practice, particularly when the teacher intends to draw students' attention to the key ideas.

5.7 Discussion: Knowledge Demands in Teaching Decimals

In this chapter, I have discussed the knowledge demands posed on a teacher as she becomes more responsive to students' thinking while teaching decimal fractions over 2 years. This section is organised around the two questions which were raised at the beginning of the chapter. First, what are knowledge demands entailed in the teaching of decimal numbers and second, how does an abstraction of knowledge demands through an analysis of teaching enrich our understanding of the specialised knowledge required for teaching.

I saw that in both years, the teachers focused on making students learn the decimal place values by connecting them with the whole numbers and fractions, but their emphasis in Y1 was on the procedures to read, write, compare, and identify place values in decimal numbers. The knowledge demands of such an approach were more restricted in comparison with the knowledge demands in Y2, where the teachers were more responsive. Teachers' responsiveness was evident in their readiness to deal with contingencies by listening to students and the ability to manage the challenges that arose thereby (Potari & Jaworski, 2002). In addition, responsiveness manifested in teachers' ability to anticipate pathways for student learning and to manage the complex relation between general knowledge and particularised knowledge that arose in the classroom through examples and students' utterances.

An examination of paired episodes helped in abstracting the knowledge required for teaching decimals. A summary of the knowledge demands in teaching decimals can be found in Table 5.10. The phrases used in the table are not precise descriptions of the knowledge demands, but are indicative. It is necessary to refer to the discussion above to appreciate their richness and situated nature.

Table 5.10: Summary of knowledge demands in teaching decimals

	Paired Episodes		
Sub topic	**Year 1**	**Year 2**	**Knowledge Demands in Decimal Teaching**
Whole numbers and decimals	Place value in whole and decimal numbers	Why are there no oneths?	(a) Affordance of the mirror metaphor in place values. (b) Extent of analogy between whole and decimal numbers. (c) Use of fraction representation in place value based explanation.
Position of zero	Position of zero in a decimal number	Position of zero in relation to place value	(a) Cases of position of zero in a decimal number. (b) Comparison of numbers with zero at different place values. (c) Offering and sequencing examples and counter–examples or cases.
Measurement	Conversion of measurement units	What is one division after one centimeter?	(a) Flexible movement between representations. (b) Consistent use of linear representation for different place values. (c) Appropriateness of the length measurement context for decimal place values.
Introduction to decimals	Guess and Measure	Guess and Measure Length of the frog	(a) Use of referents (or markings) in the process of measurement. (b) Variations in tasks which invoke relevant prior knowledge and make it accessible. (c) Affordance of the length measurement context for teaching decimals.
Place value	Money context and place value	Place value and the measurement context	(a) Purpose of using a context to show relevance of the topic or for application of learnt topic. (b) Affordance of the context with respect to the mathematical idea being demonstrated or used. (c) Analogy between whole numbers and decimal numbers.

Table 5.10: Summary of knowledge demands in teaching decimals

Sub topic	Paired Episodes Year 1	Year 2	Knowledge Demands in Decimal Teaching
Fractions, decimals and the grid representation	Fractions and decimals	Fractions, decimals and area representation	(a) Using the fractions representation for the purpose of justification. (b) Focus of variations in a problem (familiarity with the situation and the sub–topic).

Teaching of decimal numbers requires an awareness of the ways in which students' understand, interpret, and extend their understanding of whole numbers while making sense of decimal numbers. Here, I recall Doerr's (2006) dimensions of teachers' knowledge needed for responsive teaching, which include knowledge of pathways of student learning, ways of listening to that development and ways of responding and supporting such development. I stress that the interaction of prior knowledge with new knowledge underlies all three dimensions. It is necessary for the teacher to engage with the possible intersections between whole numbers and decimal numbers (position of oneths, counting the digits strategy, annexing zeroes, longer is larger, smaller is larger, etc.), and between fractions and decimals. Further, the teacher needs to support students in developing the tools to formulate conjectures, justifications, and generalisations using their existing knowledge resources. As the analysis of the episodes reveals, a rich knowledge base underlies responsive listening and teaching.

In the introductory sections, I discussed examples of knowledge demands for teaching decimals that could be gleaned from the research literature on students' difficulties and teaching approaches. The analysis revealed that some of this knowledge is invoked in the course of the teacher managing classroom interaction. The constraints imposed by students' prior knowledge about natural numbers on the development of the rational number concept are well acknowledged (Vamvakoussi & Vosniadou, 2007) in the research literature, as is the need to design specific diagnostic tasks to understand and address students' conceptions. I found that although teachers were reportedly not aware of the literature on students' conceptions, they could identify some common errors in students' ways of solving problems as in the episode on the effect of the placement of zero in a decimal number. However, this was not enough to lead teachers to orchestrate discussions around the errors or to ways of dealing with them effectively. A part of the knowledge that teachers need, therefore, is an awareness of research findings on topic–specific student conceptions. The discussion in this chapter focuses on the initial teaching of decimal numbers. Ways in which teachers can support students in building explanations for other conceptual aspects like rounding off, recurring decimals, density, rational approximations of irrational numbers and operations with decimal numbers would need further investigation.

The analysis reveals the strengths as well as the limitations of the mathematical knowledge for teaching (MKT) framework (discussed in Section 2.3.4 of Chapter 2). Some of the teachers' actions in responding to the students can be described generically as common tasks of teaching that are not specific to the topic at hand. These include recognising the students' mistaken lines of thinking, identifying a student's utterance as important and revoicing it for reflection, accepting a student's argument as correct, reformulating a student's argument, calling attention to specific aspects of the mathematical statement, posing questions for further thinking, challenging students' existing notions, generating a sequence of examples and counter–examples, and leading the class towards generalisations or conclusions. Underlying these generic actions are aspects of knowledge specific to the topic of decimal numbers that have been identified in our analysis. The construct of ‘‘common tasks of teaching'' is methodologically powerful, the categories identified above direct attention to the mathematically significant moments in classroom interaction. However, the analysis also shows that the teachers' actions are guided by an interaction of the various dimensions of knowledge identified in the MKT framework. The construction of knowledge in the classroom is about creating pathways of learning in the topic domain, where elements of common content knowledge (CCK), specialised content knowledge (SCK), and knowledge of content and students (KCS) interact with one another. It is not possible to separate knowledge that is exclusively the province of the teacher from the knowledge that is projected as content to be learned by the students, a distinction that is implied between CCK and SCK in the MKT framework. The knowledge that emerges in the course of the interaction is initiated as much, if not more, by the students' utterances as the teacher's.

Some of the knowledge demands that have been identified above may be suitably organised for mathematics teacher education or teacher professional development. However, I stress on the dynamic, rich, and situated nature of the knowledge demands as revealed by the analysis. Along with Hodgen (2011) and Adler (1998), I note that knowledge that emerges in situ in the classroom is dynamic in its specific forms – the examples, the lines of reasoning, the formulation of questions and sentences, all have a degree of contingent variability. Such variability is difficult to capture in terms of a well demarcated and mapped body of knowledge that can be canonised for teacher education. At the same time, it is valuable in supporting a teacher's journey into a culture of responsive teaching. While the teachers' abilities to deal with contingencies are extremely important in shaping students' learning, the knowledge demands that this entails cannot be easily systematised. It is likely that case–based or episode–based learning, rather than knowledge organised merely in the form of principles or assertions, is a better way of gaining the rich and situated knowledge that is needed to deal with contingencies that arise in the course of classroom teaching. In the course of analysing the knowledge demands posed by contingencies arising in classroom situations, teachers may abstract the topic–specific knowledge required for teaching in explicit and implicit ways. The mathematical knowledge demands for the teaching of decimal numbers that have been described through this analysis are particularised and contextualised for the classrooms being studied. However, the recurrence of contingencies points to their generalisable character and are significant moments for which (both pre– and in–service) teachers can be prepared. Through the analysis, I have argued that although knowledge that is manifested in actual teaching is bound to particular examples, reflection on such knowledge is likely to lead to more generalised knowledge that is useful in responsive teaching, because it allows for better anticipation and resources in responding to students. This is an important implication for how we design teacher education inputs that are focused on subject matter. I envisage discussions around the complex and varied paths that are taken in particular lessons, informed by systematically organised topic–specific subject matter knowledge specialised for teaching as forming the core of such inputs.

Chapter 6

TEACHER LEARNING FROM TEACHER–RESEARCHER MEETINGS

Architecturally many elementary schools are "egg crate" structures, where individual classrooms are self–contained. Teachers spend most of their time with students in their own classrooms and as a result have little interactions with other teachers. When teachers do interact there tends to be a norm against asking one another for help: to do so admits failure. (Liston & Zeichner, 1990)

6.1 Abstract

In this chapter, I discuss the nature of support provided to the teachers through teacher–researcher meetings (TRMs), which happened in the after–school hours. The chapter discusses the kinds of knowledge that were focused during these meetings and how this knowledge might have been helpful to teachers. Tasks planned to orchestrate teacher discussions have been used as nodes to organise the nature of knowledge that was elicited and developed during the meetings. It is argued that together these classroom–based tasks have the potential to offer a structure for organising teacher professional development initiatives aimed at enhancing teachers' knowledge and sensitivity to students' mathematical ways of thinking. The analysis reveals that through an engagement in these tasks, teachers and researchers became a part of an organically emerging community, which utilised artefacts from classroom practice for reflection on and improvement of teaching practice. An important way forward, implied from this study, would be to initiate the development of teacher–researcher communities around students' artefacts to impact classroom practice.

6.2 Central Questions

In Chapter 5, a comparison of paired lessons (of two years of teaching), revealed that there was an increase in participating teachers' responsiveness towards students' mathematical ideas, from the first to the second year of teaching. The question that arises is, what could have possibly led to this change? In this write up, I attempt to analyze the data from the support provided to the teachers through teacher–researcher meetings to answer the following questions:

1. What kind of knowledge was addressed during these meetings?
2. How was this knowledge useful in handling demands arising from contingent classroom situations in the second year of teaching?

In Section 2.5 of Chapter 2, I presented three kinds of relations between knowledge and learning, proposed by Cochran–Smith and Lytle (1999). In this study, teacher–researcher meetings were aimed at developing knowledge *of* practice through inputs from the community of teachers and researchers. The inputs from teachers included recalling and interpreting their experience of working with students, their own relation with the subject and their views about how students learn mathematics based on the experience of teaching and learning of mathematics. The researchers brought in their knowledge of the literature in mathematics education, particularly, focusing on students' errors, thinking, conceptions and learning.

In this chapter, the data from discussions during TRMs is used to unpack how certain aspects of teacher knowticing (Even, 2008) were developed. This was done by building on teachers' existing knowledge (developed over time from their experience of teaching), bringing in relevant topic–specific literature for deepening teachers' knowledge and seeking connections with their classroom practice. The analysis shows how an engagement with the knowledge situated in teachers' practice enabled an environment for challenging deep–rooted teacher beliefs and created opportunities for collaborative learning.

6.3 About Teacher–Researcher Meetings

Apart from the researcher–teacher interactions before and after the lessons, an important part of the research design was meetings among teachers and researchers. These after–school meetings happened roughly once in a week and were located in the mathematics laboratory of the school. The meetings were planned in the second phase of the study which overlapped with the third phase. To recall, a brief description of the Phases of the study can be found in Figure 6.1. The details of each phase can be found in Section 4.6 (of Chapter 4).

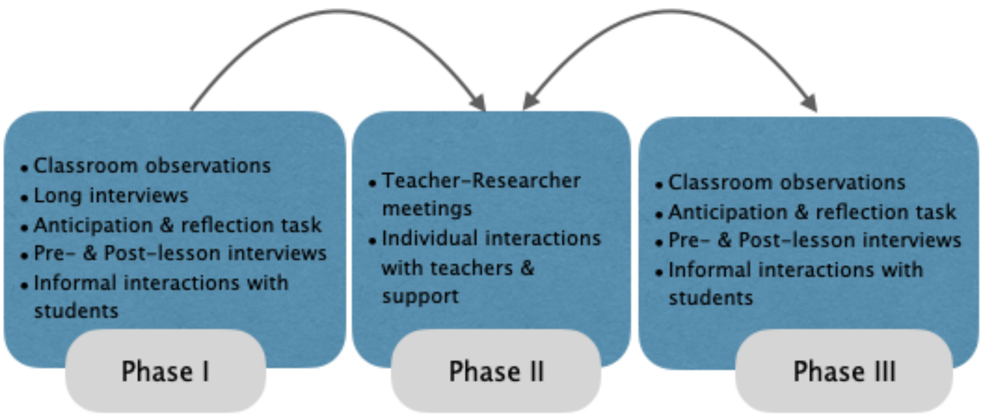


Figure 6.1: Phases of the Research Study

6.3.1 Planning of TRMs

The classroom observations and interviews from the first phase helped in developing a sense of teachers' practices, particularly those, which were related to interpreting and drawing on students' ideas. The artefacts of students' work selected for discussion during TRMs (in Phase 2) were taken from these classroom observations (of Phase 1). Students' work was selected based on its mathematical salience to the topic being discussed, for instance, the questions about oneths or position of zero. Other areas of difficulty in teaching a topic were identified through discussions with the teachers, for instance, which is a better representation for decimal numbers. A summary of the tasks, planned for the twenty TRMs held during the period of July to December 2013, is listed in Table 6.1. Since the data collected from the first phase influenced the design of tasks used during TRMs, these tasks are called classroom–based tasks.

Table 6.1: Summary of Teacher-Researcher Meetings

TRM	Duration	Content
1	00:54:38	Content for the preparation of booklet for teachers.
2	01:05:00	Relation between whole numbers and decimal numbers: Focus on student errors.
3	00:56:15	Relation between whole numbers and decimal numbers: Focus on the content.
4	01:03:46	Digit–based approach to comparison of decimal numbers.
5	01:02:05	Meanings and representations of integers.
6	02:22:33	Contexts and representations for decimal numbers.
7	01:20:54	Contexts for decimals and different types of numbers.
8	00:48:29	Grid model and money context.
9	01:09:56	Number line model and temperature context.
10	01:21:47	Length measurement as a context and a model.
11	01:06:45	Segmenting decimal lessons for Grade 5.
12	01:09:07	Discussion on one lesson in detail.
13	01:16:07	Segmenting decimal lessons for Grade 6.
14	00:41:55	Use of area and linear models for decimals.
15	01:09:46	Decisions for lesson planning and creation of decimal problems.
16	00:49:52	Reflection on classroom teaching of decimals.
17	00:39:48	Reflection on TRMs and connections with classroom teaching.
18	01:03:23	Discussion around artefacts from classroom teaching.
19	01:09:17	Discussion on student responses to decimal worksheet.
20	00:47:55	Emergent questions from classroom teaching: A discussion

The teacher–researcher meetings were organised around tasks. These tasks were designed by the researchers using the existing literature in the field and a reflection on the classroom observations from the first phase of the study. Consider the task on analysing students' responses to ordering of decimal numbers. It was noted in Phase 1 that students extended their whole number thinking to compare and order decimal numbers. For some problems, this led to errors. When probed, teachers seemed to recognise the student error, but did not understand its source and therefore dealt with it in the lesson by correcting it. This created the need for designing a task around the influence of whole number thinking on the learning of decimals. The literature suggested that the whole number thinking influences students' engagement with other sub–topics such as when multiplying decimal number with powers of 10. Students' responses noted in the first phase and the relevant literature on students' conceptions was used to design the task on the influence of whole number thinking on decimal learning (for TRM 2). While analysing each task, these considerations are discussed under the design rationale.

The tasks varied in nature and teachers were expected to engage with them in different ways. For instance, in a task teachers were asked to answer some questions in the form of a worksheet and then each response was discussed in the meeting. In another task, the purpose was to support teachers in evaluating the use of a linear model for representing decimal numbers (in TRM 14). For this task, teachers were asked to analyse an example given in the textbook, where the linear representation was used to show conversion of length measurement units. (Both these tasks have been discussed in detail in the Sections 6.4.2 and 6.5.3 respectively.)

The aim of the discussions during TRMs was to encourage teachers to articulate their ideas, knowledge and beliefs; and in the process challenge or support them. A deliberate attempt was made to create this space in a way that teachers could share their opinions and experiences freely and challenge or revisit their existing ideas, without being judged for their content knowledge.

6.3.2 Data sources and analysis

During teacher–researcher meetings, the data was collected through video and audio records, along with written records and a summary of each meeting. The written work of all the participants was collected. A transcript of each meeting was prepared using all these records. The meetings happened roughly once in a week, for a period of six months. The average duration of each meeting was 62 minutes (ranging from about 40 to 140 minutes). Four teachers and three to six researchers participated in the meetings. Not everyone was present for all the meetings. The sessions were facilitated either by the researcher or the researcher's supervisor (the latter has been referred as the facilitator in the analysis).

What was important for teacher learning was not merely the completion of these tasks but the discussions around them. The discussions afforded by tasks had different foci. For example, some tasks gave rise to discussions centered around students' thinking. These included deliberation on the sources of students' responses, the knowledge needed to handle such responses and potential ways of dealing with them. All such tasks were classified under the theme of "Developing an awareness of what students can think and do". Similarly, other themes were identified based on the nature of discussions afforded by a different set of tasks. The analysis of tasks is organised around these themes (refer Table 6.2). The tasks discussed in this chapter were selected based on the nature of changes observed in the classroom observations in the second year. They are not exhaustive of the variety of tasks that were done in the teacher–researcher meetings.

Table 6.2: Classification of TRM tasks for analysis

Task type	Task description	Theme
Task 1	Identifying students' errors and their sources.	Awareness of what students can think and do
Task 2	Anticipating and understanding students' responses.	
Task 3	Modelling teaching decisions based on understanding of students.	
Task 4	Connections between decimals, fractions and whole numbers.	Affordance of representations
Task 5	Decimal and non–decimal contexts.	
Task 6	Use of linear and area model in teaching decimals.	
Task 7	Identifying key ideas in teaching decimals.	Coherence in teaching
Task 8	Designing and sequencing decimal problems.	
Task 9	Deeper connections in the curriculum.	

6.3.3 Organisation of themes

The analysis section is organised around themes. The individual tasks within each theme are discussed under these three rubrics – their design rationale, the nature of engagement when implemented in teacher–researcher meeting and a discussion of the key ideas learnt through an engagement with the task (see Figure 6.2).

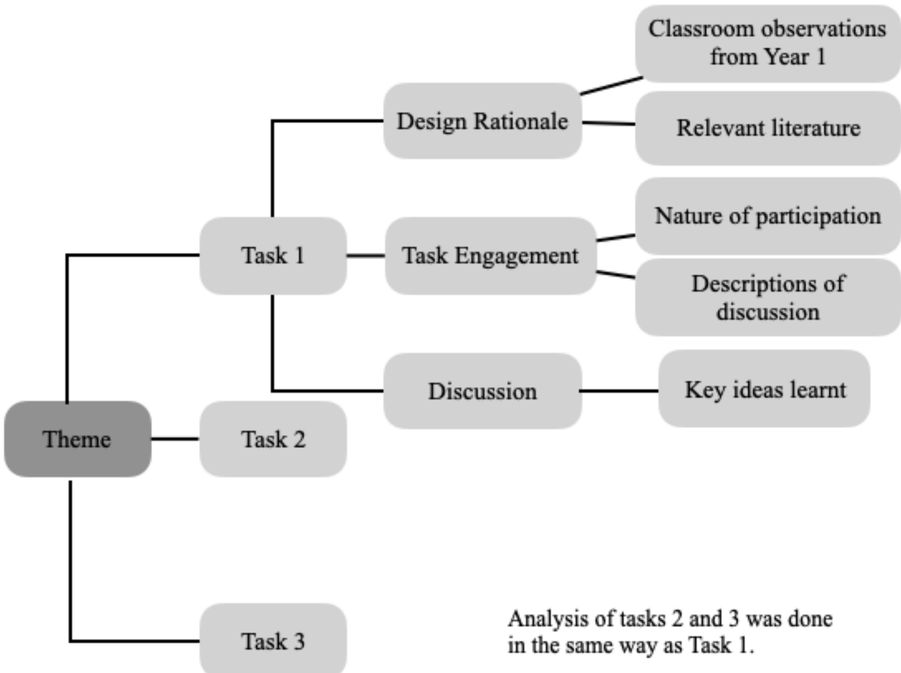


Figure 6.2: Analytical Organisation of TRMs

The observations from the first phase of the study served as a rationale for the design and development of the tasks, within each theme. Further, insights from the supporting literature, which helped in understanding the nuances of the teaching practice, or in defining the task objectives, are also discussed. While articulating the rationale for each theme, it became evident that some deep rooted beliefs and practices of teachers were being challenged. The tasks were planned to generate a discussion on some of these teacher beliefs about students' mathematical capability, and these then became an artefact for reflection, on teaching decisions, during TRMs. Thus, tasks were designed with the objective of enabling teacher reflection on the existing practices and trigger images of alternate practices. The purpose and nature of engagement planned for a series of tasks is discussed under the theme, followed by the details of specific tasks. The design rationale for each task includes a discussion on what was learnt from reflection on the classroom observations from Year 1 and the relevant literature, the artefacts that were used and the plan of the nature of interactions between teachers and researchers. The nature of engagement includes the range of discussions afforded by each task. These include the nature of beliefs and knowledge, which were triggered, and ways in which these were challenged or supported in the process of engagement in a task. Further, ways in which teachers made connections between the ideas discussed in the meetings and their classroom practices are detailed. The knowledge that was addressed during these meetings connects these different aspects of the discussion.

Although each task is discussed separately, the tasks are interlinked. Their cumulative influence on each other is only indicated and not analysed in detail. The analysis identifies aspects of teacher knowledge about students' thinking for the specific topic of decimal numbers.

6.4 Theme 1: Awareness of What Students Can Think and Do

The first theme includes a discussion on students' work aimed to develop teachers' appreciation of the mathematical nuances of students' thinking. The mathematical knowledge linked to an awareness of students' thinking informs teachers' decisions such as examining which meanings or responses to be pursued while teaching and how.

6.4.1 Task 1: Identifying students' errors and their source

While teachers recognised common student errors, they found it difficult to identify the mathematical thinking or mis–understanding underlying student responses. This task was planned to support teachers in identifying the mathematical sources of students' errors and develop the knowledge required for dealing with them in class.

6.4.1.1 Design Rationale

The classroom observations from the first year revealed that teachers often missed a student's question or utterance, which was mathematically salient, according to the observer. A mathematically salient idea expressed by the student included ideas which were common student errors, a unique mathematical conception, a new observation based on the existing information, an attempt to make connections across topics, or an (incomplete) attempt towards justification. These utterances appeared in students' oral or written work but were not discussed in class. When probed for reasons, teachers often mentioned not hearing such utterances. It was noted that the practice of not paying attention to what students were saying was linked to the teacher's dilemma about whether and how students' intervention can contribute to the ongoing discussion in the classroom (a dilemma articulated by Vindhya to seek researcher's support in the second year, refer Section 7.5.3 in Chapter 7 for details). Additionally, teachers attributed student errors to the non–mathematical factors such as, students' attention span, background, etc. (described as unproductive framing by Jackson, Gibbons & Dunlop, 2014). Therefore, encouraging teachers to notice students' utterances and identifying the mathematical sources became a salient objective to be addressed during TRMs.

Shulman (1986) classified an awareness about students' conceptions at different age groups as a sub–set of pedagogical content knowledge. Ball, Hill and Bass (2005) assert that seeing a mathematical error and identifying where the student has gone wrong or "error analysis" is a recurring task of teaching. The literature on knowticing (discussed in Section 2.4 of Chapter 2) informs us that listening to (and perhaps probing) students in ways that helps in understanding their thinking is connected to the teachers' knowledge of the subject–matter. Such an awareness or noticing the mathematical salience of students' (partial or unclear) responses can be developed.

In this task, teachers were requested to use their knowledge to brainstorm the list of errors that students make when learning decimals. They were encouraged to identify the sources of these errors, that is, identify the mathematical thinking underlying students' responses. A discussion on different student responses was aimed to challenge teachers' common response to students' utterances, which included qualifying them as correct, incorrect and attributing them to non–mathematical sources. The aim was to help teachers in exploring the possible range of responses that might come from students and their mathematical salience.

6.4.1.2 Engagement in task

In the first meeting of teachers and researchers, significant themes within decimal teaching were identified. These included reading of the decimal number, expressing a whole number or fraction using decimal, place value, comparison, etc. During this meeting, all the teachers listed some common student errors that they had observed in their classroom. These errors included – (a) reading the fraction part of the decimal number like a whole number, for instance, reading 0.25 as zero point twenty five, (b) incorrect placement of the decimal point, considering $\frac{3}{100}$ is the same as 0.3, (c) difficulty in comparison of equivalent decimals, identifying 0.30 as greater than 0.3, and (d) incorrect comparison of decimals where the whole number part is the same, considering 3.17 as greater than 3.5. The researchers probed teachers for the mathematical reasons underlying such student mistakes. It was essential to draw teachers' attention to the mathematical reasons, since teachers tended to focus on non–mathematical aspects such as, students' attention and ability to follow instructions as reasons for their responses.

Nandini proposed that students might focus on the digits of the number and ignore the length of the decimal number. Pallavi added that students tend to make more errors where the "numeral part" (referring to the whole number part) of the two decimal numbers to be compared is the same, but the other (fraction) part is different. When the researcher asked her for an example, Pallavi mentioned the comparison of numbers 3.17 and 3.5, which have varying lengths. Reema predicted that when comparing these numbers, students might treat 3.17 as a three–digit number and 3.5 as a two–digit number. The researcher distinguished between the two kinds of explanations, which could possibly lead to the same student error, while also encouraging teachers to see the similarity in these explanations. Gradually, the group distinguished between these two kinds of thinking. The first kind of explanation was the digits–based approach, where the students overgeneralise that more the number of digits, greater is the number. This thinking explains why students might think that $0.30 > 0.3$ and $3.17 > 3.5$. Linked to the first, was the discussion that students might omit the decimal point, and read these numbers like whole numbers. In this sense, 0.30 is read as 30 and 0.3 as 3. Similarly, 3.17 is read as 317 and 3.5 as 35. While there was some discussion to understand such sources of students' mistakes, Vindhya shifted the discussion to what the teachers could do to address these errors in the classroom. Pallavi offered a procedural explanation of dealing with a specific kind of error (refer Excerpt 6.1).

Excerpt 6.1: Comparing decimals: Pallavi (TRM 1, 74)

Speaker	Utterance
Pallavi	But whenever this type of problem comes no, what I do is, I just ask them to first compare the numerals [means whole number part]. So, if the numeral is same, I just ask them to cut it so, they are left only with the decimal part.
The legend TRM1, 74 indicates the first Teacher–Researcher Meeting, followed by the utterance number (in this case 74) from the transcript.	

It was noted that Pallavi's explanation does not necessarily challenge the longer is larger conception. It seemed to be a technique to compare decimals. Teachers were asked whether they use this technique to address any particular student conception while teaching. Vindhya stated that, in Grade 6, they offer an explanation based on comparing place value of each digit of the decimal numbers to be compared. This was done by placing the decimal numbers to be compared in a place value table. (The confusions with respect to the place of decimal point in the place value table were discussed in Table 5.6 of Chapter 5.) Pallavi offered another explanation that she used in her classroom teaching (refer Excerpt 6.2).

Excerpt 6.2: Annexing zeroes: Pallavi (TRM 1, 96)

Speaker	Utterance
Pallavi	Okay. So, when you have to equalise all these things. So, in that case what they do is they add one zero to it. So, everything becomes two two. Now, you think this is just a number. This is seventeen [referring to 3.17], this seventeen, this is fifty [refers to 5 in 3.5]. So, which is, which is the smaller one, that you take it.

Her explanation involved telling the students to make the length of decimal numbers the same by annexing zeroes, that is, 0.30 has two digits after the decimal, in order to compare it with 0.3 the latter should have the same length, so annexing a zero to 0.3 will make the length same and the decimals comparable. Interestingly, the explanation by Pallavi assumes a connection between whole numbers and decimal numbers evident from her reading of the fractional parts of the two decimal numbers as seventeen and fifty. Vindhya proposed that if students know how to convert decimals into fractions, and then reduce it to the lowest term, that is another way of comparing decimals. Pallavi responded that this explanation was "higher level" and hence unsuitable for primary grade students to which Vindhya concurred.

An emergence of some conceptual explanations (comparing place values and conversion to fractions) is evident in this meeting. These explanations were marked for a detailed discussion in the next TRM. The concerns with the pseudo–explanations such as, deleting the whole number part if is same and the nuances involved in the annexing the zero explanation also needed further discussion.

6.4.1.3 Discussion

One of the objectives of the teacher–researcher meetings was to help teachers articulate their existing knowledge of students, techniques or procedures used while teaching decimals. Further, the focus was on mathematical aspects of the students' talk. In this meeting, teachers were able to list some of the key difficulties, similar to those identified in the research literature on students' thinking in decimals. Examples of these include, longer is larger (3.17 > 3.5), difficulty in identifying equivalent decimals if they do not look the same (0.30 and 0.3), and conversions from fractions to decimals and vice versa. When probed to think about the reasons, together the group identified an important source of students' conceptions when comparing decimal numbers, that is, the influence of whole number thinking. This explanation enabled teachers to see the reasons for different kinds of students' responses.

Two important aspects of teachers' knowledge about procedures and students' thinking emerged in this meeting. First, both the explanations offered by Pallavi are procedural and visual in nature. The first explanation invokes a strategy of breaking the numbers into two parts – whole number and fraction, and treating them separately for comparison. Similarly, Pallavi's other explanation of "equalising" the length of the decimals by annexing zeros did not focus on looking at the number as a whole. While both these explanations work, the reasons underlying them were not discussed during this meeting. Linking these explanations to place value of the digits in the decimal number or considering the fractional equivalent of these decimal numbers could have been possible justifications. It was noted that the procedural explanations offered by the teachers can be linked to the sources of students' errors, such as, a focus on the digits of the decimal number or treating the decimal number like a whole number by omitting the decimal point. Second, teachers qualified some explanations as"higher level", which were postponed from the teaching of mathematics at the primary grades. When examining, which explanations were considered as higher level, it was evident that such explanations were more conceptual in nature. For instance, the visual explanation of cancelling the whole number part of the decimal number (in cases where it is the same) offered by Pallavi was considered understandable by young students, and therefore appropriate for Grade 5 students. While the explanation of comparing the place value of each digit in the pair of numbers was considered to be of a higher level. This observation is consistent with the classroom observations, where it was noted that Pallavi did not emphasise place value in Grade 5, while the Grade 6 teachers began the chapter on decimals by placing the numbers in a place value table, as shown in the textbook.

In this meeting, the teachers explicated their knowledge about the common student errors and discussed ways in which they deal with these errors while teaching. Teachers had also begun identifying the sources of these errors. I identified that teachers needed to extend their knowledge of specific student errors by understanding students' potential modes of thinking, particularly for the topic of decimal numbers. The underlying assumption was that as teachers recognise these errors as lines of students' thinking, it might support their anticipation of the difficulties faced by these students when solving different kinds of decimal problems. Also, I found that teachers needed support in deepening their knowledge of the justifications underlying the procedural or visual explanations. This was also meant to influence their choice of explanations at particular grades.

6.4.2 Task 2: Anticipating and understanding students' responses

After extending teachers' knowledge and awareness of students' errors to identifying their source, teachers were often encouraged to anticipate their students' responses. It was believed that an anticipation of students' mathematical responses would direct teachers' attention to the mathematical aspects of students' talk.

6.4.2.1 Design Rationale

In the first year, teachers were requested to anticipate students' responses to a set of problems designed by the researcher with help from the existing topic–specific literature. This anticipation–reflection task involved teacher anticipation followed by asking students to solve these problems and then reflecting on students' responses along with the teacher (for details refer Section 4.6.1 in Chapter 4). Teachers' initial responses to this task were generic such as "we will see how they solve it" or "some problems they will solve, others they will not". After some probing teachers began articulating their anticipation of students' responses (refer Excerpts 6.3 and 6.4).

Excerpt 6.3: Anticipation-Reflection Task in Year 1: Pallavi

Question		48 + 97 = _____ + 99
Anticipation	P	They will not be able to do it because this type (of problem) we don't do in class.
Reflection	R	A majority of students have filled the blank by writing 145.
	P	No, it is wrong, no? Actually, they split out, that is why. What they have done is add these two (points to 48 and 97) and write the answer here (in the blank). That I am sure now.
	R	Some students have extended the problem like this 48 + 97 = 145 + 99 = 244 So why do you think they would do that?
	P	No, no. I think they would simply not see this one (99). It (244) is the sum of 145 and 99. As such this type of a problem is difficult for them. See now they have made mistakes like this. But in the textbook there is nothing like this. See they have to find out 15 + ____ = 27 or 23 + ____ = 30. Only these problems would be given in class. Ok. So one (number) in the mind, and the other on fingers. We go on repeating this in the class. When two numbers are given like this, they (students) will easily do it…. In this question, the finger number is missing. So how much so ever you tell them, they will make a mistake. Because see children are not so much concentrating on their work.
Legends	P – Pallavi, R – Researcher	

Excerpt 6.4: Anticipation-Reflection Task in Year 1: Nandini

Question		48 + 97 = _____ + 99
Anticipation	N	You are thinking that out of them how many should be able to do this. They might not be able to do it. If at all any one student would have done it (correctly), it would be by logical thinking.
	N	Commutatively they would be able to do. If they see the commutative property. Since one number has increased here (pointing to the left), it will increase on the right.

Reflection	N	(Sees the students' response as 48 + 97 = 145 + 99 = 244 in the worksheets.) They (students) thought that the answer should come here (in the blank). Then, the next step is to add these (145 and 99). They didn't consider these (sides) to be equal. They just thought that these two (48 and 97) need to be added and not that some extra piece of plus two needs to be adjusted. That extra piece needs to be added to 48. They (students) don't take the equal–to sign with its meaning actually. Sometimes, in writing also they would place the equal–to sign even though the left and the right side are not equal. They use it (equal to) as a connection between steps.
Legends	N – Nandini	

While anticipating students' responses teachers under–estimated students' capabilities as evident in phrases such as "they might not be able to do it". Also, the belief that unless taught, students will not be able to solve problems is evident in Pallavi's remark, "because this type (of problems) we don't do in class". Interestingly, teachers' reflection on the actual student responses was nuanced. For instance, consider Nandini's observation that "they (students) don't take the equal–to sign with its meaning" and "they use it (equal to) as a connection between steps". Pallavi's reflection was mixed, she refers to students' concentration as a criteria but she also recognised that the problems are given in a particular format in the textbook and class. I realised that the anticipation–reflection task could potentially be used to direct teachers' attention to the mathematical thinking underlying students' actual responses.

An anticipation of students' difficulty and ways of dealing with the problems posed are considered an important part of lesson planning. Further, one of the arguments for making teachers aware of the topic–specific research literature on students' conceptions has been to develop stronger anticipation and therefore careful planning. A stronger and nuanced anticipation of students' responses and ways of dealing with it is in turn considered helpful in handling "contingent" (Rowland & Ruthven, 2011) classroom situations more effectively.

In this task, teachers were encouraged to predict students' responses to a particular problem, think about the mathematical salience of students' talk, and develop an awareness of the various ways in which whole number thinking might influence decimal learning. Such conversations included going beyond the student utterance and trying to identify the mathematical source of their conception. The teachers used their existing knowledge to interpret student conceptions and researchers' awareness of the literature on topic specific student errors was used to expand teachers' knowledge.

6.4.2.2 Engagement in task

As discussed above, teachers began unpacking the sources of students' errors from the first meeting. They identified that an important source of students' difficulties in learning decimal numbers is the analogy drawn with whole numbers. Evidences of the prevalence of such thinking among students were gathered from classroom observations in Phase 1. It was observed that (a) several errors made by students when reading, comparing or performing operations on decimal numbers were guided by their overgeneralisation of whole number thinking, and (b) teachers referred to whole numbers while teaching the names of place values in a decimal number, comparing decimals, and operating on decimal numbers. Since this analogy was common to the teaching of all teachers, ways in which it might reinforce students' misconception needed detailed discussion. In TRM 2, teachers were given a worksheet (refer Figure 6.3), to understand the student difficulties arising from the analogy between whole numbers and decimal numbers. The errors chosen for this worksheet were taken from the students' oral and written work, collected from the first phase of the study. The selection was guided by some of the errors that have been identified in the literature on students' difficulties in the learning of decimals. The purpose of the worksheet was to engage teachers into a variety of ways in which whole number thinking might influence students' responses to decimal comparison problems.

Q1. When students are asked to arrange the following decimal numbers in descending order their responses vary. 0.658, 3.7, 2.45, 5.63

(a) What are the possible (correct and incorrect) ways in which they will solve this problem.

(b) Here are the responses of two students.

Response 1: 0.658, 5.63, 2.45, 3.7 *Response 2*: 3.7, 5.63, 2.45, 0.658

What do students think when they make these errors?

(c) Can you devise some problems to check whether your students are making these errors?

(d) Do students use whole number thinking in learning other sub-topics? Give examples.

Figure 6.3: Worksheet on Students' Errors (TRM2)

After a brief reading of the first question (part *a* in Figure 6.3), Pallavi's immediate response was that students often confuse between the labels of ascending and descending order. Vindhya added that it is important for students to learn the mathematical vocabulary. The facilitator provided the Marathi (language) translation of the two words to emphasise that the students need to understand the meaning underlying the mathematical language. He indicated that in some languages the meaning of the words used to order decimals is familiar and evident, while in other languages, like English, it may not be so familiar. In response, Nandini shared how her explanation in class is based on the context in which such vocabulary is used, for instance, "ascend means climbing up". The group concluded that students might not remember the labels for the mathematical actions.

The researcher then asked whether such a difficulty could be classified as a conceptual or a language related difficulty. Vindhya argued that this is a conceptual difficulty as when asked students will "write the descending order…as ascending order". The researcher and Nandini contested Vindhya's reasoning. The researcher questioned whether comprehending the meaning of the word "ascending" can be differentiated from the students' knowledge of the mathematical idea. To support this Nandini used the example of how students often confuse between the words "yesterday and tomorrow", which is a language related difficulty, but when asked to use the meanings of these words, they can distinguish between past and future.

Offering a conceptual explanation, Reema anticipated that students might just count the number of digits and arrange the numbers as 0.658, 2.45, 5.63 and 3.7. When probed further, she stated that students might overlook the decimal point. She explained that students know that 563 is greater than 37, so ignoring the decimal point helps them in doing this comparison correctly. The facilitator proposed that a related explanation could be that students compare only the first digit of every decimal number. So the group noted that both these reasons might lead to the same response from students. Vindhya explained that it is likely that students are focusing on the first two digits for comparison (that is 65, 56, 37 and 24) since that is the shortest length of the decimal number (3.7) in the given set of numbers. Such a comparison would also lead to the answer, 0.658, 5.63, 3.7 and 2.45. Pallavi proposed that students would reach the same response if they compared only the fractional parts of the given numbers (658 > 63 > 45 > 7). However, these reasons would lead to two different responses (note the placement of 2.45 in the two responses). Then Pallavi distinguished between Reema's reasoning where students treat the number as a whole while ignoring the decimal point, and what she proposed, that is, comparing only the fractional parts of the given numbers.

Pallavi identified that comparing the numeral part (whole number part) will help students in finding the correct response to this ordering problem. While the strategy would work, I noted that despite noticing the student difficulty arising from comparing the fractional part of the given numbers, Pallavi proposes a strategy aimed at breaking the number into whole and fractional part. In other words, an anticipation of whether this strategy (of comparing the numeral part) might fail in comparing some numbers seemed missing.

The facilitator observed that students might not use decimal knowledge to answer this question. The researcher emphasised that without probing students' reasoning it is difficult to distinguish what they are thinking, and how this thinking influences their response. Pallavi resisted probing students' reasoning and reinforced learning the correct way of comparing decimal numbers (refer Excerpt 6.5).

Excerpt 6.5: Cognitive maturity: Pallavi (TRM 2, 619)

Speaker	Utterance
Pallavi	We won't go deep into the reasoning and all. First, they should learn the correct answer and then that is not the stage for them to deeply go into the concept and understand. So, once they reach sixth, eighth standard or ninth standard then we explain it to them at least they will get some parts.

To conclude this meeting, the researcher proposed thinking about whether the kind of problems posed by the teacher would differ if we identify that different sources underlie the same student response.

6.4.2.3 Discussion

In the classroom observations done as part of Phase 1 of the study, it was noted that teachers did not probe students' responses, whether correct or incorrect (also reinforced in Excerpt 6.5). The worksheet in TRM 2 was designed to invite teachers to begin thinking about the mathematical aspects of students' responses. The interpretation of the difficulty, faced by the students when comparing decimal numbers, was contested as being linguistic or conceptual. Further discussions helped in anticipating different reasons leading to the same student response to the ordering problem. Thus, the question about how to identify what is the students' thinking based on this response, was foregrounded. An anticipation of possible students' responses made the teachers revisit their experience of classroom teaching and bring forth more conceptual explanations for students' (incorrect) responses.

A discussion on different student responses and possible underlying thinking was oriented to draw teachers' attention to the mathematical aspects of students' errors. It was observed that teachers were using the knowledge from the previous meetings to identify the reasons for students' responses, for instance, the explanation of counting the number of digits, separating the whole number and the fraction part of the decimal, and using only the first digit to compare. Pallavi made attempts to distinguish between different kinds of students' thinking, while separating her and Reema's anticipation. Also notable is that teachers were beginning to distinguish between different kinds of thinking that might lead to the same student response (an objective also addressed in Task 1). Through this worksheet, teachers recognised that students' line of thinking where they draw analogies between whole numbers and decimal numbers might manifest in their responses to different kinds of problems.

These tasks supported teachers' anticipation of students' modes of thinking and ways of responding to problems by focusing on (a) the variety of ways in which students can respond to a mathematical problem, (b) different reasons underlying a response, and (c) the need to unpack mathematical explanations underlying students' (correct and incorrect) responses. Evidence for increasingly nuanced anticipation, focusing on the mathematical aspects of students' thinking, was noted in the teachers' engagement with the anticipation–reflection task in the second year. For instance, Nandini anticipated three ways in which students might think about the problem, that is, by counting the digits, by removing the decimal point, and by comparing the place value. Similarly, her reflection on students' responses is also mathematically nuanced, as she "knowtices" the thinking underlying student's response and anticipates the student's next response based on this (refer Excerpt 6.6).

Excerpt 6.6: Anticipation-Reflection Task in Year 2: Nandini

Question		Arrange these in descending order. (a) 0.658, 3.7, 2.45, 5.63 (b) 0.248, 0.85, 0.63, 0.4 (c) 3.03, 3.003, 3.303, 3.303, 3.33 (d) 5.5, 5.55, 55, 555
Anticipation	N	A few of the students will count all the digits and decide which number is greater. Others may ignore the decimal point and treat the whole number as one. A few who will think logically will compare the digits at the same places and then arrange them.
Reflection	N	These questions are different. I mean we can find how they are thinking based on their answers. Like this student did 0.658, 5.63, 2.45, 3.7. She is considering these numbers as one, like a whole number. So she is going by the more the number of digits the greater will be that number. If she thinks like that, then her second answer should be yes, see it (points to the student's response as 0.248, 0.85, 0.63, 0.4) matches.

A belief that remained unchallenged in this task was that conceptual reasoning is exclusively for the higher grades.

6.4.3 Task 3: Modelling teacher decisions

The knowledge of the subject–matter interweaves with the appropriate pedagogical moves in order to inform classroom practice. As teachers develop a deeper insight into students' ideas, supporting them through offering imaginations of alternate practices seemed necessary.

6.4.3.1 Design Rationale

An ability to handle students' utterances (Ball & Bass, 2000) in specific topics has been recognised as an important part of teacher's knowledge of mathematics teaching. This knowledge manifests itself in ways in which a teacher handles a student question or response such as, probing their thinking while teaching, and hearing (noticing) the mathematically salient aspects of students' talk. In the TRMs, several tasks focused on developing this kind of hearing (or noticing) of students' responses.

While the sensitivity can be developed through an awareness of students' ways of thinking, dealing with it in the dynamic classroom environment are occasions for in–the–moment decision making. Observing such practices and reflecting on them develops imagination of alternate practices and their affordance. Such alternate practices are evident in the writings which focus on the work of teaching. The thick descriptions, for example, of Lampert's (2001) teaching helps in zooming into teachers' thinking and identifying the interplay of considerations that underlie the decisions made in the ebb and flow of teaching. While analysing the recurrent tasks of teaching, such as identifying ways of representing an algorithm for multiplication of two whole numbers, Ball, Hill and Bass (2005) unpack the knowledge required for teaching specific topics. Ma (2010) takes a different route. She characterises deeper knowledge of the subject–matter by examining teachers' reported ways of dealing with mathematical situations arising in classroom.

In the prior meetings during the discussion on student errors, teachers often responded by offering a procedural explanation to correct the mistakes (for example, refer to the discussion on Task 1). Therefore, along with developing an understanding of the ways in which students think, it became important to discuss how to deal with a student question or response when it arises during teaching. In this task, teachers were expected to think about how to deal with a student's question and then provide them with a glimpse of practice, which is different from their routine teaching and discuss it. The alternate ways of dealing with the unanticipated students' responses was demonstrated through a discussion on a teaching video.

6.4.3.2 Engagement in task

In TRM 6, teachers were shown a video clip of a teaching camp organised for Grade 6 and 7 students of their school, a few years ago. The camp focused on the topic of fractions and was held in the researcher's institute. To introduce the video clip, teachers were briefed about the background of the students and the objectives of the camp. The video clip showed classroom discussion around an unanticipated student's question. The student's question was, "If we divide 3 by 4 the fraction will be $\frac{3}{4}$, but when we divide 3 by 4, we get 0.75; how is that [possible]?". In the video, the teacher revoiced the question and invited other students to respond to it. At this moment, the video was paused and the teachers were asked about how would they deal with such a situation in their classroom.

Vindhya noticed that the student's question was about the connection between fraction and decimal, while a researcher and Nandini thought that the student found it difficult to accommodate two different representations for the same fraction. Adding to this conversation, Pallavi asserted that the connection between division, fractions and decimals is important. In another meeting, she had raised how the treatment of all these topics in *silos*, refrains students from understanding that a fraction could be represented differently. While Pallavi asserted that the students must be told that these are three different ways of representing the same expression $a \div b$, Nandini emphasised that merely telling this to students might not help. Nandini advocated for the need to provide an explanation, which will help in convincing the students. The discussion moved to how teachers can respond to this situation. Reema expressed that students might not need an explanation for why 3 divided by 4 is the same as $\frac{3}{4}$ but for why $\frac{3}{4}$ is equal to 0.75. She proposed dealing with it using the long division algorithm by placing a decimal in the quotient. Pallavi demonstrated the long division, suggested by Reema, and explained it the way she would do it with her students. Vindhya proposed that another explanation could be given using equivalent fractions, where the students can be made to see that $\frac{3}{4}$ is equivalent to $\frac{75}{100}$. While agreeing with Vindhya, Nandini stated that she would prefer to deal with it in a reverse manner, that is, by moving from 0.75 to $\frac{3}{4}$, such that the students can see that 0.75 is the same as fraction $\frac{3}{4}$. The group came to a consensus that it was important to make the connection between fractions and decimals visible for students. Pallavi remarked that the connection, between whole numbers and decimals also, needed to be explicated. When probed further, she gave an example of how whole number 3 can be written as

3.0 in decimals, thus treating them as two different representations. After watching the complete video, the teachers noticed how students were probed and acknowledged for their responses, identified similarities and differences in the students' responses and remarked about how the connections between students' explanations can be used to build a response (from the teacher).

6.4.3.3 Discussion

In this task, the attempt was to help teachers understand that a student's question can be investigated and pursued from a mathematical viewpoint. The teachers seemed to understand that the student's question did not belong to a particular topic (that is, either fractions or decimals) in the manner that topics are seen in isolation from each other. They identified that the connection between the two representations of fractions and decimals, was crucial. The student's struggle to equate the two different looking representations ($\frac{3}{4}$ and 0.75), was possibly an unanticipated question, but the teacher was required to respond to it while in classroom. The teachers thought about how they would deal with such a situation in their classroom, and listened to each other's ways of dealing with it.

The different approaches shared included – (a) using long division algorithm to divide 3 by 4, (b) finding equivalent fractions of $\frac{3}{4}$ such that the denominator is a power of 10, (c) converting the decimal to a fraction and reducing it to its lowest term, and (d) using a sharing situation to shade $\frac{3}{4}$ and then dividing the same area into 100 parts and recording the shaded part. It was interesting to note that teachers did not just suggest these approaches but were constantly explicating the prior knowledge that students would need to be able to engage with each of these explanations. For instance, Nandini mentioned that, in order to completely divide 3 by 4 students needed to know the division of $\frac{m}{n}$ for $m < n$. In another meeting, while doing textbook analysis, she had noticed that division of this kind was missing in the textbooks of Grades 5 and 6, and therefore remains untaught. She noticed that students learn division of $\frac{m}{n}$ for $m > n$ in the primary grades. In the middle grades, teachers expected students to know the division for $m < n$, for instance, when dividing decimal numbers like 0.7 by 6. However, differences between doing whole number division and decimal division were missed in the textbook. In the former, remainder can be interpreted but the latter is a complete division with no remainders. The purpose of this meeting was also to initiate teachers into thinking about alternate ways of dealing with students' unanticipated questions or responses. The teachers noticed the similarity in their ways of addressing the question and the explanations given by the students in the second half of the video. Further, teachers appreciated practices such as acknowledging students' contributions, identifying similarities and differences in students' responses and the teacher building an explanation after carefully examining the existing responses.

6.5 Theme 2: Understanding the Affordances of Representations

The literature on teacher knowledge required for teaching mathematics suggests that teachers need content knowledge to teach. This knowledge is different from the knowledge of a mathematician or a user of mathematics. In an attempt to unpack what constitutes knowledge required for teaching mathematics, different frameworks for teacher knowledge have been proposed. While these frameworks (discussed in Chapter 2) identify the integration of knowledge of students and content as an important part of teachers' pedagogical content knowledge, I have argued that in the dynamic context of teaching, knowledge of content and students includes and is connected with other components of knowledge. Further, in the existing literature, the nature and form of this knowledge becomes explicated only through some specific examples. Researchers in this field have identified topic specific tasks of teaching, for instance, the significance of understanding meanings of integers (Kumar, 2018), preparing teachers to use specific problem types within arithmetic (Carpenter, Fennema, Franke & Empson, 2000), making sense of students' response to problems of subtraction with regrouping (Ball & Bass, 2000), understanding the changing relation between area and perimeter of a rectangle (Ma, 2010), etc. While these researchers identify specific tasks pertaining to different topics, a few attempts have been made to detail what knowledge (integration of SCK, PCK, SMK) underlies the teaching of specific topics. In this section, the attempt is to discuss how connections among and between key ideas, representations and contexts were discussed during TRMs. I also attempt to show that in the context of discussions during TRMs, several of these knowledge components appear in a connected manner (A similar argument has been made by Carrillo, Climent, Contreras & Muñoz–Catalán, 2013).

The knowledge specific to different topics is an important part of mathematical knowledge for teaching. This topic–specific knowledge might include knowledge about ways in which students might think and learn the topic, connections within and across topics, use of representations and contexts within the topic, etc. Based on the existing literature, it was hypothesised that topic specific knowledge required for teaching, has a role in supporting an understanding of students' mathematical thinking. A deeper engagement with the topic helps in expanding the knowledge base by strengthening the connections between and across concepts, processes, and representations. One of the aims of the meetings was to expand teachers' knowledge of decimal numbers and help them make connections between this knowledge and comprehension of students' utterances.

6.5.1 Task 4: Connecting division, fractions and decimals

In Section 6.4.1, I discussed the affordance of relation between whole numbers and decimals. As teachers examined this relation, the other salient connection of decimal numbers was with fractions. Teachers seemed convinced of the movement from division to fractions and then decimals. The nature of this connection between division, fractions and decimals required further discussion.

6.5.1.1 Design Rationale

It is recognised in the literature on teacher knowledge and in the teacher preparation programmes that teachers need to be aware of the broad curricular trajectories of a topic across different grade levels and the connection between different topics. Sometimes these connections between topics are assumed and not mentioned explicitly in the textbooks. However, as Ma (2010) points out, superficial knowledge of such connections manifests in fragmentation of the knowledge pieces when they are presented to students. Teachers' knowledge of connections between topics can help in supporting building on students' prior knowledge to learn new knowledge.

It was noted that although teachers recognised the relation between division, fractions and decimal numbers; such knowledge was used in the introductory lessons for each of these topics. However, such connections were less explored for building explanations or engaging with the sub–topics which were beyond the primary curriculum. The purpose of this task was to engage teachers to identify the less visible aspects of curricular connections and encourage them to examine the connections between the content taught at primary and upper–primary level.

6.5.1.2 Engagement in task

In TRM 2, there was a discussion among teachers and researchers on the student difficulties observed while teaching. Nandini pointed out that students face difficulty in dividing two whole numbers, even at Grade 10 (15 year old). In other words, if the divisor is a multiple of the dividend, then students find it easier to divide; but when this is not the case, students struggle to divide completely. She gave the example of the lens formula ($\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$, where f stands for the focal length, u for the object distance and v for the image distance), where usually calculation becomes difficult for students due to their struggle with the complete division of whole numbers. To this, Pallavi suggested that students should memorise the decimal equivalents of some common fractions like half, quarter, three–quarters, perhaps for an early introduction to decimals. This was followed by a discussion on the division of whole numbers, with and without remainder. Through a sustained dialogue between Nandini and Pallavi, they figured that students are not being taught complete division with a decimal quotient in any of the grades. They validated this observation by systematically looking at the textbooks of Grade 5 and 6. They concluded that the primary (Grade 4 and 5) teachers assumed that the division of $\frac{m}{n}$ where m is not a multiple of n would get covered after the introduction of decimal numbers (that is, after Grade 5), and therefore would leave it for the middle school. The middle school teachers presumed that students were taught this division at the primary grades, and therefore do not discuss this division in any of the later grades. Together, they noted that the division of whole numbers without remainder is not taught in the teaching of division, fractions, and decimals for any of the grades. The facilitator probed the reasons for this missing link.

In the discussion, teachers' attention was drawn to the difference between the whole number division and decimal division. Thinking aloud, Pallavi said, "if remainder comes, we add zero, and keep on dividing". Nandini added that it does not make sense to leave a remainder in decimal division, the division can continue. The facilitator added that a possible reason for missing the decimal division is that within the set of whole numbers, which is a part of the primary school math curriculum, it makes sense to leave a remainder and then interpret it. In other words, $\frac{m}{n}$ can be expressed as $m = np + r$. While for the set of rational numbers, $\frac{m}{n}$ is an entity in itself, different from a whole. This helped the group in justifying the curricular decision of doing complete division after the set of rational numbers has been introduced to students.

Connecting this discussion with a classroom experience, the researcher shared a classroom episode (from Phase 1) where Reema was discussing a word problem on dividing the given number of students equally into some number of buses for a school picnic. Reema added that such a situation requires an interpretation of the remainder, for instance, if 3 students are left, after distributing the given number of students into buses equally, then they need to be adjusted in the given buses. It was concluded that complete division is meaningless in situations such as these. So then the question was in which situations does it make sense to do complete division.

Pallavi proposed currency conversion as a context. She elaborated that it might be difficult to interpret getting a cent (hundredth part of a dollar) in return if an item which costs some dollars and a few cents. The facilitator added two more examples for consideration of the group. These included dividing a piece of cloth into n equal pieces, or making x number of students sit in a given area and finding the area covered by each student. Connecting it with the discussion that took place in Reema's class (in Phase 1), the researcher discussed how students made sense of finding the exact area required for seating each person. These situations were identified as requiring rational number division.

6.5.1.3 Discussion

In this meeting, teachers realised a missing link in the teaching of division of numbers of a particular kind. They explicated their assumptions and realised that the teaching of division where the numerator is not completely divisible by the denominator is not covered during teaching in any of the grades. However, this did not help them realise what could be the mathematical reasons for such a miss. In this meeting, teachers' existing knowledge was extended to make two important points – first, that the context of division needs to be interpreted to decide whether complete division makes sense; and second, when n is not a factor of m for $\frac{m}{n}$, then complete division can be discussed only within the set of rational numbers. Within the set of whole numbers, leaving a remainder needs to be interpreted for a given word problem (or context). To foreground this discussion, teachers were asked to think of contexts where the remainder made sense, and distinguish it from those contexts where complete division is necessary. The currency exchange, cutting the length of a cloth piece, and finding the area were identified as contexts where complete division would make sense. The distinction between the division of whole numbers (with and without remainder) and the division of rational numbers (complete division) was clarified and linked with the suitable contexts of application. These two pieces of knowledge helped teachers in selecting suitable contexts when dividing whole numbers.

6.5.2 Task 5: Decimal and non–decimal contexts

Another aspect of topic–specific knowledge is identifying relevant contexts that can be used for a topic. Further, teachers make decisions about when and how to use these contexts in their teaching trajectory depending on their affordance and purpose of use.

6.5.1.1 Design Rationale

In the tasks of the previous kind, the group had begun discussing the selection and sequencing of content in mathematics curriculum by going beyond the textbook content or by discussing the less visible aspects of the connections between topics. Teachers often relied on the decimal contexts given in the textbook for discussion in class. Also the use of the contexts while teaching was limited. For instance, in the first year, almost all the teachers introduced decimals using the money context, where they used the relation between Rupees and paisa (the denominations in Indian currency). Teachers used students out–of–school knowledge of handling Indian currency. The two concerns in the selection of this context are – the ‘paisa’ is obsolete in the Indian currency and the relation between the two denominations is a factor of 100. In other words, the context cannot be extended to the other powers of 10, as in the case of a decimal. Expanding and deepening teachers’ knowledge of decimal contexts was considered important so that they could select and use contexts based on the nature of discussion while teaching. While it was difficult to find literature on the appropriateness of the contexts used for decimals, Kumar (2018) has argued for the connection between integer meanings and contexts used. The selection of decimal contexts and their relevance was discussed in this TRM.

6.5.1.2 Engagement in task

In the previous meetings, teachers had mentioned using the contexts such as length, currency and area; in this meeting the objective was to reflect on the ways of using contexts and expanding teachers’ horizon of contexts that can be used for the teaching of decimals. In the discussion, an attempt was made to figure whether all notations with a point (or dot), that we see around us, can be classified as decimals. Further, connections were made between the meaning of decimal (base ten) with the contexts where it was being applied.

In TRM 6, after a brief discussion about the relation between fractions and decimals (refer Task 3), the facilitator asked teachers to identify all the situations where they have seen or used decimals. The following examples came up – composition and cost of a drug, billing of items, overs in a cricket match, length (or height, distance, depth) measurement, currency transactions, temperature conversion, and measurement of weight and capacity. Reema mentioned how the need for smaller units, when measuring a length of more than 1 meter, can act as a useful context for introducing decimal numbers. The facilitator asked teachers to think about real life situations where students "see" decimals, encouraging them to also think of the out–of–school examples. To this, Pallavi mentioned the weather forecast that appears daily in the newspaper. The measure of rainfall was also considered as an example. A researcher mentioned that students see marks and percentage at the end of each academic year. Another researcher mentioned use of length measurement units in tailoring as an example. The facilitator asked whether a tailor writes the measures in fractions or decimals. The group agreed that the tailors' used feet and inches as measurement units. This led to the question of whether placement of a dot (or point) between the two measurement units (feet and inches, in this case) would make this measure a decimal number.

The group re–examined the contexts that were listed at the beginning of the meeting about whether they are decimal contexts. The first non–decimal context identified from those listed above was, overs in a cricket match. When probed for the reasons, a researcher argued that the number of balls in an over is 6 so 5.4 overs means 34 balls (5 times 6 added to 4). Pallavi reiterated that a decimal number system is a base ten system and explained it as an increasing or decreasing power of 10. After a pause, she argued that the relation between measurement units feet and inches is also not a decimal number as the relation between these units is that of 12 times. Recognising that it is not a base ten unit, the facilitator tried to differentiate between the function of a dot as a "separator" and that depicting a base ten relation between units. The group identified that writing a date is a non–decimal context, where even though the number of days, months and year were related to each other, the relation varied. This was then classified as a point–as–separator only context.

The discussion moved to the similarity in the use of separators (whether decimal or non–decimal). The similarity was that there is a constant relation between the two units on the either sides of the separator. The group agreed that the exercise on re–writing the date such that it represents a single quantity or is expressed in a single unit is not meaningful or relevant. After distinguishing between the decimal and non–decimal contexts, the group members evaluated all the examples listed in the beginning, using base ten relation as a criterion. Later, the discussion was extended to the naming of units and subunits. The discussion on base ten also extended in later meetings to the significance of positionality in the Hindu–Arabic system and to examining the affordance of each of the decimal contexts.

6.5.1.3 Discussion

The listing of different contexts, where decimal numbers can be seen, pointed to the lack of clarity on the distinction between a decimal and a non–decimal context. For instance, among the listed contexts overs in a cricket match, relation between feet and inches, etc., were non–examples; while measures of length, mass, and capacity in metric units and relation between currency units were examples of decimal contexts. The explication of the rationale for qualifying a context as a decimal context was the base ten system. It was explained by Pallavi and refined in subsequent meetings that in a base ten system, every consecutive place is related by an increasing or decreasing power of ten. The group made a distinction between use of a point as a separator (with examples such as date, exercise numbers), a point depicting non–decimal relation between units (for example, overs in a cricket match, relation between feet and inches), and a decimal point where each place value is related by the (positive and negative) powers of 10. Teachers were encouraged to think about why the placement of the dot in a context is misinterpreted as a decimal context. The beginning of this discussion, which was to identify decimals around us, was important to expand teachers' horizon of distinguishing between decimal and non–decimal contexts.

6.5.3 Task 6: Linear and area representations

Representations are tools used within a discipline to model an idea. The use of a variety of representations to promote students' access to mathematics has been a central idea in curriculum reforms (Subramaniam, 2019). Developing an awareness of the kind of representations used and their affordance in teaching of decimals was the object of discussion here.

6.5.3.1 Design Rationale

In the classroom observations in the first year, it was noticed that the teachers used both linear and area representations when teaching decimals. However, the use of these representations was inconsistent and limited. All the teachers used a linear representation to show tenths and an area representation for the hundredths place value. A number line with equal divisions between two consecutive whole numbers was used to show the decimal numbers with the tenths place value. The same number line with whole numbers was not used to represent the hundredths place value. Instead, a 10×10 grid was used to show decimal numbers with the hundredths place. The thousandths place value was taught as an extension of tenths and hundredths, that is, without a model. The inconsistency in the use of representations was also evident in representing some fractions, which can be expressed as powers of 2 (such as fractions of the type $\frac{1}{2^n}$ or 2^{-n}), using a circle while the others using a bar. Also, the use of these representations was often tied to particular problems where the students were asked to show a decimal number and the representation was specified. For instance, consider problems of the type "Show 1.6 using a number line". Additionally, representations were used only for introduction and not for other purposes such as justifying procedures like multiplication of a decimal number with ten.

Ma (2010) argues that selecting correct representations and using them as explanatory tools is an important task of teaching. Subramaniam (2019) extends this argument to assert that the representational coherence is an important way of enabling students' access to mathematics. A careful selection of models, which mediate between contexts and symbols can support students in providing warrants for their reasoning.

6.5.3.2 Engagement in task

In TRM 14, teachers began by reflecting on whether and how discussions during TRMs influenced their practice. Pallavi reflected on how her routine of making the students rote memorise the formula using a specific example of conversion of units has changed. She had challenged her own practice of telling students the formula and using it to solve problems. For instance, while teaching conversion of measurement units Pallavi emphasised identifying and using the relation between units using measurement scales.

The facilitator linked the use of measurement tools with their representation, using a linear model. When probed about the use of linear and area models, the teachers mentioned using them for teaching of decimal numbers. Vindhya referred to her classroom experience of introducing the decimal numbers using a number line (a linear model) by linking it with students' prior understanding of the part–whole meaning of fractions. Similarly, a 10×10 grid (an area model) was connected with students' prior knowledge of counting the number of squares to find the area of regular and irregular closed figures. The facilitator probed whether and how the linear and area models are connected, in the course of decimal teaching. Further, he raised a question about how students make sense of these models, when learning decimals. The teachers stated that while linear model was used for tenths, an area model was used for the hundredths place value. When probed for the thousandths place value, Vindhya mentioned using the place value chart for three and more places after the decimal point. The teachers did not see the inconsistency in using different models for different place values, and did not seem to attend to the question about making students understand the links between these models. In order to get teachers' attention to the specificity of these models, a solved example from the Grade 6 textbook, was selected for discussion (refer Figure 6.4).

After reading the example, Vindhya stated that the representation provided was useful for learning conversion of units. Also, that a teacher can extend the discussion from the relation between centimetre and meter, as stated in the example, to conversion between other units, such as, millimetre and centimetre, kilometre and meter, and kilometre and centimetre. Pallavi contested the usefulness of the figure provided in the example. She expressed how the representation was misleading as it referred to the area, while the problem dealt with the length of the tabletop. Pallavi's observation initiated others to notice some other aspects. For instance, the phrase 'length of the tabletop' could be loosely used for the perimeter or to represent the breadth of the tabletop. It was also explored whether calling it height was a more appropriate formulation. Pallavi redirected group's attention to her discomfort with depicting the area while the problem was about the conversion of length measurement units. A brief

8.5.2 Length

Mahesh wanted to measure the length of his table top in metres. He had a 50 cm scale. He found that the length of the table top was 156 cm. What will be its length in metres?

Mahesh knew that

$$1 \text{ cm} = \frac{1}{100} \text{ m} \quad \text{or} \quad 0.01 \text{ m}$$

$$\text{Therefore, } 56 \text{ cm} = \frac{56}{100} \text{ m} = 0.56 \text{ m}$$

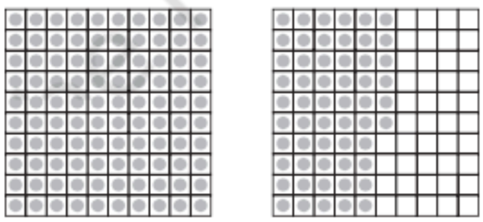
Thus, the length of the table top is 156 cm = 100 cm + 56 cm

$$= 1 \text{ m} + \frac{56}{100} \text{ m} = 1.56 \text{ m.}$$

Mahesh also wants to represent this length pictorially. He took squared papers of equal size and divided them into 100 equal parts. He considered each small square as one cm.

Try These

1. Can you write 4 mm in 'cm' using decimals?
2. How will you write 7cm 5 mm in 'cm' using decimals?
3. Can you now write 52 m as 'km' using decimals? How will you write 340 m as 'km' using decimals? How will you write 2008 m in 'km'?



100 cm 56 cm

Figure 6.4: Length Measurement Example From Textbook (NCERT, 2006*c*, p.176)

discussion on whether the students might get confused between area and length followed. Vindhya and Pallavi had a difference of opinion regarding this confusion. The difference was mainly, whether the part of the figure to be highlighted, would be the side (length) of each square block or the area of the square block. In her justification, Pallavi proposed that such a representation directs students' attention to the square blocks, which have been earlier used for finding the area of shapes. The facilitator remarked that there is a need to be careful in the choice of representations used for decimal numbers.

Considering this as an opportunity to discuss the consistency in the use of representations, the facilitator asked the teachers about the possibility of representing different place values; tenths, hundredths and thousandths; using the same number line. Promptly all the teachers showed the tenths place value using a number line with whole numbers. When asked on how to show hundredths using the same number line, Vindhya's immediate response was that this was "higher level" thinking. After some time, she divided each tenth part into ten equal parts, and showed hundredths using the same number line. The facilitator showed a meter strip (made using paper) and discussed how it can be used to show different place values and the corresponding relation between the different measurement units. The place values and the corresponding relation between conversion units shown were – tenths for decimetre and meter, hundredths for centimetre and meter, and thousandths for millimetre and meter. Vindhya pointed out that the meter strip could be used for other relations such as that between 50 centimetres and the length of the meter strip. She seemed convinced about using the meter strip for different problems within decimal numbers. To conclude this discussion, the facilitator asked the teachers whether the grid could also be used consistently for all place values.

6.5.3.3 Discussion

All the teachers used a linear model to represent the tenths place value and an area model for hundredths place value, as observed in the Phase 1 of the study and stated during this meeting. The practice of using different models for different place values was challenged in this meeting. First, the teachers' attention was drawn to the inaccuracy in the example from the textbook. While the question was about the conversion of length measurement units, the figure referred to the area. Pallavi seemed to attend to the discrepancy and was trying to argue that such a mathematical inaccuracy might create confusion among students. Vindhya opposed Pallavi by stating that she was attributing higher order thinking to students. The discussion was extended to the use of meter strip, where the teachers observed that all the place values can be shown as relations between different measurement units, thus connecting an earlier discussion on the use of linear model and length measurement context. Vindhya was beginning to think how a tool (meter strip) could be used for different kinds of problems. Through this conversation, the teachers could ‘see’ that different place values can be linked and shown using the same tool. Also, they explored how the tool can be used as a justification for a procedure, as opposed to the practice of making students memorise the conversion tables, which emerged from Pallavi’s reflection on her teaching.

6.6 Theme 3: Coherence in Teaching Mathematics

Ma (2010) defines coherence in teachers’ knowledge as knowing how and why an algorithm works, justifying an explanation using symbolic derivation, being flexible in conceptual understanding leading to an awareness of multiple approaches (standard and non–standard) to problem solving, and an awareness of connections among different operations. She distinguishes between longitudinal and vertical coherence. Longitudinal coherence refers to an understanding of the connections between topics within a curriculum. These constitute for the breadth of teacher knowledge. The connections between subtopics within a topic is defined as vertical coherence*,* indicating depth in knowledge. An example would be connecting the procedures of conversion from decimal to fraction with place value of digits in a decimal number. If teachers are aware of the longitudinal connections, they can use these to lay the foundation of further learning opportunities.

6.6.1 Task 7: Linking key ideas and conceptual explanations

Classroom observations and interviews from Phase 1 of the study revealed that teachers provided procedural or visual explanations to students. The visual clues were used to support students’ memorisation of procedures. This made us question whether the teachers themselves were, in fact, aware of the justification of these procedures or as they stated (and believed) that students of this age group would not be able to deal with them, or both. It was hypothesised that identifying the key ideas in a topic and the connections between them might support teachers in building justifications for procedures. Further, linking basic and powerful ideas of concepts and principles in mathematics affects teachers' mathematical attitudes which in turn encourage students to solve problems and use conceptual explanations (Ma, 2010).

6.6.1.1 Design Rationale

In Phase 1, it was observed that teachers used procedural explanations for various problem types. These procedural explanations were not supported by reasons and at times had a visual clue attached to them. For instance, Pallavi used the following visual explanation to reinforce the procedure of conversion from fractions to decimals (refer Excerpt 6.7).

Excerpt 6.7: Visual explanation for converting fractions to decimals: Pallavi (Y1DL3)

Line No.	Speaker	Utterance
42	Pallavi	$\frac{1}{10}$ → after the decimal part you write the unit part. .U (point unit part). So $\frac{1}{100}$ → . 01 Tens place will have a zero and ones place will go to last. Two zeroes are related to two places. So you have to write N.D like this always. The numeral part point the decimal part. So, $\frac{9}{100}$ → N.09
43	Pallavi	Circle the fractional part. Write the decimal part. Numeral part will follow. 4 .36 N.D
44	Pallavi	$\frac{1}{10}$ is 0.1, $\frac{2}{10}$ is 0.2, $\frac{3}{10}$ is 0.3 and so on. So $\frac{9}{10}$ is 0.9. But when there are ten parts, you have tens and ones so it is 1.0. Till 99 parts it will go like this. 1 to 9 → follow same pattern. Have only ones place. 10 to 99 → follow same pattern. Have both ones and tens place.
58	Pallavi	1 paise is equal to one by hundred rupee. [Writes on board 1 paise = $\frac{1}{100}$ rupee] Actually speaking, it is one divided by hundred. To find value of one paisa we want a lesser value so we divide. [On board: T O [refers to tens and ones] 0 1 = 0.01 rupee] So now you directly tell me how you will write 5 paise = 0.05 rupees = Rs 0.05 [P asks students to observe 5 and then tell this.] 8 paise = 0.08 rupees = Rs 0.08 10 paise = 0.10 rupees = Rs 0.10

After a few months of observations in Phase 1, teachers began discussing some of their difficulties with the researcher. Some of their questions included "why do we need to teach the chunking method for division" which later on was made more explicit to "how does this method work" (refer Chapter 7 for a detailed analysis of this discussion). Similar questions were asked about "how to teach multiplication with decimals", "why doesn't the book provide a reason for writing $\frac{1}{10}$ as 0.1?", etc. It was noted that some of the teachers' questions were about (a) understanding the content per se for instance, questions such as – are there different ways of multiplying or dividing decimal numbers, how can different methods used to solve a problem be connected, are there negative decimal numbers; while other questions were about (b) how procedures work, for instance, how does the chunking method for division work, would it work for different kind of numbers like an algorithm, and so on. During these one–to–one discussions with the teachers, it was also found that they treated procedures as they are, that is, they did not talk about the underlying conceptual reasons for the procedures or algorithms.

In her research on teachers' knowledge of standard topics in elementary mathematics, Ma (2010) identified coherence in knowledge as significant to developing Profound Understanding of Fundamental Mathematics (PUFM). Linking teachers' knowledge to their explanations, Ma found that teachers who relied on the procedural knowledge of the algorithms tended to provide "pseudo conceptual explanations", that is, either verbalised algorithms as explanations or invented arbitrary explanations. On the other hand, teachers who provided conceptual explanations tended to be aware of the connections between topics and their relation with the structure of mathematics. Further, teachers' attitudes that supported their conceptual understanding included "justifying a claim with a mathematical argument, knowing how and why, keeping an idea consistent in various contexts, and approaching a topic in multiple ways" (Ma, 2010, p.103).

Based on the interactions in Phase 1 and the literature on the knowledge that teachers need to teach, the researcher intended to create an opportunity where teachers could discuss their conceptual difficulties with each other and talk about the mathematics that they were teaching. The tasks, which fall within this category, served the purpose of developing conceptual understanding of the procedures in decimal numbers. The task was to examine the use of digits–based approach, identify instances where it works, and discuss the conceptual explanation underlying this procedure.

6.6.1.2 Engagement in task

In TRM 3, teachers proposed a digits–based approach for the comparison of decimal numbers. A digits–based approach refers to using digits, not their place value, when operating on decimal numbers. For instance, consistent with her (and other teachers') classroom observations, Pallavi asserted that students must be taught how to make the length of decimal numbers (to be compared) the same by "adding some number of zeroes". She stated that the comparison of numbers should start from the left–most digit of a number. For instance, for the problem used in TRM 2, the numbers to be compared were 0.658, 3.7 and 2.45. In this case, she suggested making the length of the decimals same by "adding two zeroes to 3.7 and one zero to 2.45". Then, comparing the digits from the left of the decimal number. When probed for the reasons of making the length of decimals the same, Pallavi responded that although the decimals can be compared without making the length the same, what is more important is comparing the digits. In her view, annexing the zeroes made the numbers "convenient for comparison".

Pallavi's explanation of annexing the zeroes to compare the decimal numbers digit–wise, led to two kinds of discussions. First, the researcher intended to extend the annexing the zero discussion to examining cases involved in the changing position of zero in a number (refer Section 5.5.3 of Chapter 5 for a detailed discussion of the cases).Second, Vindhya had been suggesting that the explanation of telling the students to make the comparison of digits from left to right was imprecise. Instead, she had recommended placing the digits of the decimal numbers in a place value table and then comparing them. Pallavi resisted this explanation by recalling to Vindhya's comment in the previous meeting, that such an explanation is of a "higher level for students of primary grades".

In order to develop a better understanding of the digit–based approach among teachers by reflecting on how it used in the teaching of decimal numbers, a task in TRM 4 was organised around this theme. The discussion was on the affordances and constraints of using a digit–based approach for comparison of and operations on decimals. Teachers articulated that the decimal numbers with the same length can be compared by comparing the digits without considering the decimal point. Teachers' attention was drawn to students' strategy of counting the number of digits (longer is larger thinking identified by Resnick et al., 1989; Steinle & Stacey, 2004) when comparing decimal numbers. Nandini reflected that the teachers' explanation of comparing the digits ignoring the decimal point was connected with this particular student strategy. All the teachers seemed convinced that the explanation of comparing the digits of the decimal numbers to be compared by ignoring the decimal point was consistent with the whole number comparison. Teachers were asked to consider the case of comparing 2.45 and 1.789, where the number with more digits (4 digits in 1.789) is smaller than the number with lesser digits (3 digits in 2.45). Recognising that the digits–based approach will not work here (unless zero is annexed to make the lengths of the decimals to be compared the same), Pallavi commented that the rules of the comparison of decimal numbers change depending on the kind of numbers. When probed to identify these different kinds, she gave some examples.These examples included – (a) numbers where the "first digit is the same" referring to the whole number part, as in 1.24 and 1.78), (b) numbers where the "first digit is different" (3.4 and 5.7), etc.. It was noted that in the given examples the length of the decimals was the same. However, Pallavi's argument that within the comparison of decimal numbers, pairs of numbers which are different from each other need to be identified was valid. In fact, a detailed set of problem types and the corresponding nature of students' thinking have been recorded in the research work of Steinle and Stacey (2004). However, what needed to be challenged in Pallavi's explanation was "different rules for different problem types". In order to challenge this practice, teachers were requested to think of a consistent explanation for the comparison of decimal numbers. Several pair of decimal numbers of varying length and size were noted after some brainstorming. Teachers were asked to identify different kinds of problems from those recorded. Then, teachers were encouraged to think of a conceptual explanation that might support students in comparing all these kinds of decimal numbers, with varying lengths and size.

After some deliberation, two explanations emerged. The first explanation was based on using the fractional equivalents of the decimal numbers. It included representing the decimal numbers of the same or varied length as fractions. Then, using the equivalent fractions, if required, to compare the fractions and identify the greater number (refer examples given in Excerpt 6.8).

Excerpt 6.8: Comparing decimals: Fraction explanation (TRM 4)

Compare 0.78 and 1.23. $\frac{78}{100} < \frac{123}{100}$	Compare 0.78 and 1.2. $0.78 = \frac{78}{100}$, $1.2 = \frac{12}{10}$ or $\frac{120}{100}$ $\frac{78}{100} < \frac{120}{100}$
(a) Conversion to fractions	(b) Annexing zeroes and comparing fractions

The second explanation was comparing the digits with the same place value. It included Pallavi's observation of beginning with the highest place value but with an additional condition (in order to respond to Vindhya's critique) that digits with the same place value needed to be compared. This was done through the use of a place value chart (refer Excerpt 6.9).

Excerpt 6.9: Comparing decimals: Place value explanation (TRM 4)

Number	Ones	Tenths	Hundredths
0.78	0	7	8
1.23	1	2	3
	1 ones > 0 ones so 1.23 > 0.78.		

The group checked for why both these explanations were consistent, first by checking it for specific problem types identified earlier and second by reasoning which focused on the nature of explanation (and not specific cases of numbers). Teachers were also asked whether there would be a set of numbers for which these two explanations (represented in Excepts 6.5 and 6.6) might not work. Teachers reasoned that these explanations are consistent for they use re–representing the decimal number and changing it to a form where it can be compared based on students' prior knowledge (fraction comparison and place value comparison). The task concluded with recognising the fraction representation of a decimal number and place value of digits in a decimal number as important ideas in the learning of decimals.

6.6.1.3 Discussion

From Phase 1 observations and what teachers had shared in TRM 3, it was clear that teachers worked mainly with the procedural explanation for comparing decimal numbers. They used some visual clues to support students' recall of these procedures. The discussion on the procedure of making the decimal lengths equal for the convenience of comparison revealed that Pallavi emphasised the digits–based explanation for comparison of decimal numbers. Additionally, even after Vindhya stated the place value explanation, Pallavi called it "higher level" for students and did not explicitly acknowledge that, it was indeed the underlying reason for why the procedure of making the lengths of decimals and then comparing each digit works. At the same time, it is important to note that all the teachers understood Vindhya's explanation, which indicates that they were familiar with the conceptual explanation. Interestingly, an awareness of the conceptual explanation did not translate into applying it to explain the procedure of comparing decimal numbers.

In this task, teachers explicitly identified different problem types for comparison of decimal numbers, that is, with varying lengths and sizes. When teachers were made to think of an explanation which will be consistent for different sets of numbers to be compared, two explanations: one based on fractional equivalents, and another based on place value of each digit, emerged. These explanations were conceptual in nature, and included connection with other knowledge pieces, namely, fractions and place value. Fractions and place value (from whole numbers) were identified as key pieces of knowledge, connections with which needed to be explored deeply, while dealing with decimal numbers. In the later meetings, these two explanations were used to justify the conversion of measurement units and operations on decimal numbers. Teachers realised that the advantage of using conceptual explanations was that these were consistent for different sets of numbers. In other words, rules are the same for different kinds of numbers. This, in a way, challenges the pedagogical approach of changing the rules for different sets of numbers. Teachers were becoming more sensitive towards the extension of whole number thinking to decimal number comparison. For instance, a detailed discussion on the specific case of "longer is larger" helped teachers in noticing that the rules learnt during whole numbers might not always work in the case of decimal numbers. A detailed discussion on the affordances of whole number thinking (that is, in which cases and conditions does it work and where it does not work) for the learning of rational numbers, followed from this meeting.

6.6.2 Task 8: Designing and sequencing decimal problems

One of the important tasks of teaching is designing, selecting and sequencing problems. A decision on problem selection is guided by the purpose for which it is intended to be used. As teachers design and select problems in their everyday teaching, it is reasonable to assume that a rich repertoire of problem space can support teachers in making such decisions. The following task encouraged teachers to create questions and their variations, by keeping the purpose of designing the problems at the core.

6.6.2.1 Design Rationale

Similar to the use of the contexts and representations, teachers' use of the problems given in the textbook in the first year, was found to be limiting. While teachers designed problems in the classroom, they were similar to the questions given in the textbook. Thus, additional problems were used to reinforce the use of a rule or procedure, which implied selecting examples or confirmatory cases only. Teachers encouraged students to follow the taught procedure to solve these problems. Teachers did not draw students' attention to different cases of a problem. In this task, teachers were encouraged to design non–routine problems, depending on the purpose.

Watson and Mason (2006) use the "dimensions of possible variation" and "range of permissible change" to argue that carefully designed exercises have the potential to create mathematically supportive environments for learners. Making patterns within variations explicit and drawing students' attention to such patterns helps in engaging them with the mathematical structure.

The following task invited teachers to design different kinds of problems and articulate the purpose that the problems addressed. It was also a way of understanding how discussions on students' thinking and key ideas in the topic influence the design of problems.

6.6.2.2 Engagement in task

In this task from TRM 3, following from the discussion on the influence of whole number thinking on the decimal learning, teachers were asked to design some decimal comparison problems, which would help diagnose specific students' misconceptions. The problems needed to be designed for addressing a specific alternate conception where the student ignores the decimal point while comparing decimal numbers. Different teachers suggested the following problem types, with examples (refer Table 6.4).

Table 6.4: Comparison problem types (TRM 3)

Problem type	Proposed by	Description	Example
I	Vindhya	The same whole number part but varied decimal part, but dealing with only tenths and later extending it to hundredths.	3.0, 3.9, 3.10
II	Reema	Take the same set of digits and change the position of decimal point.	3642, 364.2, 36.42, .3642
III	Pallavi	Take different three digit numbers. Compare them with decimal point at different positions.	(a) 465, 599, 436 (b) 46.5, 59.9, 43.6 (c) 4.65, 5.99, 4.36 (d) 0.465, 0.599, 0.436

Here, the teachers seemed to have begun thinking about specific problem types. Teachers discussed how a variation of a problem could generate other problems for addressing specific students' thinking. The task of designing problems was extended in TRM 15. The discussion began with identifying the sub–topics which needed to be covered. These included place value and expanded form, comparison and ordering, representing numbers on a marked ruler, conversion between measurement units, and addition and subtraction of decimal numbers. While creating problems for each of these sub–topics in pairs, there were detailed discussions among teachers about the selection of numbers, influence of whole number thinking, opportunities provided for using fractions and place value – both as explanations, position of zero and its influence on the value of a number, variations in the same problem type, representation of a decimal number using different ways, and so on. The problems designed in TRM 3 were recalled (refer Table 6.4) and there was a discussion on which problem type addresses what kind of students' thinking. For instance, the rationale for Problem type 2 was to direct students' attention to the position of the decimal point and how it changes the place value of each digit and therefore the quantity represented by the number. It was noticed that during these discussions, teachers began anticipating students' ways of thinking and solving these problems. For instance, let us take a case of discussion on the following problem designed by the teachers (refer Figure 6.5).

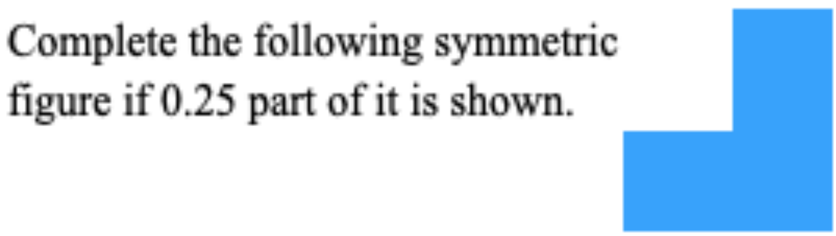


Figure 6.5: Complete the Figure (TRM 15)

Pallavi's remark on the problem was that students might treat it as the fraction one–fourth to solve this problem. She asked about how to make students solve this problem using the decimal representation. Nandini added that students might extend their knowledge of fractions, that is, four one–fourths make a whole, and extend it to repeat the figure in different orientations to complete it. Pallavi added that the figure requires students to make a connection between 0.25 and a whole. Reema stated that the task required students to draw on their prior knowledge of finding part of whole and then extend this knowledge to do the reverse. She added that exercises such as this, that is, complete the figure when half is given would support problem solving of this kind. For this particular problem, she mentioned that using a square grid in the background might support students in completing the figure and seeing the relation between fractions and decimals. The teachers decided to support students through a series of carefully designed problems or an exercise, beginning from the use of square grid, where the whole is given and students are asked to identify parts of it to the level of solving a problem where students need to construct the whole based on a given part (for the problem given in Figure 6.5) without the grid.

Another problem, which was created for by a different pair of teachers, was to ask students to state the difference between 0.02 centimetre and 0.02 meter. This problem is linked to a task from TRM 6, where the facilitator showed a short video of a telephonic conversation between a telecom company and a user. While the user was assuming the pulse rate to be charged at the rate of 0.05 cents, he was actually being charged 0.05 dollars for the same. Tracing the origin of this problem to an earlier discussion, teachers started anticipating how students would approach this problem. Almost an immediate response was visualising the lengths, 0.02 cm and 0.02 m, and representing their magnitude using a representation. Pallavi stated that students could show 0.02 meters on a 10×10 grid. Nandini reminded Pallavi of her contention, stated in an earlier meeting, about the confusion arising from representing a length measurement context using an area representation. Pallavi then corrected herself and drew a number line to show meters and centimetres. She drew a scaled down length for one meter on a sheet of paper. The teachers were visualising how 0.02 part of a centimetre would look like on a scaled down version of one meter length drawn in students' notebooks. The teachers discussed whether such a problem should be posed in a worksheet where one meter length is drawn to support students' visualisation of the given lengths (0.02 m and 0.02 cm). Pallavi suggested that the problem could be reframed as – use the diagram to find whether 0.02 cm and 0.02 m are the same. In both these cases, it was interesting to note, how Reema and Pallavi began thinking about providing the necessary mathematical supports to students in order to solve these problems.

6.6.2.3 Discussion

While engaging with the task of creating problems it became evident that teachers were making connections with the discussions from previous meetings as well as with their classroom practice. There were also some explicit references made to prior meetings. The considerations, which guided the design of questions, had changed considerably from the initial meetings. For instance, in the first meeting, teachers listed sub–topics such as expanded form, shifting the decimal point, fraction to decimal conversion, etc and resisted the inclusion of comparison of decimal numbers as a topic since it was not mentioned in the textbook. In contrast, in TRM 15, when teachers are asked to create problems they listed the sub–topics, which they thought needed to be included in the teaching of Grades 5 and 6. Here, the teachers had a shared understand that the comparison of decimal numbers was significant for building explanations based on fractions and place value. Also, major differences in the framing of questions can be noticed. In Phase 1, Pallavi and other teachers had explicitly objected to the researcher's framing of questions (given as part of a worksheet to diagnose students' understanding) where students were asked to explain the reason for their response in writing. Pallavi had argued that such an addition to the questions was non–mathematical and non–routine. In the problem of comparing 0.02 m and 0.02 cm, Pallavi suggested that the phrase "using a diagram" be explicitly mentioned in the question so that students are encouraged to use a representation. The teachers now seemed more aware of their students' thinking, as it was noted that (a) they were anticipating students' responses to the problems being designed, and (b) explicitly thinking about providing supports by appending instructions such as draw a diagram to compare decimal numbers, while recognising that some students might use fractions for the same. I also note that questions on decimal numbers designed by the teachers were different from the textbook questions and quite a few of them addressed specific students' misconceptions. Further, at this stage, thinking about the mathematical variations of a problem and anticipating students' ways of thinking while designing problems, was almost unguided by the researcher or facilitator.

6.6.3 Task 9: Deeper connections in the curriculum

As teachers became cognisant of the connections between mathematical ideas, they became sensitive to the use of specific contexts, meanings of numbers, etc. This task was initiated by a teacher and lead to discussions on the different meanings and functions of numbers in general and decimals in particular.

6.6.3.1 Task Design

The discussion in this TRM was initiated by a teacher. The salience of the situation emerged from her examination of the meaningfulness of a context. In the first year, some of the problems that were taken up in classroom included – (a) which is greater 6.2000 dm or 6.200 cm, (b) who has more money: Anu has Rupees 5.500, Shashi has 5500 paisa and Rajan has 0.0550 rupees. In such situations, it was expected that students convert (preferably) the measure in larger unit to a smaller unit, that is, convert 6.200 dm into cm and 0.0550 rupees into paisa. The meaning of the quantity or amount that was being represented, was missing. For instance, it was not considered whether there would be a situation in which Rupees 0.0550 makes sense. In fact once Reema had this discussion in her class (very briefly) that in Rupees 499.99, 99 paisa are not asked and therefore the customer gives one rupee extra to the shopkeeper. Two kinds of considerations can be noted here (a) some units like paisa are obsolete denomination or rarely used in an everyday discourse, and (b) the quantity represented by measures such as 5500 paisa and 6.2000 decimetre is difficult to imagine. Such word problems develop a strong tendency among students to dissociate the real–life contexts from the content learnt in mathematics classrooms (Verschaffel, De Corte & Lasure, 1994). The task focused on unpacking the connection between the unit measure and the quantity being represented. The discussion moved from measure numbers to how numbers are used in different ways.

6.6.3.2 Engagement in task

In TRM 6, there was a detailed discussion on the contexts used in decimal teaching. These contexts included length, weight, area and volume measurement, currency conversions, temperature, etc. In TRM 7, Pallavi raised a concern about the relevance of teaching equivalent decimals using currency context (refer Excerpt 6.10).

Excerpt 6.10: Meaningfulness of a context: Pallavi (TRM 7)

Line No.	Speaker	Utterance
22	Pallavi	See, that equal and decimals we were just discussing on that day. See I have seen that they are equal. As per as I have seen the equal and decimal we do it only for the plain numbers not with the measurements. See, you asked whether forty five point, Rupees forty five point five zero, can we, I mean, can we take it equal to forty five point five zero zero, forty five point zero zero zero?

She discussed that rupees forty five point five zero (Rupees 45.50) cannot be considered equivalent to rupees forty five point five zero zero (Rupees 45.500) since the latter is incomprehensible using the given units. The reason is that an Indian rupee is divided into 100 paise and Pallavi is concerned about the thousandth part, which can not be comprehended using the context of Indian currency. She added that 45.50 and 45.500 are equivalent decimals only as numbers. In terms of currency, their equivalence does not make sense. She distinguished the money context from the length measurement context where the relation of thousandths can be expressed using the measurement units of kilometre and meter. Pallavi seemed to have pointed out the limitation of the context based on its meaningfulness. The facilitator drew teachers' attention to the difference in the two notations, that is, Rupees 45.50 and 45.50. They were identified as measure number and plain number respectively. Reema used another example to instantiate the difference. She mentioned that, "three point five zero is an example of plain number", that is, without a unit and a measure and "three point five zero meters is a measure".

Teachers were encouraged to think about the contexts in which measure and plain numbers are used. The group identified contexts such as money, length measurement, weight, capacity, speed, area, volume, etc., for measure numbers. Teachers seemed unsure about whether the number of overs in a cricket match and time measurement are examples of measure numbers. Teachers had learnt that these two were not decimal contexts in Task 5. The facilitator proposed that some numbers are used for the purposes of labelling, for instance, naming the exercises of a chapter (where

Exercise 8.9 is followed by Exercise 8.10 and the latter is read as eight point ten). In this notation, the first number (8) represents an object or an item and the second number (9 or 10) represents its part. The relation between the object and the part does not signify a quantitative relation but an order relation that is, 8.9 always precedes 8.10. The numbers used for labelling were distinguished from plain numbers. The group identified other contexts where plain numbers are used, for instance, counting the number of people or objects, students' scores, average of a set of numbers, ranks in a test, etc.. The working definition of plain numbers was formulated, which was, quantities which can be represented using whole numbers only. For instance, number of students or coins. The facilitator asked teachers to examine whether number of marks scored in an exam is an example of plain numbers. Nandini mentioned that marks can be given in half and sometimes a quarter. She also said that "average marks is not always a whole number". She connected it with the context of attendance of the students present in a class, where although the number of students who are present will always be a whole number, average attendance need not be a whole number. Later, when the facilitator pointed to some examples of the use of numbers, Nandini promptly distinguished between ranks and marks. She identified that whole numbers are used in ranking, but decimal numbers are used in marks. The facilitator classified the ranking as an example of ordinal numbers.

Returning to Pallavi's question about the relevance of the money context, the group discussed the history of how the value of money has changed over a period of time. The group noted that the need for precision in currency was declining with time (earlier different parts of a rupee were used). It was also noted that the demand for precision in measurements in astronomy and molecular biology had led to the evolution of larger (light years) and smaller units (nanometer). Pallavi gave another example from the length measurement, 5 m and 200 mm, suggesting that writing it as 5.200 does not make sense. She recalled that 5.200 m means that the numeral and the fraction part of the number are expressed in the same unit, that is, meters in this case. Struggling to make sense of 0.200 m, she was confused whether it was 2 mm or 20 cm. Nandini proposed that time measurement context is completely different from a decimal context such as length measurement, since 10.5 hours does not mean 10 hours and 5 minutes, but 10 hours and thirty minutes. She pointed that even though the two units – hours and minutes, are related to each other, the relation is not of ten times. She used this example to recall the discussion on non–decimal contexts such as the relation between number of overs and balls in a cricket match. Teachers began differentiating between the label numbers, such as those used as bus or vehicle numbers, and ordinality represented in exercise numbers, ranks and counting in a set. There was a brief discussion on cardinality, that is, the quantity represented by the number of elements in a set. Teachers were asked what would be the cardinality of the set of whole numbers, and asked to identify its relation with the cardinality of the set of rational numbers. Nandini observed that decimal numbers are used to measure and can be ordered. Reema added that in the example of time measurement, the measure signifies both order and quantity, but the relation between units is different. This discussion was extended in another meeting where the teachers examined the meaningfulness of operations for different set of numbers, for instance, while it does not make sense to add ordinal numbers like ranks, cardinals can be added. Further, while deconstructing multiplication $m \times n$ it was recognised that m represents the number of groups of a set with n elements each.

The task concluded with (a) identifying the property of the numbers being used to represent a quantity, and (b) making connections between the measurement unit and the quantity being represented for its meaningfulness.

6.6.3.3 Discussion

Using teachers' knowledge of seeing numbers in a variety of situations, they were supported in identifying ways in which numbers are used for quantification. They examined the relevance of contexts for using specific kind of numbers, particularly, measure numbers. Going deeper into the measure numbers, they brought in their prior knowledge of defining decimal numbers, decimal and non–decimal contexts to distinguish between situations where the measures can be used as examples of decimals (such as currency and length measurement) and non–examples (overs in a cricket match, time measurement). It was interesting to note that the discussion in this meeting covered depth and breadth almost simultaneously. Attempts made by Pallavi and Nandini, to make sense of the quantities, helped in gaining clarity on which contexts are suitable for teaching decimals. Pallavi's perseverance in trying to understand the meaning of the quantity, Rupees 45.500 where 0.500 rupees does not make sense, made her bring in another example of measurement, 5.200 m. Using her prior knowledge, she recalled that a decimal number which represents a measure, has only one unit. She used this knowledge while she was struggling to make sense of the digits after the decimals in the above mentioned contexts. Similarly, Nandini pointed out that discovered the use of time measurement context like the case of the number of overs in a cricket match, was not a decimal context. She also noted that while all decimal numbers are measure numbers, not all measure numbers are examples of decimal numbers. While these discussions helped in understanding contexts used for decimal numbers in depth, the expansive use of numbers in different contexts (labelling, ordering, cardinality, measure) helped them understand how some types of numbers can be operated upon, while others cannot. For instance, adding or dividing the ranks does not make sense.

6.7 Discussion: Learning From Participation in TRMs

In the beginning of the chapter, I had raised two questions. First was about the kind of knowledge that was addressed during the teacher–researcher meetings and second, how this knowledge was useful in handling demands arising from contingent classroom situations in the second year of teaching. This section is organised around these two questions. It concludes with some notes on the design and use of classroom–based tasks in teacher development initiatives in order to impact teachers' practice.

6.7.1 Teacher knowledge and learning

The kinds of knowledge that was focused in TRMs was influenced by the classroom observations from the first phase of the study and the topic–specific literature on decimals. First, it is important to note that any such knowledge is difficult to separate from the associated practices, beliefs, skills, and knowledge about other ideas. Second, a focus on teaching practice provided a rich context for knowledge to be discussed meaningfully. Third, it is difficult to argue whether any specific kind of knowledge (identified in the frameworks on teacher knowledge) was present or absent completely. The reasons are (a) traces of knowledge were visible as prior knowledge, which then became available for building new knowledge, and (b) not all kinds of knowledge gets triggered in a situation, that is, an explicit manifestation of knowledge is situation specific. Fourth, while the study focused on discussions around the topic of decimals, there were evidences of the influence of this learning on the teaching of other topics by the teachers.

The kinds of knowledge that was addressed during meetings is evident from the three themes that emerged from a classification of tasks done in TRMs (summarised in Figure 6.6). The knowledge includes (a) developing mathematical sensitivity to students by understanding what they can think and do, (b) identifying suitable contexts, representations and explanations used for decimals by examining their affordances and limits, and (c) making connections between key ideas, explanations and topics. These three knowledge types are important for developing teachers' topic–specific knowledge of teaching decimals.

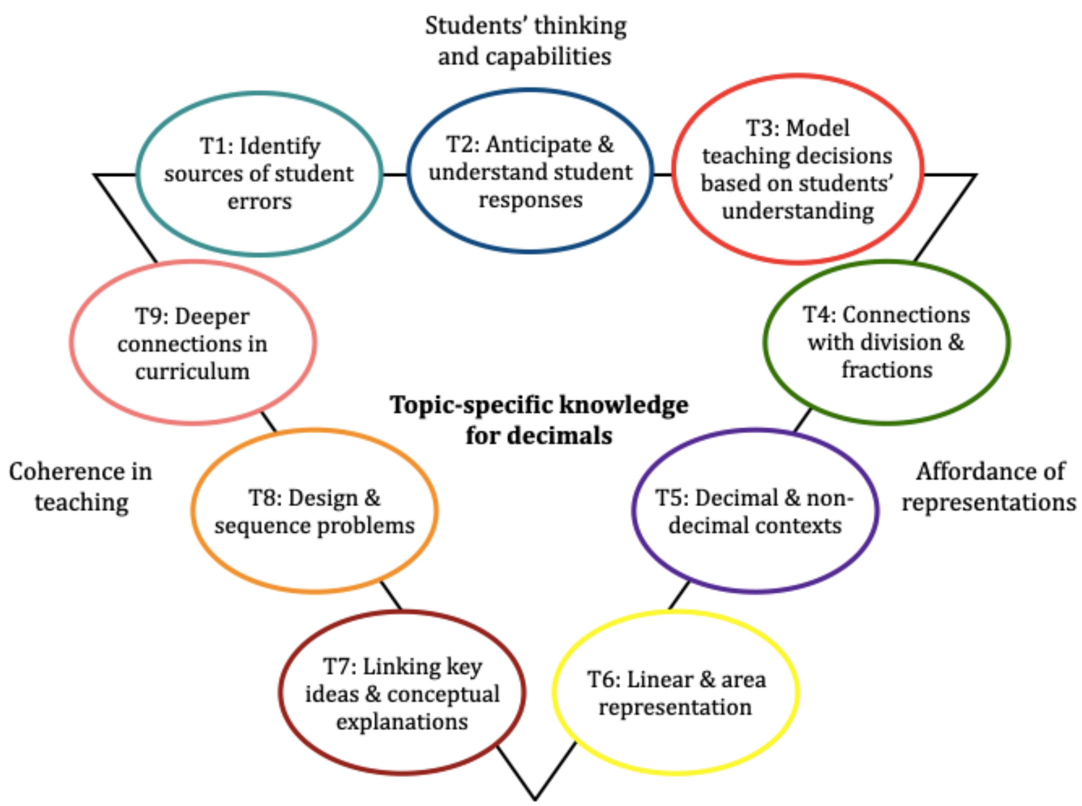


Figure 6.6: Aspects of Topic–Specific Knowledge for Decimals

Knowledge of what students can think and do mathematically was developed through engaging teachers in analysing the mathematical sources of students' errors, anticipating students' responses to problems and the underlying thinking that might influence their responses, and responding to it conceptually. In other words, developing teachers' knowticing of students' work. The research literature on students' conceptions, particularly on comparison of decimals and students' work from teachers' own classroom were used to develop teachers' awareness of students' mathematical thinking.

Teachers' knowledge of the content for teaching was developed through examination of a variety of contexts and representations used for teaching decimals. The repository of the contexts was expanded beyond those given in the textbook and the meaning of decimal notation was used as a rationale for selecting these contexts. After identifying the relevant context, its affordances in terms of relation between units and the appropriateness of the representation used for the context (the Indian currency involves numbers up to the second decimal place) were examined, for instance, relevance of the number line to represent a length measurement context. This knowledge helped teachers in examining the meaningfulness of the quantities being represented with the units, independent of the context (for example, Rupees 2.500 is meaningless). Similarly, the linear and area representations were examined for their affordances, which included identifying connections between different units. For example, in dienes blocks, a tile is 10 times a strip, and a strip is 10 times a cube. Also, their appropriateness for the context being represented was examined through questions such as, can a length measurement context be represented through a grid. The relation between units in each representation and context was explained through students' prior exposure with fractions (and division). These connections were invoked to identify continuity in students' knowledge of rational numbers.

Consistency and coherence in explanations and representations was emphasised through identifying key ideas in the teaching of the decimals and using connections between them to form conceptual explanations for procedures. Teachers were supported in creating a variety of problems (and variations) to diagnose students' thinking and support it through ways which engage students' productively. Teachers designed non–routine problems and identify sub–topics which were missing in the textbooks but were crucial for the learning of decimals. Deeper connections between how numbers are used in mathematics was identified and the property of decimal numbers (positionality and relation between units as powers of 10) became explicit.

While all these three themes have been presented separately for the purposes of analysis, I noticed overlaps and connections between the knowledge that was being discussed in each of these. Figure 6.7 shows the connections between tasks that indicate the knowledge that was invoked as teachers engaged in each task. T1 to T9 indicate the tasks discussed under the three themes in the chapter.

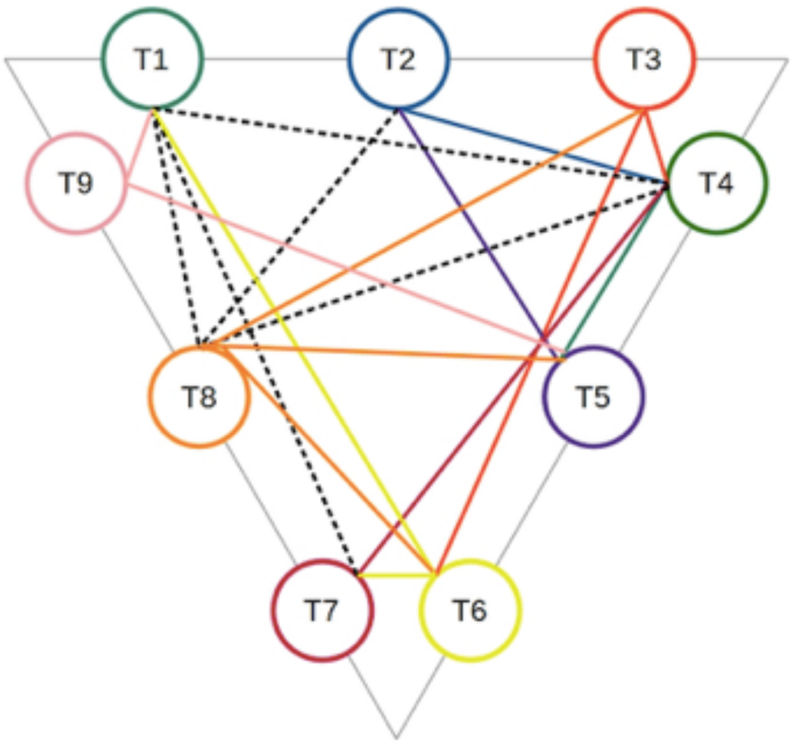


Figure 6.7: Connections Between Topic–Specific Tasks on Decimals
(T1 to T9 indicate tasks, see Figure 6.6 for elaboration)

This representation was refined to connect the themes, that is connections between the knowledge kinds, that constitute topic–specific knowledge. For instance, an awareness of the sources of students' errors (Theme 1) was crucial to designing problems for diagnostic purposes (Theme 3). Similarly, understanding students' mathematical thinking, particularly connections between students' prior knowledge and the learning of decimals (Theme 1) was useful in identifying connections between the topics of whole numbers and fractions (Theme 2).

The complex connections between the knowledge underlying each of these tasks shows how aspects of topic–specific knowledge are interconnected and difficult to separate. The map of inter–connected knowledge also indicates the difficulty in discerning knowledge that underlies teaching, unlike that identified by the existing frameworks on teacher knowledge.

6.7.2 Knowledge and practice

The aim of the teacher–researcher meetings was to develop teachers' knowledge *of* practice. The process of teacher learning was supported by establishing connections between the research literature on decimal teaching and learning, and teachers' existing practice. The interweaving of the knowledge of research literature and knowledge in practice supported teachers' learning by helping teachers in anticipating and handling contingent moments arising in their classroom. An analysis of the paired episodes from the two years of teacher, reported in (Sections 5.5 and 5.6 of) Chapter 5 indicate how teachers used the knowledge from teacher–researcher meetings to respond to the contingent moments differently.

Teachers used discussions during TRMs to make teaching decisions in the second year. Some pedagogical decisions in which teachers' knowledge was visible include – (a) dealing with students' questions and detailing their strategies, for instance, discussion on the question of oneths in Nandini's class (discussed in Section 5.5.2 of Chapter 5), partial quotients identified by students while solving division problems in Pallavi's class (will be discussed in Chapter 7), using prior knowledge to find the number of frogs of a given length on a meter long wire in Reema's class (discussed in Section 5.6.2 of Chapter 5), and Vindhya's attempts at probing students' responses and encouraging them to articulate their thinking (will be discussed in Chapter 7); (b) using consistent representations and identifying the affordances of contexts, for instance Reema and Nandini's decision to use linear representation for all sub–topics within decimal learning (discussed in Chapter 5), Pallavi identified that money context can be used to deal with the hundredths relation only and concluding that the length measurement context has larger affordances (refer Section 6.5.3); and (c) emphasizing key ideas, explanations and connections, for instance Reema and Nandini attempted to identify conceptual explanations using the key idea of place value and used it in their teaching (discussed in Chapter 5), Pallavi examined the link between division, fraction and decimals in her class, and Vindhya attempted to link students' varied responses to the key ideas such as place value, position of zero, in class.

The interweaving of knowledge and practice was evident from the ways in which teachers used reflection on their practice during discussions in the TRMs. Some of the ways in which practice seeped into the meetings are identified below.

(a) *Sharing knowledge gained from experience* – Teachers shared their prior experience of handling a sub–topic by articulating the procedure and the sequence of teaching, identifying common student errors and in situations when students make them, sharing their choice of explanations and representations used, etc. The examples of teachers' sharing their existing knowledge gained from experience of teaching can be found in the engagement with all the tasks discussed in this chapter.

(b) *Connecting teachers' explanations with the sources of students' thinking* – While reflecting on the sources of students' errors, teachers identified how some of their explanations might reinforce students' mis–conceptions. For instance, Nandini and Vindhya identified how their explanation of "adding the zeroes does not change the value of the number" without identifying the cases where it does so might reinforce students' responses, such as, 3.06 is the same as 3.6 and 3.60. Pallavi had noticed that her explanation of counting the digits of the decimal numbers to be compared, might have influenced students' responses of the kind that 3.600 is greater than 3.6. Some of these connections might be implicit but are evident in the teaching decisions mentioned earlier. For instance, Reema and Nandini decided to use the linear representation consistently for different sub–topics in decimal teaching unlike using linear and area representation intermittently as suggested by the textbook. These evidences indicate how teachers' knowticing of not just students' ideas but also about their own practice was improving in the process.

(c) *Using practice as a site for experimenting with alternate pedagogies* – I observed that teachers were becoming more careful in selecting the content to be taught. For instance, teachers decided to deal with some of the ideas (contexts, methods) from the textbooks that they had omitted or paid less attention to in the previous years. When planning to teach these ideas in the second year, teachers asked the researcher for individual support. Chapter 7 presents an analysis of the nature of *in–situ* support offered to the teachers, when they identified the content and an alternative pedagogy of dealing with it in classroom.

6.7.3 Reflections on the design of tasks for teacher learning

The tasks designed for TRMs were based on the classroom observations of teaching in the first year and the research literature on decimal teaching and learning. An overlap between the two phases of the meetings and classroom observations (Phases 2 and 3) helped in interweaving the connection between the knowledge that was being discussed and teaching practice. The analysis of teachers' engagement in and their learning from TRMs suggests that tasks centred around the interweaving of research literature with actual practice (using actual student data) have the potential to draw teachers' attention to the nuances of the knowledge that is needed to teach responsively. Further, a discussion forum where teachers and researchers bring in their knowledge about the topic served as a support structure for teachers who are struggling to engage with reforms in practice. The discussion forum in the form of TRMs was used in this case to identify aspects of teachers' knowledge base, as envisioned by Cai et al. (2017).

Chapter 7

TEACHER KNOWLEDGE AND LEARNING *IN–SITU*: CONTINGENCIES IN TEACHER–RESEARCHER COLLABORATION

If teacher knowledge is supported by social structures and relationships, then it is likely to be productive to focus on developing shared expertise rather than individual 'knowledge'. (Hodgen, 2011, p.38)

7.1 Abstract

This chapter is broadly organised into two parts. The first part deals with the detailed case study of Pallavi's teaching. The purpose is to discuss the mathematical challenges faced by her in the second year of the study, as she became more responsive to students' mathematical thinking. Through this case study, I exemplify how a focus on mathematical knowledge for teaching 'in situ' helped in triggering a change in the well–formed teacher knowledge and beliefs about the teaching and learning of a specific topic and the related students' capabilities.

The analysis of this case led to theorising the construct of contingent situations, that is those situations in the teacher–researcher collaboration, which reconfigured the relation between the researcher and the teacher, in order to address the knowledge demands arising from teaching. In the second part of the chapter, I use the construct of contingent situations to analyse the collaboration with all the participating teachers.

An analysis of contingent situations across different cases reveals the processes involved in the transformation of such contingent situations into learning opportunities for teachers. The chapter concludes by arguing that a situated approach of working with teachers and a deeper engagement with their practice, provides opportunities to challenge teachers' knowledge and beliefs in order to create

possibilities for reformed practices. The analysis also reveals the situated dimension of teachers' specialised knowledge of mathematics. In the context of educational reform, an analysis of contingent situations helps in both understanding and supporting teachers' work.

7.2 Central Questions

In Chapter 5, I had identified the knowledge demands posed on the teachers as they became more responsive to students' thinking. As the study progressed, there was a noticeable change in the teacher's noticing of the mathematical aspects underlying students' utterances. As teachers become more sensitive to students' mathematics, decisions such as the nature of mathematical explanations, connections between topics and dealing with a variety of methods, became salient for them. Consequently their teaching became more demanding. Noticing the mathematical aspects underlying students' responses while teaching in the classroom, also affected the way teachers examined the textbooks. All the teachers started reading the textbook more carefully and made detailed notes (or plans) of their lessons. In this chapter, the focus is on the nature of *in–situ* support provided to the teachers and how it influenced teachers' knowledge of the subject matter. It was noted that all participating teachers demanded individual support for specific topics and/or tasks of teaching. Since the nature of support offered to teachers was at an individual level, situations where they struggled and sought the researcher's support have been discussed and analysed. The chapter addresses the following questions.

(a) What were the mathematical challenges explicitly identified (as distinguished from knowledge demands discussed in Chapter 5) by the teachers while teaching, as they became more responsive to students' thinking?

(b) How were they supported *in–situ* in handling these challenges?

(c) What was the nature of teacher learning from this support?

In the first part of this chapter, the case study of a teacher, Pallavi, is presented to explicate the nature of support demanded and offered. The reason for selecting Pallavi's case is that she was resistant to any change in her classroom practice. She took a longer time (as compared to the other participating teachers) to appreciate the diversity in her students' responses and continued to use procedural explanations and support for rote memorisation as common practices while teaching in the second year. The case study reported here raised some questions about the challenges faced by teachers who are in transition and ways in which they can be supported. The lessons selected from the two years of Pallavi's teaching focus on the topic of division of whole numbers, so a review of literature on the topic and a contrast of how it is dealt in the old and new textbooks has been done in Section 7.3. Since the teachers aligned with the old textbooks and critiqued the new textbooks, a discussion on how the topic has been dealt within these textbooks helps in locating teachers' existing pedagogies and understanding the struggle involved in imagining alternative pedagogies. This is followed by a discussion on the process of identifying and articulating the moments of challenges, the nature of *in situ* support provided to Pallavi, and how the teacher and researcher engaged with the practice in Section 7.4. An analysis of this case study helped in identifying significant moments in the teacher–researcher collaboration which triggered a change in teacher's practice. Such moments were identified in all the four cases, and are presented in Section 7.5, to analyse how the nature of support offered to the teachers manifested in practice. The chapter concludes with the reflection on the process of using contingent situations as learning opportunities by outlining the kind of learning that was enabled through an *in situ* support and how it paved the way for deeper engagement with the knowledge required for teaching and the pedagogy in Section 7.6.

7.3 Background

Pallavi strongly and on multiple occasions expressed her appreciation of the approach of the old NCERT textbooks and the textbook designed by her school system *Ladders*, both of which emphasised repeated practice and rote memorisation. Several times during the study, she suggested reinforcing the rules in order to deal with students' errors. She explicated how students' inattention and lack of motivation undermine their success in mathematics. Although resistant to alternate pedagogies, such as conducting a discussion on various students' strategies in class, she engaged with the discussions during teacher–researcher meetings by participating in the tasks and articulating her opinions. For instance, in one of the meetings on using consistent representation, Pallavi mapped the tenths, hundredths and thousandths place on the same number line with whole numbers. In classroom teaching, on the contrary, she had used different ways of representing each of these place values. In another meeting, she initiated the idea of creating a question bank, with a variety of questions, so that teachers can refer to it during the class. A study of her case raised the following questions.

(a) How do knowledge, beliefs and practice interact as a teacher in transition struggles to implement curricular reform in the classroom?

(b) How does knowledge of "why an algorithm works" lead to productive ways of engaging students' thinking in the classroom?

An analysis of her case study, through answering these questions, will help in unpacking the challenges faced by the teacher, the nature of support offered by the researcher and how the teacher engaged with the *in situ* support.

7.3.1 Teacher knowledge in arithmetic

Mastery of the four basic operations of arithmetic is considered central to the primary school mathematics curriculum. Students are expected to "know" the algorithm for each operation and use it fluently to solve problems. Kamii and Dominick (1997) probed students' understanding of arithmetic operations and found that an excessive emphasis on the teaching of conventional algorithms (a part of social–conventional knowledge of mathematics) was constraining students in developing an understanding of relationships between numbers (logico–mathematical knowledge). Further, Khan (2004) noted that an overemphasis on the techniques used for memorisation of algorithm, makes it difficult for students to reflect on the problem, and check the appropriateness of their solutions. Despite such criticisms, the knowledge and successful application of the learnt algorithms is considered an important goal of school mathematics. Students' performance in algorithmic knowledge satisfies the dominant societal conceptualisation of what it means to do mathematics (Ebby, 2005).

The significance of teaching operations using only algorithms was challenged recently in the Indian mathematics curriculum. The change in the curriculum, however, has not changed the parental or school expectations that accord primacy to the fluency with algorithms. The knowledge of algorithms and the ability to manipulate symbols is considered as a marker of school learnt mathematics and is often used to differentiate it from out–of–school knowledge (Khan, 2004).

Students find the division algorithm difficult as it builds on their knowledge of number facts learnt during addition, subtraction, and multiplication (Anghileri & Beishuizen, 1998). Subramaniam (2003) discusses an error frequently made by students as well as some teachers in solving the division problem 981÷9, obtaining the quotient as 19. Such difficulties with long division arise from an emphasis on the inflexible procedural way of solving the problem (Windsor & Booker, 2005). The procedure of division involves remembering each step, forgetting any of which leads to errors. The misplaced emphasis on rote memorisation does not support students' understanding. Thus, even those students who use the division algorithm correctly to solve problems may not understand the meaning of the algorithm and why it works.

Anghileri, Beishuizen and van Putten (2002) conducted a comparative study of written solutions to division problems of Grade 5 students from England and the Netherlands. In England, students were being taught the division algorithm from an early age. An over–reliance on the procedures did not allow students to see the structure underlying the procedure or take the numbers into account. Evidences such as these can be found in the Indian mathematics classrooms, where students often multiply, for instance, 40 with 10 using the standard algorithm without considering the numbers or evaluating the need to use the algorithm. In contrast, the Dutch approach based on realistic mathematics education focused on eliciting students' intuitive strategies and building progressively on them. This meant beginning from repeated subtraction to increasing the number and size of chunks and flexible use of multiplication facts. The study concluded that it is meaningless for students to reproduce the taught methods mechanically while being unaware of the links between the procedure and the meaning of the division operation. The approaches of the two countries roughly correspond to the ways in which the division algorithm is dealt in the old and the new NCERT textbooks in India. I will take a closer look at these textbooks in the next section.

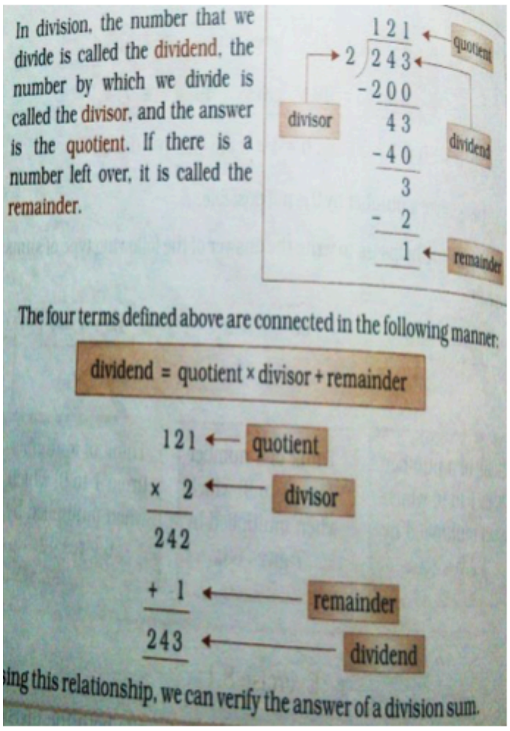
In a study with Grade 6 Government school students of rural Madhya Pradesh in India, Khemani and Subramanian (2012) reported a lack of understanding of the process of division. In their teaching experiment, the students were introduced to division as equal distribution or sharing. Students were taught to represent the process of equal distribution in a way that was visually similar to the division algorithm. The teaching trajectory for division included the physical act of distribution, using partial quotients to represent the stages in the process of distribution, and then movement to the long division algorithm. The principle of choosing an interpretation that is intuitive for students makes this approach similar to the Dutch approach.

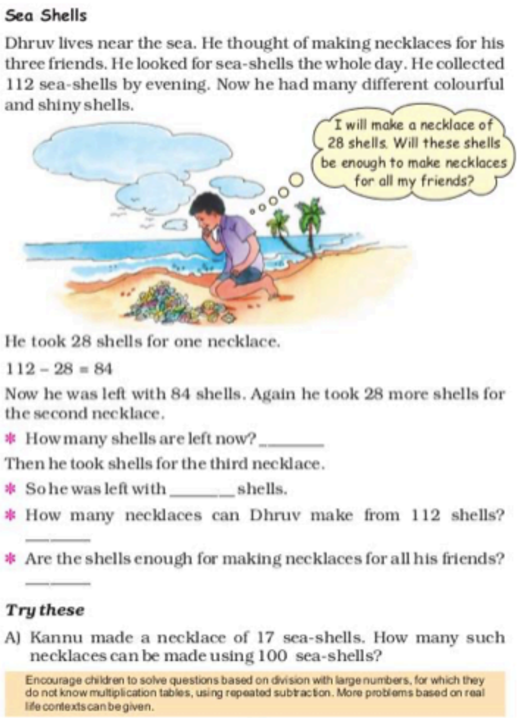
Informal strategies used by students in equal sharing or division contexts invite multiplicative thinking. Such contexts frequently call for chunking objects into equal sized groups and keeping track of the number of groups as well as the number of items accumulated, which involves multiplicative reasoning. Thus, as Lampert (1992) argues, division can be used as an opportunity for “cognitive reorientation” from additive structures to multiplicative structures and proportional reasoning. Development of multiplicative thinking is cognitively demanding but a valuable goal of learning mathematics (Subramaniam, 2003).

In summary, the literature on teaching and learning of the long division algorithm raises two important issues: formulation of a teaching approach for long division that focuses both on conceptual and procedural understanding of the algorithm, and the importance of using the context of learning the division algorithm as an opportunity to develop multiplicative thinking in students. In Pallavi’s case study, I will discuss the challenges faced by an experienced mathematics teacher while trying to unpack the structure of the division algorithm by relating it with multiplicative thinking involved in using the ‘chunking method’ of solving division problems.

7.3.2 Topic of division in the textbooks

This section presents an analysis of the way division has been dealt with in the old and new national level textbooks of Grade 4. These textbooks are designed by the National Council of Educational Research and Training (NCERT), an apex body which holds the responsibility of designing national level school textbooks to be followed by all central government run and affiliated schools. Discussion of the division trajectory in the two textbooks is necessary to understand the perspective of the teacher, whose case study is being discussed in this paper. The analysis indicates the differential nature of knowledge demands placed on the teachers when using textbooks written with different perspectives.




Sea Shells

Dhruv lives near the sea. He thought of making necklaces for his three friends. He looked for sea-shells the whole day. He collected 112 sea-shells by evening. Now he had many different colourful and shiny shells.

He took 28 shells for one necklace.

112 – 28 = 84

Now he was left with 84 shells. Again he took 28 more shells for the second necklace.

* How many shells are left now? ______

Then he took shells for the third necklace.

* So he was left with ______ shells.
* How many necklaces can Dhruv make from 112 shells? ______
* Are the shells enough for making necklaces for all his friends? ______

Try these

A) Kannu made a necklace of 17 sea-shells. How many such necklaces can be made using 100 sea-shells?

Encourage children to solve questions based on division with large numbers, for which they do not know multiplication tables, using repeated subtraction. More problems based on real life contexts can be given.

Figure 7.1 (Left): Division Algorithm (NCERT, 2003, p.30)
Figure 7.2 (Right): Repeated Subtraction (NCERT, 2007, p.125)

The earlier Grade 4 NCERT (2003) mathematics textbook, introduced division using multiplication facts, which involved division of a single digit number by a single digit number. The text gave a few examples of these facts and then introduced the algorithm for long division. As shown in Figure 7.1, the long division algorithm was introduced using the terms associated with it and the procedure to verify the answer (quotient and remainder) using multiplication. The description of the procedure, was followed by an exercise, where students were asked to solve the numerical problems (called "sums") using the algorithm. The algorithm was extended to the division of two, three and four–digit numbers by a single digit number. The successive exercises included the use of algorithm for division by 10, 100, 20, and other multiples of 10. Then, students were taught the algorithm for division by a two–digit number. The old textbook provided several numerical problems for students to practice the long division algorithm. Except the long division algorithm, no other method or ways of solving were suggested or exemplified in the text. Further, there were no word (or contextual) problems included in the chapter on division.

In the Grade 4 NCERT (2007) textbook, which is currently in use, the chapter on division begins with making a rectangular array arrangement for 18 plants. Students are expected to identify different ways in which 18 plants can be arranged. This is followed by an exercise on creating multiplication tables using the distributive property. Students are shown how to use the table of 2 and 5 to create a table of 7. The reason for why these two tables combine to give a table of 7 is not discussed. The contexts used in the text suggest the methods of repeated addition, repeated subtraction, making groups, and sharing to solve division problems. Each of the methods suggested by the textbook is appended with a note to the teacher (refer Figure 7.2). The note for the teacher, at the bottom of the page in Figure 7.2, suggests the use of large numbers to make the shift from using multiplication facts to repeated subtraction. The note mentions the ideas to be emphasised, suggests further exercises that teachers can design, and sometimes provides the justification for the activity or method discussed by the textbook writers. Similarly, other methods are introduced using a real–life context and problems are given to practice the method.

The textbook expects the teacher to know different methods and help students use these methods as well as the algorithm, which is given at the end of this chapter. However, teachers struggle to understand the significance of teaching different methods and handling students' responses navigating between these methods while the goal remains the teaching of the long division algorithm. The knowledge of why the division algorithm works, connecting different strategies of solving a division problem, and identifying links between the algorithm and these strategies, constitute an important part of teacher knowledge required for teaching the long division algorithm. These are also the areas where teachers might need support and they have been addressed in the study reported in this chapter.

7.4 Teaching Division of Whole Numbers: Pallavi's Teaching

In this section, I discuss the episodes from Pallavi's classroom teaching of the division algorithm and interactions related to the topic in the two years of the study. Pallavi's initial resistance as well as the process of change in her teaching through constant dialogue about the issues of practice is noted. The reasons for change in Pallavi's teaching through this process are analysed.

7.4.1 Teaching division in the two years

Year 1: "Different methods confuse, students should be 'taught' the division algorithm"

The new textbook expects a teacher to consider different strategies like repeated addition, repeated subtraction, use of multiplication facts, and partial quotients for solving division problems with sharing (partitive) and grouping (quotitive) interpretation. For instance, consider the problem of Gangu's sweets shown in Figure 7.3.

In the problem context, the grouping meaning is indicated by the image of 80 sweets in a box, and small boxes with 4 sweets each. The question posed is whether 23 boxes are sufficient to pack all the sweets. The problem can be solved using multiplication facts (taking products with convenient numbers 10, 5, 20), repeated addition or subtraction. The note to the teacher suggests encouraging students to use their own methods – making groups in the tray, using multiplication, or repeated subtraction, etc. The selection of a strategy by the student can indicate his or her understanding and use of additive or multiplicative thinking.

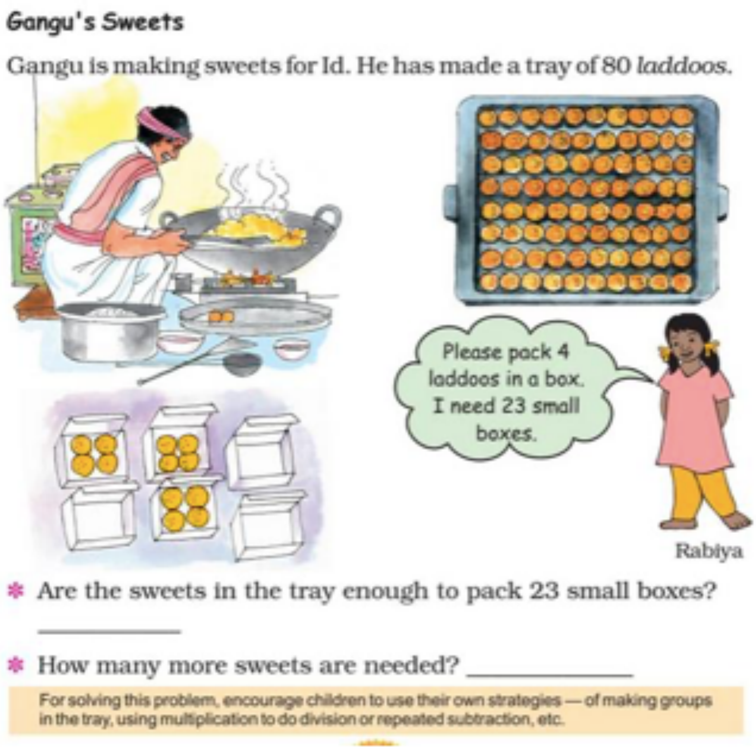


Figure 7.3: Grouping of Gangu's Sweets (NCERT, 2007, p.126)

Pallavi's interpretation of dealing with different strategies as proposed in the new textbook was to 'teach all the methods' to students. Pallavi indicated that the burden of teaching all these methods was on the teacher and consequently her concerns were guided by the difficulty of teaching them to students (refer Excerpt 7.1).

Except 7.1: Teaching multiple methods (Y1DV3)

Researcher notes	I was observing Pallavi's lesson in Grade 4, where she was teaching the division algorithm. The lesson was about to end. She came to me with the textbook and started talking about it. I think what she said is linked to the question I asked her yesterday about the difference between the old and new math textbook.
Pallavi	You can't expect them (students) to learn so many methods like the new textbook gives. It says you teach this method also, that method also. It is very confusing for students and then (when) you ask a question, which method do you want them (students) to use? They should use the long division (algorithm). It is what we have been doing for ages. And it is the systematic way.

Pallavi did not seem to associate the choice of 'method' with the problem context. Her emphasis on teaching all the methods overrides the discussion on the choice of method. Observations over several lessons show that she explicitly taught students each of the methods and then gave practice problems to use the same method repeatedly. She did not allow for students to use their own strategies or discuss why some strategies are more efficient than the others. While teaching one of the methods or strategies, Pallavi insisted that students use the same method to solve the problem and avoid thinking about any other ways of solving the problem (refer Excerpt 7.2).

Excerpt 7.2: Division as repeated subtraction (Y1DV10)

Speaker	Utterance
Researcher (R) notes	Pallavi writes the question on the black board and students copy it in their notebooks. Board Work: *Dhruv lives near the sea. He thought of making the sea shells. He took 28 sea shells for one necklace. How many necklaces can he make using 112 sea shells?*
Pallavi	(to whole class) Read the problem.
R Notes	Students read aloud the problem.
Pallavi	Total?
G St1	112 shells.
Pallavi	Method?
G St1	Division.
Pallavi	One necklace is equal to?
G St2	28 shells.
G St4	Number of necklaces is 112 ÷ 28.
Pallavi	Here comes the problem, how will you divide? Okay, you know how to divide. Tell.
R notes	Pallavi points to a girl student to come to the board.
GSt3	28)112 ((G St pauses after writing this on the board.)
Pallavi	For this type of division, I already told you the method.
Some Sts.	Minus.
Pallavi	What is it called?
Some Sts.	Subtraction
Pallavi	We have to do minus minus minus.
G St6	Repeated subtraction.
Pallavi	Okay so you do. All of you do it by repeated subtraction. Don't do long division. Do repeated subtraction. Don't think anything else. Just do repeated subtraction.

Pallavi's decision to break down the problem context into procedural steps (classifying the given information, stating the operation and method, using the method to find the unknown), and emphasising the use of one method at a time was consistent across problems and lessons. Pallavi's concern (Excerpt 7.1) that the teaching of several methods leads to confusion among students is noteworthy. Pallavi explicitly discouraged students in relating this method to the other methods. Her belief that students should not experience confusion is a strong one, also evidenced in Excerpt 7.2, where she says, "Don't think anything else. Just do repeated subtraction". A similar concern has been expressed in Excerpts 7.3 and 7.4 below. Moreover, the cause of confusion is seen to lie in the varied and multiple responses from students. Pallavi prefers students to be clear about which method to adopt when faced with a problem, which essentially forecloses any variation in student responses. If students are allowed freedom to think about a problem, then it is inevitable that multiple approaches will arise. It is not clear at this point whether Pallavi is against allowing variability in the students' response per se, or whether she feels ill confident about dealing with such variability.

Further, although problems were solved using each of the methods – repeated subtraction, grouping, and multiplication with convenient numbers, these methods were not connected with each other or the algorithm. The teaching of the long division algorithm, at the end, was given more attention and practice. Pallavi taught different methods following the textbook but held a strong belief that students must know the algorithm. The teacher's emphasis on the learning of the algorithm is a reality of Indian classrooms, as it is considered to be an important goal of 'school' learnt mathematics and is used as a differentiator from the 'out of school' mathematical knowledge. The legitimacy of the algorithm comes from the authority of the content in the school textbook and the experience of learning and teaching the same method for several decades. When Pallavi was probed about the teaching of justification of an algorithm in class, she expressed that students were not developmentally capable of understanding the reason for why a method works and therefore her decision to avoid teaching it in class (refer Excerpt 7.3).

Excerpt 7.3: Why an algorithm works? (Y1DV12)

Speaker	Utterance
R notes	I had one of my regular conversations with Pallavi. I wanted to know the reason for her emphasis on teaching the algorithm and her views on why the algorithm works. I also intended to know about her thoughts on using different methods.
Researcher	There must be a reason for why an algorithm works. Don't you think it is important for students to know why this method works?
Pallavi	They (students) are very young. Telling them what lies behind this concept or you had done that, remember? We (teachers) can't do that. Their (students') brains are not that developed. When they grow up, go to class 7 or 8, you can tell them, see this is why we did that, but not now. They are too young. They will get more and more confused.

Pallavi attributed the decision of not teaching the justification of the method when discussing the algorithm to the developmental incapacity of students. She consistently maintained that young students are incapable of handling multiple methods and representations, independent problem solving, and reasoning about why something works. Like other participating teachers, she believed that students face difficulty in understanding the justification of why an algorithm works. This led to lowering the cognitive demand of the task by demonstrating the procedure (also noted by Jackson, Gibbons & Dunlap, 2014) and asking students to follow the procedure to solve problems.

It is noted that although Pallavi believes that all methods proposed by the textbook need to be taught, she does not pay attention to the connections between these methods and their relation to the problem situation. Pallavi could not anticipate the possibility that students might use these strategies or methods when given an opportunity to solve problems by themselves. She seemed to be underestimating student capabilities by thinking that they cannot deal with different methods. It was also found that placing a low cognitive demand in problems and methods is done to avoid confusion in students, which in turn is not considered as contributing to their learning.

Year 2: "I don't understand how this method works, why don't you teach?"

In the second year, after teaching and providing practice on solving division problems using repeated addition, repeated subtraction, and use of multiplicative facts, Pallavi intended to teach the chunking method, identified in literature as working with 'partial quotients'. In this method, convenient multipliers are chosen and the multiple is subtracted from the dividend. In other words, in a quotitive interpretation where the divisor is interpreted as the fixed size of a group or share, one has to reach the maximum number of groups/shares of divisor that can be taken away from the dividend. (Alternatively, in a partitive interpretation where the divisor indicates the fixed number of equal groups, one needs to arrive at the maximal size of a group.) The number of groups may be decided by the ease of arriving at multiples using doubling, multiplication with ten and its multiples, etc. For example, Figure 7.5 shows how the chunking method is used to solve 585÷16. Literature (Anghileri, Beishuizen, & van Putten 2002; Khemani & Subramanian, 2012) suggests that partial quotients builds on students' intuitive strategies and allows for greater flexibility in the choice of chunks unlike the standard division algorithm. Although the partial quotients method is described in the textbook, and Pallavi was following the textbook closely, she had avoided introducing this method in the previous years. In Year 2, Pallavi worked with the researcher to understand the partial quotients method before teaching it in the classroom. She struggled to use the method with different numbers and while trying she remarked that the method is confusing. In the excerpts below, the process of Pallavi's gradual negotiation with the method and it's teaching can be noted.

Pallavi was struggling to use the partial quotients method to solve division problems (refer Excerpt 7.4). Her difficulty seemed to stem from the fact that the partial quotients method lacks the procedural clarity that is found in the long division algorithm. The standard algorithm works implicitly with place value, dividing one digit at a time. Each step of the algorithm repeats the same logic consistently. Pallavi's comfort with the long division algorithm came from her confidence in using the method for a long period of time, following the steps sequentially, and its efficiency.

Excerpt 7.4: Efficiency of long division algorithm (Y2DV9)

Speaker	Utterance
R notes	This is one of Pallavi's Grade 4 classes where she teaches regularly. When I asked her about her plan for the lesson, she showed me the textbook and started talking about the partial quotients method.
Pallavi	Now I have tried this method given in the book but see it is confusing… (I) have always done long division only with children. So I am not sure how to introduce it, how to actually do it in class. I am comfortable in long division and it is shorter you know. It is a step–by–step process, taking one digit at a time so they (students) can easily divide.

The division algorithm has an underlying structure. It looks at the place value of the digits in the number to be divided. The dividend is not operated as a whole but by breaking it into parts according to place value units and the left overs are transformed into the next unit (Lampert, 1992). To keep track of the place value of digits in the quotient, students are often given a clue, that is, to write the digit of the quotient just above the dividend over the same place value. Although the visual clue helps in identifying the quotient correctly, it does not explain why such an orientation must be maintained. Deconstructing the division algorithm would mean understanding the implicit place values in the number to be divided, finding the chunks of the divisor that are closer to the dividend, and distributively dividing the dividend.

In contrast, in the partial quotients method the number as a whole is taken and chunks are identified that can be safely taken away from the whole number, recording the number of chunks taken each time (called partial quotients), and finally adding the number of chunks to obtain a quotient. Structurally, partial quotients can be seen as intermediary between students' intuitive strategies and the division algorithm (van Putten, Brom–Snijders & Beishuizen, 2005; Khemani & Subramanian, 2012).

Pallavi's motivation to explicate the difficulty in using partial quotients and in seeking support from the researcher probably arises from the pressure of teaching the method, being a part of the textbook. She approached the researcher to seek support in teaching of the method to the students (refer Excerpt 7.5).

Excerpt 7.5: Researcher as teacher (Y2DV10)

Speaker	Utterance
Pallavi	Why don't you (researcher) take this (division by chunking) in my class? Tell them what this method is. (After a pause) Yes we can see how they (students) pick it and decide then only which method. I don't know if they will understand. I tried around 8 to 10 numbers, dividing them using that method. The bigger the number, the more confusing it was. I think it can confuse. But you try and let me see how they try to do it.
R notes	Pallavi asked me to teach in her class today. I am thinking several things – whether I should teach because my role is to do classroom observations, what will I teach which will encourage students to think about chunks, how will the change in the teacher affect students' response, how will Pallavi observe and interpret the classroom interaction.

Pallavi's suggestion of switching the role of the teacher and researcher marks an important event in the research. She suggested that the researcher take a more 'active' role in teaching a difficult topic. The goal of the researcher (who became the teacher) changed to thinking about a problem context that would elicit the meaning of division and will provide students with an opportunity to build on their own strategies. Along with the identification of problem context and learning goal for students, Pallavi's understanding of the method also needed scaffolding.

Year 2: "I understand why the algorithm works!"

In the second year, Pallavi introduced the researcher as a teacher in one of the division lessons. The researcher posed the following problem to the students in the class.

Grandpa wants to distribute Rupees 75 among three of his grand children equally. Can you help him in doing this? Explain your reasoning.

The rationale for beginning with a sharing context was that students might relate to this meaning of division intuitively. Also, the money context offers a potential to see the place value structure in the denominations of powers of ten. As soon as the problem was posed, students began to propose how to distribute the money to arrive at the share of each grandchild. With some guidance from the researcher on how to record the amount to be distributed to each grand child at every step, students were encouraged to come up with different ways in which the money could be distributed. They began with distributing "10 to each grand child", to which another student suggested "20" and a third student "25" or, the student said, "10, 10, and 5". When all students solved the problem, the next problem posed was, "what if there were 5 grandchildren?". Before the whole problem was restated, several students responded that the share of money would reduce. When asked why, students responded by saying that the money was the same but the number of grand children had increased, so each of them would get less money when compared with the previous distribution. Noticing the relation without solving the problem or finding the quotient for $\frac{x}{a}$ and $\frac{x}{b}$, and comparing marked an important step towards thinking proportionally (Lampert, 1992). To justify their responses, students used the sharing interpretation to find the exact share of each grand child, for the second case. In this situation, students were able to see that $\frac{x}{a} > \frac{x}{b}$, when $b > a$. As the lesson progressed, Pallavi took over the teaching and gave students the problem of distributing Rupees 127 among 5 friends equally. The choice of these numbers by Pallavi is interesting because 127 is not evenly divided by 5. I also note that Pallavi preferred to retain the number 5 as the divisor. As students proposed chunks of 10, 10 and 5; she recorded these on the blackboard labelling the number of friends as the divisor, the total amount as the dividend, and pointing to the partial quotients as the share of each friend. After the money context, students were asked to divide 89 by 4.

Pallavi's decision to switch the roles while the lesson was in progress was an in–the–moment decision. Her choice of numbers 127 and 5 seemed deliberate as she intended that students focus on the act of distribution and discuss convenient combinations. The decision to shift from a contextual problem to a bare number problem (divide 89 by 4) indicates the shift from dependence of students' reasoning on the context of sharing, while it still acted as a reference or an anchor.

As students were engaged in the problem context of distributing money, Pallavi came up to the researcher and made two observations about the partial quotients method (refer Excerpt 7.6).

Excerpt 7.6: Thinking about partial quotients (Y2DV10)

Speaker	Utterance
R notes	Pallavi gave students the bare number problem 89 divided by 4. She gave students time to think and solve the problem. And during this time she came to me and started talking about the way of recording partial quotients.
Pallavi	This way of grouping works, as it tells you each time what you are distributing In (old) textbook all of this was at the top. In fact this (horizontal) way of writing is better than this (writing above) because they can not keep track and the place value is there.

First, she noticed that the horizontal recording of the partial quotients is important to keep track of the number of chunks that have been taken away from the whole and the changing whole ("what you are distributing"). And second, she observed how the place value of each digit plays a role in the division algorithm. When Pallavi remarks that the horizontal way of writing is better, she may have been referring to the practice of writing the quotient digits to the right of the dividend rather than above the dividend.

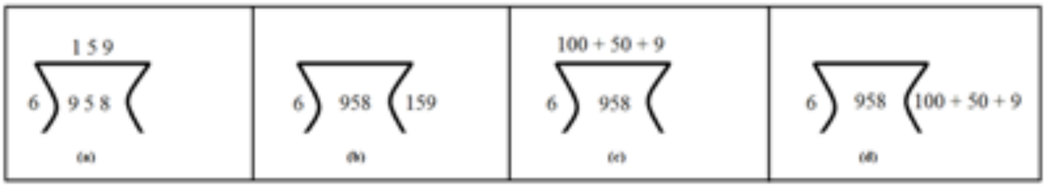


Figure 7.4: Ways of Recording the Divisor

The textbook uses both ways (shown in Figure 7.4, c and d) of recording partial quotients, and Pallavi may have been concerned about this inconsistency. After working individually on the problem, students suggested different combinations for dividing 89 with 4. Pallavi listened to these variations, each of which allowed students to arrive at the correct answer, and then closed the day's lesson. Pallavi and the researcher continued the discussion about the partial quotients after the lesson.

Excerpt 7.7: Reflection on using partial quotients (Y2DV10)

Speaker	Utterance
R notes	Today I did not have to ask Pallavi about the lesson. She was excited to talk about it with me. So as soon as she finished teaching, she started talking to me about the method.
Pallavi	I think the method is good. They (students) can use different ways to get it (answer). Also it is very clear, this vertical arrangement of numbers. And grouping by tens, they are aware also. Then slowly they can move to choosing bigger numbers. Actually you know the number of steps increases if you take small numbers (multiples). But it doesn't matter because they anyway get it. They can use 8 directly or if not, 4 and 4, or 5 and 3, it doesn't matter. This method is better and they picked it up faster also. As a teacher, I can see how they are liking it. Taking it as a full number (number as a whole) is clear to them. They find it more easy. Easy only, no? They can make as many groups and how much they want. This also tells us about the multiplication knowledge. But you know one more difference is there. In long division, I have to teach them for each increasing digit like dividing by one digit, then two [digit number] and three, all are different. But in this they have to use the same method for big numbers, by themselves and they can do also.

While reflecting on use of partial quotients, Pallavi seemed to be unpacking the structure underlying the division algorithm and related student capabilities (refer Excerpt 7.7). She noticed that the method revealed students' multiplication knowledge expressed through their choice of convenient numbers for chunking. Different students used different sequences of partial quotients, while arriving at the correct answer. As indicated in Excerpt 7.7, she noted the flexibility in the choice of the size of chunks as well as the relation that smaller chunks lead to a larger number of partial quotients. She made an interesting distinction between the way she taught the long division algorithm and partial quotients. It was the difference between a digits–based approach versus treating numbers as a whole. The reliance on the face value of the digits of a number takes away the attention from the place value. Pallavi also remarked that she does not need to teach the partial quotients method separately for one–digit, two–digit or three–digit divisors. In contrast, she mentioned that earlier she needed to teach the standard algorithm differently for divisors of different digit lengths, a view that suggested again the highly prescriptive, step–wise approach to teaching a procedure.

The data is not sufficient to conclude that Pallavi's belief about the lack of students' ability to discover methods by themselves has been challenged. But it was evident that she had begun thinking about building on students' prior knowledge. In this case, she considered that students used their knowledge of multiplication with convenient numbers to solve a division problem using partial quotients. She was engaging with the aspects of multiplicative thinking involved in the process of chunking.

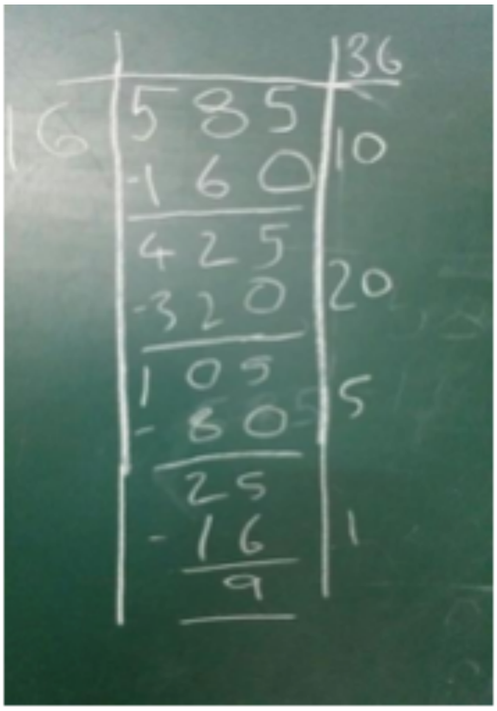


Figure 7.5: Partial Quotients

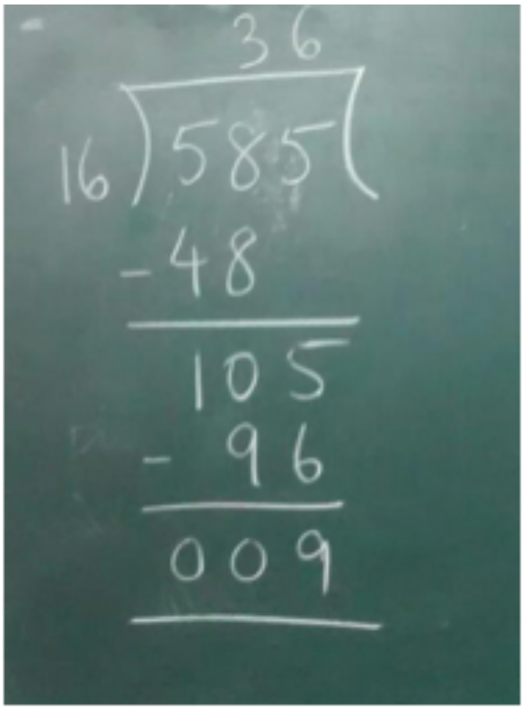


Figure 7.6: Long Division Algorithm

In the lessons that followed, Pallavi explicitly dealt with the relation between using partial quotients and the long division algorithm. She gave students the following division problems to solve: 115÷3, 236÷11, 427÷13 and 585÷16. She noticed that a majority of students used chunking to solve these problems by themselves. She found that students were extending the chunking to numbers for which they had not memorised the tables (for instance, division by 13 and 16). She was excited to notice this and shared the observation with the researcher. Later in the lesson, she brought students' attention to the relation between chunking and the long division algorithm. While teaching in class, she gave a division problem and asked students to solve it using both methods: partial quotients and long division algorithm (refer Figure 7.5 and 7.6).

Through the presentation of both the methods, Pallavi tried to engage students with the links between finding partial quotients and the long division algorithm (refer Excerpt 7.8). While teaching in the class, she figured that the place value structure is implicit in the division algorithm. The contrast between taking a digits–based approach and the number as a whole was triggered by a student's explanation. It was during teaching that Pallavi noticed and explicated that the underlying structure of the division algorithm is in finding the greatest partial quotient or with the highest place value. Although not all students could explicate the relation between the two methods sufficiently well, Pallavi reported in the post–lesson interview that the conceptual knowledge of 'why division algorithm works' must be included as an important part of the teaching of division and she would like to henceforth discuss the link between the two methods when teaching division.

Excerpt 7.8: Connecting chunking and long division (Y2DV11)

Speaker	Utterance
R notes	Teacher asked the students to solve 585 ÷ 16. After giving students some time to solve this problem, she starts talking. She asks students how they have solved the problem and records it on board (refer Figure 7.5).
Pallavi	Now, same thing, let us try to do using long division. You have to tell me what's happening?
Board	Refer Figure 7.6.
Pallavi	So what do you see? What is the difference?

Speaker	Utterance
G St	In long division, we are multiplying the number.
Pallavi	Here (pointing to chunking) also we do.
B St	In long division, we don't have to plus (add) the tens.
G St	Teacher we are not taking the full number for division.
Pallavi	Good. In long division, we are not taking the number as a whole but the digits. In grouping method, we take the whole number together. Since in long division we take one digit at a time, the number of steps is less as we look for the biggest multiple.
G St	We take 10, 20, 20 in (long) division also.
Pallavi	Yes you can reduce the number of steps in grouping also. If you are thorough with your multiplication you can take bigger multiples.

7.4.2 Reflection on teaching division

Pallavi was a confident and an articulate mathematics teacher. Her classroom observations and interactions from the first year revealed that she valued "clear" procedural answers from students. She did not appreciate multiple methods as she believed that they confuse students. Further, according to her, the goal of teaching mathematics was to "teach the algorithms" so all her teaching was aligned to this goal. From the first year analysis, it was found that, Pallavi used several visual clues to help students remember the rules (refer Excerpt 6.7 from Chapter 6). For instance, when teaching fraction to decimal conversion, she taught students that the denominator determines the number of places and the numerator fills up those spaces. So, a hundred as a denominator means that the decimal equivalent will have two places after the decimal point and the numerator will be placed such that two of the places are after the decimal point. If there are not enough digits (as in the case of 4 as a numerator), then the missing place can be filled with zero and if there are more number of digits (as in the case of 436) then the extra digits (4 here) will have a position before the decimal point. She called this a (visual) pattern and extended it for non–zero single digit numbers with ten as the denominator, and two digit numbers with hundred as denominator.

As stated earlier (refer Excerpt 7.1), Pallavi believed that multiple methods confuse students and must be avoided in class. A contingent moment with Pallavi arose when, in the second year of the study, she stated her discomfort in comprehending the partial quotients method for division, given in the textbook. In order to deal with Pallavi's difficulty, the researcher discussed the rationale for the method and solved a few problems with Pallavi using the method. While Pallavi managed to solve a few division problems with the researcher using the method, she requested the researcher to teach this lesson. She expressed being unsure about how students would respond to this method. Partly, the discomfort seemed to be emanating from her lack of confidence in dealing with the method that she had just learnt. After planning the lesson, the researcher taught this lesson in Pallavi's class. Interestingly, while the researcher was teaching, Pallavi decided to co–teach the lesson and gradually took over. She noticed the varied responses from students when partial quotients were introduced. Pallavi's decision to take over the teaching showed her interest in working with the method with the students and probably added to her conviction that students could make sense of the method and use it.

While working with and reflecting on the students' use of partial quotients, Pallavi engaged with the conceptual structure of the division algorithm. The students' responses led Pallavi to see the possibilities inherent in using the new method. An important aspect of the knowledge–in–play was the variations in students' responses to the problem posed. As evident from the classroom excerpts, this variation helped Pallavi in noticing different "correct" responses emerging from the students. The variations in the choice of chunks seemed to provide a direction to the complexity, which was difficult for her to anticipate in isolation from the classroom. The variations in examples and choice of chunks observed by Pallavi supported the insight that the partial quotients approach allows for such variations and gives an insight into the structure of the algorithm. This may have led Pallavi to take over the teaching and to introduce her own examples by way of variation. The sequencing of examples provided the scope for students to utilise their multiplicative knowledge and make connections between different ways of solving the division problem. Students' responses to the variety of examples which go beyond the knowledge "taught" to them may have led to Pallavi designing more challenging tasks for them.

As it became a part of Pallavi's explicit knowing, she decided to include a discussion of why the division algorithm works in her teaching and make the structure of the long division algorithm transparent for the students. In the next lesson, Pallavi engaged students in comparing the chunking method with the algorithm to identify the differences and similarities in them. The design and conduct of this mathematical task contrasts with her belief that the discussion of more than one method creates "confusion" among students and is beyond their cognitive ability. The links between teacher's actions, students' engagement at different levels, and teacher's responses to students are contingent to the classroom and are specific to the situated experience of learning from teaching. It was the situated nature of this experience that led to the beginnings of a deeper understanding of the mathematical structure underlying the long division algorithm. The attempts made by Pallavi in linking the partial quotients and the division algorithm was a change triggered partially by discussions with the researcher about the mathematics underlying different methods of teaching division and with the students in the classroom while solving problems using the partial quotients. Additionally, the variation in student responses triggered Pallavi's imagination of a pedagogy where the straight–jacketed approach to teaching and reproducing the algorithms was challenged. Earlier, Pallavi tended to see variation as a source of confusion among students and as impeding their learning. After a deeper engagement with the mathematical structure of the algorithm in the classroom context, she remarked on the variations afforded by the partial quotients approach. Engaging with the mathematics of the algorithm and how it played out in the classroom addressed both Pallavi's knowledge and belief; knowledge about how and why the partial quotients method works and belief about the desirability of allowing variations in student responses.

It is claimed that without the situated nature of this experience, this simultaneous addressing of knowledge and belief would have been difficult to achieve. This may explain why Pallavi resisted including the teaching of the method for several years. Also, it needs to be acknowledged here that the intervention in the form of teacher–researcher meetings focused on the topic of decimal numbers, played a significant role in orienting Pallavi to be more sensitive to student responses and in priming this change. Lastly, it was only *in–situ*, through the act of teaching and responding to the students that the possibilities inherent in the partial quotients approach opened up. It is a powerful corroboration of the situated and dynamic nature of teacher knowledge.

7.4.3 Analytical construct of contingent moments

A careful analysis of Pallavi's teaching of specific topics over two academic years indicated the ways in which knowledge and beliefs interplay when a teacher makes decisions in the classroom. A focused engagement with the topic of division helped in analysing the complex character of the teacher's work. Note that Pallavi was teaching the new textbook for several years before this research study was conducted. She used the "new" methods of division, described in the textbook, in her teaching. In the first year, she explicitly taught each of these methods while being worried about the possible confusions arising from the use of multiple methods in students' minds. However, she had omitted the partial quotients method because, as she admitted, it was confusing to her. She needed topic specific support to engage with the trajectory suggested by the textbook. In particular, she needed to understand the mathematical significance of different methods and connections between them.

It is important to note that working with a few examples using the partial quotients method along with the researcher, while planning the lesson, was not sufficient for Pallavi to develop an understanding of the method or to convince her to teach it to her class. So, the question is what kind of an engagement with the teachers creates possibilities of changed practice. It was found that Pallavi's initiative of articulating her struggles with the partial quotients method and seeking support from the researcher while teaching it in the classroom, marked an important shift allowing for a re–examination of existing beliefs and practices. Pallavi's discussions with the researcher changed considerably, after this particular interaction. In this interaction, Pallavi requested for a specific kind of support, which was unanticipated by the researcher. Such an interaction is characterised as a contingent situation. Rowland, Huckstep and Thwaites (2003) use contingency to refer to the classroom events which are difficult for the teacher to anticipate or plan. I extend the notion of contingency to refer to those situations in the teacher–researcher collaboration, which were unanticipated by the researcher but demanded an actionable response in the moment.

The analytical construct of contingent situations, theorised from Pallavi's case study was used as a lens to identify if similar instances can be found with other teachers (participants). A look back at the data revealed that there were specific instances of support demanded by individual teachers, which witnessed an actionable response by the researcher, and that they were followed by changed classroom practice. Responding to such moments helped in challenging teachers' knowledge and beliefs in the context of their practice, and strengthened the relationship between the researcher and the concerned teacher. Although, it seems that contingent situations are specific instances, I conjecture that there is a process that characterises the learning from such moments. In the following sections, in addition to analysing the contingent situations in the practice of all the participating teachers – Pallavi, Reema, Nandini, and Vindhya; I will attempt to unpack the process of converting these contingent situations, arising *in–situ,* into learning opportunities for teachers. It is important to mention here that such contingent situations were a learning experience also for the researcher, for instance, in identifying the nature of support that teachers need and about the change in practice. However, in order to maintain the focus of this chapter on teacher learning, the researcher's insights will be peripheral to this discussion. In the following sections, I will try to unpack the process involved in the transformation of contingencies into learning opportunities by focusing on the following questions.

(a) What was the nature of the contingencies that arose and what changes ensued in the interactions between the teacher and the researcher, and

(b) In what ways does responding to such contingencies enhance teacher learning?

7.5 Contingent Moments and Teacher Learning

This section discusses the contingencies arising in the teacher–researcher collaboration, while working with the participating teachers. For each case, I briefly recall the the teacher's knowledge and responses to students, particularly those, which were re–examined by the teacher later in the study. This is followed by a description of the contingent situations where the teacher sought support from the researcher. The support offered by the researcher in the moment is discussed. At the end, I attempt to extract the critical features of the support in order to gain insight into the process of identifying and responding to such situations in the context of practice.

7.5.1 Case of Reema

Reema was less confident about her teaching and opinions when compared with the other participating teachers. In her classrooms, she would get worried if a student did not understand what was being taught. She would try different ways – individual attention, seeking help from the peer, repeating an explanation, etc., to help a struggling student. However, all her efforts involved repeating a procedure. Like the other teachers, she expected the students to copy the correct solutions from the board, where she would write. Reema appreciated the real–life contexts used in the new textbooks. She used the suggested contexts and activities in her lessons, although she would teach them mechanically or procedurally. In the second year of the study, Reema was found to be studying the textbook more carefully.

On a particular occasion before teaching a lesson on decimals, Reema insisted on having a discussion with the researcher. She referred to the currency context (refer Figure 7.7) given in the textbook and expressed her discomfort in discussing one of the questions in class despite its relevance. Her concern was that the second part of the question (part B in Figure 7.7) expected the students to multiply a whole number and a decimal number, which was not discussed, anywhere in this chapter on decimals. She noticed that the textbook did not deal with the multiplication of a decimal number with a whole number before this context and here it was expected as prior knowledge to solve this problem. Therefore, in previous years she had decided to omit this question from discussion in class. Her conflict between recognising that the context was important but not knowing how to deal with it in the classroom, initiated this conversation (refer Excerpt 7.9).

Practice time

1) Money from different countries

Have you seen any notes or coins used in any other country?

Shivam Bank has a chart to show us how many Indian rupees we can get when we change the money of different countries.

Country	*Money*	*Changed into Indian Rupees*
Korea	Won	0.04
Sri Lanka	Rupee (SL)	0.37
Nepal	Rupee	0.63
Hong Kong	Dollar (HK)	5.10
South Africa	Rand	5.18
China	Yuan	5.50
U.A.E.	Dirham	10.80
U.S.A.	Dollar	39.70
Germany	Euro	58.30
England	Pound	77.76

(This is the rate on 15-2-2008)

A) The money of which country will cost the most in Indian Rupees?

B) Mithun's uncle in America had sent him 10 USA dollars as a gift. Mithun used 350 rupees for a school trip. How much money was left with him?

Children are not expected to carry out long multiplication involving decimals. Instead, encourage them to think in terms of currency. For example, 75 paise × 2 can be thought of as two 50 paisa coins and two 25 paisa coins.

Figure 7.7: Currency Context (NCERT, 2007, p.143)

Excerpt 7.9: Pre-lesson discussion: Reema (Y2DL9)

Speaker	Utterance
Reema	You see this currency problem (refers to the question given in the textbook). Every year I see it and then don't do it (in class). I mean how can I do it. Before this they are talking about fractions and decimals, place value and suddenly they (textbook or writers) expect multiplication of decimals. We have not done that, so how can students answer this question. And it is not connected. I mean multiplication of decimals comes in higher grades. Here we are just introducing decimals. Isn't it too much? (pause) I think this problem (context) is real application of decimals, so I thought I will talk to you. What do you think? Should I do it in class?
Researcher	Do you want to teach it in the class?
Reema	Haan (Yes) but they (students) don't know multiplication.
Researcher	So this requires multiplication of a decimal number with?
Reema	Of decimal numbers, but we don't do multiplication of decimals at this grade.
Researcher	For this, decimal number multiplication with whole numbers.
Reema	Haan haan whole numbers.
Researcher	Here it is basically multiplication with powers of 10. Isn't it?
Reema	Haan yes.
Researcher	You can consider connecting it with the place value explanation that you have given in class.
Reema	Place value of digits in a decimal number, it is about decimal number. This is operations with decimal numbers.
Researcher	You had discussed in class, how the place value of each consecutive digit is related by ten times.
Reema	Yes, yes.
Researcher	We can connect that with multiplying a decimal number by 10 or its powers. And then division would just be an inverse of it.
Reema	Wait, (takes a paper and pen, writes) 0.4×10, now here the point gets shifted to the right.
Researcher	Yes, consider it as 0.1×10. So basically what we are doing is multiplying the tenths by 10, which means making it ones.
Reema	So it actually means $\frac{4}{10} \times 10$, which is changing the place value of the number to be higher. And then division would mean going down a place value.
Researcher	By powers of 10.
Reema	Yes, so we can remember the place value relation. And then multiply by 100 means going up two places and division going down two places. Once I had seen this pattern, you know multiplying with 1, 10, 100. May be we can do that.
Researcher	You mean multiplying the same number by 1 then 10 and then other powers?

Speaker	Utterance
Reema	Haan (Yes) so at every step place value changes by one (level). Like see, 0.4×1 0.4×10 0.4×100 0.4×1000
Researcher	Good idea. And we can consider doing division alongside. $0.4 \times 1 = 0.4$ $\quad$ $0.4 \div 1 = 0.4$ $0.4 \times 10 = 4$ $\quad$ $0.4 \div 10 = 0.04$ and so on.
Reema	Haan, so we can begin the class like that, plan this. (After a pause, she writes) $39.70 = [3 \times 10 + 9 \times 1 + 7 \times \frac{1}{10} + 0 \times \frac{1}{100}] \times 10$ So 30 becomes 300, 9 becomes 90, 7 tenths becomes 7 and 0 hundredths becomes 0 tenths. Hmm, can try that. Do you think it might be difficult for them (students)?
Researcher	Would you like this as a worksheet, a small worksheet, at the beginning of the class for students to see the pattern and extend?
Reema	Yes, like that another worksheet can be given as homework.
Researcher	Okay, so let us discuss the questions for it.

While teaching this lesson, Reema began with a recall of the relation between place value of digits in a number. She reminded the students of the continuous relation of powers of ten as we move from left to right in the place value table. She asked students how to get tens using ones, to which the students responded ten times. She recorded "10 ones = 1 tens" on the board. Similarly, relation between other place values, such as those mentioned below, were discussed.

1 hundred = 10 tens = 100 ones

1 hundredths = $\frac{1}{10}$ tenths = $\frac{1}{100}$ ones

1 ones = 10 tenths = 100 hundredths

Then, Reema asked students what is ten times of zero point one, that is, "0.1×10 = ?" Some students referred to the relation recorded on the board to say that "tenths ten times is ones". Reema then changed the question to "0.2×10 = ?" followed by "0.5×10 = ?" and "1.1×10 = ?". Reema asked the students to complete the following worksheet (refer Figure 7.8), which was prepared along with the researcher. She asked the researcher to move around and help students individually or in groups to identify the relation using place values.

Question: Solve mentally.	
$0.1 \times 1 =$	$10 \times 100 =$
$0.1 \times 2 =$	$1 \times 100 =$
$0.1 \times 10 =$	$0.1 \times 100 =$
$0.2 \times 10 =$	$0.2 \times 100 =$
$0.5 \times 10 =$	$0.5 \times 100 =$
$1.1 \times 10 =$	$1.1 \times 100 =$
$1.2 \times 10 =$	$1.2 \times 100 =$
$2.5 \times 10 =$	$2.5 \times 100 =$
$0.01 \times 10 =$	$0.01 \times 100 =$

Figure 7.8: Worksheet on Multiplication of a Decimal Number With Powers of 10

After helping students identify the pattern in multiplication with powers of 10, the class generalised the rule as "the point shifts to the right when multiplying by 10, 100, etc.", which was further refined by a student who said, "it gets shifted by one digit for 10, two digits for 100, like that". Reema then proceeded to a discussion of the currency context along with the students for the remaining lesson. At the end of the lesson, Reema shared that teaching multiplication of a decimal number with powers of ten using the place value explanation is a key idea and will help her in teaching other topics, such as comparison of and operations with decimal numbers. She mentioned that the place value explanation allowed her to move away from the explanation of multiplication as repeated addition to grouping. In her later sessions, she extended the idea of multiplying with powers of ten using place values to expanding a decimal number and using distributive property (for example, 3.56×10) to help students understand multiplication. It is important to note that Reema's explanation for conversion from decimal to fractions by "shifting the point" also changed to identifying the fractional part of the number in the place value table.

$$3.56 \times 10 = 3 \times 10 + \frac{5}{10} \times 10 + \frac{6}{100} \times 10$$

In this case, Reema discussed the struggle of missing some content from the textbook which she believed was relevant to the teaching of decimal numbers at this grade level. While she had omitted this context in the previous years, the quest to pursue the possibility of including it could, in part, have been motivated by the need to follow the textbook content completely. She insisted on a discussion on this idea with the researcher and explained her tension with teaching it by identifying a gap (multiplication of decimal numbers) in the textbook content. She needed support in bridging this gap. Interestingly, together with the researcher she discovered the explanation of the rule of "shifting the point" when a decimal number is multiplied with powers of ten. She appreciated the explanation for this rule using place value as a key idea, tried it for a variety of numbers, then placed it in a general way using the place value chart, and planned her lesson around it. She tested the plan by teaching it in the class and made sure that the students understood it thoroughly. Her request to the researcher to move around and help students, if they are struggling to understand it, indicates her conviction in communicating this explanation to the students. The reflection at the end of the lesson indicated that Reema began seeing more possibilities of this explanation and connected it with other key ideas such as distributivity of multiplication over addition. Further, she revised her description of the concept underlying the currency conversion problem from "multiplication of decimal numbers" to teaching "multiplication of decimal numbers with powers of ten" at this grade level.

7.5.2 Case of Nandini

Nandini was a quiet person and mostly a listener in the teacher–researcher meetings. Some of her practices such as asking students to copy from the board, noting down the solutions neatly, answering a question when asked by a teacher were similar to the other teachers. However, she was sensitive to students' needs and worked towards spending extra (non–class) time with students who could not pass a test or exam. She also believed in helping students individually. She encouraged peer learning but often by asking students who score high marks to help the weaker students in order to pass the examination.

Nandini interacted with the researcher whenever her time permitted. She questioned several structural constraints such as completion of an ambitious curriculum in a given time frame, the nature and purpose of formative assessments, maintaining students' records, etc. She often conversed with the researcher about ideas such as connections between the linear and area representations, fractions and decimals, whole number thinking and decimals, etc.

Excerpt 7.10: Discussion on representations: Nandini (Y2DL9)-1

Speaker	Utterance
Nandini	You know I was showing different things (sub–topics) using the number line and grid both. It is given in the textbook also. You can do representation, comparison, addition and subtraction with both of them.
Researcher	You have been using both these representations.
Nandini	Yes, but not for all. I mean we use one for tenths and another for hundredths. But you had asked me long time ago, that in one class we are doing number line and another we are doing grid.
Researcher	In the students' worksheet also, we saw that students were using number line to show numbers such as 0.8, but showed 0.68 on a grid.
Nandini	Haan, but that is because we teach like that.
Researcher	Haan
Nandini	I was thinking, is one of them (representation) better than the other.
Researcher	In this (referring to the work she had done) you have seen that both these representations can be used to say, represent decimal numbers. Isn't it? Like you can use a number line to show decimal numbers with tenths, hundredths, thousandths and so on.
Nandini	Yes
Researcher	So one question is whether the same representation can be used consistently.
Nandini	Haan, I was thinking that we don't tell children why for tenths we use number line and for hundredths we use grid. I thought it is easier but I did not think about it. Both can be used to teach all the topics.
Researcher	You also have locating a decimal number between two whole numbers or between two higher place value numbers.
Nandini	Haan, that we do with number line only.

It was noted that all the participating teachers preferred the use of both linear and area representations for representing decimals, as suggested by the textbook. In one of the tasks during teacher–researcher meetings, teachers were asked to explore different sub–topics that can be addressed using any one kind of representation (either linear or area). A day after this meeting, Nandini discussed the task with the researcher. She had completed the task with both area and linear representations (refer Excerpt 7.10).

When she planned to teach decimals, she had a detailed discussion with the researcher on the use of the meter strip for several sub–topics within decimals.

Nandini pursued the task of using each representation for different sub–topics (refer Task 6 in Section 6.5.3 from Chapter 6). She discovered how each representation could be used consistently for different sub–topics. This discovery is important because in the first year of the study, like the other teachers, Nandini just followed the textbook and moved from one representation to another without connecting them explicitly with the students. For instance, she used a number line to introduce tenths, a grid for hundredths, then again number line for locating a decimal number, and a grid for addition of decimals. While attempting to use each representation consistently, she recalled an interaction with the researcher in the first year, where an inconsistent use of representation across different place values was brought to her notice. This inconsistency was evident in students' work when they were asked to represent different decimal numbers using a number line in the first year of the study. Students found it difficult to visualise a number with hundredths place value on a number line, unless explicitly asked to show it on a tenths number line. A reflection on students' work and the task on using a representation (through meetings and individual support from the researcher) helped Nandini in re–examining her decision on their use. The classroom observations from the second year of the study revealed that Nandini choose to teach all the sub–topics using the number line representation. Nandini taught students to represent decimals and fractions using a 10×10 grid. In one of the lessons, she asked students to represent a set of decimal numbers, using both the linear and area representation. In the post–lesson discussion with the researcher, she recalled the limitations in students' understanding (noticed in the first year) of using either linear or area representation depending on the decimal number. Additionally, Nandini referred to the Grade 5 textbook (unlike the previous years) in order to align contexts with relevant representations (refer discussions in Task 6 from Chapter 6).

While reflecting on this decision, Nandini exhibited clarity on the consistent use of the linear mode of representation (for whole numbers and rationals) by using it for the teaching of several sub–topics (representing, locating a decimal number, addition and subtraction, comparison, etc.). Nandini was cognisant of the difficulty caused due to the use of multiple representations in disconnected ways. The considerations that guided her decision of using one mode of representation for different subtopics included consistency, appropriateness, and relevance. While she independently extended the task of using different representations, the choice of one representation and the criteria seem to have emerged through the conversation with the researcher.

7.5.3 Case of Vindhya

Vindhya was an experienced mathematics teacher. Since she had taught Grades 11 and 12 earlier, she was considered as a senior high school teacher. She was confident and often consulted by other mathematics teachers for any queries, such as what can be included in the question papers, how to solve a difficult problem, etc. Teachers and students approached her for solutions to the questions asked in entrance exams.

Vindhya's classroom routine included revision of the previously taught content at the beginning of the lesson, followed by introducing an algorithm or explaining a procedure, which the students were expected to follow while solving problems. Students were expected to listen carefully to the teacher, copy the solutions from the board neatly, and answer the teacher's questions when asked without discussing with each other. Students started following this routine within a few weeks after entering the new session. During teaching, Vindhya was often found clearly stating these norms, as expectations from students. For instance, during teaching she usually selected a student who would respond to her questions after which she would inform the student whether the answer was correct or incorrect.

Vindhya believed in giving clear explanations so that students can follow them to solve problems. Like Pallavi, she believed in providing one explanation, often the algorithm, to the students in order to avoid confusions. In the teacher–researcher meetings, it was found that Vindhya had thorough knowledge of the content – algorithms, procedures, and connections between topics. She expressed her opinions about the significance of memorising algorithms and procedures in these meetings.

Vindhya began talking to the researcher after a few months of the field work and interacted whenever her time permitted.

The contingent moment in Vindhya's teaching appeared when she challenged her core belief about students' talk in a mathematics classroom. She believed that, listening to an incorrect response or probing reasons around it while teaching in the class, reinforces the response and gives an impression to the other students that it might indeed be correct (refer Excerpt 7.11).

Excerpt 7.11: On errors: Vindhya (Y1, Long Interview)

Speaker	Utterance
R notes	Vindhya is sitting in the staff room and checking students' notebooks. She calls my attention to the mistakes made by students in their homework problems.
Vindhya	See they (students) don't pay attention to what is taught in the class. They make so many errors. Even in copying from the board, they will make so many mistakes. And the questions given are same as what we did in the class.
Researcher	What do you do when they make mistakes in class?
Vindhya	When they say a wrong answer in the class, I immediately correct it.
Researcher	When students speak a wrong answer, what is their thinking behind it?
Vindhya	Incorrect only.
Researcher	Have you considered asking them what they are thinking when they give a wrong answer?
Vindhya	See if you make them (students giving wrong answers) give reasons, other students are mostly not paying full attention, they are doing something else also, thinking many things, so they will hear half of it. They will think what this student is saying is correct. I will have a whole class who will think incorrectly. They are not paying full attention in the class, listen(ing) carefully. They can't remember one method only, can't follow it. So, many many methods, they will be more inclined to choose the incorrect method.

Her approach of dealing with students' errors was by avoiding their public hearing, and, if needed, treating them individually (often after a lesson) by telling the correct procedure. Like Pallavi, she was often worried about students getting confused from listening to multiple responses or approaches to problem solving. She adopted an approach of teaching only the algorithms so that students can clearly follow them. Vindhya believed that students' mistakes were a consequence of their lack of attention in class. Her recommendation to students often was to "listen carefully" to the teacher's explanation and "attentively follow" what has been taught. In her commentaries about other teachers' teaching (including teaching videos used during teacher–researcher meetings), Vindhya was consistent in her stance of refraining from stating aloud students' incorrect responses and recommended dealing with such students' responses separately.

In one of the post–lesson reflection sessions in the second year of the study, Vindhya mentioned that the students know more than what she taught. This thought is in sharp contrast with one of her stated belief that "unless taught students do not know" in the first year. The source of this thought seems to have initiated from the discussions during the teacher–researcher meetings.

In the second year, Vindhya was becoming certain about the presence of students' intuitive knowledge and began considering ways in which it can be brought to use or handled while teaching. In an interaction (refer Excerpt 7.12), she mentioned considering (or examining) pedagogies which encourage students to talk aloud in the class without the fear of giving wrong answers.

Excerpt 7.12: Speak aloud in class: Vindhya (Y2DL4)

Speaker	Utterance
Vindhya	You know, I have been thinking (about these meetings). I want to see how students are, when they say incorrect answers also, I mean speak them aloud in the class. Because they should not be scared. Even if it is a wrong answer they should not be scared.
Researcher	Why do you think this is important?
Vindhya	I don't want them to be scared of giving a wrong answer. I want to say, children give the answer, don't worry even if it's wrong. I want them to speak. They should say. And then I would know that many of them might be thinking like that, but may not be confident to speak it aloud in class. In this way, you (a teacher) address all of them. But what does it mean? I mean how to do that?
Researcher	How about encouraging them to speak out in class, share their responses with the whole class, give reasons for what they are thinking, and respond to each other?
Vindhya	Yes, but then after listening (to students) telling them the correct answer, is not right. Then what is the point?

It is important to note here that Vindhya is anticipating the tension of not knowing how to bridge students' strategies with the content to be taught. She recognises that after listening to students' strategies, telling them the right answer is pointless. While teaching in the classroom, she made attempts to bridge students' responses with the conceptual explanations. The classroom observations from the second year of the study show that Vindhya was experimenting with the pedagogy of encouraging students to state their responses, without filtering them as correct and incorrect. Unlike the first year, she avoided passing judgments on students' responses as correct or incorrect and encouraged them to speak what they think (see Excerpt 7.13).

Excerpt 7.13: Reading a decimal number (Y2DL4)

Speaker	Utterance
Vindhya	I want you to read this number. (She writes on board) 2.36.
B St 1	Two hundred and thirty six by ten.
Vindhya	Is that how you want to read it? (Boy nods) Okay.
Vindhya	He wants to read it as two hundred and thirty six by ten. How about you?
B St 2	Twenty three point six.
G St 3	Two point three six.
G St 2	Two point thirty six?
Vindhya	(Records all these responses on board) Any other answer? Just tell me what you are thinking. Don't worry even if it is wrong. Never mind that.
G St 5	Two thirty six by hundred or two (pause) thirty six by hundred.
Vindhya	(Writes on board $\frac{236}{100}$ or $2\frac{36}{100}$)
B St 2	Ma'am, I want to revise my answer.

In Excerpt 7.13, Vindhya asked the students to read a decimal number. She recorded all the students' responses on the board and then asked them to explain their answers. While some students (like B St 2) began revising their responses, she noted that the students convinced each other by giving reasons. Instead of telling the students that the fractional part of the decimal number should be read digit wise (as she did in the first year), she explained the reason for not reading this part as a whole number. She said that, "thirty six means three tens and six ones" while that is not the place value of 3 and 6 in the number 2.36. Through the multiple student responses, recorded on the board, she directed students' attention to the different ways in which the number 2.36 can be read and written. Since none of the students talked about place value, Vindhya posed the next question as "how to read these numbers 3.006 and 3.06. Are they the same?". One of the students responded "one is three point zero zero six, and the other is three point only one zero six". The other student extended, "it is about zero, after the point when zero comes it has a value". To this, Vindhya publicly asked students whether they agree with this explanation and why. She later posed questions on comparing decimal numbers, such as 3.06, 30.6 and 3.6000 along with these numbers for discussion. The class discussed fraction equivalents of each of these decimal numbers and the place value of each digit in the number, as explanations to distinguish between numbers (3.06, 30.6 and 3.6000) and to identify equivalent decimals (3.6, 3.60, 3.60000…). In the post–lesson reflection with the researcher (refer Excerpt 7.14), Vindhya shared the tension or difficulty in managing multiple students' ideas and focusing on key ideas that she intended to teach in this lesson. The concern raised by Vindhya of connecting students' multiple responses or ideas with the content she intends to teach or key ideas point to an important challenge routinely faced by teachers.

Excerpt 7.14: Post-lesson reflection: Vindhya (Y2DL4)

Speaker	Utterance
Vindhya	We discussed so much about decimals in our meetings. In class, we are able to do only a little bit of it. But then I think we discussed even small things like those that I emphasised in class today. Because you know when students demand then as a teacher, I have to bring my ideas.
Researcher	Can you give me an instance from today's lesson about what you are saying?
Vindhya	See, two things I was doing. The place of zero in a decimal number, I wanted them to understand that when the number changes because of that (placement of zero) clearly. You know zero can take different places but I want them to understand where does it make a difference and how. Another thing was place values, which did not come (in their responses). You saw that I was making them (students) say incorrect answers, speak aloud in class. It is difficult to manage these. But I know that I want them to speak aloud and not be scared of giving their answer. You should tell me, did you think students could speak their ideas and also got these two things that I was trying to teach.

Later in the same discussion, Vindhya shared her concern about designing assessments which allow students to articulate their thinking with reasons and whether such practices can be pursued in the given time frame where curriculum completion is demanded.

In this case, Vindhya identified the challenges entailed in the pedagogic decision of encouraging students to speak aloud in class. She recognised the tensions involved in such a decision and interacted with the researcher to articulate her struggles and explore alternative pedagogies. Vindhya needed support in believing that an alternative pedagogy, which enables students to articulate correct and incorrect responses, is viable for learning. Further, she connected different pieces of knowledge such as placement of zero in a number and place value, emphasised the key ideas in teaching decimals such as, place value, fraction equivalence, and selected problems or examples (apart from those given in the textbook) purposely to direct students' attention to these key ideas. It was found in several other lessons taught by Vindhya that she dealt with the ways in which students responded to a problem by (a) encouraging students to reason and challenge each other's explanations, (b) posing counter–examples to challenge a student's way of thinking, and (c) creating variations to a given problem. Such a change in the pedagogical practices involves revisiting the belief about how students would deal with an incorrect response and more fundamentally whether they are capable of responding to a problem, offering and challenging each others' reasons and revising their responses. It is important to note that a pedagogic change which requires listening to students and directing their attention to key ideas draws on a richer knowledge base. This knowledge is needed for examining the correctness of students' responses, identifying the underlying reasons for these responses, challenging these reasons through posing counter–examples or variations in problems, and directing them to useful representations which can be used by students to justify their responses. The knowledge base that Vindhya is drawing upon, I suggest, comes from her engagement with the analysis of students' responses and viewing and discussing alternate pedagogies during teacher–researcher meetings, and from the *in–situ* discussions with the researcher on recognising and handling the conflicts arising from the practice of such pedagogies.

7.6 Discussion: Identifying and Supporting Teachers in Contingent Situations

In the last section, I have analysed the discussions on the contingent situations that arose in the interactions between the individual teachers and the researcher. An analysis of contingent situations reveals that the trigger for these lie in the context of practice. That is, the challenge and need for support emerged from the considerations of practice. It is noted that a discussion on each contingent situation was initiated by the teacher and demanded a response from the researcher. These situations were mostly tied to a specific topic and a general underlying concern about students' capability. For instance, Pallavi's struggle in connecting the partial quotients method and the division algorithm was closely tied to her concern that students get confused with multiple methods. Vindhya believed that publicly sharing incorrect responses in class might confuse students since they might not be able to distinguish the correct reasons. Her attempt at changing the classroom pedagogy, that is, focusing on key ideas and bridging them with students' erroneous attempts was challenge that she faced.

What do we note about the process of identifying and responding to these contingent moments? First, even though the appearance of the contingent moment seems episodic, the process of utilising a contingent situation is extended. It required an extended engagement between the researcher and the teacher beyond the contingent moment. Second, the process required a shared understanding and trust between the teacher and the researcher. This trust that the shared interest of both the teacher and the researcher is in enabling students' learning and supporting the teacher in doing that, is a marked shift from the common place understanding on inspecting or judging a teacher's practice. Such trust also demands that the way teachers are expected to be responsive to the individual students' needs, the researchers who intend to support teachers be also prepared to act in–the–moment responsively.

Such exemplar moments were analysed also to understand the nature of support that individual teachers sought in the sites of their practice. How did the support offered by the researcher enable teacher learning? Some aspects of the process of identifying and transforming such continent situations into learning opportunities which can be discerned from the analysis of the four cases are as follows.

1. *Identifying the mathematical (or pedagogical) challenge in a specific context of practice* – In each case, the teacher identified the challenge faced, and articulated it as a conflict. Reema identified a missing connection in the textbook, Pallavi was struggling to connect different methods with the algorithm, and Nandini was examining the consistent use of representations for teaching decimals. For Reema,

Pallavi and Nandini, the challenge arose from a deeper engagement with the content of the textbook. Vindhya faced the challenge of using an alternate practice with the concern of connecting students' ideas with the key ideas in the teaching of a topic.

Pallavi tried to make sense of the connection between the methods given in the textbook and teach them in a way so that they are less confusing for the students. The textbook does not explicitly connect these methods and for a teacher whose belief is that all content needs to be taught, teaching multiple methods with no explicit connections poses challenges. For Reema, the challenge was a missing connection between sub–topics. Identifying such a gap comes from a close study of the text and an engagement with the students. Bridging this connection requires a sense of trajectory of a topic and how parts within this trajectory can be stitched meaningfully. Nandini attempted to go beyond the textbook. Her decision to use a representation consistently required an understanding of selecting the appropriate representation and helping students use it. Vindhya had identified key ideas in the teaching of the topic of decimal numbers through teacher–researcher meetings. She found it challenging to connect different students' responses (a pedagogical decision) to these key ideas.

In the challenging situation, it can be noted that all the teachers are attempting to make connections between ideas, representations, or explanations. I suggest thatthat a textbook poses knowledge demands on the teacher, particularly, as teachers try to make sense of the content, beyond what has been stated. Identifying the challenge is guided by a closer study of the text (interpretation and reinterpretation) and an engagement with the students, these actions happen almost simultaneously in teaching.

2. *Planning ways of responding to the challenging situation by engaging with the knowledge entailed* – After the teacher discussed the challenging situation with the researcher, ways of responding to the situation were discussed. The plan to deal with this situation included – (a) discussion on the topic–specific knowledge required for teaching the mathematical idea, (b) ways in which the teaching the teacher could deal with it in the classroom, that is, the required pedagogical

moves, and (c) an anticipation of students' ways of dealing with it. For instance, Reema acknowledged that she had omitted the teaching of the question involving the currency context given in the textbook from her teaching in the last few years. She acknowledged that the context was relevant but she had difficulty in handling it in the classroom. After deciding to deal with it, in discussion with the researcher, she unpacked the conceptual explanation for the procedure of multiplication of a decimal by powers of 10. The conceptual explanation was linked to the distributive property and extended to the procedure of converting fractions and decimals. She predicted students' prior knowledge of place value and planned her moves of building on this. Nandini examined the use of linear and area models consistently for all the sub–topics in the decimals chapter in Grade 6. During teacher–researcher meetings and before teaching the lesson she thoroughly explored the consistent use of the meter strip for several subtopics along with the researcher. She found that the linear representation was consistent and decided to introduce the meter strip and make a transition to the number line. She decided to use students' prior knowledge of relation between meter and centimeter to introduce the meter strip. Interestingly, a similar trajectory was observed in Reema's teaching as she decided to use the linear representation consistently.

3. *Experimenting with the planned pedagogy in the classroom* – In each case, it was found that when enacting the planned lessons, teachers carefully noticed students' responses and made in–the–moment decisions based on them. In Pallavi's case, for instance, it was evident that she decided to co–teach with the researcher when she observed that the students were identifying different chunks to get the partial quotients and could be scaffolded to use the maximum size of the chunks. Building on the students' prior knowledge of place value, Reema introduced multiplication of a decimal number with powers of 10 in her class. While unsure whether all the students understood the conceptual explanation, she asked the researcher to help her in providing individual support to the students by moving around and supporting them in completing the worksheet co–designed for the lesson. Nandini noticed how students used the meter strip not just to visualize the

representation of parts of a whole, but also for justification of procedures such as, to show that 0.2 is 2 times one–tenth of a whole. Since she did not want students to be restricted by her choice of the linear representation, in one of her later lessons she asked students to use both the grid and linear representation to represent some decimal numbers. Her stance was informed by the limits in students' responses from the previous year. In each case, students' response in the classroom was crucial for the teachers to make decisions about teaching of the planned idea.

4. *Reflecting to identify the affordances of such teaching pedagogy (and the knowledge it entails)* – During and after teaching the lesson that the teacher had planned with support from the researcher, teachers made explicit decisions about further teaching of the specific mathematical idea. After teaching the particular lesson planned in collaboration with the researcher, the teachers reflected on the students' responses and the flow of content. In these reflective discussions, teachers made some decisions about the content to be taught in the further lessons and changes to be made when teaching this specific lesson in the next iteration. For instance, in the post–lesson reflection, Pallavi planned to examine the teaching of the connection between the partial quotients and the algorithm, an idea that she would have otherwise considered cognitively demanding for the students. She also made some decisions about the teaching of the partial quotients early on (that is from Grade 4 onwards) in her next iteration. Similarly, Reema considered the place value explanation as crucial to teaching not just multiplication with powers of ten but also conversion between fractions and decimals. After teaching it, Reema made some changes to the worksheet that was designed for this lesson and decided to use it in future while teaching this lesson. Teachers noted down these reflections and also shared them with their colleagues in the teacher–researcher meetings.

The analysis of support offered during contingent situations in teacher–researcher collaboration suggests that noticing and identifying such challenges (explicated by the teachers) and preparing to teach them in classroom requires a rich knowledge base. In

Chapter 5, I had analysed the knowledge demands faced by the teachers as they decided to teach responsively. Chapter 6 discussed how the rationale for the design of tasks emerged from the (observed) challenges faced by the teachers in the first year of the study. This chapter reports the mathematical (and pedagogical) challenges identified by the teachers and their decision on the nature of support needed from the researcher.

An analysis of these challenges reveals that experienced teachers also struggle with the conceptual understanding of a mathematical procedure. There is a need for support structures where teachers can discuss their struggles and learn from each other. I see the importance of creating a social learning space for collaboration with researchers and peer support with a focus on classrooms in enabling such an understanding. The discussion and engagement through teacher–researcher meetings or interactions (such as problem solving or planning sessions) were helpful but not sufficient to support teachers in handling the challenging situations that they had identified. Teachers' confidence in teaching the idea required *in–situ* support in the classrooms. The nature of this support was in terms of planning a lesson around the mathematical idea that was seen as crucial for the teaching of the topic. The knowledge needed to handle this idea in the classroom was the object of discussion in planning along with an anticipation of students' responses to the plan. Further, teachers sought support while teaching through collaborative teaching. Based on students' responses, teachers made some in–the–moment teaching decisions. Evidently, it is not just being in the teaching situations, but their anticipation and the appropriate use of curricular materials, also place knowledge demands on teachers. An anticipation of the knowledge demands situated and triggered in practice allowed for an engagement with the knowledge of content, teaching, and students in an integrated manner. Teachers need support in responding to these demands posed by the curriculum and teaching. Our research also indicates that discussions centred around knowledge in play (Rowland & Ruthven, 2011) invite experienced teachers to participate in active decision making and make the discourse of professional development meaningful. Further, it is evident that an intervention grounded in practice has the potential for challenging teacher's existing knowledge, and utilising the knowledge generated through research to inform practice. The engagement with a focus on teaching practice can be utilised for building and sustaining communities of practice with teachers and researchers for continuous teacher professional development. The chapter contributes to the existing literature by detailing the aspects of collaborative learning which led to teacher learning by explicating ways in which such learning is recognised (a concern raised by Robutti et al., 2016).

Chapter 8

CONCLUSIONS AND DISCUSSION

> *The need of getting theory and practical common–sense into closer connection suggests a return to our original thesis – that we have here conditions which are necessarily related to each other in the educative process, since this is precisely one of interaction and adjustment.* (Dewey, 2008, p.10)

The research study reported in this thesis aimed to investigate and support mathematics teachers' knowledge of students' mathematical thinking. This was done by following the teaching of four experienced school mathematics teachers for two consecutive years. The conclusions section is organised around the research questions that were raised at the beginning of the study and a few others that emerged during the course of the study (refer Section 4.2 of Chapter 4).

The classroom observations revealed that teachers became more responsive to students' ideas while teaching in the second in comparison to the first year of the study. An analysis of the classroom data helped in identifying aspects of responsive teaching. Section 8.1 characterises responsive teaching drawing on both, the research literature and the changes observed in teaching during the study. It was evident that responsive teaching is challenging and poses special knowledge demands on teachers. Teachers' decisions in handling unanticipated classroom situations were informed by their deeper knowledge of the subject (an insight also reported in the literature on teacher knowledge). Thus, the questions asked were – how do we identify and study the dynamic knowledge demands placed on the teachers while teaching responsively, and what are the knowledge demands posed on the teachers while teaching a topic. Section 8.2 discusses the process of abstracting these knowledge demands from an analysis of practice, and summarises the observed and anticipated knowledge demands in the teaching of decimal numbers. A part of the study was aimed at supporting teachers to handle the knowledge demands while teaching in the classroom. So the question was – how do teachers acquire this knowledge and become more responsive to their students' mathematical ideas. Section 8.3 discusses the nature of teacher learning from *ex–situ* and *in–situ* support. The *ex–situ* support was in the form of teacher–researcher meetings organised in the school and the *in–situ* support was offered through collaboration with individual teachers in their classrooms. Some other ways in which teachers were supported (such as through anticipation–reflection tasks) are indicated but have not been analysed in detail. Section 8.4 discusses the process of the organic evolution of a community of teachers and researchers which contributed to teacher learning. It is followed by a summary of the contribution of the research study (Section 8.5), its implications for research and teacher education (Section 8.6) and the limitations (Section 8.7). The chapter concludes with some suggestions of the potential ways in which the reported research can inform further work in the area (Section 8.8).

8.1 Characterising Responsive Teaching

The observations of classroom teaching and interviews from the first year of the study revealed that teachers attributed students' mathematical abilities to students' personal traits and background. Teachers were aware of the students' backgrounds and attributed their mathematical performance to factors such as lack of student's attention in class, less time spent by parents on supporting students' learning at home, and the limited access to resources. Some of the teacher decisions such as an emphasis on memorisation of algorithms and rules, particularly by the students from the weaker socio–economic backgrounds, are complex and debatable. The teachers considered memorisation as important for students to pass the examination, in order to continue their studies. Teachers' knowledge of the students' background and the concern about their continuing schooling, interacted with the goal of learning mathematics as memorisation of procedures and algorithms. Teachers' own experience of rote–memorisation of procedures and algorithms, in their schooling and teacher education programme, reinforces the translation of such a goal in practice.

One of the commonly held perceptions among teachers was that, an understanding the rationale for the algorithms or conceptual explanation underlying the procedures is "higher level" and can only be taught when students acquire some cognitive maturity (for instance, see discussions in Sections 6.5.3.2 and 6.6.1.3). Sometimes, teachers were found to be less cognisant of these conceptual explanations themselves. However, it seemed that teachers' experience of teaching mathematics helped them in understanding conceptual explanations in discussion with the researchers and in connecting these explanation with key ideas in the teaching of a topic. For instance, when Reema was supported with the place value explanation for multiplication of a decimal number with powers of 10, she extended it to division with powers of 10, and then to conversions between decimals and fractions (refer Section 7.5.1 of Chapter 7). It was noted that both – an awareness of and comfort with these conceptual explanations manifested in teachers' use of these explanations in the classroom, particularly in the second year of the study.

Teachers were probed for their knowledge of common student conceptions. It was found that teachers identified some common student errors in decimals and in other topics such as, algebra and integers. (Such identification was difficult for the topic of mensuration.) However, they struggled to locate the sources of these errors in students' use of prior knowledge to make sense of new knowledge or the way some procedures had been taught (refer Section 6.4.1 of Chapter 6). Further, an awareness of student errors did not translate into dealing with these conceptions in the classroom in ways that supported students' (independent) thinking. A common approach of handling student errors while teaching was repeating the procedure or giving individual attention. The analysis of classroom teaching from the first year indicated that teachers either ignored students' questions (unless seeking clarifications on the procedures) or judged students' responses as correct or incorrect (refer Sections 5.5.1 and 5.6.1). When such questions were brought to teachers' attention in the post–lesson interactions, they mentioned the lack of time and the rush to complete the syllabus as factors affecting such choices.

Contrasting the data on teaching from the two years of the study, it became evident that such choices or teaching decisions depend on the knowledge that teachers bring to use while teaching. A teacher's decision to ignore students' utterances could stem from teacher's difficulty with the conceptual explanation, in accessing the conceptual knowledge in–the–moment, or from the uncertainty of dealing with such situations in class. The data suggests that teachers, despite their experience in teaching mathematics, struggled to (a) anticipate ways in which students make sense of the content being taught, (b) notice the mathematics underlying students' utterances, (c) decide whether erroneous strategies can be stated aloud considering how it might affect other students' understanding, (d) connect students' ways of thinking with the content planned for a lesson, and (e) estimate students' capabilities of engaging with the new content based on their prior knowledge. An anticipation of these challenges is also connected to perceptions about students' ability, in general, and the role of teaching formal mathematics. A general under–estimation of students' abilities was common among teachers.

In the second year, it was noted that the teachers were listening to their students, interpreting the mathematics underlying their responses – sometimes along with the students or the researcher, and brainstorming ways of handling these responses in class. An analysis of classroom practices revealed that teachers re–voiced students' responses, asked probing questions, gave students an opportunity to explain their responses (correct or incorrect), and either used or adapted students' ways of problem solving when concluding a discussion. A wide variety of teachers' responses to students, such as probing, seeking further information, connecting it with other ideas, challenging the inferences made, etc., indicate that teachers "knowticed" the mathematical potential underlying students' responses and attempted to deal with them through different ways of responding to them (refer Sections 5.5.1 and 5.6.1 of Chapter 5 for details). A stronger teacher anticipation was corroborated by teachers' responses to the task on anticipation and reflection on students' responses to a worksheet designed by the researcher, in the second year when compared with their responses to the anticipation task in the first year (refer Section 6.4.2 of Chapter 6). Further, teachers' explicit anticipation also influenced their listening in the classroom. They listened to the students in ways that helped them understand students' thinking and support it through scaffolds. An analysis of teaching practice revealed that teachers' interpretive listening (Davis, 1997) expanded their knowledge of the content and informed their decisions in the classrooms. An ongoing process of careful noticing of students' responses and making sense of them with the researcher, the students and colleagues also served a hermeneutic function (evidences reported in Chapters 5, 6 and 7). Teachers were modifying their understanding by listening to students and reflecting on such situations.

The literature on responsive teaching (Empson & Jacobs, 2008; Doerr, 2006) suggests that teachers need to be able to listen to students' thinking, and plan and execute ways in which it can be developed. The reported study adds that the act of listening and responding involves acknowledging and appreciating students' ways of making sense of the content, knowticing the mathematics underlying students' utterances, and using this knowledge to make informed teaching decisions. Knowticing requires that (and develops as) teachers listen to and interpret students' incomplete, partial, incorrect, or alternate ideas; connect students' ideas with the key ideas in the teaching of a topic; and provide support that helps students learn in the process. As revealed by the data from teacher interviews, this process is complex, as teachers think about ways of interpreting and supporting a student's idea, while managing the participation and engagement of the other students in this interaction (also noted by Lampert, 2001). To conclude, responsive teaching includes a complex mix of aspects such as challenging students in ways that help them think and articulate their ideas, at the same time offering scaffold to help them reach a conclusion, and managing these discussions with individuals and groups almost simultaneously while teaching.

The analysis also unpacked the pedagogical practices which indicated teachers' mathematical responsiveness towards students. These practices include – (a) different ways of talking about students' strategies by identifying their source, (b) connecting students' explanations to teachers' explanations, (c) linking students' explanations with the key ideas of the topic by anticipating and understanding their struggles, (d) reading the textbook more closely and becoming more agentic in the selection of content, resources and contexts to be used for teaching, and (e) using questions (or worksheets) not just to assess students' understanding or offering more practice, but also for diagnosing and probing students' thinking. Some of these practices overlap with Staples (2017) findings on the pedagogical scaffolds created by a teacher to create collaborative inquiry mathematics classroom. The analysis indicates that responsive teaching is challenging and poses knowledge demands on teachers.

8.2 Knowledge Demands in Teaching

One of the research questions that the study aimed to address was how to abstract teacher knowledge from the standpoint of practice and whether the knowledge abstracted in this manner is substantively different from the way knowledge is characterised by the existing frameworks on teacher knowledge. This section discusses the (a) process of abstracting knowledge demands posed on the teachers from practice, and (b) knowledge demands that arose from the teaching of decimals.

8.2.1 Abstracting knowledge demands from a study of practice

The contemporary literature on teacher knowledge points to the need for creating descriptions of knowledge situated in the practice of teaching. The inclination to move towards practice–based descriptions has the potential to respond to the critiques of the existing frameworks such as, lack of attention to the dynamic aspects of teaching or connections between knowing and using it in practice. The reported research study offers some methodological insights into studying knowledge from an investigation of practice. While there was an awareness of the literature on the knowledge that teachers need to teach effectively, the specifics of such knowledge were discerned from the study of practice. The construct of knowledge demands helped in analysing the mathematical challenges faced by teachers in practice.

The knowledge that teachers need to teach the topic of decimal numbers was identified through an analysis of the knowledge demands placed on the teacher in contingent classroom situations. As discussed in Chapter 5, these knowledge demands are closely tied to the decisions made by the teacher. In other words, different teaching decisions have the potential to uncover various parts of teacher knowledge. Therefore, it was proposed and exemplified that apart from a discussion on the knowledge demands unpacked through an analysis of classroom teaching, a reflection on the possible trajectories that a teacher might take, has the potential to uncover the map of teacher knowledge. This map, unlike the abstract descriptions of knowledge (reported in the literature), is topic–specific and is closely tied to the work of teaching, in particular to the teacher's in–the–moment decisions. Understanding teacher knowledge as a map or landscape (described by Wenger–Trayner, et al., 2014) implies that different pieces of this knowledge get triggered in varied classroom situations, depending on the teacher's decisions. Teachers take different teaching trajectories for teaching the same sub–topic depending on the factors arising in classroom situations. A complex interplay of factors such as the nature of students' engagement, connections triggered in the mind of the teacher through interactions in classroom and the mathematical ideas (content and processes) that are prioritised; affects these teaching trajectories.

Since the descriptions of practice are expansive, teacher's responses to contingent classroom situations, served as an appropriate "grain–size" (Cai et al., 2017) for an analysis of knowledge demands. An analysis of paired episodes in the teaching of teachers in transition offered a suitable context for the knowledge demands to become visible. To summarise, an abstraction of anticipated and actual knowledge demands from an analysis of paired episodes in contingent classroom situations served as an important methodological tool to study the knowledge manifested in teachers' practice.

8.2.2 Knowledge demands in teaching decimals

A comparison of paired episodes helped in uncovering the topic–specific knowledge demands posed on the teachers. These knowledge demands (abstracted from Chapters 5 and 7) can be suitably organised to describe the mathematical knowledge needed for teaching of decimals. A summary of knowledge demands arising from an analysis of the teaching of decimals is presented below.

1. *Affordance of the analogy drawn between whole numbers and decimals.*
   Teachers need to be aware of the extent to which an understanding gained from working with whole numbers can be extended to decimal fractions (Resnick et al., 1989; Steinle, 2004). This awareness includes being cognisant of the (a)

procedures where teachers draw on this understanding while teaching, for instance, when dealing with the claim "longer number is larger" (which always works for comparing whole numbers but not always for decimal numbers), or in naming the place value of the fractional part of a decimal number using the place value names from the whole number part, and (b) potential connections that students might make between whole numbers and decimals for instance, misunderstanding that 0.4 times 10 is 0.40. Similarly, the affordance of drawing analogies between fractions and decimals also requires examination, for instance, in handling student's responses such as $\frac{4}{10} = 4.10$.

2. *Use of the fraction representation along with the place value as a justification for procedures of working with decimals.*

   The conceptual explanations which can be used to justify the procedures used in decimals can be borrowed from an understanding of place value of whole numbers and representing decimals as fractions. For instance, when comparing decimals or operating with them, the understanding of equivalent fractions can be helpful. Procedures such as, multiplying a decimal number with 10 means shifting the point by one digit on the right, or annexing zeroes at the end of the decimal number does not change its value, can be justified using the corresponding fraction representations or place value explanation.

3. *Identifying key ideas in the teaching of decimals and connecting them to form conceptual explanations*.

   Some of the key ideas identified through an analysis of teaching decimals include place value, fraction representation, positionality of a digit in a decimal number, and relation between digits of a number in base ten. The connections between these ideas were used to formulate conceptual explanations. For example, the understanding of positionality of the digit in a decimal number and the fraction representation was used to examine different cases of identifying the influence of positions of zero on the value of a number (consider 3.6, 03.6, 30.6, 3.06, 3.60).

4. *Appropriateness of a context used for teaching decimals.*

   The process of selecting an appropriate context requires (a) an examination of contexts which use a decimal relation and distinguishing them from other contexts, and (b) the affordances of the selected context for its use. Distinguishing between decimal and non–decimal contexts on the basis of the meaning of a decimal, base ten system and positionality; helps in identifying situations where a point is used as a separator (for example, in writing a date, exercise numbers in textbooks), to show a relation different from base ten (for example, in measuring time or number of balls using overs in a cricket match), and as a decimal point (for example, in length measurement units of meter, Indian currency). This understanding can be extended to identifying contexts where both a decimal and a non–decimal relation can be found depending on the relation between the selected units. For example, the relation between parts of meters (centimetres, decametre, etc.) is a decimal context while measuring length using feet and inches is a non–example.

   A context is used to introduce or demonstrate a relation which it represents. The decision of selecting a context for introducing decimals requires that it can be used to show relation between different place values. Using the money context, that is, the Indian currency, can be limiting for it exhibits the relation of hundredths only. Alternatively, a length measurement context can be used to depict the relation of tenths, hundredths and thousandths. Thus, the choice of a context depending on where and when it is used in a teaching trajectory is important.

5. *Consistent use of representations for different sub–topics within decimals.*

   Like contexts, it is important to examine the relevance and purpose of the representations being used. The choice of appropriate representations and their consistent use helps in supporting students' understanding. For instance, using a number line representation for different sub–topics (locating a decimal number, comparing decimals, converting fractions and decimals, identifying numbers between a number, etc.) helped students in visualising smaller decimal numbers, understand density, and compare decimal magnitude. How does the use of linear and area representations affect students' visualisation and reasoning, needs further investigation.

8.3 Developing Mathematical Responsiveness

The literature on teacher knowledge and the data from the reported study show that knowticing the mathematics underlying students' responses requires knowledge. This knowledge is dynamic, that is, teachers can be made aware of students' ways of thinking using research literature or through references to experienced teachers' practice. However, it is important to acknowledge that parts of such knowledge and connections get unpacked through actual interactions between the teacher, students and the content while teaching in the classroom. Therefore, teachers need support both in (a) developing knowledge from the topic–specific research literature and from the experience of teaching, and (b) dealing with the struggles arising from the challenges encountered in teaching.

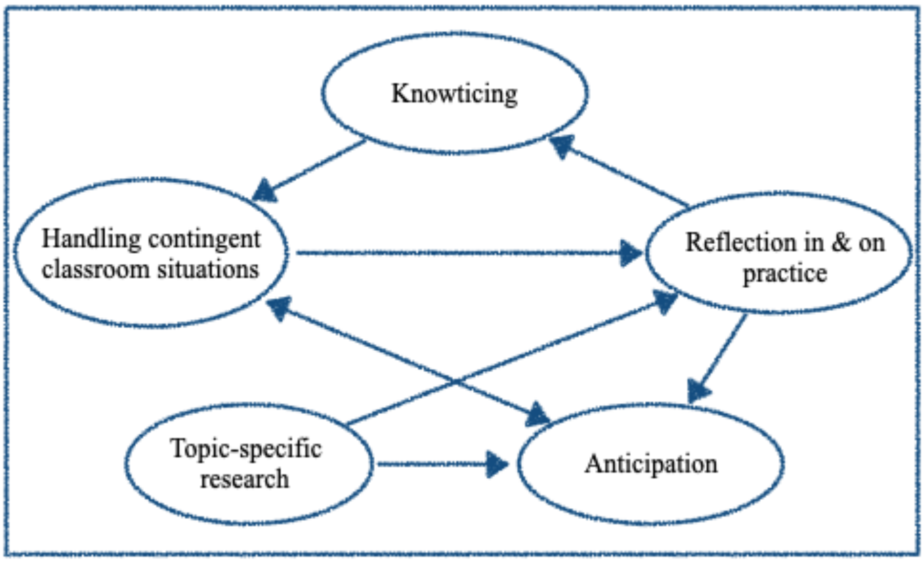


Figure 8.1: Developing Responsive Teaching

How can teachers' knowledge of students' thinking be supported? The literature on topic–specific knowledge and a close study of resources such as textbooks are sources through which such knowledge can be developed. The other sources of this knowledge are the artefacts collected from the teachers' classroom teaching. While the knowledge from the literature can help in developing anticipation, and prepare teachers in handling contingent situations, a reflection on different ways of identifying and dealing with such situations in the classroom feeds back into the corpus of knowledge required for teaching. The analysis of *in–situ* and *ex–situ* support offered to the teachers during the course of this study reveals that an interweaving of the knowledge from the literature with the actual practice of teaching developed teachers' knowledge (depicted in Figure 8.1). While the literature on topic–specific aspects was used to organise and plan teacher–researcher meetings, interactions with teachers inside and outside of the classroom encouraged sharing and reflection on the knowledge located in teaching practice. For instance, the research literature on students' conceptions in decimals was used to (a) design problems given to students in the form of a worksheet used for the anticipation task, (b) organise the actual students' responses into categories or modes of thinking, and (c) examine the similarities and differences in the way students responded to such problems. Further, teachers created adaptations of these problems to diagnose and evaluate students' thinking, to prepare a question bank for use in classroom, and for analysing students' ways of thinking. Ways in which the artefacts from literature and actual practice were interweaved to support teachers' knowledge are listed below (derived from Chapters 6 and 7).

1. *Enhancing topic–specific knowledge*: It includes knowledge of breadth and depth of modes of representation or explanations, affordances of using particular representations or explanations, identifying key ideas and formulating explanations using connections between ideas. It was addressed through supporting teachers in analysing the textbook content for its adequacy, accuracy and sequencing; discussing the literature on the use of number line and multi–base arithmetic blocks; and identifying links between arithmetic done in primary, middle and higher grades.

2. *Knowledge of students' conceptions and capabilities*: Students' responses to a variety of decimal tasks and the use of representations discussed in the literature were examined while planning content for teaching. Further, teachers used the experience from the classroom, particularly students'

responses to become confident in using some ideas, methods or representations in teaching.

3. *Using students' work as a spring board to discuss key ideas in the teaching of specific topics*: Students' work helped teachers in discussing episodes from their teaching. While analysing students' work, tasks such as identifying key ideas, using a representation consistently, reasons for a procedural explanation, discussing students' incorrect ideas and ways of challenging them mathematically, etc. were linked.

4. *Examining alternative pedagogies with their affordances and constraints*: Teachers were shown instances of alternative practices, and supported in their imagination and planning for such possibilities in their classroom. They were supported in their classroom when they wanted to experiment with a different approach or practice. Sharing such practices in the teacher–researcher meetings helped teachers in developing a space to discuss their attempts at experimenting with different ways of teaching. Supporting teachers in and outside the classroom was helpful in order to build their knowledge and support its enactment in class.

5. *Using classroom as a site of conscious practice and reflection:* In the inspectorial system, classrooms have been used as sites of judging teachers' performance. However, the study shows ways in which artefacts from classroom teaching can be used to challenge and support teachers' knowledge construction. In the process, characterising knowledge required for teaching from a practice–based perspective, has the potential for development of research on mathematics teachers' knowledge. Therefore, envisioning classrooms as spaces not just for students' learning but also for teachers and researchers' learning is significant.

In the study, teachers were provided with *ex–situ* and *in–situ* support which helped develop their knowledge for teaching. Through interactions with the researcher, teachers generated knowledge needed for practice. They also produced artefacts such as worksheets for handling specific student conceptions and a question bank for the teaching of decimals. Thus, the knowledge provided from the literature (knowledge *for* practice) and a reflection on classroom teaching (knowledge *in* practice) helped the community of teachers and researchers to generate knowledge *of* practice.

8.4 Organic Evolution of a Community of Teachers and Researchers

An intensive engagement with the teaching practice led to the evolution of a community of teachers and researchers, with a shared interest in enabling students' learning. Using the constructs of engagement, identification and boundary crossing proposed by Wenger–Trayner et al. (2014) in their characterisation of learning in the landscapes of practice, I will discuss some aspects of the development of this community.

While the teachers and researchers were aligned with their roles of teaching and data collection, respectively, an engagement with the others' role to reflect on the process of teaching helped in identifying shared interests. The sense of community developed from the researcher's engagement with the struggles faced by teachers in practice and the interest of teachers in reflecting on and learning from their practice and research literature – all these ways of engagement evolved during the course of study. This engagement was supported by a shared imagination of teaching, among teachers and researchers, which was responsive to students' ideas in the classroom. The nature and character of what constitutes responsive teaching, was learnt from observing and reflecting on teaching practice. A culture of talking about teaching evolved through discussions on the local artefacts identified from teachers' classrooms and the knowledge needed to handle unanticipated moments in teaching. This culture of talking about teaching changed the nature of interactions among teachers and with the researcher.

As teachers became more responsive in attending to students' ideas, they sought the researcher's support to handle the knowledge demands arising from teaching. The reflexivity in the researcher's role allowed taking a more participative position in teacher–researcher collaboration, leading to boundary crossing of roles. The flexibility in the researcher's role varied from being a co–teacher, a co–planner, a participant observer, and a discussant, depending on the nature of support that teachers needed in the context of their practice. Teachers' role also changed from being a participant in the research, whose teaching is being observed and analysed, to a more agentic one. Teachers demanded a change in the researcher's role from being a non–participant observer. They envisioned a change in their role by creating opportunities for students to talk about mathematics in class and re–imagined the use of material resources such as blackboard, textbook and other resources in teaching. Teachers acquired the role of observers in the researcher's and other colleagues' teaching and organised a few teacher–researcher meetings. While the exact nature of learning from such boundary crossing in the roles of the teacher and the researcher is unclear, it seemed to have created the motivation to sustain this collaboration and bring some change in teaching practice with the aim of improving students' learning.

Teachers and the researcher identified more closely with each other in tasks such as designing a worksheet, planning a lesson, analysing a teaching episode, etc. The identification was particularly stronger when the goal of students' learning was shared for instance, in lessons that were planned together. The identification was weaker in tasks such as reading of research papers, making a time–table for teaching, marking students' papers, etc. However, respectful identification and dis–identification in different tasks helped in developing a shared accountability to the act of teaching. It is important to mention that an interest in and accountability for students' learning acted as major source of identification between the teachers and researchers.

8.5 Contributions of the Study

The research study reported in this thesis contributes to the literature and practice of mathematics teacher education in the following ways.

1. The study presents a systematic account of recording and analysing teaching practice which helps in understanding how the knowledge of a teacher comes into play in the act of teaching. Thus, it offers a way of studying

knowledge from the standpoint of practice. Such detailed descriptions of practice have the potential to shed light on the complex aspects of teaching, such as, the interactive nature of talk, affordances of representations, etc..

2. Apart from an exploratory purpose, the practice–centered approach of the study extends to supporting teachers' knowledge in the contexts of their practice. The study offers an exemplar of a professional development initiative where reflexivity and participatory approach of the researcher, an interest in examining the challenges of teaching and building a support system for teachers was shared with the participants. It explicates ways in which an abstraction of knowledge ensued in teaching practice and critical reflection on it can be used to characterise aspects of mathematical knowledge required for teaching. The study offers an enactment of a teacher–researcher partnership which impacts classroom practice.

3. The study shows how research literature on students' conceptions can be used to design contextual tasks for reflection on teaching and learning. The research texts along with the actual data from classrooms was interwoven to design and implement tasks which were meaningful and contextually relevant for the participating teachers.

4. The study exemplifies how teachers and researchers can collaborate to challenge the existing practices and create a shared knowledge base to support students' learning.

5. The study demonstrates sensitive and reflexive use of research methodologies to build trust and responsibility towards reconfiguring the role of classroom observations from their inspectorial purpose to that of analysis and reflection for enabling teacher learning.

8.6 Implications

The study has implications for research in mathematics education. The analytical constructs proposed by the study, such as abstraction of "knowledge demands" from an analysis of "paired episodes" in "contingent classroom situations" have implications for further research on mathematics teacher knowledge. The constructs offer a way of characterising the topic–specific knowledge required for teaching from a close study of practice. Further, the process of analysing knowledge demands offers methodological insights for capturing the dynamic aspects of teachers' knowledge by studying "paired episodes" of teachers in transition.

For supporting mathematics teacher education and learning, the study provides an exemplar of tasks to develop the knowledge required for teaching. Discussions on specific episodes of teaching to identify the actual and anticipated knowledge demands entailed in different teaching decisions (or trajectories) is an important way of orchestrating learning in pre– and in–service teacher education programmes. Tasks on specific cases of students' work and teachers' activity to develop teachers' knowledge of the subject matter, help in understanding and analysing teaching situations where different kinds of teacher knowledge (identified in the existing frameworks) come into play and informs teaching. Such an engagement with the teaching situations makes teachers aware of the knowledge that is needed to handle challenging situations as well as offers ways of examining the affordance of the potential teaching trajectories that can be taken to deal with it in the classroom.

The study offers a way of designing and using classroom–based tasks used to develop professional knowledge required for teaching. The design of such tasks includes an interweaving of knowledge gained from the topic–specific literature on teachers' knowledge and student conceptions, with the struggles faced by teachers in actual teaching situations. The methodological and analytical insights can be used to design professional development initiatives for other disciplines.

The insights from the reported study translated into the curricular design of an innovative programme for preparing secondary school teachers[4]. The design of this teacher education programme is centred around practice communities, where pre–

[4] The research work has contributed to the design of the innovative pre–service teacher education programme of Tata Institute of Social Sciences. While the larger aim of the programme is to encourage communities, specific courses use research literature and artefacts from practice to support teacher learning.

service teachers work closely with in–service teachers and researchers to build the knowledge required for teaching mathematics. The university courses on school observations and classroom observations, in the course work of this programme, aim to develop a sense of the institutionalised systems within which teachers work. Courses on nature and pedagogy of the subject require student teachers to follow experienced teachers over long periods, record and discuss the struggles arising in teaching, read and use the existing research literature on different topics for conducting interviews with students and design a teaching trajectory for a specific topic to be tested in classroom. Student teachers develop ways of discussing teaching by participating in dialogue with disciplinary groups, different subject groups, and with the experienced teachers and teacher educators. Attempts are also made to encourage communities involving teachers, teacher educators and researchers through the organisation of weekly seminars, some of which will be planned in schools which collaborate with the teacher education institute.

8.7 Limitations

Potari and Jaworski (2002) assert that to manage the mathematical challenge of the task posed during teaching, both cognitive and affective sensitivity play a role. The cognitive sensitivity refers to acknowledging and developing students' mathematical thinking by appropriating the mathematical challenge. The affective sensitivity deals with fostering students' beliefs in their ability to do mathematics and valuing their mathematical engagement and thinking. In the reported study, as teachers became more responsive to their students' mathematical thinking, changes in both cognitive and affective sensitivity were noted. However, the affective aspects of teachers' sensitivity, the ways in which it evolved or interacted with the changing cognitive sensitivity, and manifested in the scaffolds offered during mathematically challenging situations, have not been analysed.

In each of the two years of the study, teachers worked with different cohorts of students. In order to get a general sense of the cohorts, their grades in the previous year final examination and the formative tests from the two years of the study were matched. While the average score of the cohorts was similar, the content addressed in these exams does not help in understanding the nature of students' engagement with the mathematical ideas in classroom. It was noticed that students asked a variety of questions (many a times similar across classes) but engaged with the content differently. For instance, the oneths question was discussed differently in each of the four classrooms in the second year. Interestingly, while the question appeared in all the classrooms, the engagement with it varied depending on the ideas discussed. A varied engagement with a similar set of ideas and the emergence of different questions in these classrooms helped in comprehensively capturing the knowledge demands underlying the mathematical ideas being discussed. The relation between the cohort and the nature of discussions in each classroom has not been examined.

The study indicated a change in the way teachers dealt with the contingent classroom situations while teaching decimals in the second year. The observation of teaching of a few other topics, such as mensuration, algebra, data representation also revealed changes in teachers' responses to students' ideas. While some broad changes in the teaching of these topics were observed, they were not analysed closely given the scope of the study.

In the first year of the study, it was found that teachers' interactions with each other were limited to administrative tasks, such as seeking clarification on who will prepare the question paper, the dates for examination, and so on. A strong framing of teachers' time in the school (teaching for long hours) along with the burden of several non–teaching tasks such as fee collection, restricts meaningful discussions on teaching. The post–lesson interactions with the researcher in both the years were also limited due to the lack of teachers' time. Systemic constraints, such as strong framing of teachers' time and the micro–management of their work by the school administration limited focused discussions with the teachers on the content to be taught.

The research uses a case-study methodology which opens itself to questions about generalisability and applicability of its findings to different contexts. When replicating the study, it might be important to consider the kind of knowledge that students bring to the classroom and how it interacts with the teacher knowledge in play, to analyse the knowledge demands posed on the teachers. While a part of these knowledge demands are generalisable to the topic being studied, which demands surface at a given time in classroom is dependent on the teacher-student interactions and the trajectory taken to teach the topic. The process of abstracting knowledge demands can through a comparison of paired episodes can be extended to study different topics and classrooms where teachers are making a transition to more student-centred pedagogy. An analysis of paired episodes can be a useful methodological device in studying the affordance of variations in classrooms. The aspects of topic-specific knowledge identified in Chapter 7 would need further investigation and use in planning professional development initiatives. The findings of the study offer ways of supporting responsive teaching by acknowledging the dynamic connection between knowledge and practice.

The findings of the study indicate that the cognitive issues are of serious importance in teaching of the subject matter and attention to these is critical. However, it is important to acknowledge the contexts of teachers and teaching in Indian schools, along with the layered inequalities that persist in the social and material conditions of the student population. An understanding of the role, purpose and functioning of schooling lies at the backdrop of our understanding if the objectives of the study are to bring about any change and transformation in the system.

8.8 Suggestions for Further Work

Research literature on developing mathematics teachers' knowledge suggests that teachers need to examine their practice and reflect on it in ways that improves this practice. Teachers need time and support to reflect on and learn from their practice. Challenging the existing structures in which teachers work and reconfiguring them in ways which encourages reflection on and learning from practice is important. Existing practices such as insistence on following the textbook content, expecting teachers to engage in non–teaching tasks, measuring teachers' knowledge, considering classrooms as spaces for teachers' performance, micro–managing teachers' time in the school, overcrowding of teachers' classrooms, etc.; would need to be revised so that schools become accountable for their teachers' development as professionals. Cai et al. (2017) suggest that if the accountability of teachers' continuous professional development is placed on the school administration and state systems, then effective learning environs can be created for students and teachers.

Structural envisioning of ways in which research can engage with the practice and practitioners is needed. A deeper engagement with the nuances of practice, including the institutional structure within which teaching takes place, students' backgrounds and foregrounds, goals of teaching specific disciplines, etc. might help in unpacking the complex work of teaching. Researchers can also be made more sensitive to the field contingencies by learning to use methodologies in ways that ensures learning of all participants. A deeper engagement with and respect towards the everyday work of practitioners, can inform research analysis.

Stronger teacher–researcher partnerships, where teachers and researchers take different roles, can be envisioned from the policy perspective. The reported study examines a way of working with teachers closer to the practice of teaching. Other ways in which teachers and researchers can collaborate need to be examined. For instance, teachers as researchers, researchers as teachers, teachers and researchers collaborating to teach specific topics, etc. The goal of such partnerships could be to improve students' learning, develop perspectives towards learning in specific subjects, plan research which can inform practice in direct ways, examine teaching pedagogies, and so on.

Using the analytical constructs proposed in the study, attempts can be made to characterise aspects of topic–specific knowledge in teaching of other topics in mathematics. Ways in which this topic–specific knowledge manifests in practice depending on teaching trajectories might help in creating a comprehensive map of teachers' knowledge required for teaching.

The study used a landscapes of practice framework post–facto to discuss aspects of teacher–researcher collaboration, using the constructs of "engagement, imagination and alignment". A more systematic analysis of the evolution of a community of teachers and researchers, and its impact on all participants' learning could be pursued by designing and researching such spaces.

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APPENDIX

A1. Worksheets for Students

A1.1 Worksheet 1, Class V, Year 1

Dear Students,

Read the problems carefully. This is not a test. Try to write what you think when you are solving a problem. Write all your rough work in this sheet. Ask if you do not understand anything.

1. Shalini has 17 chocolates. She wants to distribute them equally among her four friends. She wants to know how many chocolates will each of her friends will get.

   (a) Which of the following can she use to answer her question?

		Correct	Incorrect	Not sure
(a)	$17 \div \frac{1}{4}$			
(b)	17×4			
(c)	$17 \div 4$			
(d)	$17 \times \frac{1}{4}$			

   (b) How much chocolate will each of Shahni's friends get?

2. Due to a flood 785 people are stuck in an island. The government sends a helicopter to bring these people safely to the nearest city. A government helicopter can take only 10 people in one round. How many rounds will the helicopter have to take so that all people reach the city safely?

3. How many half cakes can you make from 15 cakes?

4. Ratna is a labourer. She is hired to work on a farm land for Rupees 52 per day. The government has decided that each worker should be paid a minimum wage of Rupees 71 per day.

   (a) Why do you think Ratna is being paid less than the amount she should get according to the government rules?

   (b) She works for the month of January and February. How much should she be paid?

5. Mark correct calculations with (✔) and incorrect calculations as (×). Circle the step where there is a mistake and correct that step.

(a)	2 8 9 + 4 6 3 81412	(b)	3 0 8 — 1 5 6 2 5 2
(c)	4 6 3 × 8 6 3 2 0 0 0 4 8 0 0 2 4 0 2 4 0 0 3 6 0 + 1 8 3 9 8 1 8	(d)	5 4 6 + 9 9 3 1 4 3 9
(e)	3 7 27 | 8 3 0 9 — 8 1 2 0 9 — 1 8 9 2 0	(f)	2 8 × 3 5 6 0 0 + 4 0 6 4 0

6. Frame a word problem for 846 × 29.

7. Think of the ways to solve these problems as quickly as possible. Write down how did you solve the problem quickly.
   (a) 48 + 97 = ___ + 99
   (b) 78 × 500 = _____
   (c) 67 – 58 = 65 – _____
   (d) 9853 × 50 = _____

8. Reema reads a poster outside a stationery shop. She wants to buy coloured pens.

   (a) How much will she have to pay for 8 coloured gel pens?

   (b) She checks her wallet and finds that she has Rupees 200. Can she buy more than 8 pens?

24 coloured gel pens for Rupees 270.

A1.2 Worksheet 2, Class V, Year 1

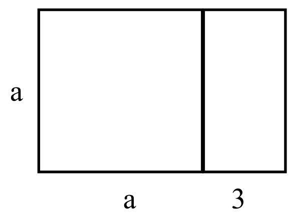
1. Chumki's teacher gave her this question: $33 \times 2\frac{1}{2} =$ ____ . Chumki says "Oh there are many ways to solve this problem".
   (a) Do you think there are many ways to solve this problem?
   (b) Can you show two (or more) ways to solve this problem?

2. Mark the correct option. You can also mark <u>more than one option</u> if you think they are correct.

		(a)	(b)	(c)	(d)	Give reasons
(a)	One–tenth of one–tenth is	100	1	$\frac{1}{100}$	one–hundredth	
(b)	$0.04 \times 10 =$	0.040	4	0.4	0.40	
(c)	$0.14 + 0.7 =$	0.21	0.084	2.1	0.84	
(d)	$1.678 > 1.8$	True	False	Can't say		
(e)	$\frac{5}{8} = 5.8$	True	False	Can't say		
(f)	$\frac{3}{4} + \frac{2}{3} = \frac{5}{7}$	True	False	Can't say		

3. I have 120 slices (pieces) of oranges. How many oranges do you think I would have peeled?

4. What is the area of the rectangle shown in the figure.



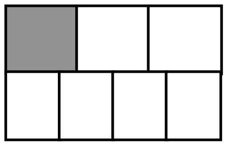
5. Match fractions and decimals which have the same value from Boxes A, B, and C and write below. One example is shown. $\frac{1}{4}$ = one–quarter = 0.25

Box A	Box B	Box C
$\frac{1}{4}$	One and a half	0.08
$\frac{8}{100}$	One and a quarter	1.5
$\frac{1}{8}$	$\frac{2}{25}$	0.25

$\frac{9}{4}$	Half of a quarter	$\frac{1}{4}+\frac{1}{4}+\frac{1}{4}$
$\frac{3}{2}$	one–quarter	$\frac{5}{40}$
$\frac{3}{4}$	Three times a quarter	More than 2

6. A car goes 55km per hour. Make a diagram to show where will it be after.
   (a) half an hour, (b) an hour, (c) two hours, (d) 15 minutes, (e) 45 minutes.

7. What is the fraction of the shaded part of the rectangle.



8. Rama thinks that if you increase the area of a rectangle, its perimeter also increases. Do you agree or disagree with her. Give reasons.

A1.3 Worksheet 1, Class VI, Year 1

1. The cost of 10 pens is Rs. 42. Find the cost of 15 and 20 pens.

2. 35 kilos of rice costs Rupees. 1160 in Mumbai. But I need only 18 kilos of rice. Approximately how much money should I have?

3. Farida says that “Sum of an even and a odd number is always odd”. Do you think she is right? Can you prove this?

4. Shalini has 17 chocolates. She wants to distribute them equally among her four friends. She wants to know how many chocolates will each of her friends will get.

   (c) Which of the following can she use to answer her question?

		Correct	Incorrect	Not sure
(a)	$17 \div \frac{1}{4}$			
(b)	17×4			
(c)	$17 \div 4$			
(d)	$17 \times \frac{1}{4}$			

   (d) How much chocolate will each of Shahni’s friends get?

5. Which vehicle has faster average speed – a truck that travels 100 kms in 1½ hours or a car that travels 140 kms in 1¾ hours?

A1.4 Worksheet 2, Class VI, Year 1

1. A group of students were practising basket ball and were trying to put the ball in the net. Here is the table of their results. According to you, who among them is the best player. How do you know?

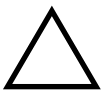
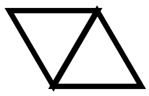
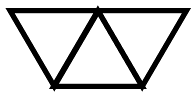
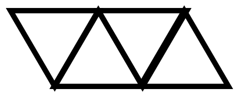
Players	Shots taken	Successes
Ankita	4	3
Siddharth	9	4
Ritu	6	2

2. Mr Short is a friend of Mr Tall. When we measure their heights with matchsticks, Mr Short is 4 matchsticks and Mr Tall is 6 matchsticks. Short measured with paper clips has a height of 6 paper clips. How many paper clips are needed to measure Mr Tall's height.

3. Saheba is thinking about multiplication and division with zero. She is confused. Help her by filling in the blanks and giving reasons for it.
   (a) a number is multiplied with zero
   (b) a number is divided by zero

4. It takes 4 people 3 days to wash all the windows of the K–Star mall. About how long will it take for 8 people to do this job.

5. Kiran and Saheb are trying to make a bridge through sticks. Observe them play and be a part of it. As they went on playing, the bridge they made looked like this.

			
Kiran	Saheb	Kiran	Saheb
1st	2nd	3rd	4th

   (a) How many sticks will be used in 5^{th} design and 100^{th} design?

   (b) Which design would require 57 sticks? Show your working.

6. Sumit thinks that 0 is an even number. Rahi thinks 0 is an odd number. What do you think? Give reasons.

A1.5 Worksheet 3, Class VI, Year 1

1. 5 + 3 = 8 and 3 + 5 = 8. Even if we change the order of the numbers the sum is the same. Do you think this is true for all the numbers? Give reasons for your choice.

   (a) Yes (b) No (c) Not sure

   Reason:

2. Chumki's teacher gave her this question: 33 x 2 ½ = ____ . Chumki says "Oh there are many ways to solve this problem".
   (a) Do you think there are many ways to solve this problem?
   (b) Can you show two (or more) ways to solve this problem?

3. Rama went to buy some cheese to make pasta for a party at her home. The shopkeeper told her that cheese costs Rupees 83 per kilo. Rama asks for 750 gms of cheese. How much money does Rama have to pay to the shopkeeper?

4. Shyam thinks that if you increase the area of a rectangle, its perimeter also increases. Do you agree or disagree with him. Why?

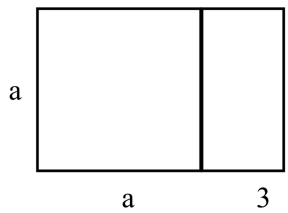
5. Sini reads a poster outside a stationery shop.

24 coloured gel pens for Rupees 270.

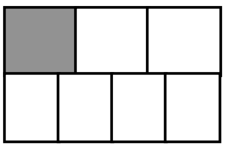
(a) She wants to buy 8 coloured pens. How much will she have to pay?

(b) She checks her purse and finds that she has Rupees 200. Can she buy more than 8 pens? How many?

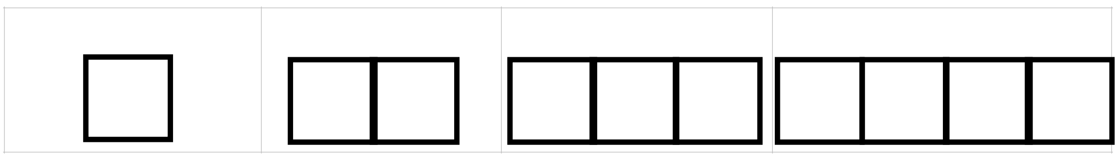
6. What is the area of the rectangle shown in the figure.



7. What is the fraction of the shaded part of the rectangle.



8. Shashi and Jugal are playing a game with sticks. First Shashi made a design and then Jugal, then Shashi again and so on.



Shashi	Jugal	Shashi	Jugal
1st	2nd	3rd	4th

(a) Complete the following table with the number of sticks used for each design number.

Design Number	1	2	5	6	50	100	p
Number of sticks used	4	7					

(b) There is a design made with 91 sticks. Who could have made it – Shashi or Jugal? What is the design number? Give reasons.

A1.6 Worksheet 4, Class VI, Year 1

1. How many half cakes can you make from 15 cakes?

2. Mark the correct option. You can also mark more than one option if you think they are correct.

3. A car goes 55 km per hour. Make a diagram to show where will it be after (a) half an hour, (b) an hour, (c) two hours, (d) 45 mins, (e) 15 mins

4. Think of the ways to solve these problems as quickly as possible. Write down how you solve the problem quickly.
   (a) 48 + 97 = ___ + 99
   (b) $78 \times 500 =$ _____
   (c) $9853 \div 50 =$ _____

5. Is $\frac{5}{7}$ different from $\frac{7}{5}$ or are they the same? Explain how.

6. Arrange these in descending order.
   (a) 2.8 , 0.43 , 1.6 , 8.7 , 12
   (b) $\frac{1}{8}, \frac{3}{9}, \frac{1}{6}, \frac{2}{8}, \frac{1}{5}$

7. 10 metre of silk cloth costs Rupees 880. Ram bought 6.5 metres of silk cloth and paid the shopkeeper a 1000 rupee note. How much money should Ram get back from the shopkeeper.

8. Match fractions and decimal fractions which have the same value from Boxes A, B, and C and write below. One example is shown. 1/4 = one–quarter = 0.25

Box A	Box B	Box C
$\frac{1}{4}$	1 and half	0.08
8/100	1 and quarter	1.5
1/8	2 /25	0.25
9/4	Half of quarter	1/4 + 1/4 + 1/4
3/2	one–quarter	5/40
3/4	3 times a quarter	More than 2

A1.7 Worksheet on Decimals, Class V, Year 2

1. Sana wants to write these as decimal numbers. Help her.

(a)	Forty five tenth	
(b)	Thirty two and six tenth	
(c)	Seven hundred and forty five hundredths	
(d)	Five and four tenths	
(e)	Ninety two and seven hundredth	

2. Write the place and place value of the underlined digit.

	Number	Place	Place value
(a)	$395.\underline{6}8$		
(b)	$562.1\underline{7}$		
(c)	$756.\underline{0}9$		
(d)	$89\underline{9}.700$		

3. Write the following numbers as decimals.
   (a) 6 (b) 37 (c) 25 ½ (d) $\frac{48}{100}$

4. Reena has written some statements after learning decimals. Identify which of her statements are correct (C) and which are incorrect (I). Write the correct statements for the incorrect ones.
   (a) 37.6 is bigger than 37.06
   (b) 57.9 = 57.90 = 57.900
   (c) 5.8 is smaller than 5.08
   (d) 37.02 is same as 37.2

5. Sri wants to arrange the following numbers in ascending order.
   (a) 5.6 , 8.3 , 7.9 , 12.6 .
   (b) 15 , 2.7 , 9.4 , 6.8
   (c) 2.75 , 8.68 , 1.79 , 14.62
   (d) 6.9 , 5.62 , 5.29 , 6.09

6. Ranjana wants to arrange these numbers in descending order. Help her.
   (a) 75 , 26.9 , 92.6 , 57
   (b) 4.86 , 9.7 , 3.2 , 6.1

7. Lalli likes to see what happens to numbers when we multiply them with 10, 100, 1000. Fill in the blanks by giving answers to Lalli's questions.

(a) $37.9 \times 10 =$ _______ (b) $37.9 \times 100 =$ _______

(c) $8.65 \times 10 =$ _______ (d) $8.65 \times 100 =$ _______

(e) $32.7 \times 20 =$ _______ (f) $3.96 \times 40 =$ _______

8. Ridhi tells Samay that $46 \times 40 = 1840$. She challenges him to use this to solve the questions below mentally. Help Samay to solve these problems orally.

(a) $0.46 \times 40 =$ _______ (b) $4.6 \times 40 =$ ______

(c) $46 \times 4 =$ _______ (d) $4.6 \times 4.0 =$ ______

9. Rahul and Deepa are writing equivalent decimals. Tick the correct pairs.

$3.7 = 3.70$

$9.65 = 9.065$

$8.79 = 8.709 = 8.790$

$4.9 = 4.09$

10. The fruit seller tells Jasmeet that the cost of one banana is Rupees 3.50. How many bananas can Jasmeet buy for Rupees 15?

A1.8 Worksheet on Decimals, Class VI (Pre–teaching), Year 2

1. Circle the smaller number? Give reasons for why you think it is smaller.
   (a) 4.63 or 4.8 (b) 0.7 or 0
   (c) 0.6 or 0.37 (d) 8.24 or 8.245
   (e) 0.25 or 0.100

2. Arrange these in descending order.
   (a) 0.658, 3.7, 2.45, 5.63
   (b) 0.248, 0.85, 0.63 , 0.4
   (c) 3.03, 3.033, 3.303, 3.33
   (d) 5.5, 5.55, 55, 555

3. Write in decimals
   (a) $\frac{14}{10}$ (b) $\frac{14}{100}$ (c) $92 + \frac{4}{100}$ (d) $\frac{1}{4}$ (e) $\frac{4}{7}$

4. Is 0.02 m is the same as 0.02 cm? Explain your answer.

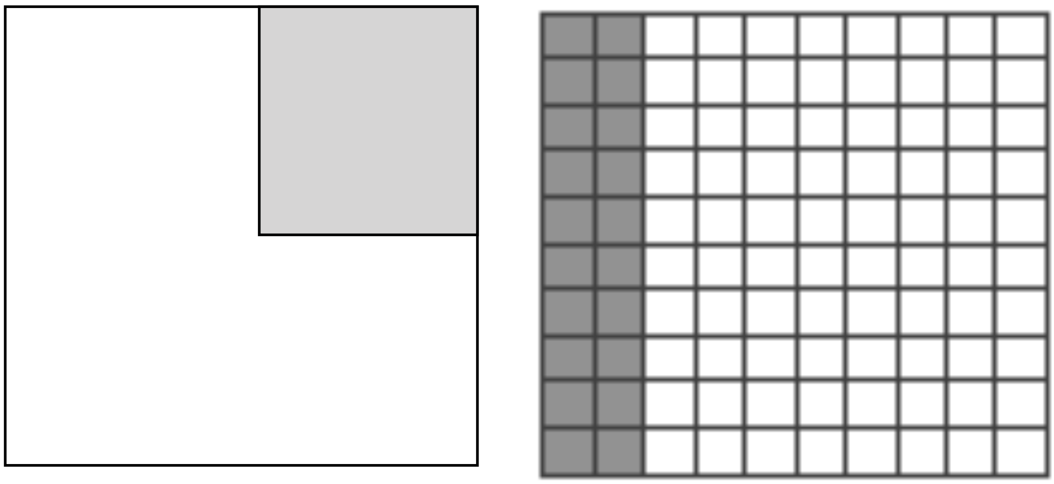
5. Circle the correct option(s).
   (i) 0.3 is same as
   (a) $\frac{1}{3}$ (b) one–third (c) $\frac{3}{10}$ (d) 0.30
   (ii) 15.72 can be written as
   (a) 15 ones and 72 tenths (b) 15 tens and 72 tenths
   (c) 1572 tenths (d) 1572 hundredths
   (iii) Circle all the numbers that are same as 7.8
   (a) $\frac{7}{8}$ (b) 78 tenths (c) 7.80 (d) 7.08
   (iv) Circle all the numbers that are same as $\frac{8}{100}$
   (a) 0.80 (b) 0.800 (c) 0.08 (d) .08
   (v) Circle all the numbers that are same as 0.51
   (a) 0.5100 (b) 0.051 (c) 0.510 (d) 51

6. Complete the sequence
   (a) 1.24, 1.26, 1.28, …....... , …....... , ….......
   (b) 0.3, 0.6, 0.9, …....... , …....... , ….......
   (c) 1.51, 1.41, 1.31, …....... , …....... , ….......

7. Do you think there are decimal numbers in between 3.9 and 4? Give some decimal numbers.

8. Reena thinks the following are true. Check whether she is right or not. Give a reason to her.
   (a) The answer for 3.7 + 8.6 is more than 1.
   (b) The answer to 4.6 + 0.6 is more than 0.6 but less than 4.6.
   (c) The answer to 16.5× 0.2 is more than 16.5.
   (d) 0.4 × 10 = 0.40
   (e) The relation between ones, tenths and hundredths is: 1 ones = 10 tenths = 100 hundredths.

9. Circle the number below that shows about how much of the shape has been shaded.
   (a) 0.20, 0.50 , 0.25 (b) 0.2, 0.6, 0.9



10. Just as you have tenths and hundredths, do you also have oneths? What is it?

11. Show these using a diagram.
    (a) 0.5 (b) 0.67 (c) 1.35

12. Add: 8 tenths and 15 hundredths, 64 tens and 73 tenths.

13. When giving injections, it is important to be accurate. If the wrong amount of drug is injected, the patient can lose consciousness, may go into a coma, or even die. Imagine you are a doctor. Note down the amount of medicine in the injections (as indicated by each arrow). Try to be as accurate as you can.

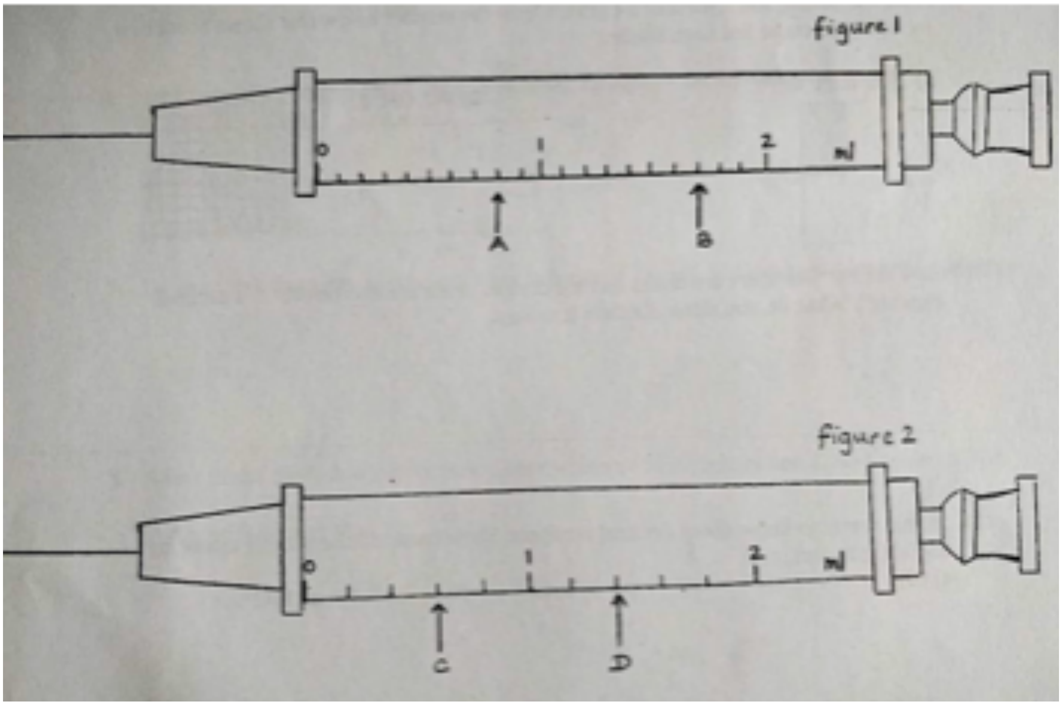


figure 1
0
1
2
ml
A
B
figure 2
0
1
2
ml
C
D

A1.9 Worksheet on Decimals, Class VI (Post–teaching), Year 2

1. Circle the greater number? Give reasons for why you think it is greater.
   (a) 7.89 or 7.9 (b) 0 or 0.9
   (c) 0.48 or 0.6 (d) 6.217 or 6.21
   (e) 0.100 or 0.25

2. Radhika wants to arrange the following decimal numbers in ascending order.
   (a) 0.768, 4.7, 3.95, 5.74
   (b) 0.564, 0.75, 0.86 , 0.9
   (c) 7.07, 7.077, 7.707, 7.77
   (d) 3.3, 3.33, 33, 333

3. Write in decimals.
   (a) $\frac{25}{10}=$ ___ (b) $\frac{49}{100}=$ ____
   (c) $89+\frac{3}{100}=$ _____ (d) $\frac{1}{4}=$ ____
   (e) 38 = _____

4. Is 0.05 m the same as 0.05 cm? Explain your answer.

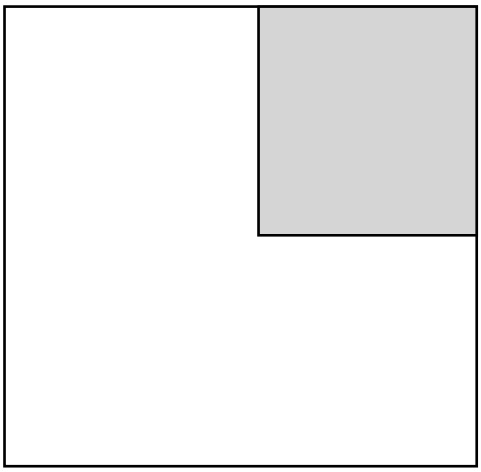
5. Saima has framed this problem. In some of the problems she has kept more than one correct answer. Think and tick the correct options.
   (i) 0.5 is same as
   (a) $\frac{1}{5}$ (b) one fifth (c) $\frac{5}{10}$ (d) 0.50 (e) five–tenths
   (ii) 13.68 can be written as
   (a) 13 ones and 68 tenths (b) 13 tens and 68 tenths
   (c) 1368 tenths (d) 1368 hundredths
   (iii) 9.3 is same as
   (a) $\frac{9}{3}$ (b) 93 tenths (c) 9.30 (d) 9.03
   (iv) $\frac{8}{100}$ is same as
   (a) 0.80 (b) 0.800 (c) 0.08 (d) .08
   (v) 0.65 is same as
   (a) 0.6500 (b) 0.065 (c) 0.650 (d) 65

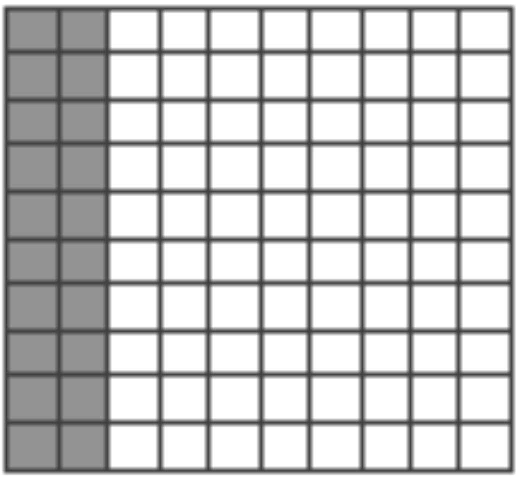
6. Sahil is creating sequences of decimal number. Complete the following sequence
   (a) 1.64, 1.66, 1.68, …....... , …....... , ….......
   (b) 0.2, 0.5, 0.8, …....... , …....... , ….......
   (c) 1.59, 1.49, 1.39, …....... , …....... , ….......

7. Do you think there are decimal numbers in between 7.9 and 8? Give some decimal numbers.

8. Reena thinks the following are true. Check whether she is right or not. Give a reason to her.
   (a) 4.6 + 6.9 is more than 10.
   (b) 7.5 + 0.5 is more than 0.5 but less than 7.5.
   (c) 14.5 × 0.2 is more than 14.5.
   (d) 0.8 × 10 = 0.80.
   (e) The relation between ones, tenths and hundredths is: 1 ones = 10 tenths = 100 hundredths

9. Ravi has shaded some parts of a figure. Circle the number below that shows about how much of the shape has been shaded.
   (a) 0.20, 0.50 , 0.25, 0.10 (b) 0.2, 0.6, 0.9



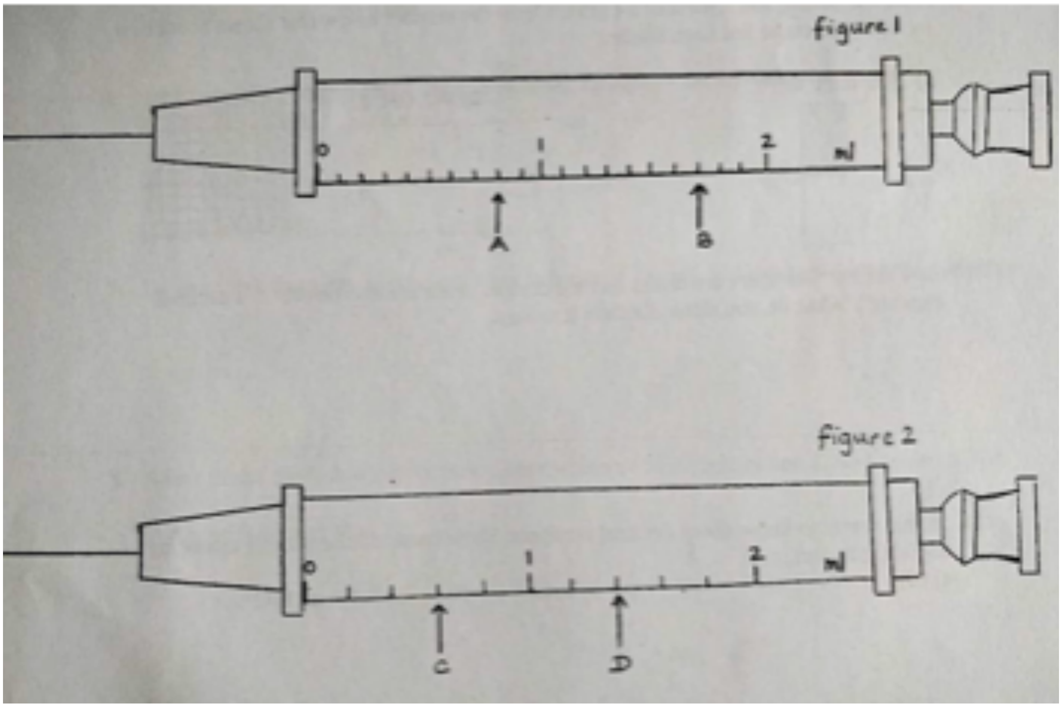


10. Sara thinks, "like there are tenths and hundredths, there are also oneths in a decimal number". What do you think. Explain it to Sara.

11. Shahid wants to know about decimal numbers. Show these numbers using a drawing to explain it to him.
    (a) 0.8 (b) 0.43 (c) 1.42

12. Giri and Saba created a problem each. Solve these problems.
    Giri: 5 tens and 17 hundredths + 43 tens and 84 tenths
    Saba: 82 tens and 78 hundredths – 75 and 24 thousandths

13. When giving injections, it is important to be accurate. If the wrong amount of drug is injected, the patient can lose consciousness, may go into a coma, or even die. Imagine you are a doctor. Note down the amount of medicine in the injections (as indicated by each arrow). Try to be as accurate as you can.



A2. Data Collection and Analysis

A2.1 Information sheet: Teacher

Dear Teacher,

Thank you for taking out time to fill this information sheet. Your identity will be kept confidential. The information will be used for research purposes only.

1. Name:
2. Designation:
3. Subject(s) taught and Grades:
4. Subject (s) currently teaching and Grades:
5. Name and Address of the Current School/Organisation:
6. Years of Experience: …...................... (elsewhere) …...................... (in this job)
7. Educational Qualifications:
8. Email id:
9. Contact No.:
10. Languages you speak: (underline the languages which you use in class)
11. Have you taught these children, whom you are currently teaching, before? When and which subject
12. What do you think about mathematics as a subject?
13. What are the common difficulties that you face while teaching mathematics to students?
14. How does experience influence your teaching? Please elaborate.
    (a) According to you, which are the mathematical concepts/ ideas that students generally find difficult to understand?
    (b) Why do you think children find these concepts/ ideas difficult?
    (c) What do you do when children face difficulty in learning these concepts?

    Date:
    Place:
    Signature:

A2.2 Information sheet: Student

1. Name:
2. Class & Section:
3. Gender:

4. Home address:
5. Father's occupation:
   Mother's occupation:
6. Contact number:
7. E–mail id:
8. Age:
9. Date of birth:
10. Language(s) you speak at home:
11. School name and Address:
12. How long have you been studying in this school?
13. Name of the school you attended before this school:
14. After school you do mathematics with: (tick the correct options)
    (a) tuition teacher (b) mother (c) father
    (d) sibling (brother/sister) (e) on your own
15. Which book(s) and other material do you use when you are doing mathematics at home?
16. Concept(s) in mathematics that you like most:
17. Concept(s) in mathematics that you do not like:
18. Marks scored in mathematics in the last exam (FA4):
19. Marks scored in final exam in Class 4 / 5:
20. According to you, mathematics is about
21. Write about your best mathematics class. What did you like in that class?

A2.3 Pointers for Classroom Observations

	Classroom Observations
	Note down the proceedings of the lessons. Make sure to focus not just on what can be heard but also the movement of the teacher, students, student–student talk, board work, gestures, etc. There might be observations that are not addressed in the scheme. Note down these observations and share with the other researchers.
1	What is the topic being taught? What were the key mathematical ideas (concepts, representations, explanations) used?
2	How is the topic being taught – introduction, explicit teaching, summary? What kind of explanations and representations were used during teaching?
3	What were the different physical spaces that were used during the lesson and by whom? (Use a sociogram to represent this.)
4	What were the resources used to teach – textbooks, teaching–learning material, students' prior knowledge, other material?
5	What was the nature of the questions posed by the teacher? Did the students get a chance to answer these questions?
6	What was the nature of questions posed by the students? Who answered these questions?
7	What were the kind of explanations valued in the classroom?
8	What was the nature of teacher's response to the students' questions or queries?
9	How was the seating arrangement of the class? How often does it change and what is the nature of this change?
10	What different interaction patterns found in the teaching: teacher–student, teacher–students, student–student, student–teacher, or any other?
11	Identify the students who interact the most, sometimes, the least in the class.
12	How often are the students given an opportunity to talk among themselves?
13	What was the kind of homework given to the students.

A2.4 Interview schedule (for teachers), Year 1

Expected time: 1 hour + 1 hour (approximately)

Mode: Oral, Written if necessary

Nature: Semi–structured

The following interview schedule is designed to probe teachers' thinking about – their own knowledge as math teachers, their awareness of students' thinking, and beliefs about mathematics teaching and learning and the goals of their own mathematics teaching. The focus is on teacher responses to the questions posed by the researcher about her explanations of classroom practices (that have been observed in Phase 1 of the study), responses to students' thinking, use of representations and reasons guiding the choice of different representations while teaching mathematics.

There are four parts to this interview. They are teacher's (1) explanations of their own (and other) teaching practices, (2) views about what is special(ised) about knowledge of maths teachers, and (3A&B) awareness , knowledge of and responses to students' thinking. The teacher will be to give examples for almost all questions. The interview will be carried out in two days. Sections (1), (2) and (3A) will be done on the first day and section (3B) on the second day of interview. The sections are sequenced in a way such that teachers begin talking about their (and their colleagues') practices that have been observed by the researcher in Phase 1 of the study. This will get teachers to talk about something familiar and concrete. It is important to understand their perspective as to why they choose to do certain practices, what are their beliefs underlying these practices. This will be done by requesting teachers to explain and reflect on their own practice. This is followed by section (2) which aims to probe teachers' conceptions on teaching mathematics and the specialised nature of their knowledge as math teachers. The idea is to know more about teacher's views about the uniqueness of teaching as a profession and of teaching math. Section (3) is intended to understand the nature of teacher's knowledge and awareness about their students' questions, errors and alternative (correct) strategies. Also, the objective is to understand how teachers anticipate and respond to students' ideas while teaching in classroom.

The approach of the interviewer will be to help teachers – explicate their thinking, give supporting arguments for why they think in a particular way, and more importantly to reflect on what they do.

<u>Section 1: Teacher Practice</u>

1. How long have you been teaching? And teaching class 5/6?

2. How long have you been teaching this (current) class? Does it help to be teaching the same class in the next grade?
3. What are the other grades you teach?
4. Have you observed some difference in teaching class 2,3,4 and class 5 students or class 7,8,9 and class 6 students.
5. Are there differences in the ways in which you started teaching math (as novice) and in the more recent years? What has caused this change?
6. Are there differences in the way novice and experienced teachers teach?
7. Can you tell me about one of your 'best class' of teaching mathematics with any of the grade level. (Later) Why do you call this class as your 'best class'?
8. So how do you generally begin your lesson, a problem, a concept?
9. Do you follow the textbook completely when teaching a lesson? You do all the exercises, solved examples? In class or homework or both? What kind of problems as homework? Do you give more problems than those given in the textbook?
10. What is your opinion about the new textbook? How do you find the new textbooks? Are they different from the old textbooks, how?
11. Do the new textbooks demand a different teaching methodology or structuring of classroom?
12. Do you think it is important for you to write the problem on board and make students read? What do you do after that?
13. What is your general approach to teaching a word problem in class? Does it differ for different students (sections, grades)? Can you think of an example of a word problem and say how would you do it in different classes?
14. On an average how many problems do you cover in one teaching period?
15. So how do you get to know what students are thinking or whether they are understanding what you are teaching? When you know that students are not understanding, then what do you do?
16. What do you think about the practice of giving homework? Do you give students homework everyday? How many problems on an average? What is the nature of homework you give?

17. How important are repetition and drilling in the learning of mathematics? Can you tell me some exercises which you do to help students memorise?

18. Are there times when you are not able to finish all of what you had planned for the class? Or you have to change your plan, when?

19. What is the language you commonly use to teach in classroom? Which language do most of the students use to talk to each other and to you?

<u>Section 2: Teacher Knowledge</u>

1. Who according to you is a good mathematics teacher or what makes a teacher a good mathematics teacher. Why?

2. Do you think teachers are different from parents, tutors, or elder siblings teaching students? Why?

3. Are there some concepts in mathematics, that according to you, all students should know? Why?

4. Some teachers feel that it is important to understand why the algorithm works, some feel it is not important and it will confuse the children. What do you think? Let's take an example of an algorithm... (teacher will be asked to give an example. If she does not then researcher will give the multiplication with a cross as an example). What is it about the method that students should know (how it works, why it works, where all it works)? Do you think that students should know why the algorithm works. Or is it enough if they know how to use the algorithm?

5. Do you think the concepts in math are related to each other or are they unique? Can you give a few examples to support what you think?

6. How is mathematics related to – (a) other subjects (disciplines), (b) daily life, (c) later life or jobs.

7. Do you think there are some short–cuts to solve some mathematics problems? Can you give an example? Do you think this will work for all the problems of the same kind? Why do you think this works?

<u>Section 3A: Teacher Responses to Students</u>

1. Which method do you encourage students to use when solving a multiplication (class 5) / proportion (class 6) problem (give an example from textbook here) and why?

2. Why do you think this method (algorithm) is important for students to learn?

3. Sometimes students do not understand what the teacher is teaching. How do you know whether students are understanding? What do you do if a student is not understanding what you are explaining to the class?

4. Do you call a student on the board? How do you decide which student to call? How does that help the student, other students? Does it help you?

5. What do you do when a student comes to the board and makes a mistake? Why?

6. Whom would you call a good student of mathematics and why? Can you name a student or two from your class who are good students?

7. What according to you are the qualities of a good student of mathematics?

8. Do you think that boys are better at learning mathematics than girls? Why?

Section 3B: Anticipation Task Interview

In this part of the interview teachers will be shown the worksheets that were given to students by the researcher. The teacher will be asked how her students would have responded to this problem, what (method) would she expect from students and why, what would the teacher do with the errors and alternative strategies used by students. If the teacher does not come up with the errors then researcher will show her some actual student responses and ask the teacher the same questions. The problems from the worksheets that will be discussed in detail with teachers are mentioned below (after the questions). The questions to be posed with the math problems almost remain the same.

This was one of questions given to the students. 48 + 97 = ____ + 99 (Class 5)

1. What are the different correct and incorrect responses that students would have given for this problem?

2. Could there be other ways of solving this problems? Can children come up with these different ways? Do you tell students what are the different ways of solving a problem. Why? Would you give students marks in exams if they use an alternative method.

3. Can you tell me one of the common mistakes that students make when solving problems of this kind?

4. Do you think this could be one of the errors that student could have made, why? (Researcher gives one of the errors that students have actually made). One of the students has solved the problem like this. What do you think about this student response? So how do you plan to deal with it? (to be done twice: one for an error and other one for alternative strategy).

5. What do you do when a student commits a mistake in solving a problem? Can you give an example of a mistake that students most commonly do and your response to it. So how will you take the student from this mistake to learning the method that you want the child to learn?

[This will be followed by sharing the selected student responses with the teacher and seeking their responses.]

A2.5 Consent Form

HOMI BHABHA CENTRE FOR SCIENCE EDUCATION, TIFR

V.N. Purav Marg, Mankhurd

Mumbai 400 088

Mumbai

October 2013

Dear Teacher

Thank you for participating in this phase of the study. We are grateful for your cooperation. Some of your classes and interviews will be audio and video recorded. The recordings will help us analyse the work and learn from the practice of teaching. To use the recordings for analysis, we need your consent. Please note that your identity will be kept confidential. Your data will be used for academic purposes only. Please mark your preference on the use of these recordings.

1. Consent for use in research
   I am aware that the sessions and discussions are being recorded for the purpose of research and analysis. The recording of sessions can be used for academic purposes of research and analysis by academic team of HBCSE.
   Agree / Disagree

2. Consent for use in Teacher Workshops
   The recordings can be used with other teachers and researchers in group meetings or teacher professional development workshops.
   Agree / Disagree

Signature of participating teacher: Researcher:
Name:
Date:

A3. Artefacts From Teacher–Researcher Collaboration

A3.1 Worksheet used in TRM2

Student Responses to Decimal Problems

1. When students are asked to arrange the following decimal numbers in descending order their responses vary.
   0.658, 3.7, 2.45, 5.63
   (a) What are the possible (correct and incorrect) ways in which they will solve this problem.

2. b) Here are the responses of two students.
   Response 1: 0.658, 5.63, 2.45, 3.7 Response 2: 3.7, 5.63, 2.45, 0.658
   (a) What do students think when they make these errors?
   (b) Can you devise some problems to check whether your students are making these errors?
   (c) Do students make mistakes in different kinds of problems following this pattern of thought?

3. $1.678 > 1.8$ (i) True (ii) False (iii) Can't say
   (a) What are the possible reasons that students think for choosing (i)?
   (b) Is there a way to distinguish between students who are choosing (i) for different reasons?

4. $0.4 \times 10 = 0.40$
   (a) Why do students multiply like this?
   (b) What part of their previous knowledge interferes here?

5. One of the students responded that $0.3 < 0$. For which all decimal numbers do students think that they are less than 0. Why?

6. 145.31 means 14 tens 5 ones 31 tenths. Why do students respond in this way? Is this response different from the responses listed above?

7. Do you see a pattern in student's thinking in all these problems? What is it? What could be some of the useful strategies to help students overcome these errors?

A3.2 Question bank on decimals

Note: This question bank was prepared by the teachers and researchers in a teacher–researcher meeting (2013, Year 2).

1. Which is these is same as 0.5?
   (a) $\frac{1}{2}$ (b) $\frac{50}{100}$ (c) $\frac{10}{200}$ (d) $\frac{5}{10}$ (e) $\frac{500}{1000}$

2. Represent $1\frac{1}{2}$ (one and a half) in decimal.

3. Write the decimal notation for the following: (a) two tens two tenths two thousandths, (b) three hundredths one tens five tenths.'

4. Write the place and place value of the underlined digits.
   a) $3\underline{6}.4$ b) $62.\underline{5}$ c) $96.5\underline{7}$ d) $4\underline{5}6.\ 2$ e) $2\underline{0}.79$

5. Which of these are same as 14.72
   (a) 1472 hundredths (b) 14 ones 72 hundredths
   (c) 147 tenths 2 hundredths (d) 1 tens 4 ones 7 tenths 2 hundredths

6. $6.7 \times 10 =$ _____, $6.7 \times 100 =$ ______

7. Where will 1.65 lie on the number line?

8. How is 7.1 different from 7.13?

9. Show the number(s) between 7.9 and 7.10 on a number line.

10. Where all do you see decimals around you?

11. Write the expanded form of
    (a) 51.3 (b) 65.72

12. Fill in the blanks
    (a) 40 rupees 75 paisa = Rs ______ (b) 2 rupees 5 paisa = Rs _______
    (c) 100 m 35 cm = ______ m (d) 25 cm 5 mm = ____ cm
    (e) 1.23 has ____ hundredths

13. Circle the bigger decimal number

(a)	85.6, 95.3	(b)	64.2, 64.8
(c)	0.51, 1.70	(d)	0.6, 0.37
(e)	0, 0.7	(f)	0.648, 0.9
(g)	0.75, 0.751	(h)	0.30, 0.030

(i)	0.2 cm, 0.9 mm	(j)	0.10 mm, 1 cm
(k)	0.10 mm, 0.9 cm		

14. Add the following
    (a) 2.3, 64.73
    (b) 45, 61.3
    (c) 8 tenths and 15 hundredths
    (d) 1.5 + 2.32 + 2.6
    (e) 3m 5cm, 8m 25cm

15. Find the difference between 74.6 and 62.9

16. Solve 135.25 $\times$ 5 in two different ways.

17. In cricket we see 0.6 overs. What does it mean? Is it a decimal number? Why do you think so?

18. What happens when you multiply or divide a decimal number with 10, 100 or 1000.

19. We know that 2.5 $\times$ 25 = 62.5. Which all number facts can you write using this? [Eg: 2.5 $\times$ 2.5 = 6.25]

A3.3 Worksheet on measurement and decimals

Note: This worksheet was prepared by Reema and the researcher for classroom teaching (Year 2).

1. Anya wants to make a line segment of length 4 cm. She wants the line segment to end at 9 cm. Draw the line segment for Anya.

2. Sahil and Rekha are measuring lengths using a broken ruler. Help them by measuring these lengths.

3. Show a length of 2.5 cm in different ways on this scale.

4. How will you read the following decimal numbers? Write it.
   (a) 8.5
   (b) 0.4
   (c) 10.01
   (d) 13.50

5. Joy is reading aloud these decimal numbers to you. Write them down as decimal numbers.
   (a) Five thousand three hundred twenty five and five tenths
   (b) Six hundred two tenths
   (c) Nine tenths seven hundredths
   (d) Ten tenths
   (e) Twenty five tens sixty six hundredths

6. How many threads of length 9.5 mm can be cut from a 1 metre long thread. Explain your answer.

7. A lemon costs Rupees 2.50. How many lemons can Sanjay buy for Rupees 50.

A3.4 Worksheet on multiplication of decimals with powers of 10

Note: This worksheet was prepared by Reema and the researcher for classroom teaching (Year 2).

Solve these mentally.

(a)	$0.1 \times 1 =$ ____	(b)	$0.1 \times 2 =$ ____
(c)	$0.1 \times 10 =$ ____	(d)	$0.2 \times 10 =$ ____
(e)	$0.5 \times 10 =$ _____	(f)	$1.1 \times 10 =$ ____
(g)	$1.2 \times 10 =$ ____	(h)	$2.5 \times 10 =$ ____
(i)	$0.01 \times 10 =$ ____	(j)	$10 \times 100 =$ ___
(k)	$1 \times 100 =$ ___	(l)	$0.1 \times 100 =$ ___
(m)	$0.2 \times 100 =$ ___	(n)	$0.5 \times 100 =$ ___
(o)	$1.1 \times 100 =$ ___	(p)	$1.2 \times 100 =$ ___
(q)	$2.5 \times 100 =$ ____	(r)	$0.01 \times 100 =$ ____