Touchy Feely Vectors: A design-based study examining the role of representational media in STEM cognition

A Synopsis of the PhD Thesis

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by

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Abstract:

Recent theories argue that cognition, in general, is 'constituted', i.e. brought into being, by sensorimotor interactions between the body and the environment. Extensions of this ‘constitutivity hypothesis’ suggest that for the phenomena and models (in STEM) not directly accessible to sensorimotor interactions, cognition is through multiple external representations (MERs). This theoretical position leads to a corollary: the understanding and processing of STEM concepts may be shaped by representational media (text, animation etc.) encoding the MERs.

To test this corollary, we examined: 1) how existing static media encoded a complex STEM modeling concept (vectors), and 2) whether the limitations of this media correlated with students' conceptual reasoning behaviour (CRB). Results indicated a possible correlation. To further investigate this, we: 1) designed a new media interface (Touchy-Feely Vectors, TFV), which compensated for the interaction limitations of textbook media, and 2) examined whether the principled design of TFV led to systematic changes in students' CRB. Results indicated a change in CRB correlated with the design. We then examined the robustness of this result, by augmenting existing textbook media using virtual lesson plans (co-designed with the teachers) based on TFV, and a larger field study (N=266) in real-world classroom situations. Results showed that both students' CRB and classroom teaching/learning practices changed.

These results, and the principled design rationale of the TFV system, together indicate that interactive affordances of representational media play a critical role in STEM cognition, thus supporting the constitutivity hypothesis, as well as recent 'field' theories of cognition. Further, our operationalization also illustrates a systematic approach to the design of digital media for STEM learning in developing nation contexts.

Keywords: representational media, digital media, vectors, paper-based media, embodied learning, model-based reasoning, developing country context

Numbering convention: The thesis has 3 parts, with chapter numbered as A,B,C... Part-1 (Conceptualisation: Ch-1A,1B), Part-2 (Execution:Ch-2A,2B,...2G), and Part-3 (Conclusion: Ch-3A, 3B, 3C). The first two characters of Figures, Tables and Appendices numbers correspond to the chapter. For example, in Ch-1A figures, tables and appendices are numbered as Fig.1Ax, Tab.1Ax and App.1Ax respectively. All the Appendices are provided as separate files in the attached folder Publications based on each of the chapters are added after the Titles (Part-2) and are listed at the end: JPx - Journal Paper; FPx - Conference Full Paper; APx - Conference Abstract or Poster.

1 Introduction

Samantha is a high school student trying to understand the abstract topic of vectors in physics and mathematics. Based on the classroom instruction, she reads many text passages, and solves many problems by writing equations. Yet, she struggles in developing a conceptual understanding of vectors and imagining vector operations. This is a general problem most students face while learning and understanding modeling topics such as vectors, which form an important component of STEM education. The experiences of teachers, and significant education research studies, testify to these difficulties.

Apart from pedagogical strategies, designing novel instructional material to support learning has been one of the key approaches towards addressing these learning difficulties, particularly with the advent of computational media. These designs seek to augment student understanding of complex modeling topics. However, it is unclear how the features of the new designs, particularly their interactive affordances, support learning of abstract models. Further, in developing country contexts, this cognitive augmentation potential of digital media is underexplored, as educational technology designs in such contexts have focused mostly on access, i.e. making learning material widely available. The research objective of this thesis is to examine these two issues (role of media affordances in learning, ways to develop new media designs in developing country contexts) in detail.

Insights into the nature of learning abstract models, as well as the learning process, are needed to develop practically-meaningful and theoretically well-grounded digital media designs. In ch-1A, we outline the nature of the learning problem in STEM, particularly highlighting the abstract nature of the topics involved. Recent shifts in our understanding of cognition ('field theories' of cognition; 4E cognition) highlight the constitutive role played by sensorimotor interactions in cognition. We draw on these insights about the learning processes, and propose a corollary of the constitutivity hypothesis, where a connection exists between the interactive affordances of educational technology and learning. This connection is poorly understood. In ch-1B, we outline how the research objective discussed above is operationalized using the topic of vectors.

In the part-2, we report a series of designs and studies across 3 years, analysing:

- existing representational media and their connection to students' conceptual reasoning behaviour (Ch-2A, 2B)
- two iterations of a design (Touchy Feely Vectors, TFV) that sought to compensate for the limitations (esp. of interactions) of the existing representational media (Ch-2C and 2E)
- two rounds of studies that tested changes in students' conceptual reasoning behaviour based

on TFV. Ch-2D describes lab studies, Ch-2F reports studies in classroom settings. Further, Ch-2G examines the way TFV changed classroom practices.

Results from these studies indicate that limitations of the paper-based medium, particularly in supporting manipulation of geometrical entities, could be a key reason for students struggling to imagine vector operations, and the resulting lack of epistemic access to such abstract models. The lab and classroom studies showed that TFV, partly co-designed with teachers to compensate for the limitations of the paper-based medium, enhanced epistemic access in students, and fostered model-based reasoning with better geometry-algebra integration, which is required to understand and use vectors.

These results indicate that the interactive affordances of representational media play a critical role in shaping students' cognition of abstract models. Further, the studies illustrate a novel design framework, where teachers participated in the design and the studies, thus paving the way for the inclusion of TFV in existing classrooms, and broader institutional frameworks for learning. These results have wide implications and further development possibilities, for researchers, practitioners, and policymakers, across cognition, educational technology and digital media design. These implications and possible future work are discussed in Ch-3B.

PART-1: CONCEPTUALISATION

2 Ch-1A: Theoretical Background

Our study is broadly situated within theoretical discussions on the nature of learning abstract models, process models of learning, and the role of representational media in learning, particularly digital media. In this section, we briefly review the relevant literature and develop a theoretical framework for the thesis, which informs its research questions as well as their operationalisation.

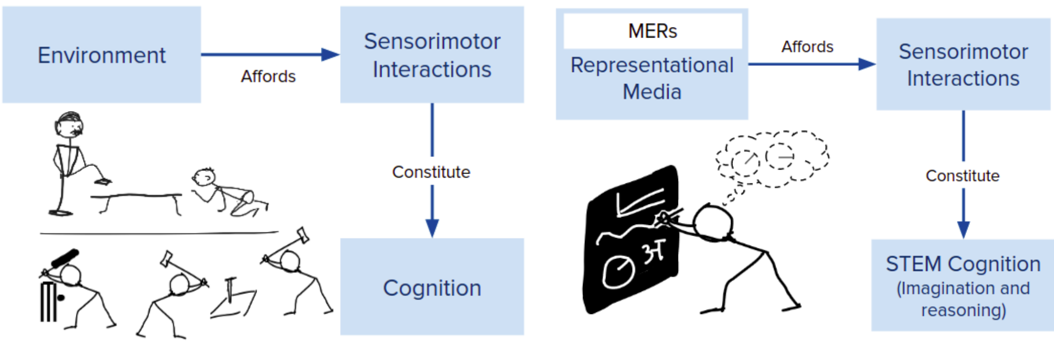
2.1 The STEM learning problem: lack of epistemic access to abstract models

Practising STEM requires engagement with abstract scientific-models. Models are essentially representations of structural relations between physical and conceptual entities (e.g. Hestenes, 2010; Suppe, 1989; Suppes, 1960; van Fraassen & Van Fraassen, 1980). Due to this abstract nature, learning in STEM involves cognitive and epistemic issues that are unlike learning other skills and topics, including language. A widely accepted requirement for a student to learn science is modelling — the ability to reason and imagine with abstract models, which involves making (constructing) and tweaking (manipulating to test and refine) model systems (e.g. Lehrer & Schauble, 2015; Ngss, 2013). Like Samantha, most students struggle to engage with (imagine and use) these abstract entities. When students lack epistemic access to these abstract models, they end up memorising and rote learning them. Thus a key question here is:

how can such abstract models become epistemically accessible (available[1]) and amenable to manipulation?

2.2 Embodied nature of cognition and Constitutivity hypothesis

Embodied and allied models of cognition, often together referred to as 4E models (e.g. Glenberg, 2010; Hutchins, 1995; Lave & Wenger, 1991; Sterelny, 2004; Thelen & Smith, 1996; Van Gelder, 1999), offer useful perspectives towards understanding this learning problem. The central claim of these models is the constitutivity hypothesis: *sensorimotor interactions between the body and the environment constitute*[2] *cognition*. These interactions in turn are based on the affordances (action possibilities) in the body-environment system. Per this view, cognition involving abstract symbolic processing, such as understanding of models, is also constituted by sensorimotor interactions (Glenberg & Kaschak, 2002; e.g. Landy & Goldstone, 2007). In line with this view, numerous studies show the embodied nature of symbol-based understanding, including language (e.g. Glenberg & Kaschak, 2002; Lakoff & Johnson, 1980) and mathematics (e.g. Abrahamson & Sánchez-García, 2016; Andres et al., 2008; de Freitas & Sinclair, 2012; Domahs et al., 2010).



(left)Fig-1A1: Constitutivity hypothesis of cognition; (right) Fig-1A2: The SCIARM Framework capturing the constitutivity hypothesis of cognition and the role of representational media.

2.3 The materiality of External Representations: A corollary and SCIARM framework

Abstract models are considered to be accessed using their multiple external representations (MERs) (e.g. Knuuttila, 2011; Morrison & Morgan, 1999). Different MERs (the symbolic entities), which are concrete inscriptions on representational media (e.g. paper and pencil, computer etc), have different affordances, and they ‘allow us to think the previously

[1]This usage is along the lines of epistemic availability (O’Donovan-Anderson, 1997) used in philosophical explorations on how a cognitive being *knows* anything; in other words, how does anything become epistemically available to the *knower*?

[2]According to this view, cognition ‘comes into being’ through sensori-motor interactions between the body and the environment, which are coupled to each other. This is in contrast to the notion of cognition as a centralised process independent of the body, the environment. We extend the notion of environment to include MERs (Pande & Chandrasekharan, 2017) by emphasizing the concreteness of MERs as inscriptions on representational media, as discussed in section 2.3.

unthinkable' (Kirsh, 2010). This view suggests that abstract models can become epistemically accessible by rich and systematic interactions with MERs, when the MERs are encoded using media that provide affordances for such interactions. Extending the constitutivity hypothesis, this view considers the nature of thinking and reasoning to be linked to sensorimotor interactions, which in turn are based on the concrete interactive affordances made possible by media form-factors. For example, diagrams have different affordances for reasoning and modelling (e.g. Dörfler, 2005), from plain text or algebraic equations. Supporting this view, studies, focussing on concrete artefacts, show that reasoning approaches are dependent on the nature of the manipulatives used to learn them (e.g. Fractions: Martin & Schwartz, 2005; Area: Rahaman et al., 2017). These observations are consistent with the proposed role of sensorimotor interactions in learning abstract models, and strengthen the embodied cognition approach towards understanding the process of learning abstract models.

The above theoretical insight, about the nature of cognition and the nature of learning abstract STEM models, help us frame the relationship between media and STEM cognition/learning (see Fig.1A2). Extending the constitutivity hypothesis of cognition, the following corollary emerges: "The action possibilities (interactive affordances: IA) of the media in which the MERs are encoded (text, computation; representational media: RM) have an effect on the understanding and processing of concepts (STEM Cognition: SC)." We refer to this theoretical position as the SCIARM (STEM Cognition and Interactive Affordances of Representational Media) framework in this thesis. The central objective of the thesis, thus, is to systematically test this corollary, and by extension the constitutivity hypothesis. This framing, thus, gives us the first research question: "RQ1: Do the interactive affordances of representational media in STEM shape learners' STEM cognition?"

2.4 Role of media in STEM cognition and learning: A review of Educational Technology Research and Design

Further, as the operationalisation of the investigation to test the above corollary involves the design of a digital media interface (as described in detail in the section-3 of ch-1B), we review the literature in research in media and learning, especially using technology, and attempt to identify key trends and insights and, gaps in the current understanding of the relation between media and learning.

Educational researchers have long noted the potential of digital media to develop novel MERs and thus improve learning (e.g. Kaput et al., 2002; Papert, 1980; Tall, 2000). Many designs have explored the computational, multimedia and communication affordances of digital media in learning contexts. However, the dominant view of technology was that they are merely 'vehicles' for delivering content (e.g. Clark, 2012; Ross et al., 2010). This view led to the interactive affordances of such media, and their role in learning, not being explored. There have been intense debates on the efficacy of media in learning in general (e.g. Clark, 1994; Kozma, 1994). The use of software seems to affect learning in unpredictable ways, and learning appears to be sensitive to small changes in technology (Hoyles & Noss, 2003). These debates indicate that systematic research, grounded in cognitive and social processes of knowledge construction, is warranted, to better understand the role of media technology in learning (e.g. Clark, 1994; Kozma, 1994; Reiser & Dempsey, 2012; Ross et al., 2010). Our work is situated within such emerging explorations, particularly examining the interactive (sensorimotor/ action-perception) affordances of media, informed by recent models of cognition. Examples with similar explorations include the Mathematical Imagery Trainer (Howison et al., 2011), Graspable Math (Ottmar et al., 2015) and TouchCounts (Sinclair & Heyd-Metzuyanim, 2014). We extend these explorations with a specific focus on the affordances of representational media for interactions with MERs.

The SCIARM framework, which is informed by recent cognitive and social theories, provides us with a strong base to investigate the relation between representational media and learning. Further, this framework also provides us a structure to explore the potential of technology in improving learning of abstract models in developing country contexts. Most explorations of educational technology in such contexts have focused on expanding physical access to learning material (driven by the technology-as-vehicle perspective), rather than the cognitive augmentation potential of technology, which is mostly underexplored.

3 Ch-1B Operationalisation using the Topic of Vectors

3.1 Research Questions and Approach

Given the gap in our understanding of the role of representational media in cognition and learning, and in line with the above explorations, we ask the following questions.

1. Do the interactive affordances of representational media shape learners' STEM cognition?
2. What does a systematic design of a media-intervention look like in a developing nation context?

To address these questions, we first examine the existing medium and its effects on STEM cognition, and then test for changes in students' cognition with controlled changes in the medium. We operationalise these investigations in the context of a specific topic, vectors, due to its suitability to examine imagination, modelling applications, and importantly the usage of geometric and algebraic MERs. We characterise the states of students' STEM cognition, as patterns underlying their utterances and behaviour, exhibited during the use of different MERs.

We term the behaviour exhibited as various concrete utterances (words, writings, diagrams, gestures etc.), in the contexts of conceptual-reasoning in STEM, conceptual reasoning behaviour (CRB). In this analysis, patterns of usage of MERs, based on these utterances, are taken to be indicative of patterns of students' ability to reason with models, and the coherence of their internal models.

3.2 The Vector Learning Problem

Understanding higher-level physics and engineering requires understanding the topic of vectors, which is taught at the pre-university level in India. Student difficulties in mechanics (Flores et al., 2004; Shaffer & McDermott, 2005; White, 1983) as well as electricity and magnetism (Pepper et al., 2012) have been attributed to the lack of understanding of vectors. Numerous studies in physics education indicate the struggles students face while learning this topic, from introductory levels of physics to postgraduate levels (Usharani & Meera, 2018). Mathematics education studies (as linear algebra) also indicate similar difficulties (e.g. Dorier & Sierpinska, 2002; Dreyfus et al., 1998; Harel, 1989; Hillel, 2000). Students struggle with reconciling their intuitions about physical contexts and judgements about resultant vectors (Aguirre, 1988; Aguirre & Erickson, 1984). Studies on formal vector operations (without explicit physical contexts) report students' difficulties to add and subtract vectors graphically (Flores et al., 2004; Nguyen & Meltzer, 2003). Other studies confirm struggles with the directionality of a vector, and report difficulties with graphical/geometrical methods, as against the use of rectangular components for addition (e.g. Barniol & Zavala, 2014; Heckler & Scaife, 2015; Knight, 1995; Shaffer & McDermott, 2005; Usharani & Meera, 2018; Wutchana & Emarat, 2011). Students lack coherent models inter-linking concepts (Dorier et al., 2000) to imagine and reason with vectors, and rely on mechanical manipulations using the $i^j^k^$ components. Broadly, students appear to have difficulty in making sense of the geometrical aspect of vectors, and integrating it with algebraic aspects of vectors (Bollen et al., 2017; Liu & Kottegoda, 2019). Overall, students have poor geometry-algebra integration.

3.3 Operationalisation of the study

The above two research problems were operationalised into four studies, involving two design iterations, and a study of changes in teaching-learning practices in classrooms (Fig.1B1). This design allowed examining further sub-questions under each major question. These sub-questions (in italics) are listed below.

1 Do the interactive affordances of representational media in STEM (illustrated using the topic of vectors) shape learners' STEM cognition?

1.1 What is the current state of representational media and its manifestations in classrooms (in the topic *of vectors)?*

1.2 What is the current state of students' CRB (for the topic of vectors)?

1.3 Is there a relation between the existing representational medium usage and students' CRB?

1.4 Would a change in the interactive affordances of the representational medium change students' CRB?

1.5 What are the larger effects on students' CRB when lessons are taught using the changed representational medium?

2 What does a systematic design of media-intervention look like in a developing nation context?

2.1 What design considerations need to be addressed, to ensure that teachers adopt the system?

2.2 What changes do we find in student and teacher practices in classrooms, when the new medium is introduced?

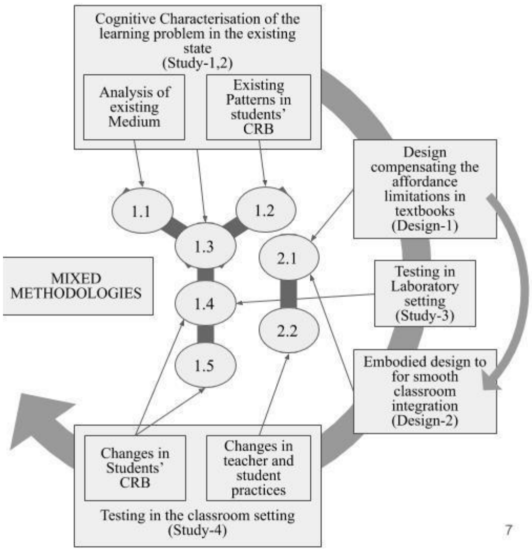


Fig-1B1: Operationalisation of the research problem in the thesis as 4 studies with 2 design iterations. (These sub-questions will be referred as RQ-x.y in the rest of the document)

The studies started with a detailed analysis of the paper-based textbooks, the existing media in classrooms (Ch2A: study-1), and students' existing conceptual-reasoning behaviour (Ch2B: study-2). These two studies address RQ-1 (1.1 and 1.2), by characterising the current state of the representational medium and its manifestations in classrooms (in the topic of vectors), and the current state of the student's CRB (for the topic of vectors), respectively. Findings from these two studies together provided an initial picture of the relation between the interactive affordances of existing media and students' existing CRB, addressing RQ- 1.3.

The findings from these two studies, along with the cognitive models under the SCIARM framework, guided the design of a new-media intervention, across two iterations (Ch2C: design-1 and Ch2E: design-2). Design-1 was implemented in a laboratory setting (Ch2D: study-3), using a small-scale pre-post study protocol. Design-2 was then implemented in classroom settings (Ch2F: study-4), using a large scale control-experimental study protocol. The findings from studies 3 and 4 helped address sub-questions under RQ-1 (RQ-1.4,1.5), showing the nuances of how interactions with designs 1 and 2 changed students' CRB. Design-2, and the practice changes that followed from it in classroom settings (Ch2G), illustrated an effective compensatory design approach, in a developing country context. These data, along with the design iterations, address RQ-2. A summary of this structure is provided in Fig.1B1.

PART-2: EXECUTION

3 Ch-2A - Study-1: Analysis of Existing Medium [JP1, FP4]

The objective of this study was to capture patterns in the way content was presented using paper-based media. This analysis was done through a detailed textbook-analysis and brief interaction with the teachers.

3.1 Textbook Analysis

We analysed grades 8-12 mathematics and science/physics textbooks based on two Indian curricula, examining patterns in the way they treated all the topics related to vectors (including prerequisites like the geometry of lines, triangles and circles, trigonometry etc). We categorised these topics into 23 units of analysis, as listed in Table-2A1, and coded the presentation of the relevant interlinks between all these units (CCLs, concept-concept links) using an 8-character scheme. We coded for the mode (explanatory or problem solving) and rigour (rated on a 5-point scale) of presentation. The coding scheme and a sample coding sheet are provided in App.2A1 and App.2A2. We looked for patterns in each of the topics that were presented, across the modes and rigour of presentation, in both the curricula. Further, we also did a quick survey of some standard physics textbooks used by students in India.

Broad category	Units of Analysis	
Definition	Direction	Magnitude
Resolution	Rectangular Components	Unit Vectors
	Non-rectangular Components	
Addition	Triangle Law	Parallelogram law
	Polygon Law	Algebraic addition
Application in Mechanics	Rotation of Frame of Reference	Resolved Forces
	Resultant Forces	Inclined Plane
Pre-requisites and related topics	Properties of Angles	Trigonometric Ratios
	Unit Circle	Polar Coordinates
	3D components	Trigonometric applications
Scalar Product	Geometric Interpretation	Algebraic Interpretation
Vector Product	Geometric Interpretation	Algebraic Interpretation

Table 2A1. The units that are used to analyse the textbooks

3.2 Teacher Interactions

Semi-structured interviews were conducted with 4 grade-11 teachers (1 math and 3 physics) with varied teaching experience. The questions were semi-structured and focused on the gaps and problems teachers found when teaching the topic of vectors, particularly the issues identified in the textbook analysis. Besides generic conceptual and procedural issues (e.g. determinants in vector products), the teachers made some specific comments about their limitations in drawing diagrams multiple times. They faced specific difficulties demonstrating changes in geometry. For instance, one teacher says -

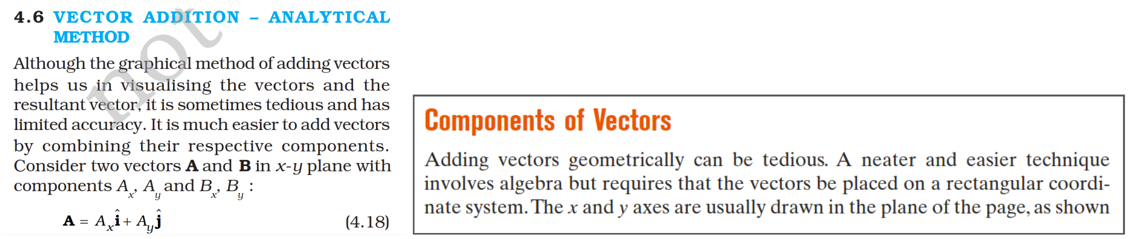
> *"...there is a constraint in visualising — the way it (content in general) is presented. Because it is verbal, no? So, from those words only you have to understand. And even if you put it in the form of a picture, it is in 2D. It is not moving in time. So it is difficult for them. Like the superposition of waves... there are students who have asked - "ma'am, in the stationary waves, does the wire split into two?" [Link]*

Further, it was pointed out that there were limited opportunities for interactions between physics and mathematics teachers, due to the tight timelines for syllabus completion etc. This made developing integrated learning experiences difficult.

3.3 Findings

After analysis of the coded patterns and qualitative aspects (described in greater detail in the thesis), some key findings emerged related to the limitations of paper-based textbooks.

3.3.1 Lack of geometric manipulation


4.6 VECTOR ADDITION – ANALYTICAL METHOD

Although the graphical method of adding vectors helps us in visualising the vectors and the resultant vector, it is sometimes tedious and has limited accuracy. It is much easier to add vectors by combining their respective components. Consider two vectors **A** and **B** in x-y plane with components A_x, A_y and B_x, B_y :

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} \quad (4.18)$$

Components of Vectors

Adding vectors geometrically can be tedious. A neater and easier technique involves algebra but requires that the vectors be placed on a rectangular coordinate system. The x and y axes are usually drawn in the plane of the page, as shown

Fig-2A1. Excerpts from (left) NCERT textbook, (right) another textbook (Halliday et al., 2013)

A key pattern found was that the curricula offered little scope for exploring geometric ways of adding vectors. For example, the triangle law of addition of vectors is presented only in the explanatory mode. There was little scope for applying it in problem-solving. Rectangular components are widely used for adding the vectors, and the textbooks explicitly justify the usage of rectangular components, ascribing limited accuracy and tedium to adding using geometric methods (See excerpts in fig.2A1). We interpret this view as deriving from the limited affordances provided by the paper-based medium to manipulate geometric aspects of vectors. This view is supported by teachers' comments.

3.3.2 Serial ordering

Secondly, the paper-based medium does not lend itself to presenting content in an integrated manner. Textbooks present content in a modular and distributed manner, across at least 8 textbooks. The effect of this limitation is worsened by the lack of explicit interlinks between these modules, as indicated by the link analysis (more details in the thesis), as well as the limited opportunity for teachers across subjects and levels to interact with each other.

3.3.3 Opaque problem-solving

Explanations, or any kind of heuristics, for applying vectors are missing from textbooks as evident from the patterns in modes of content presentation. For example, in the case of a mass on an inclined plane, the student is expected to pick from standard patterns or templates, and make decisions about the choice of: 1) the mass for which the free body diagram is made, 2) the explicit steps of making a free body diagram and, 3) the choice of frame of reference to resolve the forces. This opacity of procedures is reflected in teachers' remarks on the difficulties that students face in making sense of the determinant method of finding vector products, as well as the use of trigonometric ratio in resolution. These issues can be traced to the limitations set by the paper-based medium, particularly its inability to support the exploration of a range of problem solving approaches.

3.4 Chapter-Discussion

Broadly, the above findings indicate the limitations of paper-based media in presenting content. As the textbooks anchor most teaching-learning practices in India, where the educational system is often considered a textbook-culture (Kumar, 1988), these limitations can in turn limit students' conceptual reasoning behaviour, as the lack of affordances of the textbook media limit the sensorimotor interactions students can develop with the media and the content. The next study examined this possibility.

4 Ch-2B - Study-2: Existing Student Conceptual Reasoning Behaviour [JP2]

After the analysis of the way content is treated using the paper-based medium, we interacted with two groups of students: typical students (TS; from a typical city-school in Mumbai) and olympiad students (OS; shortlisted through a highly competitive national-level selection test to represent India in the International Olympiad on Astronomy and Astrophysics). The objective was to characterise the conceptual-reasoning behavior (CRB) of students in the topic of vectors. The rationale to probe these two groups was to capture common patterns in CRB among students with widely different scholastic abilities. Each group was given a written questionnaire, with questions that required them to apply vector knowledge (See App.2B1 and 2B2). Sub-groups 6 TSs (from 49) and 5 OSs (from 26) were selected as representative samples and interviewed (semi-structured) based on their questionnaire-responses and some prompts.

Students' written responses, video recordings and written material generated during their interviews were subjected to a thematic analysis. The focus of this analysis was to find recurring patterns in students' conceptual-reasoning-behaviour associated with vectors. The videos were analysed after meshing them with corresponding written material, thus generating coherent episodes. The diagrams, equations and gestures employed by the students while answering the questions were carefully examined. The episodes identified from the video and written data were iteratively organised into themes, from which certain recurrent patterns emerged. Three broad themes emerged while categorising these episodes of behaviour.

- Reasoning approaches with algebraic dominance
- Limitations in interconnections between conceptual aspects
- Limitations in problem-solving approaches

Each of the reported episodes were not observed in the case of every student, but collectively, the episodes cover the entire range of patterns found among the students. Here we provide brief summaries. Detailed episodes are presented in the thesis. (Also see App.2B4 for a summary)

4.1 Theme-1: Reasoning approaches with Algebraic Dominance

4 indicators of students' reasoning patterns were categorised under this theme.

4.1.1 Indicator-1: Reliance on memorised-formulae and algebraic manipulations

In their written scripts, students were found to rely a lot on memorised formulae and algebraic manipulations, even in situations where reasoning using geometry was essential or easier. Statements like "I forgot the formula...", indicating memorisation of formulae, were used by most students, across questions.

4.1.2 Indicator-2: Treating vectors as scalars (ignoring directional aspects)

Consistent with the literature, we observed students treating vectors akin to scalars, ignoring the directionality (e.g. writing $|\vec{A}|+|\vec{B}|=|\vec{C}|$ as A+B=C and confusing with the vector equation $\vec{A} + \vec{B} = \vec{C}$). Most students were comfortable with addition using rectangular components (adding like-terms) akin to the algorithm used in scalar algebra. They often ignored the unit vectors and added rectangular components of vectors like scalars.

4.1.3 Indicator-3: Preference for algebraic explanations during interviews

Similar to indicator-1, we observed that even during interviews, algebraic reasoning dominated students' reasoning-behaviour. Students tended to derive or prove using algebraic means to explain their reasoning, even when geometric means were essential and easier (e.g. to explain triangle law of addition, they derive the expression for the magnitude and use algebraic means to explain equivalence of algebraic and geometric addition).

4.1.4 Indicator-4: Underlying algebraic influences in talk

We found deeper evidence for lack of geometric reasoning, and complete dependence on algebraic reasoning, in students' utterances during interviews. For instance, the underlying algebraic influences can be inferred when students confuse terms 'rms' for 'resultant' (they share similar algebraic expressions with '√', the square root), related gestures (e.g. gesturing a '√' symbol in the air when discussing resultant), and the use of algebraic expressions and their verbalisations as actual definitions (e.g. defining dot product as sum of the products of corresponding components etc).

4.2 Theme-2: Limitations in interconnections between conceptual units

Discussions with students across both TS and OS groups indicated numerous weak spots in their understanding, and fragile interconnections between conceptual units.

4.2.1 Addition

Students had limited understanding of the geometric or physical meaning of vector addition. They often drew triangles and parallelograms with improper directions. They were unclear about the equivalence of addition using different algebraic and geometric methods. Largely, the responses were in confirmation with the findings of Nguyen and Meltzer (2003).

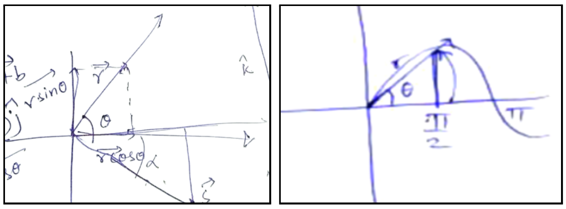


Figure 2B1. Students showing the connection between resolution and trigonometric ratios

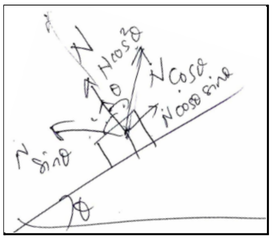


Figure 2B2. Repetitive resolution leading to the components of components paradox

4.2.2 Resolution and trigonometric ratios

Students had difficulty explaining the process of resolution using simple constructions and trigonometric ratios (fig.2B1 left). A key issue was reconciling the trigonometric ratios in a right triangle projected on the axes and a unit circle with the sinusoidal wave. For example, one student tried to construct a right triangle within the sinusoidal curve (fig.2B1 right), and then was clueless how to proceed.

4.2.3 Importance of rectangular components

When vectors are resolved in mutually orthogonal components, like rectangular components, the addition along a given direction becomes similar to scalars. The idea that a vector can be resolved into non-rectangular components as well is crucial to understanding the reverse nature of the addition and resolution processes. Students had difficulties understanding the general notion of a unit vector, their role in adding vectors, and the basis of addition using $i^j^k^$ components.

4.2.4 Components of components

Except for 2 OSs, none of the students could explain the component-of-component paradox (see fig-2B2 and App.2B3). The fact that the resultant is replaced by the rectangular components once you resolve them is at the root of this paradox. Resolving this paradox requires an understanding of the process of resolution, and the relation between the components and the initial unresolved vector.

4.2.5 Products of vectors

Students struggled with imagining vector products, and they did not have an understanding beyond the algebraic calculation, defining products using simple expressions of $abcos\theta$ and $absin\theta$. They gave explanations like 'use a vector product if the answer needs to be a vector and scalar product if it needs to be a scalar'.

4.3 Theme-3: Limitations in Problem-solving approaches

Along with indicators of algebraic dominance, the study indicated certain patterns in students' problem-solving practices.

4.3.1 Superficial and procedural description

When asked to explain addition of vectors, some students (like OS4) explained the procedure of drawing the triangle or the parallelogram. However, beyond the procedures of implementing the geometrical rules, the students did not have a coherent conceptual understanding of these relations.

4.3.2 Attempts to map to standard algorithms of problem-solving

In simple yet non-standard tasks, students attempted to match the given task with standard algorithms/ steps. They preferred to rely on this approach even at the cost of losing mathematical coherence.

4.4 Chapter-Discussion

The above findings confirm the findings from an extensive literature — both within and beyond the topic of vectors — on rote-learning and algorithmic problem-solving, and emphasize the need for spatial reasoning (geometric reasoning in our context) in students. The bias towards algebra in students' reasoning-behaviour is not problematic as long as geometric understanding is also available, and both are strongly integrated. However, the literature systematically documents students' struggle with geometric aspects of vectors and poor geometry-algebra integration. These reports, along with the evidence across the themes from our study, point towards a clear pattern in student reasoning-behaviour, which has a strong parallel to the way textbooks discuss vectors. To understand in more detail the relation between students' CRB and the role of textbook media in generating this CRB, we designed an alternate media presentation of vectors, and studied how this new media structure changed students' CRB.

5 Ch-2C: Touchy-Feely Vectors: Design Trajectory [JP1, FP3, FP6]

As the analysis of textbooks and inputs from teachers showed the limitations of static media in supporting vector related MERs, we designed a digital media system (Touchy-Feely Vectors, TFV) addressing these specific limitations. The key conceptual design requirements were: 1) allowing manipulation of geometric entities, and 2) real-time integration between geometric and algebraic representations (See App.2C1 for summary table). The central design objective was to make vectors epistemically accessible to students' like Samantha, who currently have no means to geometrically manipulate and imagine vectors. The design focused on addition and resolution of 2D vectors.

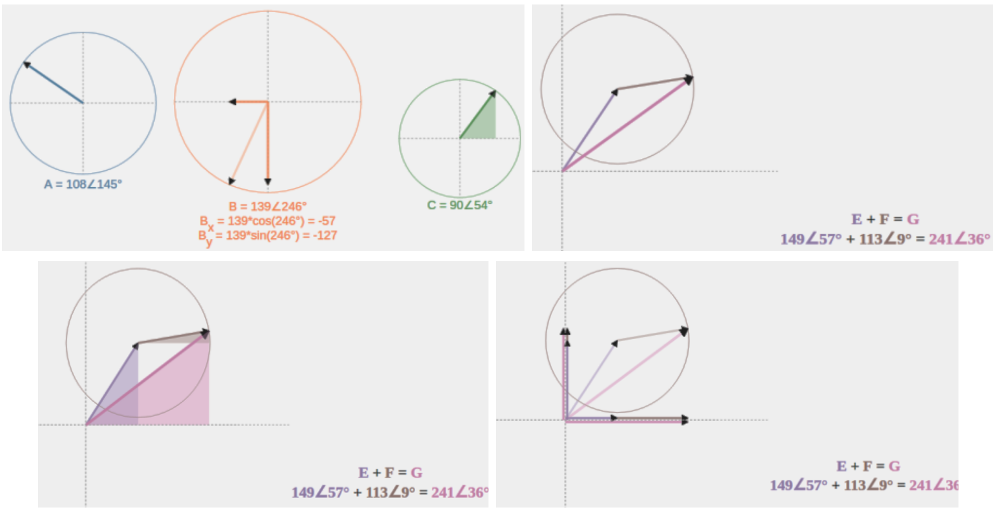


Figure 2C1. (a) Creating a vector, changing its direction and magnitude, and resolving it into rectangular components. (b) Addition of two vectors. (c) Right triangles superimposed. (d) Rectangular Components getting added. a → d (left-right from the top)

5.1 TFV-1

The initial version, built using Javascript, allows students to create, manipulate and operate (add and resolve) vectors from a browser, using a mouse-based interface. Clicking anywhere on the screen creates a vector (Fig-2C1a). The vector can be moved on the screen by right-clicking and dragging. The direction of a vector can be manipulated by click-hold and dragging the line of the vector. Its magnitude can be manipulated by click-hold and dragging inside the circle. The vector can be resolved using double-click (which displays the right triangle form, as seen in C in fig-2C1a) followed by a single click (leads to a resolved form similar to B in fig-2C1a). The related changes in algebraic equations are displayed simultaneously, capturing the relation between the direction and magnitude and the rectangular components. Moving back to the initial state of the vector requires just reversing the steps (single click to right triangle state, followed by a double click).

Addition of vectors is done by pressing Ctrl and then clicking on the two vectors to be added. This leads to addition using triangle law, as shown in fig-2C1(b). The active vector (one inscribed in the circle in fig-2C1b) can then be manipulated to see the effect on the resultant, using the same controls as earlier. The second vector can be made active for manipulation by right-clicking on the first vector and vice versa. Also, the magnitude and direction of the component vectors, along with those of the resultant, are provided as equations. This allows students to see geometrically the process of addition. Further, addition using the rectangular components can be performed using the same series of steps used for resolution. The system is available for interaction at bit.ly/tfv-1.

6 Ch-2D - Study-3: Laboratory Testing of TFV [JP1, FP3, FP6, AP3]

To pilot test the system, we built on our initial interactions with the 6 typical students in study-2, inviting them for extended interviews and tests of the system. After 4-5 days of interviews based on the earlier test, we invited them to interact with the system for 70-90 mins, where they did a set of tasks that allowed exploring the interactive affordances of TFV-1, applying their concepts related to vectors (App 2D4, 2D5, 2D6). After a week, the students were given a test similar to the one that they did earlier, to capture changes in their reasoning, as well as conceptual understanding. We video-recorded their interactions with TFV and also collected eye-tracking data (Tobii X2-60).

6.1 Analysis

Students' written scripts were analysed using a detailed framework that built on the one we used for text-book analysis. The 23 categories from the textbook analysis were refined to develop 16 concept-links related to the key aspects of addition and resolution supported by TFV (indicated by the arrows linking concept-nodes in fig.2D1). For analysis, the topics presented in Table-2A1 were categorised into three Broad Concept Areas (BCAs, represented by the dots/bold links in fig.2D1). These were then further classified into 5 sub-concept areas (SCAs, represented as 5 coloured boxes labelling the links: SCA1 - Triangle Law, SCA2 - Parallelogram Law, SCA3 - Rectangular components, SCA4 - Non-rectangular components, SCA5 - Application in the context of forces). The concept areas were constituted by 16 links between concepts (CLs). CL7 with only one CL was not considered as a separate SCA, and not used in this analysis. Table in App.2D1 provides a list of all these analysis categories.

Student responses were independently rated by 3 raters, who examined each question and the concept link that could be expressed in response. App.2D2 provides a snapshot of the rating sheet. Ratings were given to all relevant CL-question pairs. Only those CLs which could be expressed in response to a particular question (in pre or post-test) formed relevant CL-question pairs (all CLs could not be expressed in each question). A CL-question pair was deemed irrelevant if found irrelevant by at least 2 raters. A 5-point rating scale for conceptual understanding was developed for each of these concept links (1 = no indication, 5 = strong indication; see rubric in App.2D3). This structure follows studies examining levels of conceptual understanding, based on the analysis of reasons and judgments in test responses (e.g. Besterfield‐Sacre & Gerchak, 2004; Niemi, 1996). The scale does not measure correctness, but rates the conceptual clarity of that particular concept link, as expressed in the response to a given question. The mode (statistical) of the 3 ratings by raters was taken as the final score for each CL-question pair. For cases where all the 3 ratings varied, a consensus of at least 2 raters was arrived at through a discussion, thus ensuring that every rating has an agreement of at least 2 raters. These ratings were then converted into percentages, denoting the CL strengths (App.2D2) for each student. This analysis provided a comprehensive picture of each student's understanding, in the form of CL-strengths, pre and post-interaction with TFV-1.

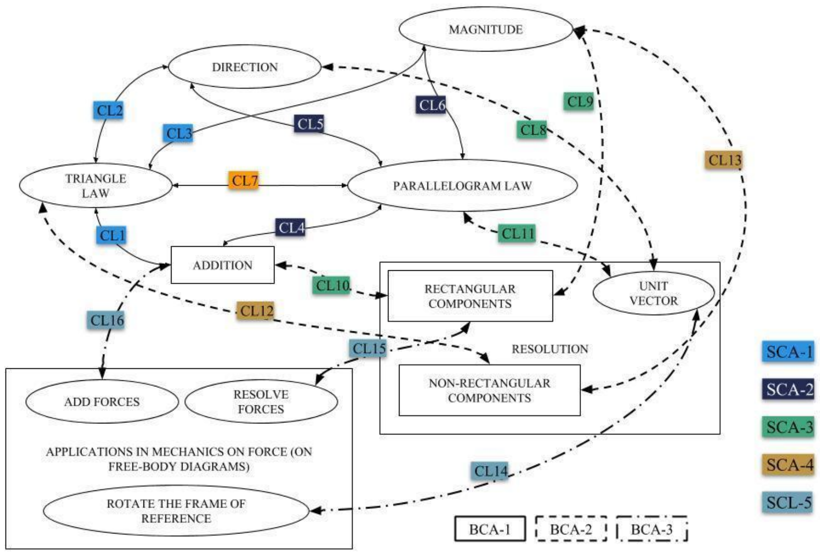


Figure 2D1. Analysis Framework with CLs (concept-links), SCA (sub-concept-areas) and BCA (broad-concept-areas) for analysing students' written scripts

6.2 Findings

For each student, the strengths of the 15 CLs were aggregated into strengths of 5 SCAs. Strengths at the SCA level gave more meaningful patterns than those at CL, which were too detailed. Fig.2D2 shows students' trajectories of strength changes in 5 SCAs. Each arc corresponds to the transition from pretest to post-test of one student. The size of the arc captures the magnitude of growth/fall in the strength of the links, in that particular SCA of that particular student. An arc above the strength axis denotes a growth from pretest to post-test and an arc below the axis denotes a fall.

Students' SCA strengths changed in different ways. At least 3 students showed growth in each of the SCAs, with more than 3 students improving in SCA1, SCA4 and SCA5. SCA1 (triangle law) improved for 4 students. This is not surprising, due to the emphasis on addition using triangle law in TFV-1. Even the 2 drops were only about 2-3% (smaller arcs). The parallelogram law (SAC2) is not explicitly expressed in the system, but 3 students improved in this. Surprisingly, only 3 students improved in the rectangular component (SCA3), even though rectangular components were part of the system. Five students showed growth in SCA4 (non-rectangular components), which is an under-explored topic in textbooks. All the three (S2, S5, S6) students, with weakened conceptual understanding in SCA3, showed growth in SCA4. SCA5 (applying vectors and vector operations in the context of forces) improved in all students. Most students whose performance dropped (7 out of 9 drops — arc below the axis) had pre-test percentages around 50-60%. A possible interpretation is that the system disrupted their existing concepts (possibly rote-learnt, as indicated in pre-test interviews).

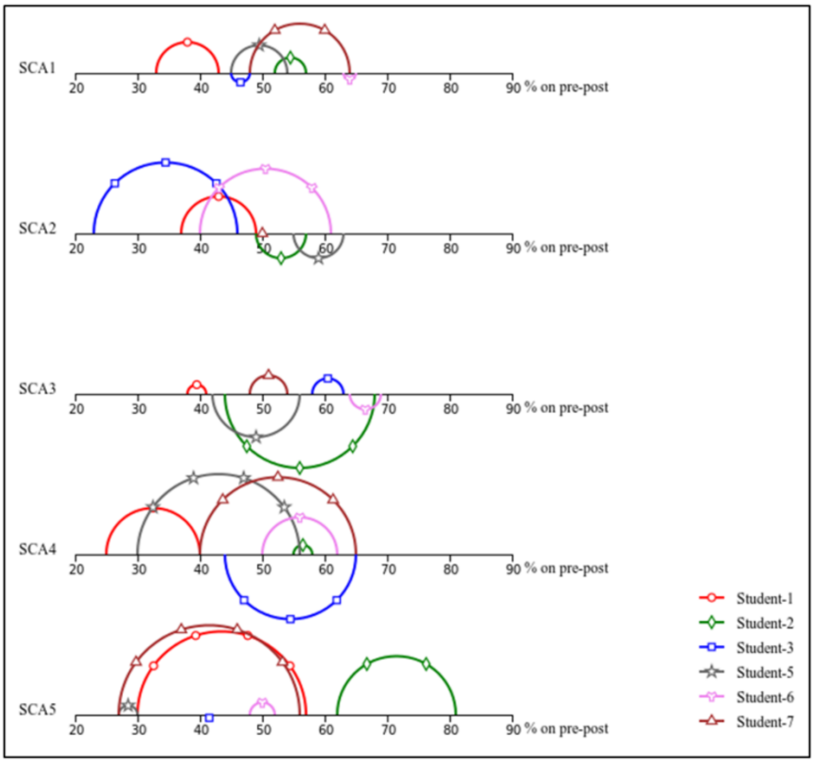


Figure-2D2. Arcs: Trajectories of change in strengths across SCAs of 6 students

Overall, this data suggests that interacting with TFV-1 compensated for students' understanding of vectors in two ways: 1) improving the understanding of addition using triangle law and non-rectangular components, and 2) disrupting their understanding of rectangular components. The case studies in the next section provide more insights into these quantitative observations.

6.3 Case Studies

This analysis captured overall changes in concepts. To understand the relationship between these changes and the process of interaction with TFV, we analysed in detail the cases of two students with contrasting patterns of interaction and learning trajectories.

For S2, who is considered a promising student by his teachers, the pretest responses indicated an algebraic dominance similar to findings from study-2 (Ch-2B). During the intervention, he tried doing the tasks entirely based on estimations driven by his conceptual understanding, and analytical methods using paper and pencil. We term these conceptually-guided interactions. He interacted with TFV-1 only when the tasks were challenging, where thinking with only paper and pencil was difficult. His post-test performances showed drops in SCA3 & SCA4 (related to rectangular and non-rectangular components). This indicates that the system disrupted his existing models of vectors, which were incomplete — as evident from pretest interviews. Further, in interviews after the post-test, we found episodes where he started gesturing, indicating dynamic and geometric imagination of vectors (fig.2D3). One of his key comments, that working with TFV was like experimentally proving theoretical aspects, indicates that TFV gave him a sense of tangibility of geometric representations. The use of algebra-based estimations and conceptually-guided interactions during the intervention, combined with episodes of gestures indicating spatial/ geometric reasoning, indicate better geometry-algebra integration, and triggering of spatial imagination. More interaction with the system would possibly help him settle some of the remaining conceptual issues.

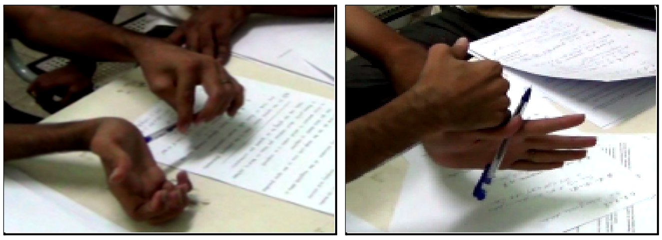


Figure-2D3. S2 using gestures to explain underlying geometric in the posttest interview

Unlike the case of S2, the case of S5 conclusively shows, based on pre-post test strength analysis as well as the interview responses, a growth in understanding of addition using triangle law and components of vectors (SCA1 and 4 respectively). Even with SCA3 (related to rectangular components), she explicitly notes that she can solve the given problem using the system, which indicates an enabling effect of the system. While there was limited clarity after the intervention, S5 was elated to see the resolution process, which allowed her to see explicitly the underlying dynamic process while arriving at rcosθ and rsinθ. This allowed her to have a better cognitive engagement with vectors and thus epistemic access. However, her post-test responses indicate that she struggled to transfer this understanding to text representations. The geometry transformations allowed her to see how the mathematical system works (unpacking the multiple states), and this, as Simon (1996) notes, could lead to an understanding different from the current one, which are based on static book-based representations. S5 considered this an exciting shift.

6.4 Chapter Discussion

In summary, the study showed that the interaction with TFV offered the students novel experiences with geometrical representations, which led to strengthened conceptual links for all the students, and disrupted models for good performers. The case studies showed details of the changed reasoning approach among traditional good-performers (illustrated by S2), and the gaining of better epistemic access to concepts, which were hitherto opaque, among average-performers (illustrated by S5).

The study also showed student limitations in transferring the new understanding to paper-based assessments. However, the indicators of specific effects are promising, and suggest that sustained interaction with careful designs such as TFV — based on the affordances of the digital medium, particularly dynamic real-time interaction with hidden states of formal representations — can trigger students' imagination, and shift their conceptual-reasoning-behaviour in the desired direction.

7 Ch-2E - Design-2: TFV-2 [JP1, FP1, FP2, FP5, AP4]

The next design iteration (TFV-2) is similar to TFV-1, but is based on a touch-based interface, which allows it to be used on any touch-enabled device (tablet, smartphone etc). This version can be accessed at bit.ly/tfv-2. A vector can be created using a simple gesture with double-consecutive taps (two taps one after another) on the touch screen (figure 2E1b). The vector can be moved on the screen by dragging the circle. This is in an inactive state. A long tap on the vector will make the vector active, i.e. making manipulation of the direction and the magnitude possible (fig 2E1d). The magnitude can be changed by holding the head of the vector and dragging (fig.2E1e). The direction can be changed by holding the line and rotating (fig 2E1f). A vector in its active state can be resolved using a pinching away action, leading to rectangular components. The process of resolution is presented using a short animation, mediated by the presence of shaded right triangles (fig 2E1g,h). The equations relating the magnitudes and directions are presented simultaneously, similar to TFV-1. Moving back to the initial state requires double-consecutive taps, similar to the creation of vectors (two taps one at the tail of the vector followed by a tap at the head of the vector; fig 2E1i).

Addition of vectors, using the triangle law, requires moving the second vector and placing its tail on the head of the first vector. The tail-head touch is confirmed by a faint circle at the contact (fig 2E2a). This provides feedback that the addition can now be performed. Tapping on the tail of the first vector, followed by the head of the second vector, creates the third side of the triangle, which is the resultant of the vector (2E2b,c). The equations related to the resultant vector are presented simultaneously.

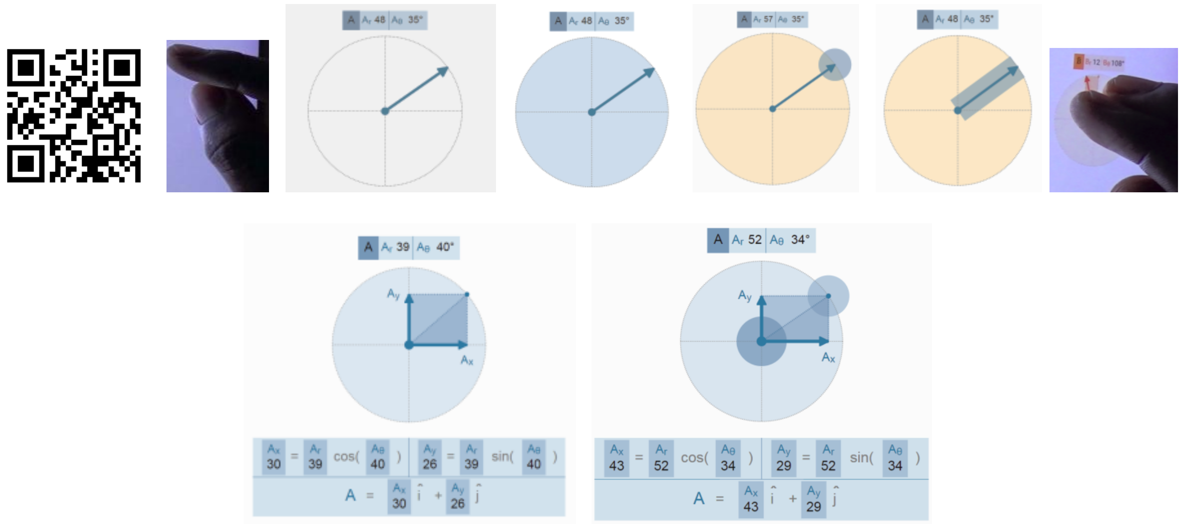


Figure 2E1. Gestures for creating a vector, and a vector (a,b,c); An active vector, changing magnitude, direction (d,e,f); Pinching away gesture for resolution and gesture for reversing the resolution of vector (g,h,i). Scan the QR code to interact with the simulation. a → i (left-right from the top)

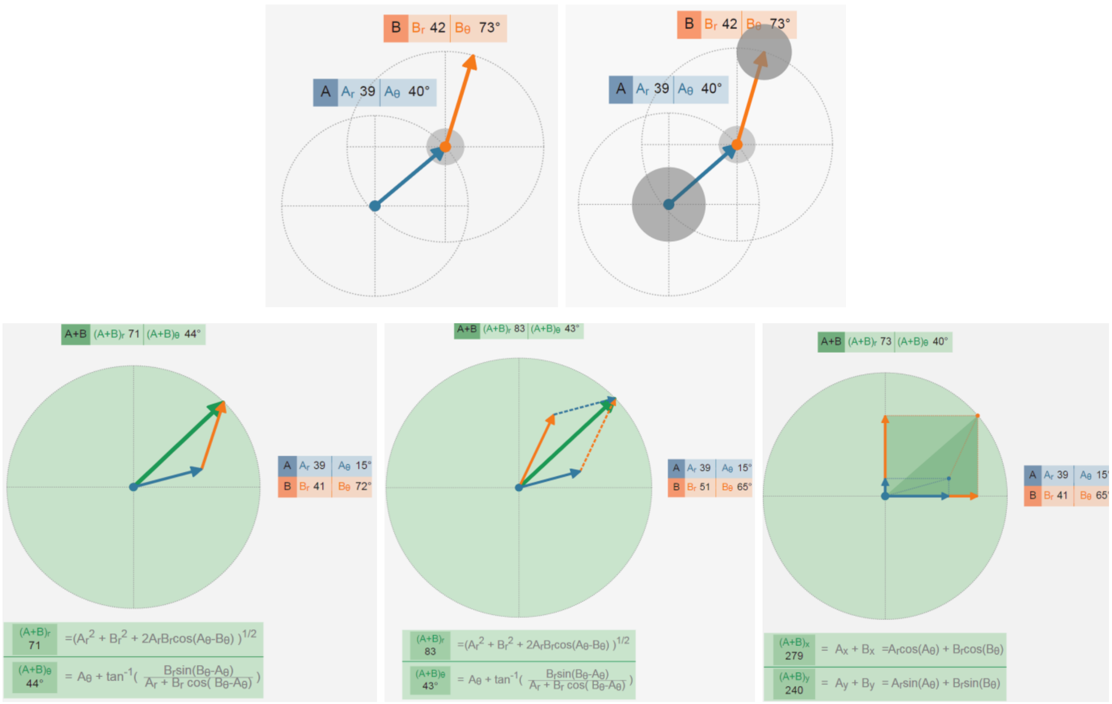


Figure 2E2. (a,b) Gesture for adding two vectors (top); (c) addition and manipulation in triangle law and, (d) parallelogram law. (e) Pinching away the resultant leads to addition using rectangular components. a → i (left to right from the top)

The parallelogram law implementation of the addition can be seen by long-pressing at the centre of the circle (fig.2E2d). The rectangular components version of addition can be seen by using the same pinching-away action used for resolution, on the resultant vector. A short animation shows the rectangular components emerging from the initial vectors, aligning themselves along the axes, thus leading to the rectangular components of the resultant vector. In both the modes (triangle and parallelogram laws, as well as rectangular components), the direction and magnitude of the initial vectors can be changed to see the effect on the resultant vector. One can move back to the triangle law mode from the parallelogram law mode, by long-pressing at the centre again; and from the rectangular component mode, by double-consecutive taps (one tap at the tail of the first vector or the resultant vector followed by a tap at the head of the second vector or the resultant vector).

Note that these operations extend the descriptions in the textbook considerably, making the dynamic behaviour and hidden states of the vector explicit and available for imagination. The visual language in the interface also builds on the representation style in the textbook, to retain familiarity and maintain consistency. App-2E3 outlines the key changes in interactions and visualisation from TFV-1 and TFV-2.

7.1 Virtual lesson plans and QR-code based integration with textbooks

TFV-2 was a learning system based on open-exploration, and teachers found it difficult to incorporate it into a timed classroom session. To address the integration with teachers' current textbook-based practice, particularly their thinking about vectors and sequencing of lessons, we decided to design tasks that could be planned for organised classroom activities. For this, we conducted a lesson planning workshop with four teachers. Based on this discussion, we developed 'virtual lesson plans' (VLP), which consisted of manipulation tasks in the TFV system that smoothly extended teachers' existing textbook-based lesson plans. The lesson plans included 4 modules with a set of tasks and a tentative duration of each module (App.2E1).

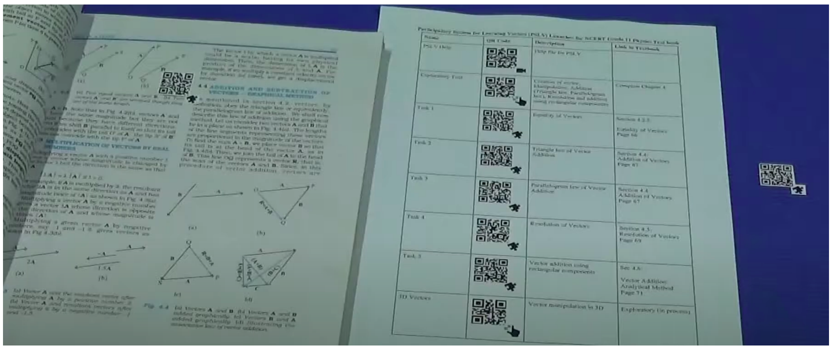


Figure 2E3. Tasks in VLP linked to QR-codes (App.2E2) which are printed, and cut and pasted in textbooks, to augment them

To integrate these lesson plans better with teachers' textbook-based classroom practice, we developed a QR-code based system that created an 'augmented textbook'. Each task in the virtual lesson was linked to a QR code label, and this label was printed and attached next to a textbook figure. The figure thus became manipulable, as scanning the QR codes took students to specific tasks, allowing them to manipulate and understand dynamically the teacher's discussion of the figure. To provide wider access to the imagination possibilities provided by the TFV system, the QR codes linked to lesson plans were compiled into a table, with QR codes in one column, and the related figures and textbook page numbers in another (See Fig.2E3, App.2E2 for freely downloadable pdf with QR codes). This makes the introduction of new artefacts into the classrooms smooth and less abrupt, as the system builds on already existing static artefacts (textbooks), and compensates for their limitations, instead of seeking to replace the textbook and related practices.

8 Ch-2F - Study-4: Classroom testing of TFV-2 [JP1, FP1, AP3]

Extending the findings from study-3, we did a larger field implementation of TFV. We deployed the system in 6 grade-11 classrooms (3 control - 135 students and 3 experimental - 131 students), with one control-experimental pair each in 3 schools (SC1, 2, 3). In the control group, teachers taught lessons on vectors (introduction, addition and resolution) in the Physics curriculum, using traditional paper-based methods (CGx). In the experimental classrooms, teachers used TFV-2 (EGx). See Table-2F1 for details of the study design. This study addresses both RQ1 and RQ2.

School (type, Curriculum)	Classroom	Teacher	No. of students
SC1 (Private, MH)	EG1	T1	63
	CG1	T2	53
SC2 (Public, MH)	EG2	T3	34
	CG2	T3 & T5*	41
SC3 (Public, NCERT)	EG3	T4	34
	CG3	T5 & T3*	41

*Table-2F1: Codes for schools and classrooms, and corresponding teachers. NCERT - national, and MH - a provincial (Maharashtra) curricula. * T3 & T5 taught in SC2 and SC3 due to inter-school teacher transfers during the study (could not be controlled).*

8.1 Methodology and Data collection protocol

The virtual lesson plans (VLPs) developed with teachers were implemented in the EG classrooms. Teachers used worksheets and tasks developed based on the VLP. About 25-30 6” Android tablets (SWIPE) — with the necessary files and QR reading applications pre-installed — were provided in the EG classrooms. Groups of 2-3 students worked on the tasks presented in the worksheet using a tablet during the lessons. The researchers’ roles were: 1) logistics related to the tablets, technical support (for the teacher as well as the students), and 2) recording the classroom proceedings. In CG classrooms, the teacher was given no inputs by the researchers. We video-recorded all the classroom sessions (5-8 35/40 minute teaching sessions in each CG and EG classroom). After about 1-3 weeks, students in all the 6 classrooms were given a questionnaire (sample questions in fig.2F2; entire questionnaire in App.2F1), which included questions that sought to capture students' conceptual understanding and reasoning approaches in the context of vectors. We also collected grade-10 (averaged math and science scores on common board examination) scores of 234 students for analysis (Fig.2F1).

8.2 Analysis and Findings

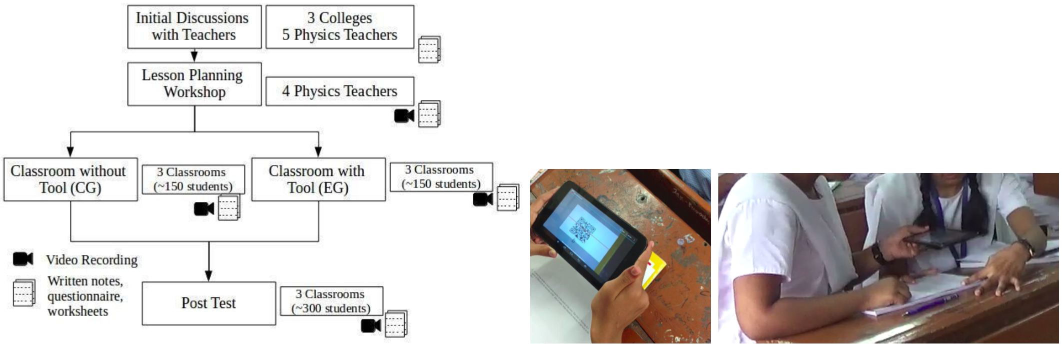


Figure 2F1. (left) The flow of the study and the sample. (right). Students scanning the QR code in an EG Classroom

Choose the correct answer (multiple answers allowed) and state the reason. (Q-6, Q-7 and Q-8)

Q-6. $\overline{P}$ and $\overline{Q}$ are two vectors at an angle of 45° with each other at their co-initial points are tails, and $|\overline{P} + \overline{Q}|$ (magnitude of their resultant ($\overline{P} + \overline{Q}$)) is 5 units. If magnitudes of $\overline{P}$ and $\overline{Q}$ are fixed and the angle between them is increased to 120°, which of the following could be the value of $|\overline{P} + \overline{Q}|$?

a) 3 units b) 5 units c) 8 units d) 9 units e) 2 units

Reason:

Q-7. If $\overline{P}$ and $\overline{Q}$ are two **collinear** vectors, and their resultant is $\overline{R}$. (**magnitudes** are represented by small letters p,q and r respectively)

1. If p = q , r = 0

a) Never b) Sometimes Possible c) Always true

Reason:

Figure 2F2. Sample questions used in the test (snapshot from the test)

8.2.1 Analysis Framework

We analysed the written responses of all the 266 students. For this, a rating scheme (App.2F2) was developed, first using a smaller sample, where 3 raters iteratively coded the material, until at least 2 out of 3 raters converged on identical ratings. This rating scheme was then finalised. The answer scripts were equally distributed among the 3 raters, ensuring equal distribution of students from all the 6 classrooms. The raters focused on the responses to questions (Q6, Q7 with 4 sub-questions, Q8 with 4 sub-questions, see app.2F1), which were relevant to the geometry-algebra integration. The rating scheme gave each student geometric and algebra scores, counting the number of instances (out of a total of 17 possible instances), and capturing the reasoning approaches. The markers for geometric reasoning were: diagrams etc. and geometric keywords like ‘collinear’, ‘lying on the same plane’, ‘parallel’ etc. The markers for algebraic reasoning were: calculations etc. and algebraic keywords like ‘solve’, ‘value’ and quantitative action words such as ‘subtract’, ‘substitute’ etc.

The rating data was collated and then analysed further, particularly to interpret the effects of the lessons with TFV. We focused on two aspects of students' reasoning behaviour with vectors: reasoning approaches with geometry-algebra integration, and cognitive engagement with the content. We analysed these patterns using two kinds of slicing of the data: (1) with all the 266 students (without incorporating grade-10 scores), and slicing them school-wise, and (2) with 234 students (with grade-10 scores), and slicing them bin wise (lower bin: below 85% as average-performers; upper bin: upper 85% good-performers). We found interesting patterns related to each of these effects, with students of different academic abilities. The following subsections provide details of the patterns found using this analysis, and their interpretations.

8.2.2 Effect on geometry-algebra integration

To understand the patterns in students' reasoning approaches between groups, we examined the difference in the distribution of geometric and algebraic scores between the groups. For each comparison between the control and the experimental groups, we present two plots (fig-2F3 and 2F4) and the results of KS tests[3] (Tables- 2F2, 2F3). Each geometry-algebra (x-y) plot has 3 embedded visualisations.

- Joint probability distributions functions (j-PDFs) were arrived at by smoothing discrete data using Gaussian kernels, using kernel density estimation[4]. The resulting j-PDFs (Fig-2F3 and 2F4) are normalised distributions for each group, and capture the probability of finding a student with a given algebra and geometry score (the shade of the colour). These j-PDFs (kernel density plots; KDE) allow us to jointly track the overall patterns in the distribution of algebraic and geometric reasoning, which provides a marker of geometry-algebra integration. In the plots, darker regions correspond to the mode of the distribution (higher probability of students being in that region). Any expansion along the x-axis indicates an improved tendency to reason using geometric entities. Similarly, expansion along the y-axis indicates algebra tendency. Expansion along the diagonal away from the origin (visually-guided by the faint diagonal split) indicates a better geometry-algebra integration, which is a desirable effect.
- The KDEs and histograms of algebra and geometry scores are individually plotted along the axes and can be seen as projections of the j-PDF on the respective axes.
- The scatter plot is overlaid on the j-PDF, with the size of the marker indicating the frequency of students with a particular geometry algebra score. As can be seen, the data is discrete, as the scatters take only integral values. The higher frequency markers can be seen to overlap with the darker regions of the j-PDF.

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[3]Establishing the difference between two KDEs is an ongoing research problem in statistics, and various approaches have been indicated in the literature, such as Maximum Mean Discrepancy (Alba Fernández et al., 2008; Anderson et al., 1994; Gretton et al., 2012), Distance-based tests (Sejdinovic et al., 2013), Kolmogorov–Smirnov (KS) Test (Pavia, 2015) and local significant differences for KDE (Duong, 2013). To measure the significance of the difference visually seen in the j-PDFs, we employed the 2-sample KS test, which is fairly well understood and sufficient for our purposes. The KS-test quantifies the difference between two continuous PDFss, and is suitable for the goodness of fit with KDE (Pavia, 2015).

[4]A quick histogram of both these scores indicated that they are not normal distributions, and we needed to compare 2 bivariate samples. We employed kernel density estimation (KDE), a non-parametric method found useful to analyze smaller sample sizes (Anderson et al., 1994; Gretton et al., 2012; Li, 1996; Ramdas et al., 2017; Silverman, 2018). Kernel density estimation is not widely used in education research but is used in multivariate nonparametric testing in genetics (Saunders et al., 2017), and in chemical engineering (Liang, 2008) among others.

Geometry-Algebra Integration: School-wise slicing (N=266)

Control Group Experimental Group

Overall

SC1

SC2

SC3

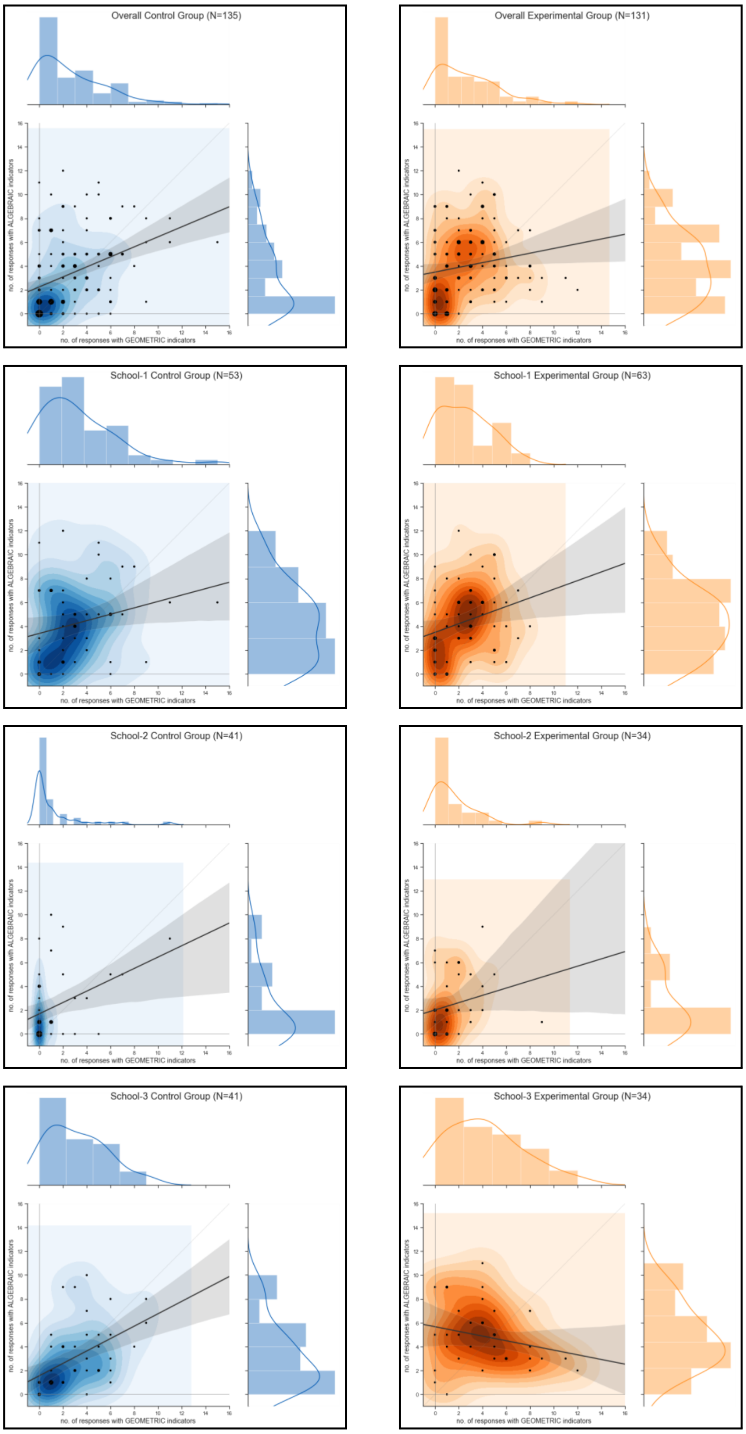

Fig. 2F3. School-wise slicing: Joint Probability Distribution plots of students' geometry vs algebra performance in the 3 schools (row-1 overall; rows-2,3,4: Schools-1,2,3 respectively).

Overall (Fig-2F3 row-1), there is a shift in the distribution, with a second mode emerging away from the origin. A school-wise comparison is more meaningful, given the direct comparability between the groups. In SC1 (Fig-2F3 row-2), the distribution does not seem to be much affected. However, a clear second and higher mode (darkest) away from the origin is seen in EG, which is a shift in the desired direction. In SC2 (Fig-2F3 row-3), the CG appears to be highly reliant on algebraic reasoning, and EG has a very slight effect in the desired direction. In SC3 (Fig-2F3 row-4), the mode for EG clearly shifts diagonally away from the origin significantly, indicating improved geometry-algebra integration. Also, expansion along the geometric axes indicates enhanced geometric reasoning. The KS tests (Table 2F2) confirm the visual interpretation, with the difference being significant overall, and for SC3 as well.

Fig. No.	**Without Grade-10**	**N(students)**		**Alg+Geo**		**Algebra**		**Geometry**	
		C	**E**	**D**	***P***	**D**	***P***	**D**	***P***
2F3.row-1	**Total**	135	131	0.139	*0.14*	0.201	*0.007**	0.059	*0.968*
2F3.row-2	**SC1**	53	63	0.105	*0.889*	0.111	*0.846*	0.131	*0.674*
2F3.row-3	**SC2**	41	34	0.144	*0.798*	0.197	*0.417*	0.232	*0.234*
2F3.row-4	**SC3**	41	34	0.364	*0.010**	0.360	*0.012**	0.149	*0.768*

*Table-2F2. Table showing the KS-test results for the above j-PDFs. D statistic (ranging 0-1), which gives the effect size and p indicates the 2 tailed p-value with 95% confidence. *p<0.05*

Given these interesting patterns in the school-wise data, we looked at effects on students with different academic abilities (Bin-wise slicing). We first sliced the sample into good (above 85%) and average (below 85%) performing groups. These samples were further examined using the jPDF (Fig-2F4). The overall bin-wise pattern (for 234 students) almost matches with the overall school-wise (for all 266 students) (Compare row-1 in Figs 2F3, 2F4). An interesting pattern was the differences in the distribution in the above 85% bin in comparison to the below 85% bin. In the above 85% bin (3rd row Fig-2F4) the mode in EG appears to shift very much away from the origin, in comparison to the CG. The distribution also spreads along the geometric axis. Whereas in the below 85% bin, the pattern does not appear to change much, except for a slight widening of the distribution. This shift, as indicated by the KS test results, again hints at an increasing algebraic representation, but contributing to better geometry-algebra integration in the upper bin (>= 85% grade 10 score). This integration does not appear to be significantly affected in the lower bin.

Furthermore, it is interesting to note that in both the slicings of data, the difference in the geometry-algebra integration is contributed significantly by algebra, not geometry. Based on the evidence from the written-scripts, we can claim an overall better geometry-algebra integration in EG: visually both in algebra and geometry from the plots (Figs 2F3, 2F4) and statistically significant only in algebra (from Table-2F2, Table-2F3).

This pattern indicates that geometry-algebra integration could happen through many complex trajectories. Establishing the nature of these trajectories, however, requires detailed investigations of student interactions with TFV (see the discussion in ch-3A), which is a future work project.

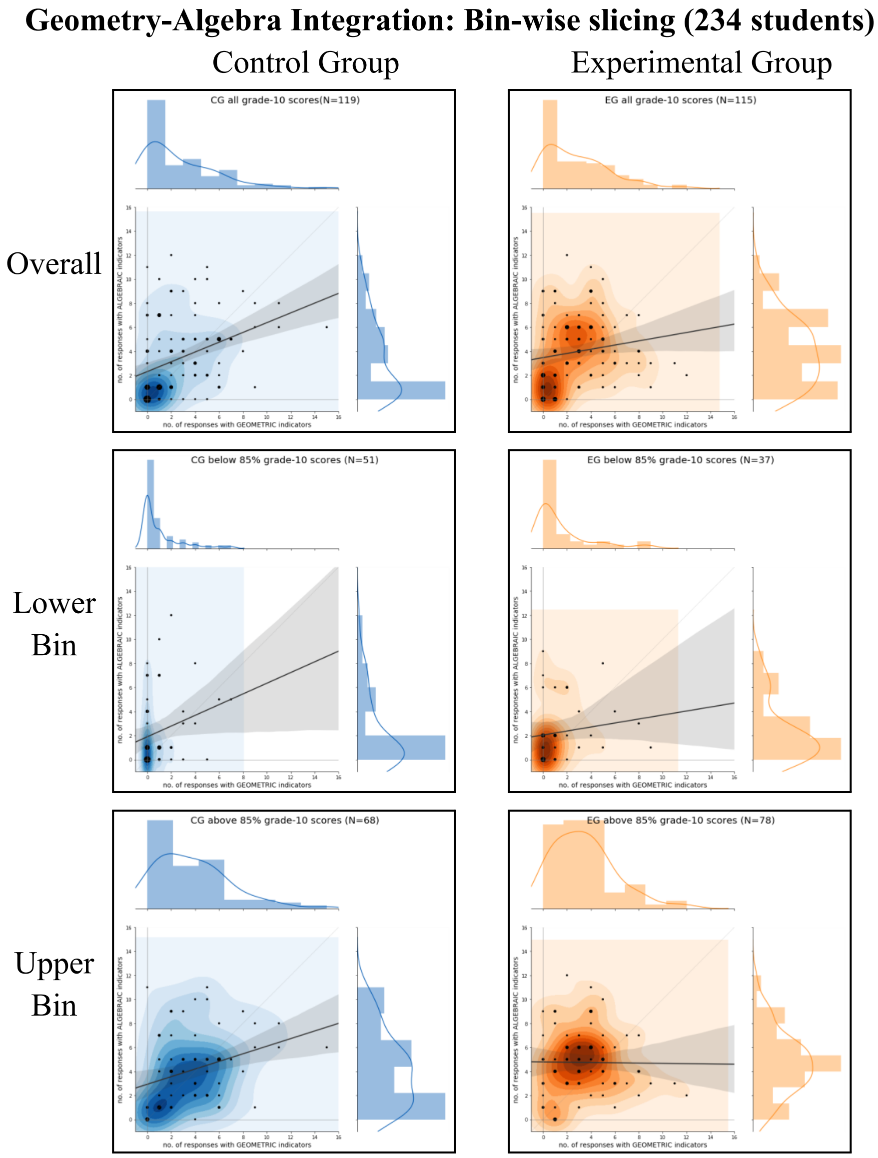

Figure 2F4. Bin-wise slicing: Joint Probability Distribution plots of students' geometry vs algebra performance, in students with <85% (lower bin Row-2) and >85% (Upper bin Row-3) grade-10 scores (row-1 overall)

Fig. No.	With Grade-10	N(students)		Alg+Geo		Algebra		Geometry	
		C	E	D	*P*	D	*P*	D	*P*
2F4.row-1	Total	119	115	0.141	*0.180*	0.202	*0.014**	0.079	*0.844*
2F4.row-2	Lower Bin	51	37	0.127	*0.86*	0.187	*0.398*	0.89	*0.99*
2F4.row-3	Upper Bin	68	78	0.108	*0.76*	0.202	*0.08*	0.09	*0.85*

*Table-2F3. Table showing the KS-test results for the above j-PDFs. *p<0.05*

8.2.3 Effect on cognitive engagement

We used two measures to capture students' cognitive engagement with the concepts: the proportion of students who responded using any of the modes (text, equations, drawings or keywords) to: (1) all-the-3 questions and (2) at-least-one question. These simple markers, measuring whether the students respond to questions, are interpreted as markers of students' engagement with the content. Their attempting a question is an indicator of them attaching some meaning to the content. As this may not necessarily be a correct meaning, it does not directly correlate with conceptual understanding. This response is thus similar to markers of academic engagement, such as completion of tasks in homework, participation, and attendance (Appleton et al., 2006, 2008; Christenson et al., 2012; Lamborn et al., 1992).

Cognitive Engagement: School-wise Slicing (N=266)

Overall School-wise Slicing

All-the-3 Questions

At-least-one Question

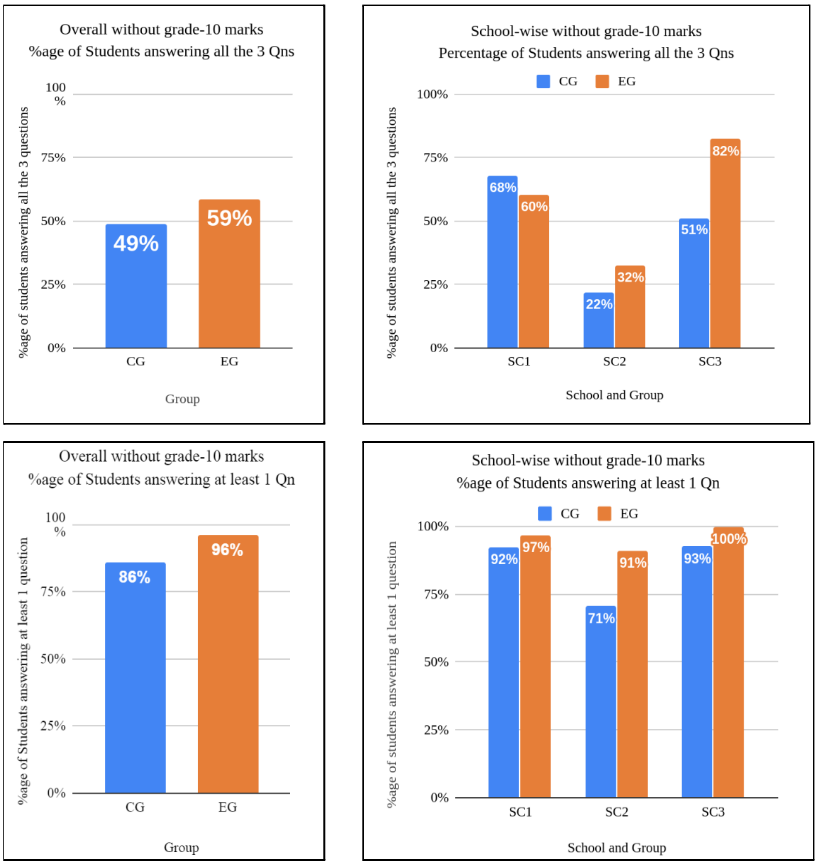

Fig 2F5. The proportion of students responding to (row-1) all-the-3 questions, and (row-2) at-least-one question, in the control and experimental group. Left figure: Proportion combining all the schools. Right figure: Proportions in individual schools.

Fig-2F5 shows the comparison with overall and school-wise slicing. Fig-2F6 shows bin-wise slicing. These patterns were tested for significance using the odds-ratio test (Tables-2F4, 2F5). An odds-ratio (r) indicates that it is *r* times more likely that an individual who completes the tests has used TFV-2, i.e. the student belongs to EG. An r greater than 1 is desirable. The greater the value of r, the greater is the likelihood of TFV-2 being effective. The statistical significance of this is provided by the Z statistic and the p-value at a 95% confidence interval (CI). The results show that a greater proportion of EG students have better cognitive engagement, in all categories in both measures, except for SC1 in the all-the-3 questions case (Fig-2F5). In agreement with this, the odds ratios are greater than 1 in all cases, except for SC1 in the all-the-3 case (Table-2F4). These effects are statistically significant in the all-the-3 case for SC3, and the at-least-one-question case for SC2.

	N(students)		All 3 Questions								
	C	E	C	E	C%	E%	Diff	Odds Ratio	Z-Stat	P-value	CI
Overall	135	131	66	77	48.9%	58.8%	9.9%	1.4907	1.615	*0.1064*	0.9181 to 2.4204
SC1	53	63	36	38	67.9%	60.3%	-7.6%	0.7178	0.848	*0.3965*	0.3335 to 1.5447
SC2	41	34	9	11	22.0%	32.4%	10.4%	1.7005	1.009	*0.3129*	0.6064 to 4.7684
SC3	41	34	21	28	51.2%	82.4%	31.1%	4.4444	2.723	*0.0065**	1.5191 to 13.0031
Without Grade-10 Scores			At least one question								
School-wise Analysis			C	E	C%	E%	Diff	Odds Ratio	Z-Stat	*P-value*	CI
Overall			116	126	85.9%	96.2%	10.3%	4.1276	2.732	*0.0063**	1.4930 to 11.4113
SC1			49	61	92.5%	96.8%	4.4%	2.4898	1.028	*0.3038*	0.4376 to 14.1648
SC2			29	31	70.7%	91.2%	20.4%	4.2759	2.09	*0.0366**	1.0944 to 16.7053
SC3			38	34	92.7%	100.0%	7.3%	6.2727	1.2	*0.2301*	0.3127 to 125.8218

*Table 2F4: School-wise Slicing: The odds ratio-test for the school-wise analysis (without using grade-10 scores). *p<0.05*

Cognitive Engagement: Bin-wise slicing (N=234)

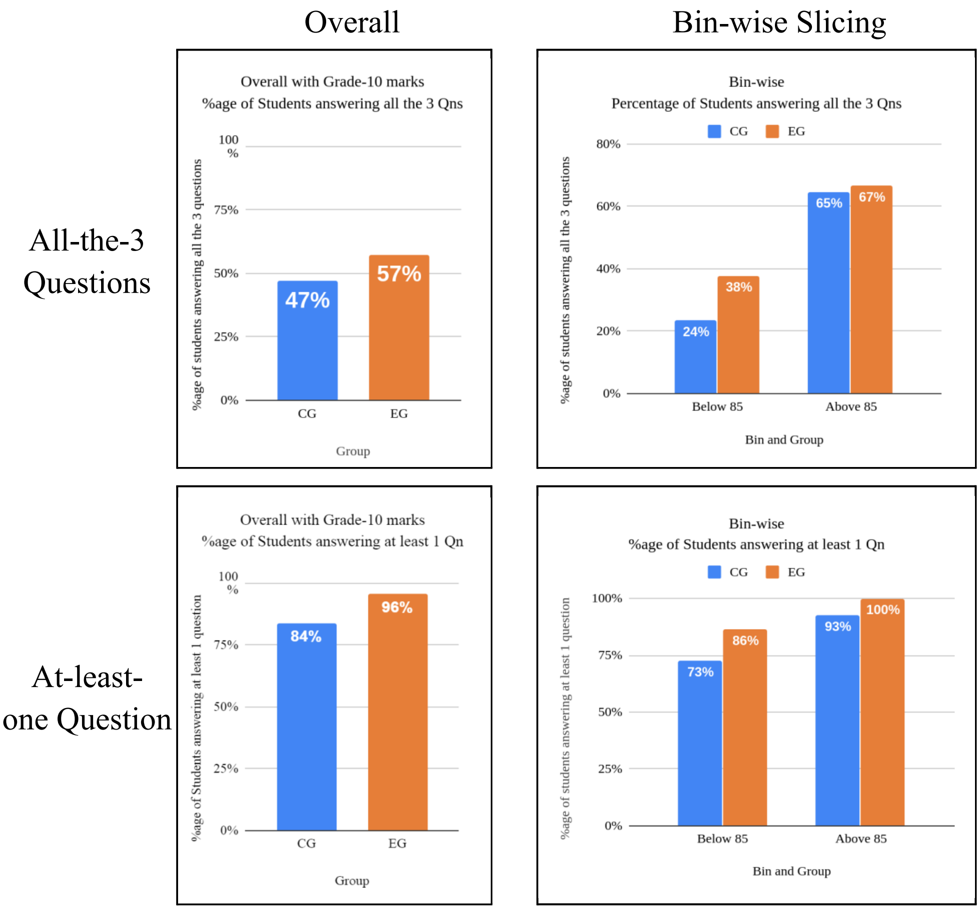

Fig 2F6. The proportion of students responding to all-the-3 questions (top row); at-least-one question (bottom row), in CG and EG students whose grade-10 scores were available. Left figures: Proportions combining both the bins. Right figures: Proportions in individual bins.

In the bin-wise slicing (N=234 students with grade 10 scores; Fig 2F6) as well, we see increased engagement by the EG group in all categories, across both measures. Furthermore, an interesting pattern, showing greater impact on cognitive engagement, was found in the lower bin than in the upper bin. For the all-the-3 case (Fig-2F6 and Table-2F5), the lower bin has a greater

(~14%) difference, compared to the upper bin, which has a smaller (~2%) difference. This pattern is similar in the at-least-one case, with ~14% and ~7% differences in the lower and upper bins respectively. This pattern indicates that lessons with TFV improved cognitive engagement more in average-performers than in good-performers. This is reasonable, because it is safe to assume that good-performers are already cognitively better engaged with the content (which is reflected in the taller bars in upper-bins than lower-bins in both CG and EG, see Fig-2F6).

	N(Students)		**All 3 Questions**								
	C	**E**	**C**	**E**	**C%**	**E%**	**Diff**	**Odds Ratio**	**Z-Stat**	***P-value***	**CI**
Overall	119	115	56	66	47.1%	57.4%	10.3%	1.5153	1.579	*0.1144*	0.9046 to 2.5384
Lower Bin	51	37	12	14	23.5%	37.8%	14.3%	1.9783	1.442	*0.1494*	0.7826 to 5.0009
Upper Bin	68	78	44	52	64.7%	66.7%	2.0%	1.0909	0.249	*0.8033*	0.5500 to 2.1638
With grade-10 Scores Bin Analysis			**At least 1 question**								
			C	**E**	**C%**	**E%**	**Diff**	**Odds Ratio**	**Z-Stat**	***P-value***	**CI**
Overall			100	110	84.0%	95.7%	11.6%	4.18	2.744	*0.0061**	1.5047 to 11.6116
Lower Bin			37	32	72.5%	86.5%	13.9%	2.4216	1.54	*0.1235*	0.7858 to 7.4623
Upper Bin			63	78	92.6%	100.0%	7.4%	13.5984	1.756	*0.0792*	0.7379 to 250.6117

*Table 2F5. The odds ratio-test results for Bin-wise Slicing: *p<0.05*

8.2.4 Qualitative episodes of reasoning behaviour

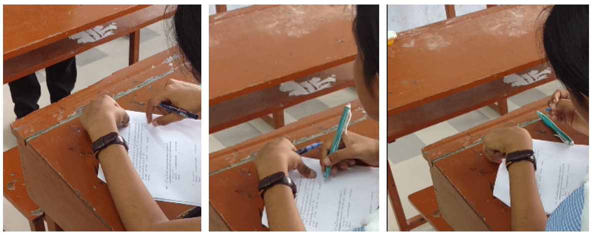

Fig.2F7 A student moving hands above the paper indicating geometric reasoning in the post-test left-finger movements, middle - raising the head during her work with pencil in hand, right- consolidated reasoning and getting ready with pen in hand

We analysed episodes during the post-test, where EG students traced their actions on paper, indicating geometric reasoning (fig 2F7). One episode, of a student from EG2, is related to Q6 (Fig.2F2), which involved changing the angle and estimating the effect on the resultant vector. Here the student starts with moving her fingers on the paper indicating the directions of the given vectors (which are at an acute angle), and then moves fingers as if she was writing numbers (making some numerical reasoning). Then she picks up a pencil (still in the air), and moves it to trace the vectors with a greater angle, and pauses and raises her head for a moment. Then she makes the initial set of movements of acute angle again, and traces the longer diagonal (diagonal represents the resultant). Then repeats the tracing of vectors in obtuse angle and a shorter diagonal and makes arc-like movements indicating an obtuse angle, indicating a dynamic change of angle this time. Then immediately, she picks her pen, pauses for a moment, attempts to begin to write, slightly picks up the paper and changes her posture and responds to the question. These set of movements appear to be more confident as she consolidates her reasoning. A combination of geometric as well as algebraic reasoning is expressed in this episode, indicating better geometry-algebra integration in the student.

8.3 Chapter-Discussion

The results from the field studies show that TFV improved geometry-algebra integration, and influenced reasoning approaches, more in good-performers than in average-performers. It also increased cognitive engagement, but more in average-performers than in good-performers. Overall, these results indicate that learning with TFV changed reasoning approaches (esp. geometry-algebra integration), but with no big effect on cognitive engagement in high-performers. In the average-performers, TFV only enhanced cognitive engagement, but could not make any effect on the reasoning approaches. Both these changes are meaningful shifts in the desired direction. Also, the episodes like the one reported above indicates geometric reasoning well integrated with algebra.

9 Ch-2G - Report of changed Classroom practices [FP2, AP5]

There were numerous qualitative changes in the classrooms observed in the field study. The presence of a new medium in the classroom afforded new kinds of actions and new classroom practices. In this chapter, we report broad changes and invariances in teacher and student practices in the experimental and control classrooms. The final reasoning changes found in students could be ascribed to the holistic changes in classroom practices, brought about by the presence of a new media system, and not merely the TFV interface.

9.1 Learning practices

The main differences in the learning process between the CG and EG classrooms, particularly related to imagination, were in terms of the use of gestures, multiple-trajectories, collaboration and flow of learning.

9.1.1 Gestures

Gestures can be key indicators of geometric imagination. In CG classrooms, students were often passive observers of teacher-enaction. Their gesture activity, besides taking notes, was limited to nodding heads in agreement or tracing pens remotely over diagrams seen on the board (fig-2G1 left). In EG classrooms, more active engagement was reflected in sustained gestures, embedded in conversations, indicating conceptually deeper engagement with the geometric aspects. For example, in fig-2G1 (right) a student is seen actively gesturing when trying to explain his imagination. The differences in gestures indicate the different opportunities for interactions afforded by the two classrooms, and the higher levels of interactions afforded by the EG classroom leading to geometric imagination.

9.1.2 Multiple trajectories

In the CG classrooms, learning opportunities were driven by the teachers' enaction of concepts and problem-solving approaches, and there was limited scope for student exploration. In the EG classrooms, the tasks were exploratory and open-ended, and this structure led to a surprising number of problem-solving trajectories. For example, student approaches to a task to find 2 vectors, whose resultant is a target vector (magnitude 60 at 40° from the x axis), included:

- Trial & error (the most common approach, where students manipulated the magnitude and direction of vectors randomly).
- Calling x and y components estimated using paper-pencil as 2 vectors. These students struggled when asked to create another set of vectors without TFV.
- Creating two vectors with magnitude 30 at 20° from the x-axis, and then adding to find resultant of magnitude 60 at 20° instead of at 40°. This challenges direct algebraic addition of magnitude and direction — a strategy emerging from addition using the like terms.
- Creating a vector of magnitude 60 at 40° from the x-axis, and arguing that the other vector is a zero vector. One student pair tried this for a long time using trial and error, and eventually applied the idea of zero vectors.

This diversity of approaches are strong indicators of active and meaningful engagement with content, and the imagination triggered by the affordances of TFV, which have enriched students' learning experience.

9.1.3 Collaboration

In CG classrooms, there was very limited scope for students to engage in collaboration and discussions with peers, except for practical necessities like asking for an extra pen, for clarification in note-taking, or playful giggles (disinterested in the content of the lesson). In the EG classroom, there were many episodes of inter-group interactions grounded in the tasks, in addition to the collaboration that was required by the class structure, with the lesson plan and 2-3 student-groups. Though student collaboration could lead to a lot of social and cognitive loafing (O'Donnell & O'Kelly, 1994), which was also seen in EG, there was enough scope for teachers to monitor student progress and trigger discussions (as can be seen later) (Fig.2G2).

9.1.4 Flow of learning

In CG classrooms, the teacher controlled the flow of learning, leaving little scope for students to intervene in the flow and build their own knowledge. Typically the teacher delivered content as lectures, and dictated notes that students took down. The teacher decided when to listen to him/her, which book was to be taken, who should respond to questions etc. In developing countries with large classrooms and limited systemic support for collaboration, this authoritarian stance may be needed to some extent for smooth classroom management. However, students find this structure often intimidating, and they are reticent to interact with teachers.

Figure 2G1. Student in CG using gestures to remotely access the content on the board (left). Student in EG using gestures to explain to the peer in the context of an activity on TFV (right)

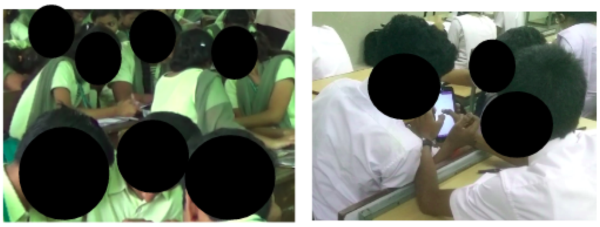

Figure 2G2. Students in EGs collaborating within and beyond their designated groups

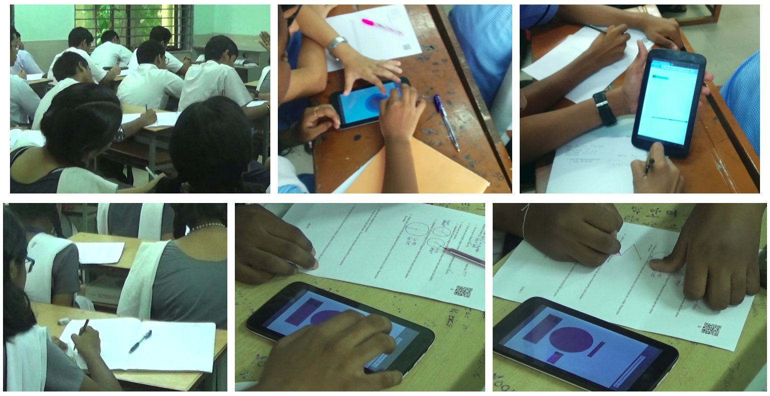

Figure 2G3. Students in a CG taking notes in their notebooks (left). Students in EG interacting with the tablet (middle) and taking note of it in the worksheet (right)

In EG classrooms, the lesson was still driven by the teacher. However, there was some transfer of agency to students, as they actively participated in the teacher's enaction. This was reflected in the free and meaningful interactions with the content: 1) deliberate activity with TFV and purposive note-taking (fig-2G3), as well as interaction among themselves (sec 9.1.3); 2) change in the nature of teacher-student interactions — with students asking questions and driving their learning and a changed teacher stance (sec 9.2.2); and 3) students' emotional connection with the tasks (sec 9.1.5). As suggested by the teachers, if tasks could be designed for students to do at home (thus flipping the classroom), the flow of learning could become even more student-centered.

9.1.5 Other observations

The CG classroom was very calm, and there were never any emotionally charged moments. In contrast, the EG classroom had a charged feeling, and there were numerous moments of excitement and disappointment. Beyond the initial excitement of seeing the tablets on the first day, by days 3-4, they started engaging with the TFV system in a more serious way, and were not excited just due to the presence of the tablet. There were multiple instances where EG students asked for similar systems for other topics in the curriculum.

9.2 Teaching practices

The integration of TFV with textbooks helped teachers to some extent, in easily, but not fully, embracing the new media interface, and integrating it into their teaching practices. Here we discuss some of the key changes, as well as a lack thereof, in the teaching practices, as captured by the video data.

9.2.1 Resistant teacher practice

In the CG classroom, the teaching narrative usually took the form of a lecture. The flow was linear and monotonic, with content introduced using drawings, enaction, and some problem solving. In EG classrooms, the narrative was similar, though there was a deviation in the flow involving tasks with TFV. Interestingly, there was a tendency for both CG and EG teachers to follow a set question and answer template for discussion (especially when stating the definitions or the laws of additions), as this helps students in answering standard exam questions. In later discussions, T1, who taught in an EG class, noted that though the teachers are given the flexibility to use new modes of teaching in principle, exploring multiple narratives in practice is restricted, due to the overemphasis on examination results and time limitations in the system.

The TFV system changed the way teachers presented the content (as discussed in later subsections), but the scope of discussions and certain subtler aspects were still driven by the textbooks. For instance, 3 out of 5 teachers made similar figures on the board (while teaching parallelogram law), and the figures mimicked the one in the textbook (fig-2G4). In EG classrooms, where teachers had the possibility of widening the scope of discussions (like discussing the patterns of changes in the algebraic expressions through geometric manipulations, the nature of the x and y components being interconnected by the circle), such topics were not discussed.

9.2.2 Practice elements that changed

9.2.2.1 Teacher utterances

In EG classrooms, TFV allowed teachers to enhance their enaction of the dynamics, beyond mere descriptions of static diagrams. This was reflected in their talk as well as gestures. For instance, to teach triangle law of addition, a CG teacher used iconic gestures (fig 2G5-left; stretched palm representing the arrow of a vector in the diagram) and described a static diagram: one arrow and another arrow forming two sides of a triangle, and the third side being the resultant. EG teachers also used similar gestures and descriptions but extended them further.

TFV's dynamic process and gestures were reflected in both their descriptions and gestures (fig.2G5 right).

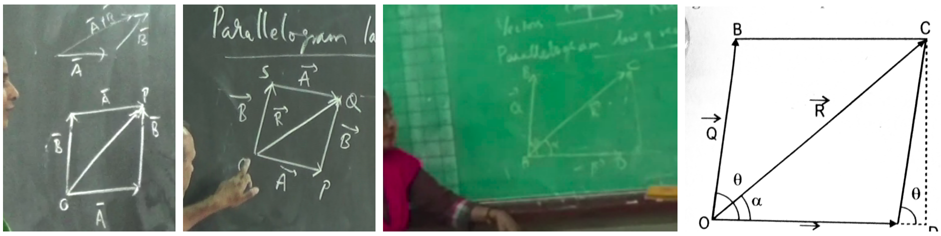

Figure 2G4. Teachers making parallelograms very similar to the one given in their textbook (right) with almost 90°angle while teaching parallelogram law of vector addition.

Figure 2G5. (Left) Teacher in CG using iconic gestures to show a vector, which is still a diagram. (Right) Teacher in EG using a picking action, reflecting that the vector is a touchable entity.

Figure 2G6. (Left and Center) Teachers engaging in discussion with student groups in EG. (Right) Teacher showing a group's work on the tablet to the entire classroom (a potential space for initiating a discussion)

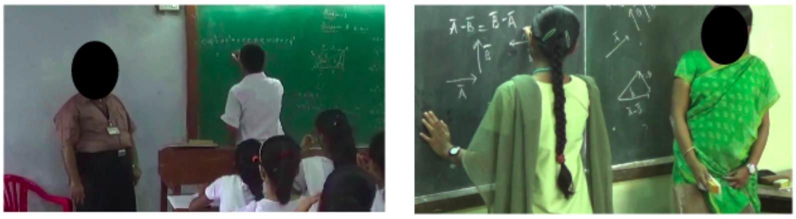

Figure 2G7. Teachers calling students to the board to answer or solve some questions in CG

9.2.2.2 Physical Movement of Teachers

As the TFV system allowed all EG students to enact vector operations, there was no need for the teacher to demonstrate or lecture all through the class, standing at the blackboard. S/he had a lot of time to move around in the class, and teachers identified and connected interesting student-strategies through discussions with individual groups (fig.2G6 right and centre) or the entire classroom (fig.2G6 left), thus enriching the collective learning experience for the students. The role of the TFV in generating this behavior is clear from the contrasting behaviour of T3, the only teacher who taught in both CG and EG classrooms (fig.2G6 middle and fig.2G7 right).

9.2.2.3 Teacher questions

In CG classrooms, teachers' monologues were interspersed with questions — mostly factual and close-ended — to check for understanding (Rosenshine, 1983), and students were cold-called (fig.2G7). In contrast, in the EG classroom, after initial instruction, most of the teachers' time was spent answering students' questions — which were open-ended (with no single answer) and exploratory, grounded in their experiences doing the tasks.

9.2.2.4 Other changes

Besides the above changes, a few other episodes indicated significant changes in teacher thinking and practice, and these were induced by the TFV system.

- As teachers started discussing possible lesson plans during the lesson planning workshop (LPW), they immediately suggested that we could design tasks for students to do at home (as they get more time to engage with the TFV system) and then discuss their findings and responses in the classroom next day, effectively flipping the classroom.
- After one of the lessons, T3 said - "besides helping the students in visualising and understanding the concepts, it has clarified and made imagination easier for topics for me too". This shows how systems like TFV could also help teachers understand complex concepts better, and also help develop content knowledge as well as pedagogical content knowledge (Shulman, 1986), particularly given the acute shortage of good teacher trainers and programs in the developing world.
- Teachers, like the students, appreciated TFV and have repeatedly requested extensions to the TFV system, particularly to include 3D vectors, products of vectors, as well as waves. This is because it is difficult to draw repeatedly to explain these formal systems using the blackboard.
- A physics teacher who was not part of the study sat through an entire classroom session, to see how the system was being used by students.

9.3 Chapter Discussion

Though appreciative of TFV, teachers were not comfortable using the system in their teaching at first. This changed after co-designing the lesson plan and integrating it with the textbooks. The conditioning by textbooks is still evident in teacher practice, but their resistance eased after the workshop and they smoothly embraced TFV, as reflected in the changes in their practices. Their imagination about classroom structure also changed, to the extent that they naturally moved towards suggesting novel models like flipping the classroom. The new classroom-practices, along with novel sensorimotor interactions for students, led to changes even in teachers' STEM cognition.

PART-3: CONCLUSION

10 Ch-3A - Discussion

10.1 Summary of results

The 4 studies addressed RQ-1 (whether affordances of representational media shape STEM cognition) and the 2 design iterations addressed RQ-2 (illustrate a design approach in the developing nation context). The study results suggest that media do affect STEM cognition. The success of the 2 designs in changing student imagination and teacher practice, and the careful process of designing the system and integrating it with existing classroom practice, illustrate a workable approach to design digital media to augment model-based reasoning in developing country classrooms. We discuss the more detailed results below.

From the analysis of textbooks (study 1), to understand how they treat the topic of vectors, we found that they manifested three limitations of static paper-based media.

- Limited affordances for geometric manipulations, manifested in the limited scope to use and understand geometric aspects of vectors.
- Limited affordances for integrating content, manifested in the serial ordering of vector topics.
- Limited affordances for exploring multiple problem approaches, manifested in opaque problem solving.

Study-2 probed students' existing conceptual reasoning behaviour, through their reasoning and problem solving approaches, and their explanations using written, verbal and gestural utterances. It showed the following reasoning patterns in students (across a wide range of scholastic abilities):

- Dominant algebraic reasoning
- Weak conceptual links and incomplete understanding
- Algorithmic problem solving

Study-3 examined student reasoning changes after interaction with TFV-1. It revealed the following patterns.

- After the interaction, students' conceptual link strength was enhanced in ~70% of the cases. The interaction also disrupted their existing models, as illustrated by the case study of S2.
- The two detailed case studies, integrating interviews and interaction data, indicated a change in student reasoning approaches and increased epistemic access.

Study-4 examined the effects of TFV-2 in a field study, where lesson plans designed with TFV2 were taught. It revealed some of the finer effects of TFV interactions on students' conceptual reasoning behaviour. A detailed analysis of written scripts showed that TFV-2 changes reasoning behaviour in good performers and enhances cognitive engagement in average-performers. This evidence, along with some qualitative episodes that showed students using dynamic geometric reasoning even when answering a paper-based test, strengthens the initial finding that TFV triggers geometry imagination. TFV thus enhances cognitive engagement/ epistemic access, and shifts student reasoning towards better geometry-algebra integration.The classroom implementation indicated qualitative changes in teaching and learning practice, and illustrated the following patterns.

- Teachers could adopt the system smoothly into their practice, but their enaction of content was still influenced by the textbooks.
- The organisation and flow of knowledge in the EG classrooms became more decentralised and distributed, through the interaction possibilities of TFV-2.

The 2 design iterations of TFV illustrate a novel approach to design digital media, to support model-based reasoning in developing nation contexts. The designs show:

- how the interactive affordances of computational media could be productively used systematically to support model-based reasoning, particularly to support real-time geometry-algebra integration, through interactions that promoted embodied learning.
- how digital media could play a compensatory role, augmenting textbooks, rather than replacing them.
- how co-designing of lesson plans with teachers, in ways that augment their existing practices, could enable the teacher adoption of digital media technologies, leading to collaborative learning and teaching practices.

10.2 Discussion and broad claims

The results from the studies 1 and 2 (the parallels between patterns in textbook content and the conceptual reasoning behaviour in students) together provide indicative evidence that the affordances of representational media play a role in students' STEM cognition (Karnam, Mashood, et al., 2020). An analysis of textbooks does not directly reflect the state of practices and sensorimotor interactions of students, and we don't overlook the possibility of teachers improvising on the content and thus enriching the geometric experiences of students. However, the parallel is interesting, and the possible relationship between the patterns is particularly relevant for Indian classrooms, where textbooks wield immense authority — as the state manages both the assessment and textbook making bodies — and the classroom culture and the conceptual ecosystem is very tightly bound by the textbooks (Kumar, 1988). In this scenario, it is highly plausible that the limitations of the dominant paper-based medium could be limiting the nature and substance of classroom practices. This problem is thus wider than vectors and model-based reasoning, and affects all learning based on paper-based media.

The 4E cognitive models provide a theoretical basis for understanding how limitations in sensorimotor interactions limit cognition (here imagination and reasoning using models) as seen in students' behaviour. The design of TFV tried to specifically compensate for these media limitations, by making geometric aspects of vectors manipulable, and real-time integration with algebraic aspects. If the interactive affordances of the representational medium play a role in shaping students' STEM cognition, then interactions with a system like TFV, based on a representational medium providing wider affordances, should change students' STEM cognition. The changes found in students' CRB, after they interacted with TFV-1 (study-3), and when taught lessons using TFV-2 (study-4), together provide a nuanced picture of the effects of such interactions on STEM cognition. The presence of TFV changed classroom practices and allowed students to have richer geometric experiences, both in sensorimotor and collaborative terms. This led to changes in their STEM cognition. These together support the corollary linking the interactive affordances of representational media and students' STEM cognition (fig 3A.1). This result, as well as the nuanced way in which the design was integrated into current practices, indicate how computational media could be designed to augment, and not replace, textbook media and associated practices in developing nations.

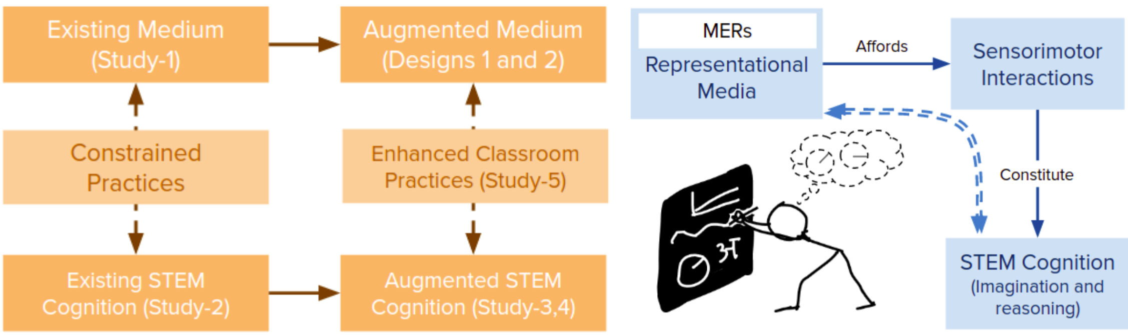

Fig 3A1. The studies together support the corollary of the SCIARM framework (the 2 dotted links).

10.3 Limitations and Future Directions

The above discussion provides an overview of the contributions of the study. However, the study also had a set of limitations. We discuss some of these issues, and future work to address them, below.

- The surprising finding that algebra contributed to the growth in geometry-algebra integration (study-4), as well as the conceptual disruptions shown by the pilot study (study-3), indicate that learning of vectors can occur through many complex trajectories, and we need to attend to these processes. Our study has not explored these underlying processes, but only the overall cognitive effect of representational media on students' cognition, brought about by new sensorimotor interactions. Establishing the underlying mechanisms that lead up to this change requires controlled laboratory experiments, examining cognitive processes. The factors involved in these changes are not just the interactive design features, but a whole set of factors triggered by the novel affordances of TFV. Technology features are thus one component of a holistic learning experience, which is shaped by social interactions, including pedagogy, as discussed in Karnam et al. (2019).

- The claims of the study are based on a pluralistic methodology, combining content analysis, qualitative analysis, design and quantitative analysis. The study's results are thus not fully evaluable from standard frameworks based on quantitative analysis. We consider this a strength of the study, as recent developments in educational technology have also led to shifts in study methods, where researchers combine different approaches (methodological pluralism), and develop thick descriptions to develop insight into technology-mediated cognitive processes. There is thus a shift away from the exclusive focus on statistical significance of effects (Hew & Brush, 2007; Papert, 1988; Reiser, 1994; Ullmer, 1994). The coherence of our quantitative data (with larger patterns), particularly findings of the textbook analysis, students' CRB, the design and design-based changes in CRB, as well as the case studies from the pilot study (Karnam et al., 2018), provide wide-ranging support for our claims, across methods. This convergence compensates for the limited statistical significance of some findings in the field study.
- The evidence we present (for the role of representational media in STEM cognition, particularly model-based reasoning) is based on a single case, on the topic of vectors. Though we have provided justifications for why this topic is suitable for the required investigation, we have not explored modelling aspects of vectors in this thesis. We hope to address this issue in future extensions of this investigation.
- Though TFV provides some meaningful context to learning of vectors, its scope is limited to enriching experiences with vector abstractions within classrooms. TFV does not provide contexts of physical applications of vectors, say to model forces etc. It is confined to creating a world of vectors, where the tangible objects (the arrows and others on screen) obey vector laws, and hence through interaction, concrete ways of learning abstract mathematical ideas become possible. It is also confined to triggering imagination and providing meaning through islands of mathematical experiences created in classrooms using the interface. As this is still disconnected with the real-world — e.g. there are no connections to the action one performs on TFV with any relation or effect in the real world like forces, etc., — these experiences are still limited. The real-world contextualisation needs to be and could be done by the teachers. Once there is such a learning experience of vectors, by which an imagination is triggered in the learners, and the students gain epistemic access to the abstract modelling capabilities, the

teachers can use this structure, and extend these to physical contexts. There are complex pedagogical and learning issues in making this connection — where learners use the mathematical tools (like vectors) to model the physical world — as well, which is beyond the scope of the thesis, but need to be examined in future. TFV could definitely play some role in this process.

- Though the thesis adopts the position that science curriculum should reflect science practice, it examines a specific topic in the curriculum, and the study is thus situated within the traditional content-based paradigm. This is a deliberate choice, based on the acknowledgment that the educational system, particularly in India and other developing nations, is still in a transition from the content-based paradigm. Nevertheless, to focus on practices within this limited framework, our study paid more attention to overall changes in practices with MERs, like reasoning using algebraic and geometric representations, instead of just specific changes in conceptual understanding of say resolution, addition etc.
- The reach of TFV, and other similar digital media systems, could be limited in rural and remote parts of developing countries like India, which may not always have physical-access to computing devices. Textbooks are still important artefacts for a majority. However, given the trend of increasing smartphone usage in India, we hope systems like TFV-2, embedded in textbooks, could foster epistemic-access in motivated children, even if one person/ teacher has a smartphone, until universal and equitable physical-access to open and free computational technology is available. The findings of the study thus generalise in principle to most contexts where static-media use is dominant (Karnam, Mashood, et al., 2020; Karnam & Sule, 2018).

11 Ch-3B - Implications

The study results and the design framework reported in this thesis have wide implications, across various stakeholders, and could thus trigger a range of deeper investigations. We examine some of these implications below.

11.1 For Researchers

11.1.1 STEM Education Researchers

- The media-based cognition framework to understand STEM learning (the SCIARM framework) allows revisiting many learning issues from a new perspective.
- The enhanced role of actions and practices in cognition, and also the ongoing larger transition towards practice-based approaches to science and science education, allows reimagining new kinds of educational frameworks. This requires education researchers to engage more meaningfully with science practitioners and teachers, making them active collaborators and

co-designers of educational designs.

- Besides the above larger implications, the thesis also illustrates how methodological pluralism (diverse qualitative, quantitative and design approaches) could be used as a fruitful approach to address STEM education issues. Extensions of this approach would allow education researchers to investigate new and wider kinds of learning, and develop meaningful insights for practice.

11.1.2 Educational Technology Researchers

The thesis provides a stable theoretical ground (SCIARM), supported by promising empirical evidence, to revisit debates around the role of media in learning debates. In particular, the thesis illustrates productive ways to develop technology based on mixed media, which seeks to compensate/augment existing media, rather than replace them. This structure provides a new lens to understand the role of media in learning and the adoption of digital systems by institutional frameworks.

The current narrative in the EdTech industry appears to favour a radical restructuring of the entire educational ecosystem, replacing existing practices with purely digital technology and its practices. Though such a shift may be needed eventually, particularly to support abilities like computational thinking, the thesis illustrates how a more smoother, rather than disruptive, transition of the educational system towards such approaches is possible. This approach requires educational technology researchers to develop deeper and more meaningful engagement with the stakeholders in the educational system, and develop theoretically grounded designs and studies that examine the cognitive augmentation potential of different media. We discuss below some detailed implications for educational technology practitioners and researchers.

- Not every learning problem needs digital technology and digital learning tools may not be needed for each and every topic. Some learning objectives, such as the understanding of the area-topic, can be better addressed perhaps using a combination of various representational media and physical artefacts (Rahaman et al., 2017).
- Every medium has its place and relevance, and careful examination of the affordances of different media is needed before advocating the adoption of particular technologies. Critically, this analysis requires understanding the cognitive facilitation each media provides.
- Laboratory explorations of prototype systems can be technocentric, but such prototypes need to be systematically integrated with existing practice for applications of technology to social enterprises such as education to succeed. This requires shifting to a systems approach to design once the prototype stabilises, to ground the technology in larger social processes related to education, starting from classrooms.

11.2 Practitioners

11.2.1 STEM Education

Teachers who teach vectors can now use TFV to advance their knowledge and student learning. Beyond this immediate application, the collaborative design practices supported by the thesis project, and the papers co-authored with teachers (Chauhan et al., 2019; Karnam et al., 2020) could inspire and encourage more future teacher-researcher collaborations.

11.2.2 Designers

Design is at the heart of this thesis. The design considerations we outline have many contributions and implications for designers. We discuss two key aspects below:

- Embodied interactions for teaching and learning formal systems: The now widely-accepted theoretical model of embodied cognition, and the parallel development of embodied interaction systems, offer new and exciting avenues for designers of learning media. The interaction design illustrated in this thesis provides two key design constructs that can be built to develop such new media systems: (1) concept-laden interactions (for instance, double consecutive taps for creating and adding vectors and pinch-away for resolution) that draw on the embodied meanings of these gestures to advance learning of formal systems; (2) tangible concepts (conceptual entities like unit circle — integrating geometry of lines, circles and triangle and trigonometry — also used as interactive entities), which allow interacting with formal systems in an embodied way, and thus learning their possibilities.
- Design for technology adoption: The two design iterations illustrate systems-level design strategies such as platform independence (browser based design), co-design (with teachers) and practice augmentation (textbook-linking) that ensure wider adoption of new technologies. These specific approaches could seed wider models of technology adoption, systems-level design.

11.3 Policy

- Educational Technology policy in developing countries currently lacks a clear direction on the way to promote computational thinking, while also simultaneously addressing access issues and technology adoption in classrooms. This thesis illustrates a new approach to address this complex problem (fig 3B1), particularly in developing countries. This model could be extended to develop new ways to address other such complex policy problems, such as integration of data science and sustainability into the curriculum.
- It is well-known that technology systems change organisational structures and institutions (Shah, 2020). It is currently unclear how embodied digital media technologies will change institutional structures, and how such structures need to adapt to support new technologies.

The changes in classroom practice enabled by TFV provides an initial guideline for developing policy frameworks to address this complex issue.

- Policy frameworks have struggled to develop clear ways to enable productive collaborations between education researchers, teachers and technology designers. The collaboration model illustrated in this thesis, where lesson plans were used as a key construct to develop such a productive collaboration, provides a framework to develop policies and systems that support such rich collaborations.

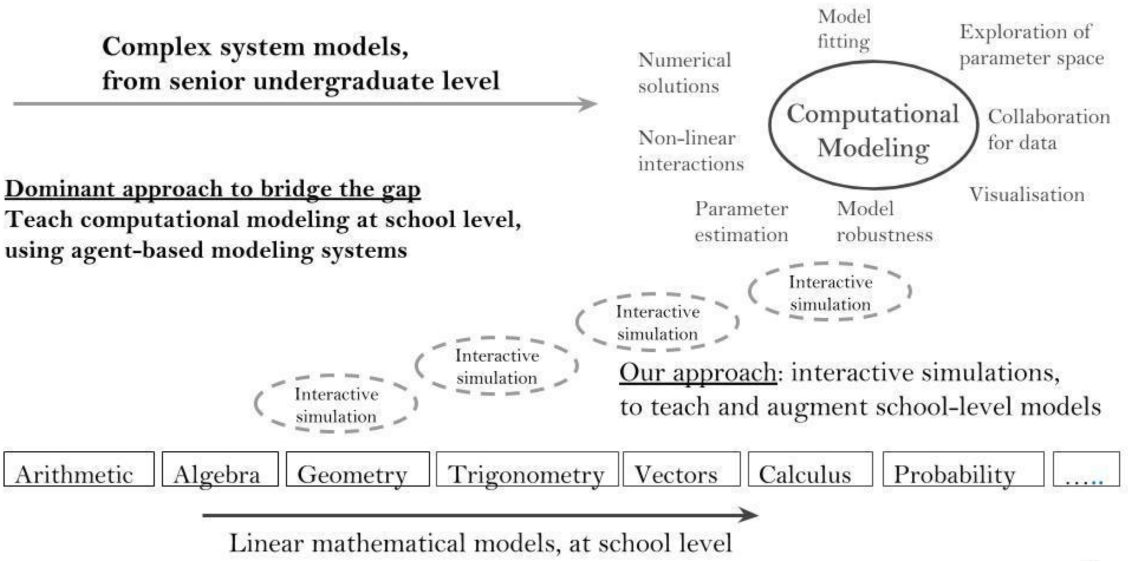

Fig 3B1. Our approach, of interactive simulations that build on existing curricula (linear mathematical models taught at the school level). This approach allows a gradual transition towards Computational Thinking (CT: complex system models at senior UG level). In contrast, the currently dominant approach to promote computational modeling, based on agent-based modeling systems, requires learning a completely new practice. A combination of these two approaches may work best to help develop CT.

12 Ch-3C - Conclusion

From oral to written and now to digital media, humanity has gone through many transitions in the way knowledge is represented. In the transition from oral to written cultures, human civilisation witnessed tectonic shifts, in knowledge systems and their resulting social orders. We are currently witnessing how digital technology is reshaping our world -- the way we travel, transact, and do science -- at a breathtaking pace. STEM education is a crucial cultural activity, which will also change soon, dramatically, with the introduction of digital media. This thesis provides initial directions towards both initiating and managing this critical and drastic change. It also could help us understand the cognitive realities of millions of children like Samantha and advance these using novel perspectives based on representational media. Meaningful collaboration between stakeholders, an instance of which is illustrated by this thesis work, is the way ahead to reach, and manage, this radically different future.

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Publications based on the thesis-work [Relevant Chapters]

Journal Publications

JP1. **Karnam, DP.,** Agrawal, H., Parte, P., Ranjan, S., Borar, P., Kurup, P., Joel, A J., Srinivasan, PS., Suryawanshi, U., Sule, A., & Chandrasekharan, S (2020). Touchy-Feely Vectors: a compensatory design approach to support model-based reasoning in developing country classrooms. *Journal of Computer Assisted Learning*, 446-474, 37(2). https://doi.org/10.1111/jcal.12500 [Ch-2C, 2D, 2E, 2F]

JP2. **Karnam, DP.**, Mashood, K. K., & Sule, A. (2020). Do student difficulties with vectors emerge partly from the limitations of static textbook media? *European Journal of Physics*, 41(3), 035703. https://doi.org/10.1088/1361-6404/ab782e [Ch-2A,2B]

Peer-reviewed Conference Proceedings

Full-Papers

FP1. **Karnam, DP.,** Agrawal, H., Parte, P., Ranjan, S., Sule, A., & Chandrasekharan, S. (2019). Touchy Feely affordances of digital technology for embodied interactions can enhance 'epistemic access' In M. Chang, R. Rajendran, Kinshuk, S. Murthy, & V. Karnat (Eds.), *Proceedings of the 10th IEEE International Conference on Technology For Education (T4E) 2019*. (pp. 114-121). Goa, India. [Ch-2E, 2F]

FP2. **Karnam, DP.,** Agrawal, H., Borar, P., & Chandrasekharan, S. (2019). The Affordable Touchy Feely Classroom: Textbooks embedded with Manipulable Vectors and Lesson Plans augment imagination, extend teaching-learning practices. In Lund, K., Niccolai, G., Lavoué, E., Hmelo-Silver, C., Gweon, G., and Baker, M. (Eds.). *Proceedings of 13th International Conference on Computer Supported Collaborative Learning (CSCL) 2019, Volume 1*. (pp. 488-495) Lyon, France: International Society of the Learning Sciences. [Ch-2E, 2G]

FP3. **Karnam, DP.,** Agrawal, H., & Chandrasekharan, S. (2018). "Touchy Feely Vectors" changes students' understanding and modes of reasoning. In J. C. Yang, M. Chang, L.-H. Wong, & M. M. T. Rodrigo (Eds.), *Proceedings of the 26th International Conference on Computers in Education (ICCE) 2018* (pp. 143–152). Philippines. [Ch-2C, 2D]

FP4. **Karnam, DP.,** & Sule, A. (2018). Vectors in Higher Secondary School Textbooks. In S. Ladage & S. Narvekar (Eds.), *Proceedings of epiSTEME 7 — International Conference to Review Research on Science, Technology and Mathematics Education* (pp. 159–167). India: Cinnamon Teal. [Ch-2A]

FP5. Borar, P., **Karnam, DP.,** Agrawal, H., & Chandrasekharan, S. (2017). Augmenting the Textbook for Enaction: Designing Media for Participatory Learning in Classrooms. In R. Bernhaupt, G. Dalvi, A. Joshi, D. K. Balkrishan, J. O'Neill, & M. Winckler (Eds.), *Human-Computer Interaction – INTERACT 2017* (Vol. 10516, pp. 336–339). Cham: Springer International Publishing. [Ch-2E]

FP6. **Karnam, DP.,** Agrawal, H., Mishra, D., & Chandrasekharan, S. (2016). Interactive vectors for model-based reasoning. In W. Chen, T. Supnithi, A. F. Mohd Ayub, M. Mavinkurve, T. Kojiri, J.-C. Yang, … S. Iyer (Eds.), *The Workshop Proceedings of the 24th International Conference on Computers in Education (ICCE) 2016* (pp. 401–406). Mumbai, India: IIT Bombay. [Ch-2C, 2D]

Abstracts (with oral presentations) & *Posters

AP1. ***Karnam, DP.,** Agrawal, H., Sule, A., & Chandrasekharan, S. (2019). Touchy Feely Vectors - Material Experiences of Geometrical Representations of Vectors. In Graven, M., Venkat, H., Essien, A. & Vale, P. (Eds). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 148). Pretoria, South Africa: PME. [Ch-2G]

AP2. *Chauhan, P., Joel, A.J., Kurup, P., Srinivasan, P.S., & **Karnam, DP.** (2019). Experiences of teaching Vectors in Indian pre-university classrooms: An account by Teachers. In *Proceedings of the Inaugural Conference of the Mathematics Teachers' Association* - India. (pp 126-127) Mumbai: HBCSE. [Ch-2A]

AP3. **Karnam, DP.,** Agrawal, H., Sule, A., & Chandrasekharan, S. (2019). Need to explore affordances of technology for better learning and teaching interface designs. In *The Future of Learning Conference - Learning 4.0: Connecting the Dots and Reaching the Unreached.* Bangalore: IIMB. [Ch-2D, 2F]

AP4. ***Karnam, DP.**, Agrawal, H., Borar, P., & Chandrasekharan, S. (2018). Touchy Feely Vectors: exploring how embodied interactions based on new computational media can help learn complex math concepts. In *Proceedings of the 5th ERME Topic Conference MEDA, 2018.* Copenhagen: University of Copenhagen, 313-314. [Ch-2E]

AP5. **Karnam, DP.**, Borar, P., Agrawal, H., & Chandrasekharan, S. (2018). The Affordable Multitouch Classroom. In *The Future of Learning Conference - Pedagogy, Policy and Technology in a Digital World.* Bangalore: IIMB. [Ch-2G]

Other Publications

1. Kiri, H., Agrawal, H., Forte, S., **Karnam, DP**., & Chandrasekharan, S. (2020). Unpacking students' modelling behaviour in the Sun-Earth system: Use of digital media tool-based epistemological resources. In So, H. J. et al. (Eds.), the *Proceedings of the 28th International Conference on Computers in Education (ICCE) 2020* (pp. 99-108). Darwin, Australia (Online). Nominated for Best Technical Design paper.

2. ***Karnam, DP.**, Sinha, R., & Pardeshi, A. (2020). Figuring the 'making' with Indian teachers: Can we reimagine STEM learning?. In Gresalfi M., & Horn, I. S. (Eds.). *Proceedings of The Interdisciplinarity of the Learning Sciences, 14th The International Conference of the Learning Sciences (ICLS) 2020,* Volume 4. (pp. 2339-2340), Nashville, Tennessee: (Online): International Society of the Learning Sciences.
3. Kiri, H., Agrawal, H., Forte, S., **Karnam, DP.**, & Chandrasekharan, S. (2020). New Media Design offers novel Learning Experiences of the Sun-Earth System. *In The Future of Learning Conference - Learning to Learn, Unlearn and Relearn.* Bangalore: IIMB, 33-34.
4. Nagarjuna, G., **Karnam, DP.**, Sanyal, M., Agarwal, H., & Yadav, G. (2019). An Architecture of a Social Agent for Grounded Cognitive Semiotics. *In the Conference on Complex Systems 2019.* Singapore.
5. Gupta, T., Chandrasekharan, S., Forte, S., & **Karnam, DP.** (2019). Impact of simulations on student models of acids and bases. *In the 54th Midwest Regional Meeting 2019* by the American Chemical Society Wichita Section, Kansas, US. Abstract ID 362334.
6. **Karnam, DP.,** & Rahaman, J. (2019). Turbulence in the nature of mathematics: Way ahead for Math Education Research. In Graven, M., Venkat, H., Essien, A. & Vale, P. (Eds). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 54). Pretoria, South Africa: PME.