Touchy Feely Vectors: A design-based study examining the role of representational media in STEM cognition

A Thesis

Submitted to the

Tata Institute of Fundamental Research, Mumbai
for the degree of Doctor of Philosophy
In Science Education

by

DurgaPrasad Karnam

Homi Bhabha Centre for Science Education
Tata Institute of Fundamental Research
Mumbai

October, 2021

Touchy Feely Vectors: A design-based study examining the role of representational media in STEM cognition

A Thesis

submitted by

DurgaPrasad Karnam

advised by

Aniket Sule & Sanjay Chandrasekharan

Submitted in the fulfilment of the academic requirements for the degree of Doctor of Philosophy in Science Education

Homi Bhabha Centre for Science Education,
Tata Institute of Fundamental Research, Mumbai.
October, 2021

Humbly dedicated to

All the children who are excluded from learning, and

All those whose knowledge remains ‘powerless’...

నేర్చుకునేందుకు నోచుకోని పిల్లందరికి,
నేర్పు ఉన్నా ... దుర్బలులుగా మిగిలినవారికి

అంకితం

जिन बच्चों को सीखने न मिला, और

जिनका कुशलता कमजोरी ही रह गया,

उनको समर्पित

DECLARATION

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions.

The work was done under the guidance of Professors Aniket Sule and Sanjay Chandrasekharan, at the Homi Bhabha Centre for Science Education, Tata Institute of Fundamental Research, Mumbai.

DurgaPrasad Karnam

In our capacity as supervisors of the candidate's thesis, we certify that the above statements are true to the best of our knowledge.

Aniket Sule

Sanjay Chandrasekharan

Thesis Supervisors

Date: 18th May 2021

Place: Mumbai, India

INDEX

PART-0

PART-1

PART-0

Abstract

Recent theories argue that cognition, in general, is 'constituted', i.e. brought into being, by sensorimotor interactions between the body and the environment. Extensions of this constitutivity hypothesis suggest that for the phenomena and models (in STEM) not directly accessible to sensorimotor interactions, cognition is through multiple external representations (MERs). This theoretical position leads to a corollary: the understanding and processing of STEM concepts may be shaped by representational media (text, animation etc.) encoding the MERs.

To test this corollary, we examined: 1) how existing static media encoded a complex STEM modeling concept (vectors), and 2) whether the limitations of this media correlated with students' conceptual reasoning behaviour (CRB). Results indicated a possible correlation. To further investigate this, we: 1) designed a new media interface (Touchy-Feely Vectors, TFV), which compensated for the interaction limitations of textbook media, and 2) examined whether the principled design of TFV led to systematic changes in students' CRB. Results indicated a change in CRB correlated with the design. We then examined the robustness of this result, by augmenting existing textbook media using virtual lesson plans (co-designed with the teachers) based on TFV, and a larger field study (N=266) in real-world classroom situations. Results showed that both students' CRB and classroom teaching/learning practices changed.

These results, and the principled design rationale of the TFV system, together indicate that interactive affordances of representational media play a critical role in STEM cognition, thus supporting the constitutivity hypothesis, as well as recent 'field' theories of cognition. Further, our operationalization also illustrates a systematic approach to the design of digital media for STEM learning in developing nation contexts.

Keywords: representational media, digital media, vectors, paper-based media, embodied learning, model-based reasoning, developing country context

Extended Abstract

A majority of students struggle with reasoning and imagining with abstract models, particularly constructing and manipulating such models, while learning STEM topics. Analyses based on cognitive theory indicate that external representations (ERs), and representational media (RM: form factors used to inscribe the ERs), contribute significantly to the way we gain epistemic access to abstract models. In particular, the constitutivity hypothesis, put forward by recent 'field theory' models of cognition (4E cognition), suggest that material interactions, and the affordances of the learning environment, play critical roles in the understanding of science and mathematics.

These views provide a good starting framework to examine the relationship between STEM cognition, particularly learning of abstract models, and the interactive affordances of representational media. If the interactive affordances of representational media shape learners' STEM cognition, new designs, based on digital media, could help students learn such abstract models. However, work in educational technology has not examined in detail the role that media plays in learning. Addressing this issue requires developing a grounded design-based research approach, informed by cognitive theory. Such a research approach is developed here, where we seek to understand the relationship between media and learning, by developing a digital media solution to the problems students face while learning abstract models. As this design study is based in India, it also outlines a way to design such systematic media in developing nation contexts, to address the learning of abstract models.

This research approach is operationalized using the topic of vectors, which has wide modelling applications. Learning vectors requires understanding multiple ERs with spatial (geometric) and algebraic components, and studies on vector learning show that students struggle with geometrical aspects of vectors. Specifically, the integration of geometry and algebra is difficult to master. The model-based reasoning involved, and the well-documented learning difficulties related to vectors, make the topic ideal to address the question of what roles media play in STEM cognition, particularly imagination using models, and the way affordances of RM (used to inscribe ERs) contribute to this cognitive process. Towards addressing this research question, we first examined the affordances of

paper-based textbooks, which are critical and essential to classroom-practices in India, by studying students' conceptual reasoning behaviour (CRB) in relation to textbook representations. An analysis of student utterances (including gestures) while reasoning and solving problems showed a clear pattern between students' CRB and the way vector content was represented in textbooks. For instance, paper-based textbooks provide limited affordance for geometric manipulation, and they thus explicitly underplay geometric addition of vectors. In parallel, we see dominant algebra-based reasoning in students.

To examine this correlation in detail, as well as to address students' limited geometry-algebra integration, we designed an interactive digital media system, Touchy-Feely Vectors (TFV), to specifically compensate for the interactive limitations of the paper-based medium. Results of studies based on this system showed that students moved to a more integrated understanding of algebra and geometry. This result further indicated that representational media play a role in STEM cognition. A second version refined this system (TFV-2), and it was embedded in textbooks using QR codes, which allowed the system to be integrated smoothly with existing classroom practices. Virtual lesson plans were then co-designed with teachers using workshops based on this system. Field tests (N >250) showed that the use of TFV enhanced students' reasoning approaches, improving geometry-algebra integration (in good-performers) and cognitive engagement (in average-performers). The affordances of TFV also changed classroom practices.

These studies (textbook analysis, design studies), and the design rationale of the TFV system, together indicate that interactive affordances of representational media play a critical role in STEM cognition. Further, the new media designs and the field studies illustrate a compensatory design approach that seeks to augment, and not replace, existing learning media and practices, particularly based on textbooks. This approach thus illustrates how such a media design could be developed to suit the classroom conditions of developing nations. The promising evidence, of a strong link between representational media and STEM cognition, could be extended to develop theoretically grounded and more rigorous STEM cognitive studies, as well as to design novel approaches towards embodied learning in STEM.

Elaboration of some terms repeatedly used in the thesis

Affordances: Action possibilities allowed in a given body-environment system. For example, a 2 ft high surface affords sitting action for an adult and climbing action for an infant.

Representational Media: The technological form factor on which representations are inscribed. (For example, paper and pencil, coal and wall, chalk and board, a computer, etc.

Epistemic Access: This is in contrast to physical access, which is the connotation that people usually have when referring to "access". For us, physical access refers to mere material access to something. This usage is along the lines of epistemic availability (O'Donovan-Anderson, 1997) used in philosophical explorations on how a cognitive being knows anything; in other words, how does anything become epistemically available to the knower? For instance, in STEM contexts, when one encounters an equation "4 x 3 = 12" inscribed on a blackboard, physical access refers to material access to such an inscription. In contrast, epistemic access corresponds to an access where one is able to cognitively engage with such an inscription. Books and other MOOCs like usage of representational media primarily provide physical access and need not ensure epistemic access.

Conceptual-reasoning-behaviour (CRB): Various utterances (through writing, speaking, gesturing) that one makes in the context of reasoning using concepts, like while solving problems, reasoning, or explaining one's reasoning approaches.

Acknowledgement

This thesis is an outcome of efforts of tens of people directly involved in the execution of the project, and even more number of people creating a space — intellectual (the researcher community on whose shoulders this thesis stands) and institutional (all the organisational and practical conditions enabling the execution) — conducive to a smooth completion.

In Aug 2014, after tireless sculpting by Prof. MC Arunan for a 6-month period, I began my journey of PhD, with naive curiosity in human cognition behind learning STEM. I was not only limited in foreseeing the terrains I would walk into but also deeply handicapped in many facets which later were to become crucial parts of this thesis. I must say, Prof Sanjay Chandrasekharan's style of working with what the circumstances throw at us (challenges or opportunities) rubbed off on me. For instance, interactions with Prof. Aniket Sule, not considered until demanded by certain circumstances, have turned out to be very valuable in my thesis execution and shaping my academic persona. Both Sanjay and Aniket had different shades of pragmatism, a unique combination of which built a spirit in me — captured aptly by these lines from a bollywood song "मैं ज़िन्दगी का साथ निभाता चला गया... बरबादियों का जश्न मनाता चला गया[1]" — to smoothly handle the usual ups and downs in a PhD journey. Another example of this happenstance is the component of digital media in my thesis. Until I began my proposal writing, I did not foresee digital media being a key part of my thesis project, nor did I have any aptitude or ability to program. Harshit Agrawal's walking into our lab for his internship and presence of Dr David Landy and Dr Natalie Sinclair during the epiSteme-6 conference in Dec 2015 triggered us to explore technology, perhaps also triggered by Prajakt Pande's (fellow research scholar) work using digital media interfaces in his thesis.

These happenstances, of course, were not baseless, but were embedded in the rich and fertile conversations that I was part of, around learning, knowledge and cognition during the courses both as a student and tutor at HBCSE and IIT-Bombay. Delightful experiences of philosophical reflections

[1] A hindi song written by Sahir Ludhianvi and sung by Mohd. Rafi. Meaning: "I kept abreast with the flow of life... I kept celebrating the failures (of course am referring to numerous rejections)"

during courses covering History and Philosophy of Science and Education, epistemology by Prof G Nagarjuna (GN), Prof K Subramaniam (KS) and Prof Karen Haydock; overcoming my struggles with articles in social sciences through a wonderful series of readings and meticulous support by Dr Shekhar Krishnan; and a reading course on representations besides courses on cognition (taken 4-5 times) by Sanjay, which I attended as a student as well as a tutor, had all deeply nurtured my thinking and provided strong foundations to my future work.

These conversations often continued with members of the Learning Science Research group, Math-Ed Group and Gnowledge Lab — Harshit, Prajakt, Jeenath Rahaman, Prateek Shah, Geetanjali Date, Deborah Dutta, Ganesh Shinde, Mukesh Makwana, Milind Khadilkar, Swetha Naik, Harita Raval, Gaurang Yadav, Biswajit Boity. Besides rigorous coursework, numerous platforms in the form of Thursday talks, Annual Research Meets, episteme conferences, and frequent guest courses have provided ample opportunities for inspiring, thought-provoking, safe yet challenging encounters ensuring that I try, err, learn, aspire and grow. Also I had interactions in the later stages of PhD in the form of chatShaalas with MakerSpace, CUBE, and barefootChat of Gnowledge Lab, especially working with Ravi Sinha, Ashish K Pardesi, Jude T D'souza, Drishtant Kawale and Kiran Yadav among others. I shall be deeply indebted to all these people, all the participants in these courses and platforms, for inspiring and introducing me to some of the key ideas and the thinkers and for enabling an enriching intellectual experience.

Besides inputs from Sanjay and Aniket, my regular and timely interactions with Dr KK Mashood and KS, as part of my thesis advisory committee, and frequent philosophical discussions with GN (especially in later stages for STEM Roots project) have together guided me navigate through a tough intellectual exercise, with uncompromising expectations on the quality of work as well as ever-raising expectations on the qualities in me. These have made me realise I am as important an outcome of the PhD as this thesis itself. These interactions have immensely helped me in nurturing different qualities of being a scholar — clarity in thinking and concision in articulation, scholarship in the concerned research domain beyond just the specifics of the thesis, a collaborative spirit (as against the competitive academic culture dominant from school till research) etc. I am extremely grateful to these people for upholding these expectations and I believe I put sincere efforts to build a quality thesis and I am still evolving on these scholarly qualities.

Consequently, glimpses of this quality were evident from the appreciation of our work by scholars across the world during numerous conferences — both domestic and international — and university visits. At this juncture, I must also acknowledge the inputs from brief yet very focussed interactions with Dr. Jana Traglova, Dr. Sophie Soury Lavergne, Dr. Gilles Aldon, Dr. Ton de Jong, Dr. Paul Drijvers, Dr. Anna Shvarts, Dr Dragan Trninic and Dr Manu Kapur, Dr Ferdinando Arzarello, Dr Fransesca Ferrara, Dr Ornella Robutti, Dr Pratim Sengupta in embracing this work and widening my perspectives. Dr Bjorn de Koning enabled a very enriching 15-day visit at the Rotterdam university to collaborate on an extension of the thesis and let me meet various scholars. Also, special guidance provided by Dr Arthur Bakker, Dr Jeremy Roschelle and Dr Jan van Aalst in writing and publishing was invaluable.

Equally important participants in my PhD journey have been the teachers and students that I interacted with. Numerous conversations I had with highly capable and experienced teachers — Prasanna teacher, Jebin sir, Srinivasan sir, Suryawanshi sir, Lekha teacher (and brief interactions with few other teachers) — since 2016 have helped me to stay grounded and anchor the thesis to practice. These interactions, along with my prior experiences in teaching[2], have deeply and implicitly shaped my understanding about education — not only conceptual and pedagogical, but also practical, systemic and institutional aspects — which I hope to explicate at the earliest opportune moment and conditions. Above experiences and the guidance of the afore-mentioned faculty have helped me to balance the forces of idealism of a theoretical pursuit, and realism of the practice, which also closely reflects in multiple layers of the thesis — grounding abstract in concrete, mind in body, research in practice.

Besides the above intellectual sculptors — both implicit and explicit, there were numerous people who participated directly and made significant contributions in the thesis at various stages. Harshit, besides

[2] Working as a full time teacher in a school in Dharavi Mumbai, and later in community learning centres in Dharavi and Govandi, have immensely helped me be sensitive to the teachers and students and various forces at play on the ground.

single-handedly spearheading the technical development of Touchy Feely Vectors, contributed immensely in data collection and analysis. Priyanka Borar magically enhanced the UI/UX of the Touchy Feely Vector system. Pranay Parte and Saurabh Ranjan with their unique expertise in Physics Education and Cognitive Psychology have actively helped in data analysis. Dibyanshee Mishra and Hemanth Kumar also significantly contributed to the data collection and analysis at various stages of the thesis. Contributions by all these people were recognised by co-authored publications with them. Besides these, there were brief yet very important contributions by many friends at HBCSE — Ashwini Dhanvijay, Charudatta Navare, Disha Gupta, Harita Raval, Tuba Khan in rating exercises; Shirish Pathare for graciously letting us use the tablet devices for field interaction; Math-Ed Lab and Gurinder Singh for letting us use the video recording devices.

Besides above, there were numerous people who enabled smooth execution of the thesis project. The first among these are Dr Sugra Chunawala, Dr KS, Dr Chitra Natarajan, Dr Jayashree Ramdas in their respective terms as the Directors and Deans and the ever-supportive administrative teams (including the security and cosmetic staff) of HBCSE, most of whom have become my friends, for providing a very conducive ambience and ensuring a smooth and hassle-free execution and operations all through the PhD both within and outside HBCSE, with an institutional-support and financial-aid (Govt. of India, Department of Atomic Energy, RTI4001), and more importantly their warmth. Likewise, managements of Swami Vivekanand Junior College - Chembur, Atomic Energy Central School-4 and Atomic Energy Junior College - Anushaktinagar, had been very cooperative partners in allowing and supporting interactions with teachers and students and data collection (for several weeks across a year) of the thesis project. I also thank Asia-Pacific Society for Computers in Education for APSCE Merit Scholarship (2018, 2020), Technical Committee on Learning Technology for TCLT IEEE Student Travel Grant (2019), and International Society of the Learning Sciences for ISLS Travel Grant (2019) for enabling me to participate in the conferences, which proved very helpful.

Finishing a PhD Thesis, I come to realise, has been a very demanding endeavour at various levels not just to me but also to close members in my family. This was possible only with constant understanding, emotional and moral support from Keerthi KRD, my wife, and we drew a lot of moral strength from my parents, siblings and the trust of my in-laws.

This thesis is made possible with the direct and indirect support of many people like the ones listed above, at multiple levels and stages. This document is an outcome of all their efforts and support. I shall be forever indebted to all of them.

Numbering convention

The thesis has 3 parts, with chapters numbered as A,B,C... Part-1 (Conceptualisation: Ch-1A,1B), Part-2 (Execution:Ch-2A,2B,...2G), and Part-3 (Conclusion: Ch-3A, 3B, 3C). The first two characters of Figures, Tables and Appendices numbers correspond to the chapter. For example, in Ch-1A figures, tables and appendices are numbered as Figure 1A.x, Table 1A.x and Appendix 1A.x respectively. All the Appendices are provided as separate files in the attached folder.

Publications from this work

Journal Publications

Karnam, DP., Agrawal, H., Parte, P., Ranjan, S., Borar, P., Kurup, P., Joel, A J., Srinivasan, PS., Suryawanshi, U., Sule, A., & Chandrasekharan, S (2020). Touchy-Feely Vectors: a compensatory design approach to support model-based reasoning in developing country classrooms. Journal of Computer Assisted Learning, 446-474, 37(2). https://doi.org/10.1111/jcal.12500 [Ch-2C, 2D, 2E, 2F]

Karnam, DP., Mashood, K. K., & Sule, A. (2020). Do student difficulties with vectors emerge partly from the limitations of static textbook media? European Journal of Physics, 41(3), 035703. https://doi.org/10.1088/1361-6404/ab782e [Ch-2A,2B]

Peer-reviewed Conference Proceedings

Full-Papers

Karnam, DP., Agrawal, H., Parte, P., Ranjan, S., Sule, A., & Chandrasekharan, S. (2019). Touchy Feely affordances of digital technology for embodied interactions can enhance 'epistemic access' In M. Chang, R. Rajendran, Kinshuk, S. Murthy, & V. Karnat (Eds.), Proceedings of the 10th IEEE International Conference on Technology For Education (T4E) 2019. (pp. 114-121). Goa, India. [Ch-2E, 2F]

Karnam, DP., Agrawal, H., Borar, P., & Chandrasekharan, S. (2019). The Affordable Touchy Feely Classroom: Textbooks embedded with Manipulable Vectors and Lesson Plans augment imagination, extend teaching-learning practices. In Lund, K., Niccolai, G., Lavoué, E., Hmelo-Silver, C., Gweon, G., and Baker, M. (Eds.). Proceedings of 13th International Conference on Computer Supported Collaborative Learning (CSCL) 2019, Volume 1. (pp. 488-495) Lyon, France: International Society of the Learning Sciences. [Ch-2E, 2G]

Karnam, DP., Agrawal, H., & Chandrasekharan, S. (2018). "Touchy Feely Vectors" changes students' understanding and modes of reasoning. In J. C. Yang, M. Chang, L.-H. Wong, & M. M. T. Rodrigo (Eds.), Proceedings of the 26th International Conference on Computers in Education (ICCE) 2018 (pp. 143–152). Philippines. [Ch-2C, 2D]

Karnam, DP., & Sule, A. (2018). Vectors in Higher Secondary School Textbooks. In S. Ladage & S. Narvekar (Eds.), Proceedings of epiSTEME 7 — International Conference to Review Research on Science, Technology and Mathematics Education (pp. 159–167). India: Cinnamon Teal. [Ch-2A]

Borar, P., **Karnam, DP.**, Agrawal, H., & Chandrasekharan, S. (2017). Augmenting the Textbook for Enaction: Designing Media for Participatory Learning in Classrooms. In R. Bernhaupt, G. Dalvi, A. Joshi, D. K. Balkrishan, J. O'Neill, & M. Winckler (Eds.), Human-Computer Interaction – INTERACT 2017 (Vol. 10516, pp. 336–339). Cham: Springer International Publishing. [Ch-2E]

Karnam, DP., Agrawal, H., Mishra, D., & Chandrasekharan, S. (2016). Interactive vectors for model-based reasoning. In W. Chen, T. Supnithi, A. F. Mohd Ayub, M. Mavinkurve, T. Kojiri, J.-C. Yang, ... S. Iyer (Eds.), The Workshop Proceedings of the 24th International Conference on Computers in Education (ICCE) 2016 (pp. 401–406). Mumbai, India: IIT Bombay. [Ch-2C, 2D]

*Abstracts (with oral presentations) & *Posters*

*Karnam, DP., Agrawal, H., Sule, A., & Chandrasekharan, S. (2019). Touchy Feely Vectors - Material Experiences of Geometrical Representations of Vectors. In Graven, M., Venkat, H., Essien, A. & Vale, P. (Eds). Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education (Vol. 4, pp. 148). Pretoria, South Africa: PME. [Ch-2G]

*Chauhan, P., Joel, A.J., Kurup, P., Srinivasan, P.S., & Karnam, DP. (2019). Experiences of teaching Vectors in Indian pre-university classrooms: An account by Teachers. In Proceedings of the Inaugural Conference of the Mathematics Teachers' Association - India. (pp 126-127) Mumbai: HBCSE. [Ch-2A]

Karnam, DP., Agrawal, H., Sule, A., & Chandrasekharan, S. (2019). Need to explore affordances of technology for better learning and teaching interface designs. In The Future of Learning Conference - Learning 4.0: Connecting the Dots and Reaching the Unreached. Bangalore: IIMB. [Ch-2D, 2F]

*Karnam, DP., Agrawal, H., Borar, P., & Chandrasekharan, S. (2018). Touchy Feely Vectors: exploring how embodied interactions based on new computational media can help learn complex math concepts. In Proceedings of the 5th ERME Topic Conference MEDA, 2018. Copenhagen: University of Copenhagen, 313-314. [Ch-2E]

Karnam, DP., Borar, P., Agrawal, H., & Chandrasekharan, S. (2018). The Affordable Multitouch Classroom. In The Future of Learning Conference - Pedagogy, Policy and Technology in a Digital World. Bangalore: IIMB. [Ch-2G]

Chapter-wise List of Figures and Tables

Chapter-1A

Chapter-1B

Chapter-2A

Chapter-2B

Chapter-2C

Chapter-2D

Chapter-2E

Chapter-2F

Chapter-2G

Chapter-3A

Chapter-3B

0

Preface

An 11-year-old kid struggled with picking the abilities to read simple sentences with 3 lettered words. He struggled with performing simple arithmetic. He was otherwise a smart and good chap good at singing and playing the drums, with a loving family. His friends are progressing in their grades, but he could not cope up with them and has always been a nightmare from a classroom management point of view. I repeatedly sat for extra hours, to initially motivate him to learn; he too started putting in efforts in vain. This was in 2011 when I started my teaching career with a bunch of 25 10-12-year-old kids in a school in the slums of Mumbai. We called ourselves 'the explorers'. I used to teach them all the subjects except for vernacular languages. I managed to enable decent learning among all the students, except for this one. He grew up and eventually had to drop out of his schooling after he reached grade-8.

Even among other students, I began to wonder how much of what they learn as part of formal STEM education is actually making sense to them? Suppose, I write “4 x 3 = 12”, or any similar arithmetic equation with say fractions or worse an algebraic equation. Similar is the situation with sciences, where they learn essentially questions and answers which cover definitions of concepts, laws, their examples, some taxonomy of things supposedly referring to many of both living and nonliving entities and phenomena around them. I seriously doubt how meaningful is this to these students? As teachers, some of us put in a lot of efforts to make things contextual and meaningful, but under the larger systemic pressures both curricular as well as social requirements especially tied up with the assessments, the priorities inevitably shift! The students also end up meeting these larger pressures and requirements — manifesting within their classrooms — by resorting to various techniques of rote memorisation. A whole lot of them play the game of picking up the rules and procedures, questions and answers, etc., either by rote or some mnemonics or any other way, and follow them. A few of them might excel in doing this, and a very few of them might actually find these meaningful, forming the 'meritorious' sections (whatever its essence is) among us!

This state of affairs made me wonder - How does learning happen and how does meaning emerge, especially of abstract knowledge like in STEM? All those who try to learn how to ride a bicycle, with reasonable effort, succeed to learn. All of us pick up speaking our mother-tongue, without fail. But when it comes to learning something abstract like in STEM, an awful lot of us struggle to meaningfully learn these! Is it that only some come with a proclivity to learn STEM? What does universal (science) education mean, in this context? These were some questions that I began my PhD with.

During the course work, I could anchor these questions subconsciously yet deeply and firmly in discussions during the cognitive science courses by Prof. Sanjay Chandrasekharan, Philosophy courses by Prof G Nagarjuna and K Subramaniam, broad sociology of education in courses by Dr Shekhar Krishnan, Dr Karen Haydock and Prof Sugra Chunawala. The questions sharpened: When a grade-3 student sees an equation as simple as "4 x 3 = 12", or for that matter when one says "apple", what does it mean to her? Where does one draw its meaning from? I got familiar with the reflections on meaning, ranging from Frege's distinction of 'sense' and 'reference' to Wittgenstein's idea of language game where meaning emerged from usage, to Vygotsky's notion of language as a tool. A later reading course with Sanjay on Representations and cognition, helped sharpen and ground these fuzzy questions to a concrete problem in the topics of trigonometry and then vectors, especially using the emerging paradigm of field theories or 4E (embodied, extended, embedded and enactive) models of cognition, giving the constitutivity hypothesis. According to this, one's sensorimotor interactions between the body and environment play a central role in shaping one's cognition. And this is where the entire thesis will be anchored; this shaped and refined the research questions further specifically looking at interactions that students have with representations in the classrooms.

All through the thesis, we will take Samantha, as a prototypical student, to refer to typical students in today's STEM Education, and to bring some concreteness to our reflections. Let us visualise and build what her learning experiences are when she learns, say, multiplications or trigonometry or vectors. What are the sensorimotor interactions, especially with the representations, manifesting as certain classroom practices, that the students and teachers engage in — which can deeply shape the learning experiences and hence meaningfulness of these representational activities, besides other larger social factors of meaning? Typically, her teachers teach her such topics in the school, taking the above simple example of "4 x 3 = 12", by writing on the board or narrating something verbally. Samantha sees these remotely and noting (often just involves copying) — writing or drawing, replicating the script on the board or the teachers' words — in her book. For Samantha, to begin with, it could just be an opaque string of characters strung in a particular way. This string could trigger elaborate and rich meaning or imagination in the teacher, which is what the teacher tries to enact and provide access to, to the students. Some teachers may try and bring some concrete examples to explain things like repetitive addition etc. But often, as mentioned earlier, most of the students like Samantha struggle to access this, and end up learning these by rote or memorising the procedures, whether it is a multiplication table of 4 (from 4 x 1 = 4 till 4 x 10 = 40) or trigonometric identities or expressions of the resultant or products of vectors.

Samantha, based on the classroom instruction, reads many text passages and solves many problems by writing equations. It is interesting to note here that most of her sensorimotor interactions with the representations are static and paper-based. Can sensorimotor interactions with these representations, especially with formal mathematical symbol systems like in vectors, shape their cognition about these topics[3]? And do the media (representational media: the technologies used to represent or the media in which the representations, of say, formal mathematics, are inscribed) we interact with play a role in shaping our cognition in STEM? This is the specific question that this thesis is trying to address. Extension of the constitutivity hypothesis provides us with a corollary that The action possibilities (interactive affordances: IA) of the media in which the representations are encoded (text, computation; representational media: RM) have an effect on the understanding and processing of concepts (STEM Cognition: SC). We essentially try to test this corollary in this thesis. In Samantha's case, the question turns out, does the current static paper-based medium used to learn vectors (or for that matter any formal topic in her STEM education) shape her STEM cognition (reasoning, imagining, thinking etc)?

It is here that the pandora box of technology in education opens. By asking this question, the notion of technology could now be expanded to include spoken language, to written media and to digital media. Digital technology with its novel affordances can enhance the interactions that are possible. So, can these new affordances with representations (enabled by digital technology) change the existing STEM cognition in the students? This is the entire empirical pursuit of this thesis.

As we encountered digital technology, in the pursuit, let us also quickly give a context to this thesis from this perspective. During my PhD work, I was at a conference at the Indian Institute of Management- Bangalore on the Future of learning. This included delegates from academia, the industry as well as policy. Technology, as I realised, has been a very central piece of the imaginations of future education. There has been an immense institutional push for technology in education from the policy as well as industry. These imaginations and these forces had to become real especially during the pandemic. But, during the conference, I realised there are some serious limitations in the current imaginations of what technology can do to education. First of this is limiting the notion of

[3] For time-being leaving the social emergence of meaning aside which is especially valid for picking spoken language. In STEM often even the social interactions are played out in carefully structured language, the epitome of which is formal mathematics. The drive behind the entire positivist project of philosophers of science, is a validation to special focus on the representations.

technology to digital technology, which is already widened by the way we framed the research question earlier. The second and perhaps, even an important and urgent one to be attended to, is the appreciation of the affordance of digital technology. The current dominant imagination is that digital technology enhances access to education; technology can take quality educational resources to every nook and corner (as provided by the internet connectivity). I realised that only physical access is being referred to as 'access' in this popular imagination. But there is another aspect of epistemic access that is important, given the context of learning in the educational sector and given the novel affordances of digital technology to represent and know. Technology, besides providing physical access to resources, can enable fundamentally different ways of knowing and learning. Strangely, these were the debates even among researchers in educational technology and only recently, a new wave of research exploring these possibilities is emerging. The thesis, thus, also while addressing the question of whether representational media shape students' STEM cognition, can demonstrate this cognitive augmentation potential of the digital media, and specifically show how technology can play a role in enhancing the epistemic access to STEM abstractions to a lot of students like Samantha.

The organisation of the thesis

The thesis is broadly organised into three sections.

- The first section, titled Conceptualisation, with two chapters (1A and 1B), frames and operationalises the research questions. These chapters review the relevant literature and also describe the logic model of the execution of the studies in addressing these questions.
- The second section, titled Execution, with seven chapters (2A to 2G), reports the details of the design iterations and the studies. These provide detailed design principles, methodologies of data collection, analysis, results and a brief discussion of the results in each study.
- The third and final section, title Conclusion, with 3 chapters (3A to 3C), discusses all the findings of the Execution (part-2), in connection with the research questions conceptualisation (part-1), and draws implications for multiple stakeholders ranging from

researchers (STEM education, educational technology, etc), practitioners (designers, teachers) and policy.

A detailed description of the content of each of the chapters

In ch-1A, we outline the nature of the learning problem in STEM, particularly highlighting the abstract nature of the topics involved. Recent shifts in our understanding of cognition ('field theories' of cognition; 4E cognition) highlight the constitutive role played by sensorimotor interactions in cognition. We draw on these insights about the learning processes and propose a corollary of the constitutivity hypothesis, where a connection exists between the interactive affordances of educational technology and learning. This connection is poorly understood. In ch-1B, we outline how the research objective discussed above is operationalized using the topic of vectors.

In part-2 of the thesis, we report a series of designs and studies across 3 years, analysing:

- existing representational media and their connection to students' conceptual reasoning behaviour (Ch-2A, 2B)
- two iterations of a design (Touchy Feely Vectors, TFV) that sought to compensate for the limitations (esp. of interactions) of the existing representational media (Ch-2C and 2E)
- two rounds of studies that tested changes in students' conceptual reasoning behaviour based on TFV. Ch-2D describes lab studies, Ch-2F reports studies in classroom settings. Further, Ch-2G examines the way TFV changed classroom practices.

Results from these studies indicate that limitations of the paper-based medium, particularly in supporting manipulation of geometrical entities, could be a key reason for students struggling to imagine vector operations and the resulting lack of epistemic access to such abstract models. The lab and classroom studies showed that TFV, partly co-designed with teachers to compensate for the limitations of the paper-based medium, enhanced epistemic access in students, and fostered model-based reasoning with better geometry-algebra integration, which is required to understand and use vectors.

These results indicate that the interactive affordances of representational media play a critical role in shaping students' cognition of abstract models. Further, the studies illustrate a novel design framework, where teachers participated in the design and the studies, thus paving the way for the inclusion of the design in existing classrooms, and broader institutional frameworks for learning. These results have wide implications and further development possibilities, for researchers, practitioners, and policymakers, across cognition, educational technology and digital media design. These implications and possible future work are discussed in Ch-3B.

Please enjoy reading the thesis, and be inspired to enhance STEM learning in our society.

- DurgaPrasad

Mumbai, 2021

PART-1

CONCEPTUALISATION

1A

Theoretical background, SCIARM framework and the Research Questions

The objective of the chapter: To review literature related to STEM cognition and the role of representational medium, and frame the research questions in the STEM education context.

Key points

- Constitutivity Hypothesis: Sensorimotor interactions between body and environment *constitute* cognition.
- STEM cognition involves reasoning using abstract entities like models. This becomes possible due to the concrete affordances of the multiple external representations (MERs).
- A corollary, framed using SCIARM (STEM Cognition and Interactive Affordances of Representational Media) framework, follows - *The **action possibilities of the media** in which the MERs are encoded (text, computation) have an effect on the understanding and processing of concepts (**STEM Cognition**).*
- Debates on media and learning are not grounded in recent shifts in theories of cognition, and computational media design frameworks for cognitive augmentation in developing nations are limited.

Research Questions

- Do the interactive affordances of representational media in STEM shape learners' STEM cognition?
- What does a systematic design of media-intervention look like in a developing nation context?

Practising STEM requires engagement with abstract scientific-models. In STEM education there is a learning problem where most students, like Samantha, struggle to engage with (imagine) the abstract entities. Students lack epistemic access to these abstract models, and they end up memorising and rote learning them. Broadly, the following pressing questions emerge here. How does any human being manage to epistemically access and engage with these abstract entities beyond everyday action and perception? And why is it that many students struggle to meaningfully understand these abstract entities in STEM? How can such abstract models become epistemically accessible (available[4]) and amenable to manipulation in imagination (e.g. mental rotation etc.)? In connection, we also ask, what entails STEM cognition and how is it made possible? These broad questions are at the root of what this thesis attempts to address.

Ways of thinking and addressing these questions are recently emerging from cognitive sciences informed analysis of STEM. Our study is broadly situated within the paradigm shifts in theories of cognition and learning leading to the constitutivity hypothesis. We extend these reflections to the nature of learning abstract models in STEM (STEM Cognition) and examine the role of representational media, particularly digital media, in learning. In this section, we review the relevant literature and develop a theoretical framework for the thesis, which informs its research questions and their operationalisation.

1 Theoretical Background

We begin with a review of the paradigm shifts in the theories of cognition — the field theories or 4E models of cognition — capturing the constitutivity hypothesis. Then we review the emerging and widely accepted notions of STEM cognition drawing from the characterisation of STEM practices. Then we outline the scope of the thesis and develop a framework that leads us to the corollary to be tested by this study and in turn the research questions.

---

[4]This usage is along the lines of epistemic availability (O'Donovan-Anderson, 1997) used in philosophical explorations on how a cognitive being *knows* anything; in other words, how does anything become epistemically available to the *knower*?

1.1 *Embodied nature of cognition: The Constitutivity Hypothesis*

Embodied and allied models of cognition, often together referred to as 4E models (e.g. Glenberg, 2010; Hutchins, 1995b; Lave & Wenger, 1991; Sterelny, 2004; Thelen & Smith, 1996; Van Gelder, 1999), offer useful perspectives towards understanding the STEM learning problem. The central claim of these models is the constitutivity hypothesis which can be understood in two levels:

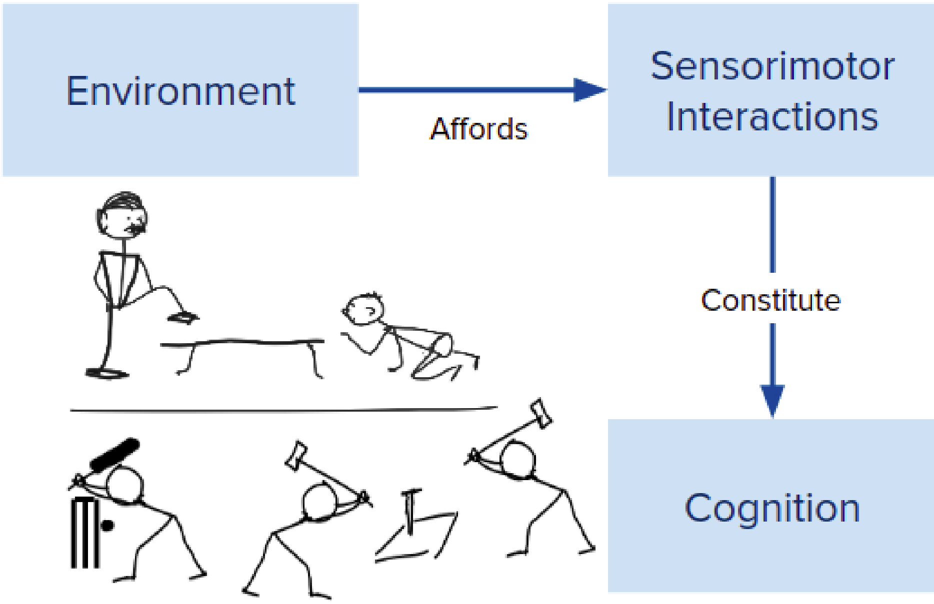


Figure 1A.1: Constitutivity Hypothesis

Cognition is linked to (constituted by) sensorimotor interactions between the body and the environment. (Vertical arm in figure 1A.1)

> *Cognition is constituted[5] by sensorimotor interactions between the body and the environment.* (Glenberg, 2010; Clark & Chalmers, 1998; Noë, 2004; Hutchins, 1995): Cognition emerges from the dynamic action-perception coupling between the body and the environment. For example, a pilot's cognition is distributed in the sensorimotor interactions s/he has with various artefacts (like speed-bugs) used in the cockpit[6].

[5]According to this view, cognition 'comes into being' through sensori-motor interactions between the body and the environment, which are coupled to each other. This is in contrast to the notion of cognition as a centralised process independent of the body, the environment. We extend the notion of environment to include MERs (P. Pande & Chandrasekharan, 2017) by emphasizing the concreteness of MERs as inscriptions on representational media, as discussed in section 2.3.

[6] See (Hutchins, 1995b) for a detailed case study.

The traditionally held notion is that cognition is information (abstracted from symbols) processing in a centralised processor — modelled closely as a computer — involving computations and heuristics like search etc leading to intelligent behaviour like problem-solving (Boden, 1988; Newell & Simon, 1976). The 4E paradigm of cognition argues for the '*constitutive*' nature of sensorimotor motor interactions with the environment in human cognition. Cognition is *constituted* in the sensorimotor interactions between the physical body and various artefacts (physical and social) in the environment. Cognition is not centralised in the brain, but distributed and extended into the environment (e.g. A. Clark & Chalmers, 1998; Hutchins, 1995a, 1995b; Sterelny, 2004); cognition is tightly connected to the contingencies of the situation and hence an embedded or situated process (e.g. Lave & Wenger, 1991); cognition is a dynamic enactive process (e.g. Noë, 2004; Port & Van Gelder, 1995; Thelen & Smith, 1996); cognition is embodied (e.g. Glenberg, 2010; Varela et al., 1991). This paradigm of models of cognition are collectively referred to as 4E cognition: embodied, embedded, enactive, and extended (4E) cognition. These together indicate that we cognise as we act (at a concrete level of sensorimotor actions).

These sensorimotor interactions are enabled by the affordances (action possibilities) of the environment (Horizontal arm of figure 1A.1)

> *Sensorimotor interactions are afforded by or enabled by the affordances (action possibilities) of the environment:* Actions are linked to the action possibilities of the body-environment system (Chemero, 2003; Heft, 1989). For instance, seeing a hammer — with its materiality of the rigid handle and a hardened head — covertly activates hitting actions. (Turvey, 1992).

Sensorimotor interactions are afforded by or enabled by the affordances (action possibilities) of the environment: Actions are linked to the action possibilities of the body-environment system (Chemero, 2003; Heft, 1989). For instance, seeing a hammer — with its materiality of the rigid handle and a hardened head — covertly activates hitting actions. (Turvey, 1992).

Gibson's ecological psychology, among others, is a key force inspiring, shaping and cohering the 4E models of cognition. The idea of affordances is also central in analysing cognition in the new paradigm, fundamentally questioning the traditional units of analysis. Here, actions are linked to the affordances of the body-environment system (Chemero, 2003; Gibson, 1977; Greeno, 1994; Heft, 1989). Affordances are properties of a body-environment system; they indicate the action possibilities primed in a body-environment system. The possibility of an action is determined by the material constraints of the subject of action (actor) and the object of the action (the object in the environment). Thus the material constraints of the actor's body and the environment's object allow only certain actions; in other words, they afford only certain actions. For example, a foot-high flat surface 'affords' a stepping-on action to an adult, but not for a toddler. Here, the stepping-on action is bound to the action possibilities of the body-environment system. In a way, this perspective enhances the agency of the material in shaping the sensorimotor interactions in the body-environment system; i.e., the objects in the environments are not passive receivers of the actions, but the affordances dictate what actions can be performed. For example, we can reach out to a distant object using a long stick but not a long rope — the rigid materiality of a long stick allows an act of reaching out to a distant object; we can wipe a surface with a cloth but not with a brick — the softness of the cloth affords a wiping action. Accordingly, our actions are oriented by affordances in the environment.

These field theories of cognition, essentially, challenge the traditionally held view of cognition as abstract processing of information centralised in the human brain independent of the environment, and present constitutivity hypothesis highlighting the centrality of actions (sensorimotor interactions between body and the environment) in understanding cognition.

Having introduced the currently accepted models of cognition, which emphasise the role of sensorimotor interactions, we shall now proceed to a cognitive characterisation of doing STEM, and understand what entails STEM cognition.

1.2 *What entails STEM cognition in the scope of the thesis?*

What entails scientific thinking and STEM cognition has been of significant interest among sociologists and philosophers of science for some time. The desired state of STEM cognition is either explicitly or implicitly articulated in STEM education research, as well as in the policy documents and curricular frameworks. In this section, we review the currently accepted notions of desired STEM cognition and outline the scope of this thesis.

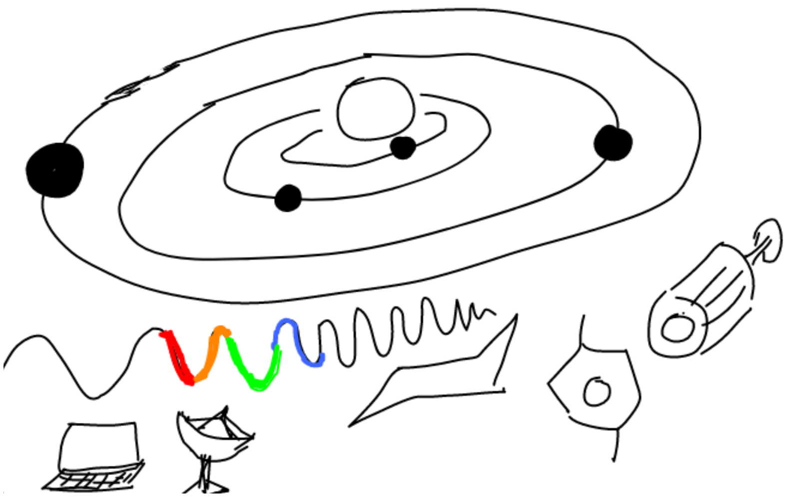


Figure 1A2: Constitutivity Hypothesis

STEM often engages with phenomena beyond everyday perception/action: big (astronomy), small (DNA, atoms); long time scales (evolution), short time scales (molecular events); feedback loops (arms races, oscillating reactions); abstract concepts (taxonomies, models, theories); opaque instruments (measuring equipment, symbols, algorithms) (Prajakt Pande & Chandrasekharan, 2017) (See Figure 1A.2). From a practices perspective, which is gaining wider acceptance, scientific thinking (STEM cognition) involves engaging in following interconnected practices characteristic to doing STEM: modelling, asking questions, planning and executing investigations, analysing and interpreting data, using mathematics and computational thinking, constructing explanations, arguing from evidence, obtaining, evaluating and communicating information (Lehrer & Schauble, 2015). Of the above practices, modelling is widely considered the defining characteristic of science, as it coherently embeds all other practices (Gierre, 1988; Hestenes, 2010; Lehrer & Schauble, 2010; Magnani et al., 2012; Nersessian, 2008).

<u>*Model-based reasoning and imagination*</u>

Post the received view of positivism, philosophers of science have characterised scientific theories as extra-linguistic entities called models: idealisations, simplifications and approximations of the target physical phenomena (Suppe, 1989; Suppes, 1960). Scientific models are essentially representations of structural relations between physical and conceptual entities (e.g. Hestenes, 2010; Suppe, 1989; Suppes, 1960; van Fraassen & Van Fraassen, 1980). Scientific models can be of various forms (e.g. mathematical and computational models, scale models, physical models, model organisms etc). They broadly could subtly shift the epistemic value of scientific knowledge from truthfulness to fitness — how best does a model fit the (data about the) target phenomena (Knuuttila, 2011).

Models, as epistemic artefacts (Knuuttila, 2011), due to their manipulable affordances (Morrison & Morgan, 1999) provide a 'machinic grip' (Pickering, 1995) on the target physical phenomenon. The models, as representations, are simultaneously decoupled from the physical world, thus helping scientists to use them to explore, examine, predict and experiment (through other practices like experimenting etc) about counterfactuals and possible worlds (Knuuttila & Voutilainen, 2003; Mäki, 2009). This machinic grip on, and simultaneous decoupling from, the concreteness of the physical phenomena makes models very powerful tools of thinking (reasoning, mental simulation, imagination etc.; cognition in general). One can safely explore and experiment in the modelling spaces, performing mental simulations, and also guide the provision of simplified and controlled conditions for careful experimentation on, and observation of, the concrete physical phenomena (ranging from subatomic phenomena to astronomical phenomena).

Models, as *imagined concreta* (Godfrey-Smith, 2006), thus, play a very significant cognitive role in enabling human beings to do science, and thus can be thought of as external tools of thinking (Knuuttila, 2011); in short, a model is a not just a component of, but an enabler of, STEM cognition (Knuuttila, 2017). Cognitively, reasoning using models involves rich imagination capabilities such as holding coherent mental models, performing spatial operations on them (like mental rotation), which are *constituted* by material sensorimotor interactions (See the constitutivity hypothesis in section 1.1). Numerous studies on mental rotation and visuospatial reasoning and related abilities of imagination, which are supported by models, have been found to augment reasoning in STEM domains (Prajakt Pande & Chandrasekharan, 2017; Ramadas, 2009; Tversky, 2005; Vosniadou, 2002a; Wai et al., 2009; Wu & Shah, 2004). Hence, a widely accepted requirement for a student to learn science is modelling — the ability to reason and imagine with abstract models, which involves making (constructing) and tweaking (manipulating to test and refine) model systems (e.g. Lehrer & Schauble, 2015; Ngss, 2013).

Scope of the Thesis: Role of Multiple External Representations (MERs)

In STEM, a very wide range of above practices, including modelling, in the form of interactions — sensorimotor interactions with physical material like laboratory equipment and social interactions mediated by language and scientific discourse — embody STEM cognition (e.g. Latour & Woolgar, 1979). We, in this thesis, confine to the practices involving engagement with multiple external representations (MERs) — activities within the symbol systems.

Doing and reasoning in science — described at a very concrete interaction level — involves engaging with multiple external representations or symbolic entities. These include conventional decoding, encoding, using and interlinking varied symbolic entities. In fact, philosophers have always tended to consider models ultimately as abstract entities, and the machinic grip afforded by the models can be seen to be due to the concreteness of the external representations[7] (Knuuttila, 2011). The varied *representational means*[8] in STEM provide varied affordances to think or reason with these abstract models (Kress & Van Leeuwen, 2001; Vorms, 2011; Zhang, 1997). External representations have unique affordances allowing certain types of cognition (Kirsh, 2010) essential for STEM, like explicitness and shareability, physical persistence and independence, manipulability.

Cognition in doing science (STEM cognition) is widely argued to be mediated by these engagements with external representations (Chandrasekharan & Nersessian, 2017; Daston & Galison, 2007; Latour, 1990). Related to this position, there is now a 'constitutive'[9] view of the role played by multiple external representations (MERs) in STEM practice and learning (Landy et al., 2014; Landy & Goldstone, 2007; Prajakt Pande & Chandrasekharan, 2017), which argues that scientific concepts are partly

---

[7] We will discuss the concreteness of the external representations in greater detail in Framework section

[8] We will discuss the *representational means* in greater detail in the Framework section

[9] Mechanisms of this ‘constitutive’ role discussed in detail in the review of cognitive mechanisms under the Framework section

brought into existence by MERs, as most entities and processes examined by current science are not readily available to perception and action. The inaccessible models and entities are therefore accessed and manipulated using imagination based on external representations. In short, the epistemic access[10] to the models and the power of imagination provided by the scientific models emerges from the concrete affordances of the external representations (Knuuttila, 2011, 2017).

1.3 STEM Cognition and Representational Media: SCIARM Framework

Concreteness of Representations – Representational means, mode and medium

As briefly indicated in the previous section 1.2, representational means of MERs provide concrete affordances in building and using (construction and manipulation of) models, thereby enabling the STEM cognition involved in reasoning with models. Any construction or manipulation of a model is materialised through representational means in a representational medium (Knuuttila, 2011; Morrison & Morgan, 1999). For example, MERs are the concrete entities that the cognitive agent interacts with when s/he reasons using abstract models while doing algebra in mathematics using concrete scribbles on paper (Landy et al., 2014; Ottmar et al., 2015; Presmeg, 2006) or doing organic chemistry using Berzelian formulas as paper tools (Klein, 2001). Presmeg (2006) using a semiotic view, acknowledges this concrete aspect of the symbols calling them inscriptions (concrete shapes created using paper and pencil/pen or chalk and board or any other medium) which give access to abstract mathematical objects (just like models), otherwise inapprehensible through senses. To unpack how the representational means enable this, it is important to dig deeper into the anatomy of representational means.

Representational modes and representational medium together constitute the representational means (Knuuttila, 2011, p. 269). Representational

[10] *Epistemic access* to something refers to *availability* of something to *knowing.* This particular notion of *epistemic access* can be traced to philosophers' reflections (O'Donovan-Anderson, 1997) on the way the human system knows anything (or the way anything becomes epistemically available to the human system). Contrasting it with physical access could be useful. Physical access is about engaging with something (as books, lectures or otherwise) physically; it is accessed physically. The epistemic access, in contrast is at a deeper level, and is about engaging with something cognitively; it is epistemically available. Mere physical access does not always necessitate that one can cognitively engage with the content.

mode corresponds to the format of the representation (Vorms, 2011); pictures, speech, written text, audio recording are all different representational modes; it is a relatively abstract aspect — the form — of the representational means. A more concrete aspect of the representational means is the representational medium; this corresponds to the material, the concrete medium — the paper and pencil, or chisel on a rock, or a computer key — with which the representation is produced or in which the representation materialises. For example, the vocal cord embodies/materialises the utterances in the form of speech; paper and pencil materialise the text and picture in the form of inscriptions or traces of writing and drawing respectively; a computer materialises a code (computer program) in the form of lines of text. But how could a representational medium relate to cognition, leave alone STEM cognition? The SCIARM framework developed here works out this possibility.

Representations, the media and STEM Cognition

To illustrate the possible relationship between the representational medium and cognition, we begin with some broad examples. In general, it is widely acknowledged that different types of representations afford different types of thinking; for example, diagrams provide affordances for reasoning and modelling different from plain text or algebraic equations (e.g. Dörfler, 2005). That the external representations shape the nature of imagination can be more clearly illustrated with the following example of operations with different number systems. See figure 1A.2. If roman numerals are used, the required imagination gets very complex, and it used to require seasoned mathematicians to perform a simple arithmetic calculation. Whereas, the same using a changed system of Indo-Arabic system, becomes so simple that a primary school student is expected to perform the arithmetic operation. This example illustrates the way different external representational systems could lead to fundamentally different kinds of STEM cognition (Wilensky & Papert, 2010). A similar argument was made by Bret Victor (2011), a designer who converts the math learning-problem into a human interface problem, and envisions alternate representational systems saying ‘math requires new interfaces’.

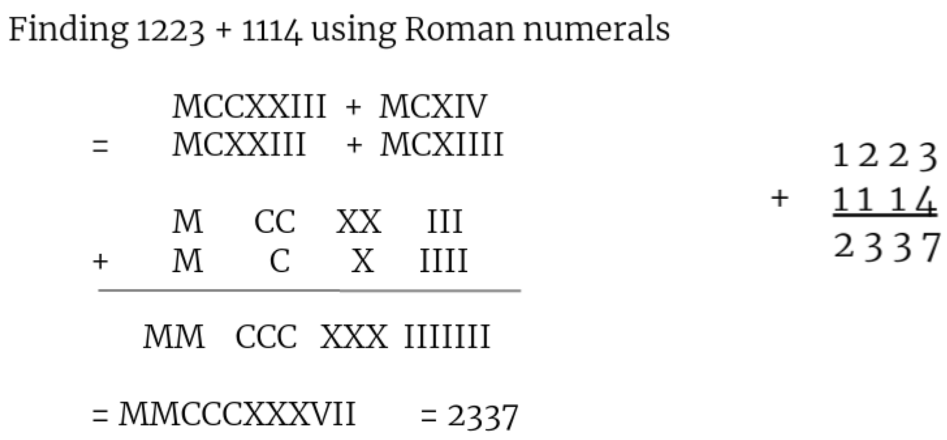


Figure 1A.3. Different representational systems require different imaginations

Deepening this further, there is also evidence emerging specifically for different sensorimotor interactions with different manipulatives and representational mediums having different cognitive processes in STEM educational contexts with learners. A recent eye-tracking based study found that interactions with tangrams trigger different problem-solving approaches in area problems (Rahaman et al., 2017), and another study reported that the interactions with different kinds of blocks and pies produce a different understanding of fractions among students (Martin & Schwartz, 2005). Evidence for such differences is present all the way at the neuronal level. One such case is presented by a study with brain imaging studies of people trained to do abacus-based arithmetic, leading to the formation of a 'mental abacus', and calculations based on this cognitive mechanism. Brain images of this group of participants doing arithmetic calculations using the mental abacus show that the tasks activate visual and motor areas more, as compared to the case of a group of participants trained to do the same calculation using pen and paper (Chen et al., 2006; Hanakawa et al., 2003). This suggests the mathematical operation is implemented differently in the brain, depending on the media through which the operation is learnt. Such results indicate that material limitations/possibilities, and the sensorimotor processes they afford (block/support), can orient mathematical thinking (de Freitas, 2016; Landy & Goldstone, 2007); and that the representational activities in STEM and underlying reasoning are not abstract but constituted by sensorimotor interactions with the concrete symbols (Landy et al., 2014).

These observations and arguments in relation to the representations and the media of interaction are consistent with the proposed role of sensorimotor interactions in learning abstract models, and strengthen the constitutivity hypothesis and embodied cognition approach towards understanding the process of learning, even the abstract models.

Towards a corollary: Extension of constitutivity hypothesis in abstract activities like STEM:

Extending the constitutivity hypothesis, cognition involving abstract symbolic processing, such as understanding of models, is also argued to be constituted by sensorimotor interactions (Glenberg & Kaschak, 2002; e.g. Landy & Goldstone, 2007). Numerous studies show the embodied nature of symbol-based understanding, including

- Language (e.g. Glenberg & Kaschak, 2002; Lakoff & Johnson, 1980): They, using some experiments and arguments, indicate that language comprehension is tightly connected with bodily actions and body-based metaphors.
- Mathematics (e.g. Abrahamson & Sánchez-García, 2016; Andres et al., 2008; de Freitas & Sinclair, 2012; Domahs et al., 2010; Lakoff & Núñez, 2000; Landy et al., 2014; Rahaman et al., 2017): They indicate that different kinds of mathematics emerge from different kinds of actions, using some detailed analysis of human bodily actions in multiple mathematical contexts.
- Similar observations were also made recently in chemistry (P. Pande, 2018)

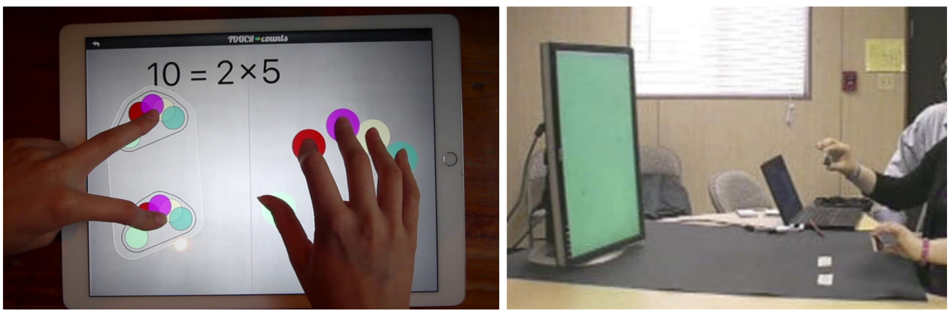


Figure 1A.4: TouchCounts (left); Mathematical Imagery trainer (right)

The specific role of the representational media is also evident in recent explorations using digital media and its affordances for new kinds of STEM (mathematical) thinking. For example, a different number sense can be found in children, when they make numbers by touching a screen, combine number groups to create new numbers, instead of learning them by merely counting objects or fingers (See Touch Counts in figure 1A.4-left) (Sinclair & Heyd-Metzuyanim, 2014). Similarly, making proportional hand movements to change a screen builds a movement-based understanding of proportionality (See Mathematical Imagery Trainer in figure 1A.4-right) (e.g. Abrahamson & Sánchez-García, 2016). These works involved designing new representational media, which allow learning concepts using new sensorimotor interactions, in turn, developing new ways to constitute STEM cognition.

<u>*SCIARM Framework and the Corollary*</u>

All the above discussions can be consolidated into a framework extending constitutivity hypothesis to characterise STEM Cognition. See figure 1A.5, with a block diagram similar to the one developed for the constitutivity hypothesis in figure 1A.1. As with the constitutivity hypothesis, this can be captured in two levels.

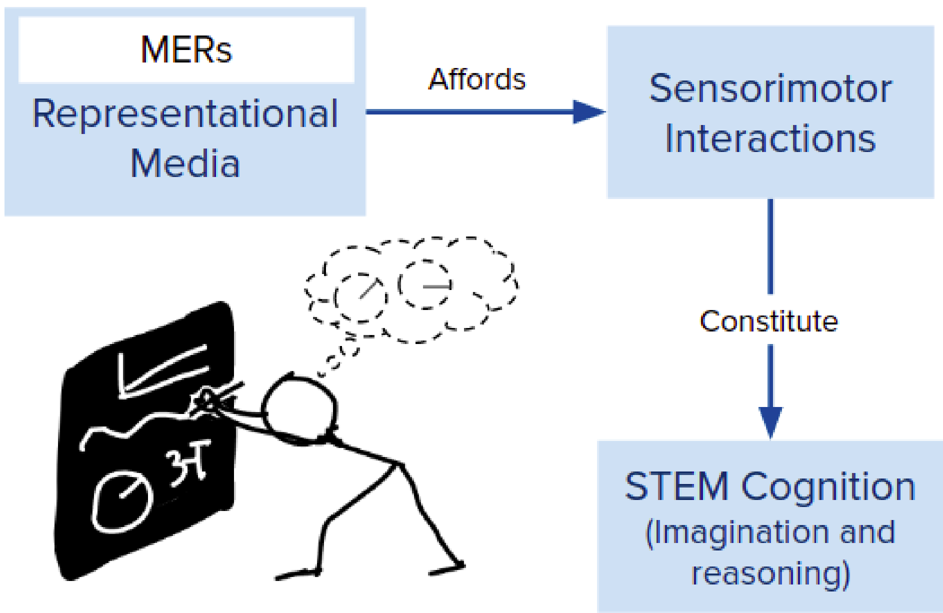


Figure 1A.5: SCIARM Framework (STEM Cognition and Interactive Affordances of Representational Media)

- *STEM Cognition constituted by sensorimotor interactions:* Extending constitutivity hypothesis, STEM Cognition (imagination and model-based reasoning) can be seen to be constituted by sensorimotor interactions between the body and the environment. See the vertical arm in figure 1A.5.

- *The interactive affordance of Representational media:* The sensorimotor interactions are enabled by the affordances of the environment. In STEM learning contexts, the environment, within the scope of this thesis, includes concrete representational media, on which MERs are inscribed and operated on. See the horizontal arm in figure 1A.5.

Overall, given the above discussions around the concreteness of the external representations as inscriptions on the representational media, the following corollary emerges: "The action possibilities (interactive affordances: IA) of the media in which the MERs are encoded (text, computation; representational media: RM) have an effect on the understanding and processing of concepts (STEM Cognition: SC)." This corollary is about a possible relationship between the STEM Cognition and the Interactive Affordances of the Representational Media, also shortly referred to throughout the thesis as the SCIARM framework. And the central objective of the thesis, thus, is to systematically test this corollary, and by extension the constitutivity hypothesis. This framing, thus, gives us the first research question:

"RQ1: Do the interactive affordances of representational media in STEM shape learners' STEM cognition?"

2 Media in STEM Education

As the operationalisation of the investigation to test the above corollary involves the design of a digital media interface (as described in detail in the ch-1B), we review the literature in research in media and learning, especially using technology, and attempt to identify key trends and insights and, gaps in the current understanding of the relation between media and learning. These will further refine the research question that the thesis tries to address.

2.1 Review of Educational technology applications

We start with a review of the various kinds of applications that are developed towards STEM learning, broadly categorised based on the affordances of the media, consistent with the larger theme of affordances under the SCIARM framework. The objective is not to do an exhaustive inventory of all the applications, but do an indicative one covering different kinds of affordances of digital technology that are used to develop learning applications. Some of the key affordances of computational media are 1) enhanced ability to perform computations, 2) ability to provide multimedia support and 3) ability to provide communication, logistical and operational support. We discuss each of these below.

Computational affordances

Computational media come with an immense potential to perform complex computations quicker. Different kinds of applications have emerged using this. Some of them support cognition, by offloading computations, while some act as objects-to-think-with (Papert, 1980). Some examples are below.

- *Direct computational applications:* CAS systems (including forms of spreadsheets), graphing calculators, dynamic geometric environments (CABRI, Geogebra etc.) and simulations platforms (MATLAB, MathWorks etc) of existing complex computations, have been widely used and adopted by students, to offload tedious computations. This strand of applications are very popular, especially in mathematics and engineering education. There are numerous studies on the effect of such platforms in helping students learn (e.g. Arzarello et al., 2002; Calder et al., 2006; Drier, 2001; Kieran & Drijvers, 2006).

- *New computational modelling paradigms:* As an extension of the above applications, affordance of complex computations have triggered new kinds of specialised simulation platforms like agent-based modelling platforms like Netlogo, turtle logo, etc., to acquire skills like computational thinking (e.g. Clements, 1985; Kalelioglu & Gülbahar, 2014; Sengupta et al., 2013; Wilensky, 1999). There are other gaming environments and microworlds like Dragon Box (Siew et al., 2016), ThinkerTools (White, 1993), etc. These applications broadly act as objects-to-think-with.

- *Artificial Intelligence (AI) and big data related computations*: Recently there has been a surge of interventions under personalised education and intelligent tutoring systems, where AI and machine learning is deployed to identify patterns in student learning and to prescribe lessons as per a validated learning trajectory (e.g. McArthur et al., 2005; Sleeman & Brown, 1982; Woolf, 2010). This learning-specific deployment of analytics is not directed at easing the conceptual aspects of STEM learning, but more at providing logistical support of STEM learning - supporting assessments, planning of lessons etc.

Dynamic and Multimedia affordance- Affordances of novel representations

Computational media offer a rich range of sensorimotor modalities ranging from visual, audio and recently kinesthetic interfaces (evolving further with embodied controllers and virtual reality (VR) based technologies). Another related affordance is the ability to 'run' graphical dynamics. The consequent potential for new kinds of representations has been in the air for some time (e.g. Kaput, 1986), and many applications emerged using multiple combinations of the dynamics and varied audio-visual affordances of a conventional computer and recently, haptic and kinesthetic affordances. Some examples are below.

- *Dynamics of the representations*: The affordance of graphical dynamics is tapped in through a range of application ranging from dynamic visualisations and animations (Ainsworth, 2008; Lin & Atkinson, 2011), and those with active manipulations (DGEs like Geogebra, Cabri etc) (e.g. Falcade et al., 2007; International GeoGebra Institute, 2002). A set of applications took special advantage of the visual affordances to present multiple representations simultaneously. Many of the applications listed earlier in computational affordances also make use of this affordance.
- Multimodal and multimedia technology: This is one of the most widely tapped into affordance. Widely used applications include using audio-visual systems like projectors to present videos, animations, slides etc. Many specialised packages aligned with curricular progressions are also widely available in the market. Besides these, some applications also tap into these affordances for audio-visual (e.g. Mayer, 1997), haptic (e.g. Sinclair &

Heyd-Metzuyanim, 2014) and VR (e.g. Bakas & Mikropoulos, 2003) interactions.

Communicative/Logistical affordance -Affordances of communication of information

This is one of the most directly apparent affordances of computational media, especially since the development of the internet, as evident during the pandemic. Central to this affordance is the feature that digital technology has allowed access to information across space and time. This affordance is widely applied through various kinds of technological solutions to the learning problem, particularly for scaling the reach of education. Information transfer is at the heart of these applications. Some examples are below.

- *Electronic versions of educational resources:* These mostly change the form of existing sources of information and overcome the barriers of physical access. These include electronic versions of material such as textbooks or reference books — the traditional centralised sources of information — taking the shape of ebooks, pdf, etc. Other forms are repositories and online platforms like Wikipedia, open educational resources (OERs) etc (Geith & Vignare, 2008; Reddy & Mukherjee, 1915; Woody et al., 2010) — the new avatars of traditional resources like encyclopedias.
- *Online video lectures:* Along with the curated resources, online video lectures have been acknowledged to have an immense potential to increase physical access to quality learning resources at scale, by bringing an additional factor of instructor enaction around the content available in live classrooms. These include centralised MOOCs like platforms e.g. edx and Coursera (Liyanagunawardena et al., 2013), and even crowd-generated videos on platforms like Youtube. These are essentially electronic versions of traditional brick and mortar classroom lessons.
- *Discourse platforms:* These include internet and intranet (e.g. Clark-Wilson, 2010) based applications for synchronous and asynchronous communication through messaging portals, emails, recorded sessions, etc. These are mostly used to coordinate

activities and deliver information or content like homework, assignments etc. These are already part of the lived worlds of the members of societies with better access to digital infrastructure, in the form of emails and social media and instant messaging applications.

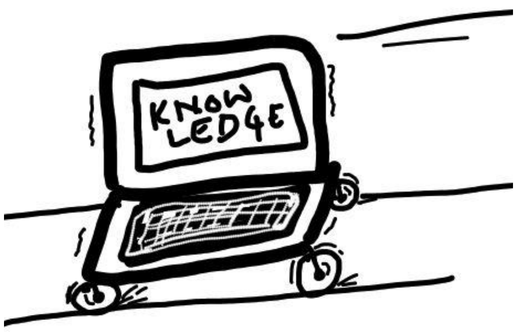
- *Miscellaneous applications:* These include aids in classrooms like clickers (Penuel et al., 2007) and learning management systems supporting data management both for schools and individual teachers at varying levels of granularity — e.g. broad classroom level indicators of student attendance, performance, or very detailed indicators of individual students. These basically ease various logistical and administrative issues in teaching and learning activities.

The above is a quick review of various example applications, using a wide range of affordances that digital technology offers. While all the above applications are intended to support learning outcomes and help shift the learning towards desired modes of STEM cognition, not everything appears to be undisputedly successful, as outlined in the next sections.

2.2 *Does Medium influence learning? Debates and directives*

Educational researchers have long noted the potential of digital media to develop novel MERs and thus improve learning (e.g. Kaput et al., 2002; Papert, 1980; Tall, 2000). However, though a range of interventions was tested by researchers, and marketed by commercial agencies, this promise has not materialised fully (e.g. Hew et al., 2007; Kirkwood & Price, 2014; Mason, 2005; Masood, 2004; Russell, 1999; Tinio, 2003, p. 17). Many reviews examining the effects of using computational media-based interventions on learning indicate mixed results (e.g. Cheung & Slavin, 2013; Cuban & Kirkpatrick, 1998; Ross et al., 2010; Sancho-Gil et al., 2019). For instance, a review of personalisation using intelligent-tutoring-systems indicated no significant or very small positive effects but had no negative effects on students' learning (e.g. Steenbergen-Hu & Cooper, 2013). A similar pattern can be found with MOOCS like platforms which are reported to struggle with student-engagement, as indicated by high dropout rates (e.g. Clow, 2013; Jordan, 2014; Phan et al., 2016; Xing et al., 2016). Hoyles and Noss (2003) summarise this groundless state well by noting that - "the use of software seems to affect learning in unpredictable ways, and learning appears to be sensitive to even small changes in technology". All these indicate a need for more careful examination and a measured application, instead of a sweeping and unconstrained imposition of the computational media for learning.

At the heart of the possible sources of these problems could be what Papert calls a 'technocentric' approach (Papert, 1987). Solving a learning problem is not merely a technological problem. While he acknowledges the potential computers have to enable learning, he critiques the perspective towards technology as a context-independent solution. He critiques the futility of dominant research questions about the efficacy of educational technology by comparing them with those like "Would wood make a good house?" or "Would a hammer help make good furniture?" A similar stance could be found emerging from the debates about media influencing learning.



The following arguments clearly capture some of the nuances in the use of media for learning. Clark argues that the media will never influence learning (R. E. Clark, 1983, 1994), as it is merely a vehicle of delivering instruction, and hence the comparative studies of multiple medium based interventions are meaningless. A systematic critique of the existing multimedia solutions also questions the efficacy of such applications, and use of such research, highlighting and dismissing the underlying assumptions of such studies (R. E. Clark & Feldon, 2005). An underlying issue behind all these critiques is common with the Papert's notion that media is not a panacea to all the learning problems, and that learning involves far more complex processes and that technology is, at best, just an instrument mediating these processes. So, any instructional medium is just as good as the way it is used, which Clark refers to as the *method*. Kozma (1994) acknowledges the validity of the concerns behind Clark's critiques and seeks to reframe the debate, as also seconded by Reiser (1994). A summary of these debates indicates that further research on media and learning requires grounding the theory of media in cognitive and social processes of knowledge construction. Also, the computational media changing the educational space is speculated to be a slower process than what most media enthusiasts currently expect (Reiser & Dempsey, 2012).

The way ahead for future pursuits should meet some of the following requirements, emerging through the arguments in the above debates. These involve developing a better understanding of the process of learning, along with clarity on the specific roles media play in shaping the learning.

- *A need for medium+method approach:* Medium is not an independent artefact, the presence of which will magically transform the learning. Clark argues that efficacy-based studies are prone to confound medium with method or content of instruction. Methodologically, it is not possible to tease out the effect of medium and the effect of the method in such simple studies. In fact, this is central to Clark's contention that a medium does not affect learning, as well as Papert’s criticism of technocentric interventions. A medium is just as effective as its usage in the instructional process. A teacher with a good method could deliver far better learning outcomes than a shabbily implemented media intervention. Future research needs to adopt a systems approach, paying attention to the contextual and implementational aspects when examining media in instructional contexts. A lot of emerging research in the field of computer-supported collaborative learning and classroom execution of technology (e.g. Kumpulainen & Kajamaa, 2019; Wilensky & Stroup, 1999) begin to pay attention to the method as well as the media, and their interactions.

- *A need to shift from the notion of mode of delivery:* Traditionally media is seen as a vehicle for delivering or presenting information (R. E. Clark, 1994; Ross et al., 2010), as can be seen in some of the listed software applications like ebooks, MOOCs, and even simulation-like platforms. The mere replication of paper-based content using digital media is not found to have any effect (e.g. in reading context: Hou et al., 2017). However, given the advances in our understanding of cognition, we can note that affordances of media can have a nuanced role in shaping cognition, and those nuances need to be incorporated when conducting studies about media and learning. Hints of this can be seen in suggestions for future directions examining digital technology as cognitive media (Hokanson & Hooper, 2000). Some recent explorations inspired and informed by 4E cognitive paradigms are in line with these recommendations (e.g. Majumdar et al., 2014; Ottmar et al., 2015; Shayan et al., 2015; Sinclair & Heyd-Metzuyanim, 2014). One must note that the acknowledgement of this need — to shift from the medium as a mode of delivery — has underlying assumptions about the nature of knowledge or the process of knowing, which is discussed next.

- *A need for a changed notion of knowing:* In recent theories of knowing, knowledge has moved from an objective transferable piece of information to a more dynamic construction by an individual student, with respect to his/her prior experiences, informed by constructivist paradigms (Duffy & Jonassen, 2013; Jonassen et al., 2003). Further, with refinement towards constructionism, this knowing process is grounded in the material interactions during the building process. This characterisation of knowing is consistent with the 4E cognitive models, which are slowly evolving towards similar models of dynamic and constructive generation of knowledge. The knowing process is grounded in a distributed mechanism of sensorimotor interactions in the body-environment system (see constitutivity hypothesis, section 1.1). Instructional designs need to be aligned with this changed notion of knowing. Triggers in this direction can be found in the form of acknowledgement of the 4E nature of cognition in the education research community (e.g. Winn, 2003) and consequent explorations including immersive environments like virtual and mixed reality systems (e.g. Gabert, 2001; Lindgren et al., 2016).

- *A need for attending to processes in learning:* In line with all the above considerations, a need to pay careful attention to the processes of learning is articulated in some studies (e.g. Kozma, 1994). In the STEM learning scenario, especially with the media in the schemes of things, attention needs to be paid to a range of processes. Researchers began to note that studies can be more insightful by delving into thicker data with methodological pluralism than being stuck only to the traditional metrics of rigour like statistical significance (Hew & Brush, 2007; Reiser, 1994; Ullmer, 1994). Beyond attending to these processes through group discussions (e.g. White, 1993) and clinical interviews (e.g. Abrahamson & Sánchez-García, 2016), a lot of studies now employ novel process data collection and analysis techniques like eye tracking and learning analytics (e.g. Frisch et al., 2010; e.g. Holsanova, 2014; Shvarts, 2018).

2.3 Technology adoption in schools, especially in a developing country context

Another related issue, also stemming from the technocentric approach, reflects in the technology adoption by the educational system, which has not kept pace with the development of the technology itself (Mason, 2005). Despite numerous applications, new media have not changed classroom practices significantly, particularly in developing countries like India (Motiwalla, 2007), as teaching and learning processes are still centred around textbooks. For instance, the Indian education system, characterised as 'textbook culture' (Kumar, 1988), has mostly remained unchanged. Unlike other domains of our lives, such as banking and hailing taxis, that have changed drastically, traditional classrooms have been in conventional instructional modes for far too long. As Kaput and Roschelle (2013) note, our ancestors from 100 years ago would not be as much shocked by our schooling system, as they would be with our overnight shipments and telecommuting culture.

Several issues of implementation and scaling-up have hindered the adoption of technology especially in developing countries like India (Naik et al., 2020). Limited infrastructure (e.g. high student-to-computer ratio) is one of the key factors especially in such contexts (e.g. Banerjee et al., 2007; Carrillo et al., 2011). Projects like one-laptop-per-child also faced similar implementation-related issues (e.g. Kraemer et al., 2009; Nugroho & Lonsdale, 2010).

Some of the possible reasons and ways to grasp this lack of integration could emerge from the examination of existing popular media artefacts, like the textbooks. In developing nations like India, textbooks are used to optimize the difficult problem of educating students at a massive scale

(~1.5 million schools, ~8.7 million K-12 teachers, 257 million students (Bhattacharya et al., 2018)) using limited resources. Textbooks are cheap and widely available, and they play a crucial artifactual role in shaping teaching practice, helping teachers organise their thinking, workflow and classroom interaction. Students too prefer textbooks (Woody et al., 2010). Teacher training is also focused on textbooks, as they are the only affordable media for millions of rural schools and students and due to the teachers' comfort in using them. Thus, textbooks are still central to classroom practices.

To address various technology adoption-related issues, like resources, attitudes and beliefs, knowledge and skills, subject culture, institutions, and assessments, many recommendations, like resource management, teacher professional development, and reframing assessments, were made by Hew & Brush (2007). Logistical and resource constraints are identified as first-order barriers and teachers' skills, mindsets and beliefs as second-order ones (Ertmer, 1999; Park & Ertmer, 2008). In short media artefacts, to be smoothly integrated into the system, one must take a systemic approach, addressing both the infrastructural constraints and teachers' existing practices and the larger ecosystem.

2.4 Conclusion

In the context of STEM learning, there has been particular interest in using media to address the learning problem, since the advent of broadcasting media like radio and television, which peaked with the development of computational media. The advent of computers has brought with it a promise of a revolution in the educational sector. However, the studies reviewed here could not claim to have solved all learning problems, but have generated active reflections and debates on the role and efficacy of the media in learning. A more nuanced and systematic approach towards the design grounded in social and cognitive processes is needed and this thesis, within the SCIARM framework, seeks to be one such endeavour. Also, limited technology adoption, especially in developing nations, points to a need to pay attention to subtle aspects related to resource constraints and other larger systemic aspects, particularly existing teacher practices. All these indicate a need for systematic frameworks to design computational media to address both cognitive augmentation and classroom adoption in the Indian context. This gives us the second research question:

“RQ2: What does a systematic design of a media-intervention look like in a developing nation context?”

3 The Research Questions

Extending the central objective of testing the corollary from constitutivity hypothesis, the above review of existing research in media and STEM learning helps us frame and provide a research-context for the question. From the application of the SCIARM framework to learning contexts, we see that there are many students like Samantha who may be struggling without epistemic access to, and abilities to imagine with, the abstract models. The SCIARM framework provides an approach to understand the underlying STEM cognition in connection to the interactive affordances of the representational medium. Though, found promising computational media had limited impact on learning, especially in developing nation contexts due to a technocentric approach and limited perspectives relating cognition and technology and limited systematic frameworks for digital designs for cognitive augmentation for smoother adoption in developing nation context. These together guide us to two broad research questions.

1. ***Do the interactive affordances of representational media shape learners' STEM cognition?***

Dominant explorations of the role of media in STEM learning have looked at technology as only a medium of delivery and adopted a technocentric research approach. Systematic research grounded in cognitive and social processes is hence warranted. The SCIARM framework — which is directly emerging from constitutIvity hypothesis about the mechanisms underlying sensorimotor interactions depending on the affordances of the body-environment system — is one such attempt. The application of SCIARM to the learning contexts gives us a corollary that interactive affordance of representational medium shapes students’ STEM cognition. The review of research examining the role of media in STEM learning validates the necessity of addressing this research question. Through this question, we seek to gather empirical evidence to understand the potential relation between STEM Cognition and the interactive affordance of representational media. Also, this can be a good trigger for other nuanced explorations, to develop such design approaches addressing students' learning problems in a grounded manner.

2. ***What does a systematic design of a media-intervention look like in a developing nation context?***

This second question emerges essentially from the limited new media design frameworks in developing nation contexts. As can be seen from the above review, technology adoption is minimal and textbooks are the dominant media artefact shaping the current learning culture. To address the limitations of technology adoption, any media design-based research needs to be sensitive to all the systemic factors, to be actually effective in classrooms. The necessity of such contextually embedded examination of technology is also reflected in the debates about the efficacy of media, and on the emphasis on the need to examine media in a holistic manner. This question, by grounding the research project in the practices of the current educational system, could help avoid the pitfalls of technocentrism as highlighted in the literature. To address this question, we document the process of a design embedded in the larger context of the Indian school system. The execution of the thesis project could serve as an illustration of addressing technology integration issues in Indian contexts like ones done in other contexts (e.g. Karaca et al., 2013).

1B

Operationalisation

The objective of the chapter: To operationalise the research questions using the case of the topic of vectors, reviewing relevant literature, into a series of research studies

Key points

- The topic of vectors is a suitable candidate case to address the research questions.
- Students struggle with the geometric aspect of vectors and show poor algebra-geometry integration.

Studies: We address the above research questions using the case of vectors through a series of 4 studies, a report and 2 design iterations with mixed methodologies.

- Study-1 analyses textbooks to capture the existing media usage in Indian classrooms
- Study-2 probes students to capture patterns in their existing reasoning behaviour
- TFV-1 is the first iteration of the design compensating the limitations in the textbooks
- Study-3 does a detailed longitudinal study for the effect of interactions with TFV-1 in laboratory conditions.
- TFV-2 is the second iteration of the design refining TFV-2 to ensure smooth classroom adoption
- Study-4 is a large scale implementation of TFV-2 as actual classroom lessons to capture the changes in students' reasoning behaviour
- A report on the changed classroom practices during the implementation in study-4

In the last chapter, we have seen that modelling, a key practice in STEM, is cognitively possible due to the affordances of MERs and that the affordances of MERs are tightly linked to the representational medium that materialises them. These along with the constitutivity hypothesis shaped the SCIARM framework and the corollary relating the STEM cognition and the interactive affordances of the representational media. We reviewed the literature on media in education, especially computational media, and identified the lack of clarity in the role that media plays in learning and limited design frameworks in developing nation contexts. These together helped us shape the research questions.

In this chapter, we shall outline how the thesis addresses the research questions, by outlining the operationalisation of the research studies. Details of the series of studies that we conducted, and the larger rationale and methodological coherence of these studies are discussed. More details about the execution and results of the studies are reported in part-2 of the thesis (the empirical part).

1 The case of vectors

The particular candidate case used to operationalise the research question is the topic of vectors. This topic is typically introduced in pre-university level mathematics or physics curricula. It is suitable for addressing the research questions for the following reasons.

- *Vectors and modelling nature:* Many physical quantities (like length, mass, volume, etc) can be described just by magnitudes, for which scalar mathematics is sufficient. However, certain physical quantities (like force, velocity, torque, etc.) are incomplete without capturing their direction. Such quantities can be mathematically handled using special entities called vectors, which capture both the magnitude and direction. Thus, vectors provide a suitable mathematical system to model various physical phenomena, ranging from those in mechanics (with physical quantities like momentum, force, displacement being vectors) and electromagnetics (with electric and magnetic forces and fields being vectors), to many engineering systems. Understanding the mathematics of vectors (including operations like addition, dot and cross products, resolution) is inevitable in advanced physics and engineering. In confirmation, studies attribute a lack of understanding of vectors as a reason to the problems students face in Newtonian dynamics (Flores et al., 2004; Shaffer & McDermott, 2005; White, 1983) and electricity and magnetism (Pepper et al., 2012). From the SCIARM framework — pertaining to the role of MERs through representational media in affording reasoning, imagination (manipulations on models) — the topic of vectors, with its modelling nature, is apt for the investigation of STEM cognition among students.

- *Vectors and multiple external representations*: Besides their modelling nature, another key feature of the topic of vectors comes from the multiple external representations used in this topic. A vector is a geometrical/spatial entity represented by a ray of a given length in a given direction, corresponding to the respective magnitude and direction of the physical quantity, like force, field, momentum etc. Besides this, a vector can also be represented algebraically, in a coordinate frame, as a linear combination of components (usually three rectangular components *i, j, k* along x, y, and z axes, but can also be of higher dimensions)[11]. The topic of vectors having these multiple external representations makes it an apt topic to examine the affordances of representational media in STEM cognition.

- *Vectors and spatial nature:* As already briefly hinted, vectors have a very strong spatial nature, as they are fundamental geometric entities, allowing the mathematical representation and modeling of physical quantities with their directions. Reasoning with vectors using these representations invokes visuo-spatial reasoning, at least up to the 3 dimensions, which are used in most beginner level physics applications. Cognitively, meaningful engagement with this topic requires rich imagination capabilities to hold and operate on mental models. This spatial nature of vectors makes them suitable to examine STEM cognition. In a sense, the topic of vectors is similar to the cases like the sun, earth and moon system, which is

[11] Often, a third representation in the form of a row or column matrix is also used by mathematicians. But in the thesis, sticking to the contexts of Indian high-schools, we focus on the geometric (arrow) and algebraic (*i,j,k*) representations.

widely studied in the context of mental models and visuo-spatial reasoning (Padalkar & Ramadas, 2011; Vosniadou & Brewer, 1992). Hence the insights from this study on vectors can have wider implications to other topics with spatial nature.

- *Vectors and learning problems in students:* The topic of vectors, similar to calculus, is critical in the transition from high school to college, leading up to higher education in STEM. Literature (reviewed in the next section) indicates that students struggle a lot in imagining with vectors. Students' difficulties with the directional and graphical/geometrical aspect of vectors, integrating geometry and algebra, and their preference for algebraic forms (rectangular components) is a recurring finding in most studies (Aguirre, 1988a; Barniol & Zavala, 2014; Dorier & Sierpinska, 2002; Heckler & Scaife, 2015; Knight, 1995; Liu & Kottegoda, 2019; Nguyen & Meltzer, 2003; Shaffer & McDermott, 2005; Usharani & Meera, 2018; Wutchana & Emarat, 2011). These studies provide valuable insights, and capture a range of misconceptions and student difficulties. They also indicate the necessity to engage with the geometrical aspects of vectors, to develop an adequate understanding. However, attempts towards addressing these difficulties have met with little success in most cases. The topic of vectors thus provides a productive context also to examine the possible role of interactions with new representational media. The empirical exercises in the thesis can thus provide numerous insights into the learning problem of vectors and could make direct pedagogical contributions.

2 The learning problem of vectors

Many studies have reported the difficulties students face in handling vectors, from introductory physics levels (Aguirre, 1988b; Aguirre & Erickson, 1984; Aguirre & Rankin, 1989; Flores et al., 2004; Knight, 1995; Nguyen & Meltzer, 2003; Shaffer & McDermott, 2005; Wutchana & Emarat, 2011) to postgraduate levels (Usharani & Meera, 2018).

2.1 DIfficulties with formal knowledge and physical intuitions

Students are reported to encounter conflicts between formal vector knowledge and their intuitions about the situations provided. Aguirre's group tested students' knowledge of vectors in the context of physical applications like a boat in a river (Aguirre, 1988a; Aguirre & Erickson, 1984). They report students' errors in estimating the resultant vector capturing their models of vector addition. To estimate the resultant of two physical quantities (say forces), students consider the two forces to compete against each other. Hence they consider the magnitude of the resultant vector (force) as either the largest of the two, or the difference between the two, or a simplistic sum of the two. Similarly, with respect to the direction of the resultant vector, some students qualitatively intuited that the resultant (its direction) is in the middle of the two component vectors, or along the one with larger magnitude. Further, irrespective of the correctness of the intuitions, students struggle in translating these qualitative judgements into a quantitative description of direction.

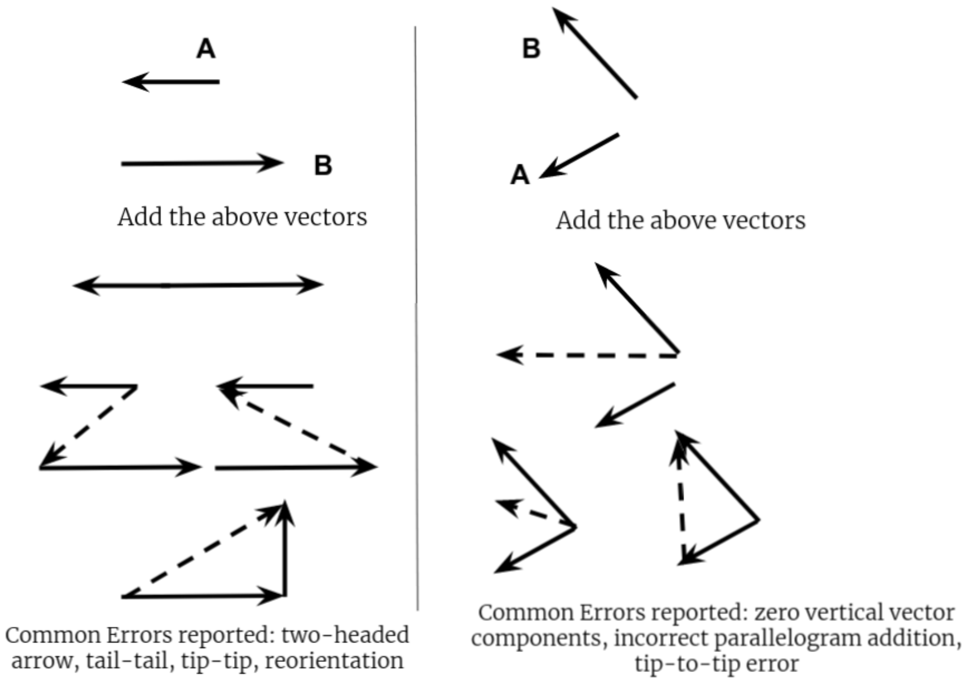


Figure 1B.1: Common errors in adding vectors, Partially adapted from the study by Nguyen and Meltzer (2003)

This translation needs working with frames of reference and the effect of components' direction. In example cases like the velocity of a boat in a flowing river students were reported to struggle with reference frames. Students confuse the very nature of vector quantities and extend their judgements of vectors like velocities to time. A significant 40% of students were found to use Pythagorean theorem for the time of the motion, resultant of two orthogonal velocities (Aguirre & Rankin, 1989). In conclusion, the authors find that about 50% of students struggle with the conflict between formal vector knowledge and their intuitions about the situations provided. This broadly is in agreement with the general finding in physics education that students come with intuitive resources to understand formal physics (Clement, 1982; Halloun & Hestenes, 1985; Vosniadou, 2002b).

2.2 *Difficulties with geometry, comfort with algebraic addition and poor geometry-algebra integration*

Other studies (Flores et al., 2004; Nguyen & Meltzer, 2003), in the context of mechanics with physical situations, have explored aspects of formal vector knowledge. They report that about 50% of students are unable to add and subtract vectors graphically after traditional teaching. Nguyen and Meltzer study around 200 introductory physics students and characterise the kind of their mistakes in adding vectors graphically. When adding two collinear vectors graphically, students err in making the resultant by joining both the tails or heads or by rearranging them as a right triangle (shown by the dotted lines in the left panel of figure 1B.1). Similarly, for a resultant of two non-collinear vectors (shown by the dotted lines in the right panel of figure 1B.1), common errors were making the resultant along the horizontal axes (with magnitude equal to the sum of horizontal components and different vertical components), parallelogram like implementations, and tip-tip method. Knight (1995) also found that students struggle in capturing the direction of a vector numerically (in situations where rectangular components were given) and with geometric addition of vectors. A common mistake was joining the heads (tip-tip) of the vectors (as reconfirmed by Nguyen and Meltzer's study). Students performed better in addition using rectangular components.

Wutchana and Emarat (2011) also found that less than 35% of high school students who have taken lessons on vectors can add vectors graphically, both in 1D and 2D. The difficulties with geometrical methods were reported even among physics PG and graduate students, and among pre-college teachers (Shaffer & McDermott, 2005; Usharani & Meera, 2018). Barniol and Zavala (2014) reviewed the literature of errors found and studied over 2000 introductory physics students, and present a range of mistakes related to the geometric understanding of vectors. The results confirmed the struggle students go through while working with graphical/ geometrical forms of vectors.

Interestingly in most of these cases, students were found to be comfortable using algebraic representations (e.g adding using rectangular components) (Heckler & Scaife, 2015; Usharani & Meera, 2018). In parallel, mathematics education research on vectors shows that learners rely on memorised formulae and mechanical use of algorithms while learning vectors, and a geometric understanding of vectors is usually lacking (Dorier, 1998; Dorier & Sierpinska, 2002; Dreyfus et al., 1998; Hillel, 2000). A recent study, confirming the above findings, reports the lack of correlation between students' algebraic and geometric reasoning in vectors (Liu & Kottegoda, 2019). Students tend to rely on rote methods to learn vectors and perform algebraic manipulations mechanically using the *ijk* components (Aguirre, 1988a; Knight, 1995; Usharani & Meera, 2018), without a coherent geometry-algebra integration. The lack of this integration is reflected in students' struggles with vectors while modelling physical phenomena (Aguirre & Erickson, 1984; Aguirre & Rankin, 1989). This integration of different representational modes is widely acknowledged as required for modelling activities (e.g. Hohenwarter & Jones, 2007; Madden et al., 2011; Van Dooren et al., 2013).

2.3 Existing explanations

Identification of the fundamental causes of student difficulties could allow pedagogically meaningful design decisions. The literature on the possible sources of student difficulties is reviewed in the next paragraphs.

Representational issues

Van Deventer & Wittmann (2007) report student difficulties stemming from differences in the representation of vectors between mathematical and physical contexts. Dray and Manogue (1999) discuss the differential viewing and handling of vectors in the domains of physics and mathematics (including the representations) and their influence on teaching and consider these as potential sources for student difficulties. Heckler and Scaife (2015) report that the arrow representation, as against algebraic (ijk) representations, could cause difficulties with vector addition and subtraction. Hawkins et al. (Hawkins et al., 2010) show that the way the arrows are positioned for geometric addition (head to tail arrangement as well as the grid shown in figure 1B.2) could affect student responses. These studies indicate possible representational sources of student difficulties.

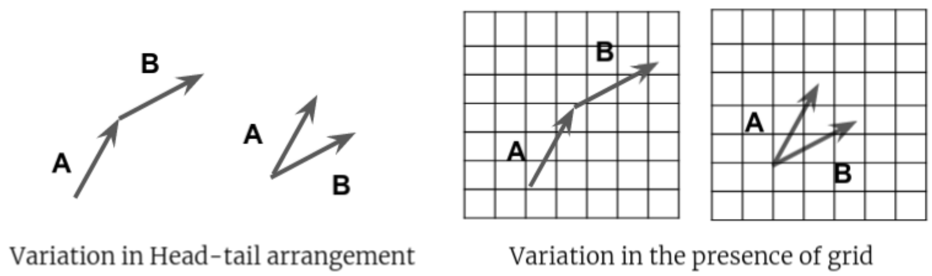


Figure 1B2. Changed representations affecting student responses adapted from ((Hawkins et al., 2010) 2010)

Lack of integration of prerequisites and prior habituation to scalars

Within physics, vectors are represented and operated both geometrically (some studies refer to this as graphical, but we use geometrical throughout this thesis for consistency) and algebraically. The translation and the resulting equivalence of the two forms (geometric and algebraic) of the same vector, appear not to be appreciated enough by students, as reported by some of the studies listed earlier. Understanding these operations requires complex combinations of prerequisites, like the geometry of lines and triangle and trigonometry of right triangle and circle, which students struggle with (Byers, 2010; Gur, 2009; Orhun, 2004). Byers (2010) notes that the disconnections between the related topics (emphasizing trigonometric ratios) in the curriculum as a possible source of student difficulties. Further, the vector topic is a new mathematical entity for students at the introductory physics stage, as they are only familiar with working with scalars, with simple arithmetic operations using magnitudes. White (1983) cites this switch in the nature of operations, from scalar to vector, incorporating directionality, as a potential source of the difficulty.

Some remedial attempts

Dubinsky (1997) reviews various sources of difficulties, including those of Carlson (1993), and identifies pedagogical limitations in allowing opportunities for interactions with mathematical concepts, as well as social agents, as a source of student difficulties, among others. In mathematics education, studies recommend the introduction of linear algebra, using geometric representations and visualisations, as this is more intuitive and concrete for students to learn (Harel, 1990, 1999; A. C. Konyalioglu et al., 2011; S. Konyalioglu et al., 2005). Courses in introductory linear algebra (Dorier et al., 2000; Harel, 1990) and physics (Flores et al., 2004) with such a modified approach have reported varying levels of impact. Alternatively, researchers have also used computer-based intervention and dynamic geometry environments to improve students' understanding of vectors (Donevska-Todorova, 2018; Dreyfus et al., 1998; Mikula & Heckler, 2017). Hestenes (1988; 2012) proposes a new mathematical system called geometric algebra (also called Clifford's algebra), which seeks to address a version of this incoherence in using vectors in physics and mathematics[12] and also address the geometry-algebra integration issue.

Flores et al (2004) note that the modifications made to vector courses resulted in moderate success but the errors are still presented by students, and the authors explicitly acknowledge that the root cause of the difficulties is not trivial. Liu and Kottegoda also note the need for an in-depth investigation into student-reasoning approaches (Liu & Kottegoda, 2019). All the above studies characterise student errors well and discuss possible ways of addressing the difficulties, but a systematic analysis for the possible sources of these difficulties is still needed, as explicitly acknowledged by some studies.

2.4 Conclusion

In conclusion, student difficulties in mechanics (Flores et al., 2004; Shaffer & McDermott, 2005; White, 1983), as well as electricity and magnetism (Pepper et al., 2012), have been attributed to the lack of understanding of vectors. Numerous studies in physics education indicate the struggles students face while learning this topic, from introductory levels of physics to postgraduate levels (Usharani & Meera, 2018). Mathematics education studies (as linear algebra) also indicate similar difficulties (e.g. Dorier & Sierpinska, 2002; Dreyfus et al., 1998; Harel, 1989; Hillel, 2000). Students struggle with reconciling their intuitions about physical contexts and judgements about resultant vectors (Aguirre, 1988a; Aguirre & Erickson,

[12] Clifford Geometry is argued to be the natural extension of real numbers to include the geometric idea of direction (Chisolm, 2012). Hestenes argues that this could provide a unified language for mathematics and physics.

1984). Studies on formal vector operations (without explicit physical contexts) report students' difficulties to add and subtract vectors graphically (Flores et al., 2004; Nguyen & Meltzer, 2003). Other studies confirm struggles with the directionality of a vector, and report difficulties with graphical/geometrical methods, as against the use of rectangular components for addition (e.g. Barniol & Zavala, 2014; Heckler & Scaife, 2015; Knight, 1995; Shaffer & McDermott, 2005; Usharani & Meera, 2018; Wutchana & Emarat, 2011). Students lack coherent models inter-linking concepts (Dorier et al., 2000) to imagine and reason with vectors, and rely on mechanical manipulations using the $i^j^k^$ components. Broadly, students appear to have difficulty in making sense of the geometrical aspect of vectors and integrating it with algebraic aspects of vectors (Bollen et al., 2017; Liu & Kottegoda, 2019). Overall, students have poor geometry-algebra integration.

3 Operationalisation - a prelude

Given the amenability of the topic of vectors to extend our testing of the corollary, based on the SCIARM framework, we will now outline a series of studies to operationalise the research project, specifically addressing both the research questions. Before we dive into the specific plans of action, we operationalise some broad aspects and break down the research questions into sub-questions.

3.1 STEM cognition operationalised in conceptual reasoning behaviour

STEM cognition involves reasoning in STEM contexts, as discussed at length in chapter-1A. This involves reasoning with models (construction and manipulation), which are constituted by concrete interactions with MERs. The research studies required to capture changes in the students' STEM cognition. Consistent with this focus on practices in the thesis, the examination and description of changes in students' states of STEM cognition are at the level of behaviour in the STEM contexts — the concrete aspects of which are directly accessible. We call this conceptual-reasoning-behaviour (CRB): behaviour exhibited as various concrete utterances (words, writings, diagrams, gestures etc.) in the contexts of conceptual-reasoning in STEM. We characterise states of their STEM cognition as patterns underlying the utterances in their behaviour, exhibited in the use of MERs to reason/think with. The rationale for this is that patterns of usage of MERs in these utterances are indicative of patterns of their ability to reason with models and the coherence of these internal models themselves. This behavioural characterisation of STEM cognition is also grounded in the practice-based characterisation of STEM cognition shifting away from traditional content-based characterizations, where patterns in conceptual understanding are examined. This operationalisation of STEM cognition into CRB has important methodological and analytic implications, as will be evident in the planning (next sections) and the execution (entire part-2) of the empirical studies.

3.2 *Operationalising the learning objectives in the case of vectors*

The review of literature about the learning problem of vectors indicated two key patterns, purely at the level of MERs. They are the limitations with graphical/geometric aspects of vectors and the poor algebra-geometry integration. Towards addressing these problems, which are related to the role of interactive affordances of the representational medium in STEM cognition, we outline the operationalization of STEM cognition through these two factors, in the empirical studies. The key two aspects that we looked for in the students' CRB are:

- *Separate usage of algebraic and geometric modes of representations:* We looked at the patterns and biases, and changes therein, in the separate usage of these representations in students' CRB. During the studies, we created situations (with direct questions and other related and meaningful contexts) probing students' tendencies to use these representations in their reasoning, captured through students' scripts and verbal/ gestural utterances. For example, the use of mathematical calculations or geometric constructions while reasoning to solve a given problem is interpreted as markers of algebraic and geometric modes of reasoning respectively.

- *Geometry-algebra integration:* We looked for indicators, and changes therein, of students' ability to integrate these two representations (algebraic and geometric) in some meaningful way. We created deliberate contexts and situations requiring using algebraic and geometric representations together to probe their abilities to integrate geometry and algebra, across the studies. The contexts involved either direct questions or meaningful contexts that generated problem-solving behaviour. In such contexts, we looked for patterns of conceptual reasoning behaviour. For example, students' simultaneous use of the two representations while reasoning to address a given problem, is interpreted as markers of geometry-algebra integration.

- *Imagining the spatial aspect of vectors:* Besides the above behaviour, related to the use of geometric and algebraic representations of vectors, we looked for other indicators of students' ability to reason with vectors. In particular, we looked for indicators of imagination, such as gestures and other utterances indicating geometric or spatial reasoning with vectors.

All the above indicators are important in examining the modelling practices of STEM cognition. As the topic of vectors has a strong spatial nature and involves the use of MERs, the above indicators of STEM cognition (the abstract process of imagining and reasoning with vectors) were invoked. Their visible (behavioural) nature allowed capturing clear data, as well as rigorous analysis, with little ambiguity.

3.3 Operationalizing the Research questions into sub-questions

A broad objective of the operationalised empirical study was examining the effects of the interactive affordances of the representational media on student cognition in the case of the topic of vectors. This empirical endeavour required developing a systemic interactive simulation design, in a developing country context. The process of developing this design also served as an illustrative model of how to go about developing such a design. The two research questions that we defined in the previous chapter are operationalised in the following section as four studies, involving two design iterations and a study of changes in teacher and student practices in classrooms.

1. RQ1: Do the interactive affordances of representational mediums in STEM (illustrated using the topic of vectors) shape the learners' STEM cognition (displayed by their conceptual reasoning behaviour - CRB)?
    1.1. What is the current state of representational media and its manifestations in classrooms (in the topic of vectors)?

1.2. What is the current state of students' CRB (for the topic of vectors)?

1.3. Is there a relation between the existing representational medium usage and existing students' CRB?

1.4. Does a change in the interactive affordances of the representational medium change the students' CRB?

1.5. What are the larger effects on students' CRB when lessons are taught using the changed representational medium in the classrooms?

2. RQ2: What does a systematic design of media-intervention look like in a developing nation context?

   2.1. What design considerations need to be addressed, to ensure that teachers adopt the system?

   2.2. What changes do we find in student and teacher practices in classrooms, when the new medium is introduced?

4 Plan of Empirical studies

This section operationalises the above sub-questions into specific research studies. This involves developing an operationalization framework, which logically binds together the different empirical endeavours with the research questions. We begin with describing such a scheme (in figure 1B.3), embedding the research studies and their interconnections with the research questions, and then work out some of the implementation considerations.

The studies started with a detailed analysis of the paper-based textbooks, the existing media in classrooms (Ch2A: study-1), and students' existing conceptual-reasoning behaviour (Ch2B: study-2). These two studies address RQ-1 (1.1 and 1.2), by characterising the current state of the representational medium and its manifestations in classrooms (in the topic of vectors), and the current state of the student's CRB (for the topic of vectors), respectively. Findings from these two studies together provided an initial picture of the relation between the interactive affordances of existing media and students' existing CRB, addressing RQ- 1.3.

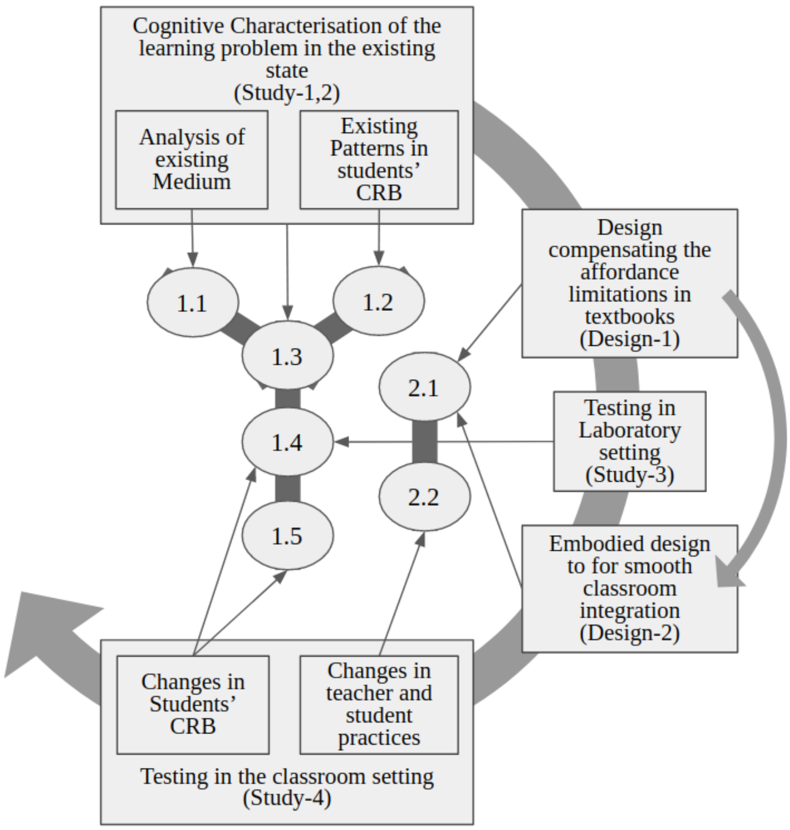


Figure 1B.3. Schematic showing the study plan and the research sub-questions

The findings from these two studies, along with the cognitive models under the SCIARM framework, guided the design of a new-media intervention, across two iterations (Ch2C: design-1 and Ch2E: design-2). Design-1 was implemented in a laboratory setting (Ch2D: study-3), using a small-scale pre-post study protocol. Design-2 was then implemented in classroom settings (Ch2F: study-4), using a large scale control-experimental study protocol. The findings from studies 3 and 4 helped address sub-questions under RQ-1 (RQ-1.4,1.5), showing the nuances of how interactions with designs 1 and 2 changed students' CRB. Design-2, and the practice changes that followed from it in classroom settings (Ch2G), illustrated an effective compensatory design approach, in a developing country context. These data, along with the design iterations, address RQ-2. A summary of this structure is provided in figure 1B.3.

Overall, we follow methodological pluralism and employ multiple methodologies ranging from content analysis (a textbook analysis), interviews (with teachers as well as students), laboratory and field studies (with pre-post, control experimental protocols and case studies) and design-based research. The following sections give a quick indicator of the nature of these studies with some of the details of the implementation, data collection, and analysis in each of the studies. More elaborate details of the actual sample, data collection and analysis protocols are described in respective chapters in part-2 of the thesis.

4.1 Study-1: Existing Medium Analysis

To address 1.1, we analysed the existing medium (dominantly static media like paper-based ones) and its affordances and limitations. This involved a detailed analysis of textbooks and a quick probing of the way these are used by teachers in classrooms. Extending the discussion in the previous chapter, it is worth noting that textbooks have very high institutional authority, and their indispensable nature orient all teaching-learning practices towards writing, such as instruction (chalkboard), classwork and homework (using notebooks), and assessments (written examinations). Even if an atypical teacher puts extra effort and creates lecture notes and exercises (rare in Indian classrooms), these notes would be strongly connected with the patterns used in textbooks to discuss content. Hence, an analysis of textbooks to understand the affordances of the paper-based medium provided a characterisation of the medium and its manifestation in the classroom.

Existing textbook analyses do not specifically look for patterns from the affordances of the medium perspective. Entwistle et al. (1999) conducted a survey of seven physics textbooks used in the US, across topics, listing various limitations in the presentation of the topic of vectors and forces (such as ignoring the notion of directional nature of vectors, among others). Another analysis (Bauman, (1992) examined textbooks to understand the conceptual consistency of definitions, laws and statements. Our study, along with the inputs from the teachers, extends these studies and provides a good indicator of the existing state of affairs in classrooms. Further, the broader interpretation emerging from the SCIARM framework adds more insight into the effect the paper-based media, especially the interactions it affords, could have on students' STEM cognition. More details of this study are provided in chapter-2A.

4.2 Study-2: Learners' Existing Conceptual Behavior

Addressing sub-question 1.2, we examined students' current conceptual reasoning behaviour. We did this by evoking student utterances, using a written questionnaire, followed by a detailed interview anchored to the responses in the written test. We looked for patterns in students' use of representations, while reasoning to solve a problem, explaining their response to a question, and in general discussing some conceptual issues. More details of the student sample, the data collection protocol, and the analysis are presented in chapter-2B.

Addressing sub-question 1.3, these patterns were interpreted, along with the patterns found in the earlier analysis of the existing medium. Based on this analysis, we argue, given the strong theoretical basis for the possible relation between the interactive affordance of representational medium and STEM cognition, that there is a strong overlap between the media and the CRB.

4.3 Design-1: Touchy-Feely Vectors (TFV)-1

There are many existing technology-based interventions for learning vectors, including the popular and research-based ones like Geogebra (International GeoGebra Institute, 2002) and PhET (Perkins et al., 2006). These do address some of the conceptual issues related to the topic of vectors. However, when examining the interactive affordances of representational media, we needed to have more control over some of the fundamental design features, especially those relating to learner interactions and conceptual features. Hence we developed our own in-house design of a digital interface called Touchy-Feely Vectors. The design process is very tightly interleaved with the remaining studies all through the thesis.

Addressing sub-question 2.1, the design of the system was very informed by the analysis of the existing state of media in the classroom, and students' related conceptual-reasoning-behaviour. We designed a new media system systematically addressing the interaction and conceptual limitations identified in the earlier two studies. The design process itself involved taking inputs from 4E cognitive models (constitutivity hypothesis), underlying the SCIARM framework. In this iteration, special attention was given to conceptual issues and the interactive features, through a detailed consultative approach, based on discussions and deliberations between learning science experts, programmers, designers and teachers. More details on this design and analysis can be found in chapter-2C.

4.4 Study-3: Laboratory Study

Addressing sub-questions 1.4 and 1.5, we first looked for the influence of TFV-1 on students' CRB in a lab setting. This study followed a pre-post protocol with a smaller sample of students but captured all the processes — interviewing the students after the pre and post-test, capturing their interactions with the interface. The analysis involved looking for deeper patterns in students' conceptual structure, using a detailed rating method. We also did a qualitative analysis of a limited set of cases, for a better understanding of change in students' reasoning behaviour. Eye-tracking data and video recording of the interactions were also collected, but detailed analysis of these are not included in this thesis. More details of the student sample, the material used, the data collection protocol, and the analysis are presented in chapter-2D.

4.5 Design-2: TFV-2

Addressing sub-question 2.1, we redesigned TFV-1 to better fit the existing educational system, particularly smoother implementation in classrooms. This required seriously considering factors related to accessibility and teacher adoption, which is a difficult problem (Ertmer, 1999; Fabry & Higgs, 1997; Tsai & Chai, 2012). We followed the teacher co-design principle (Tsai & Chai, 2012) to address some of the technology adoption barriers, by involving teachers in the process of designing the lesson plans. Also, this iteration gave us a chance to iron out some of the issues related to the first iteration, as well as incorporate both direct and indirect feedback received from the laboratory study (study-3).

The design involved co-designing the system with teachers, developing a virtual lesson plan and embedding them in the textbooks using the QR codes. These features addressed some of the limitations identified in the literature review, in terms of technology integration and the need to take a systems approach while developing the designs. This design addressed sub-question 2.1. More details of this design can be found in chapter-2E.

4.6 Study-4: Classroom Study

Addressing sub-question 1.5, we looked for the influence of TFV-2 on learners' conceptual behaviour when lessons are taught in real classroom settings. This study involved teachers executing their virtual lesson plans, in a series of classroom sessions, based on a control-experimental group protocol. Both the groups were administered a test triggering the students to use various representations related to vectors while solving or responding to certain questions. Written scripts and other utterances of students were analysed, to develop a clear picture of the nuanced changes in students' CRB, specifically for markers related to geometry-algebra integration, and the epistemic access to abstract models. This analysis required developing our own data analysis framework. More details of the student sample, the material used, the data collection protocol, and the analysis are presented in chapter-2F.

4.7 Touchy-Feely classrooms: Effect on Teaching-learning practices

Addressing sub-question 2.2, we looked for the influence of the design on teaching-learning practices in the classroom setting, and on students' CRB. This involved analysing the video recordings of the classroom session, to identify the differences and similarities in practices, in both students and teachers. This provides a sense of the way the interactive affordances of the media could change the classroom practices and also provides insights into smoother integration of designs into the educational system. More details of this analysis are presented in chapter-2G.

PART-2

EXECUTION

2A

Study-1: Existing Medium Analysis

The objective of the chapter: To capture the *content treatment* by existing paper-based medium using the SCIARM framework.
Methodology: Textbook Analysis and interactions with teachers

Key findings: Three limitations of the paper-based medium manifest in the way the textbooks treat the topic of vectors and the teachers enact or practice.

- Lack of geometric manipulation
- Serial ordering
- Opaque problem solving

Content Treatment is the way a particular topic is presented and treated by an artefact of any medium like the paper-based textbooks. As indicated in chapter-1A, in the Indian formal school setting, the textbooks work as anchors that shape the classroom ecosystems and culture. In this context, this chapter reports an analysis of textbooks revealing the patterns of content treatment, by virtue of them being paper-based, consistent with the STEM Cognition and interactive affordances of Representation medium (SCIARM) framework. We analysed the treatment of vectors in textbooks prescribed under two Indian curricula: (1) NCERT curriculum, one of the well-researched curricula, followed popularly in India, (2) Maharashtra State Board curriculum followed in one of the provinces of India. It is important to note the position of these textbooks as artefacts in the educational ecosystem. These textbooks are written under the purview of and prescribed by the respective boards of education (state-owned agencies). All the schools and colleges (till grade-12) under a board use the corresponding textbooks. The assessments, also done by the board, are based directly on the content in these textbooks. Other materials such as guide books, workbooks, etc produced by private agencies are also primarily based on the content in these textbooks.

Additionally, we reviewed an internationally recognised standard physics textbook at the pre-university level for the treatment of the topic of vectors. Lastly, we also report some relevant findings from the interactions with the teachers indicating the treatment of the content from SCIARM framework.

1 Textbook Analysis: Methodology and Analysis

In the two Indian curricula, the topics related to vectors (including direct prerequisites) span in the textbooks of mathematics and physics (science in lower grades) of grades 8-12. We began with a listing of various concepts that are directly related to vectors and their applications in physics of grades 11-12 (the higher-secondary school level or pre-college level). These include the definition of vectors, denoting them geometrically and algebraically, operations of addition and resolution (and products), and certain applications in mechanics. Then, we identified prerequisites, such as the geometry of lines, angles, triangles and trigonometry (usually covered in the lower grade level mathematics textbooks). We have categorised these vector related broad concepts into 23 units of analysis. See table 2A.1.

*Table 2A.1: The units that are used to analyse the textbooks**

Broad category	Units of Analysis	
Definition	Direction	Magnitude
Resolution	Rectangular Components	Unit Vectors
	Non-rectangular Components	
Addition	Triangle Law	Parallelogram law
	Polygon Law	Algebraic addition
Application in Mechanics	Rotation of Frame of Reference	Resolved Forces
	Resultant Forces	Inclined Plane
Pre-requisites and related topics	Properties of Angles	Trigonometric Ratios
	Unit Circle	Polar Coordinates
	3D components	Trigonometric applications
Scalar Product	Geometric Interpretation	Algebraic Interpretation
Vector Product	Geometric Interpretation	Algebraic Interpretation

1.1 *Coding Schema*

Table 2A.2: ABCDEFGH coding system (#In grades 11 and 12 textbooks are in two parts)

Character	Details	Code	Description
A	Subject (textbook)	1	Physics (Part-1)# /Science
		2	Mathematics (Part-1)#
		3	Physics (Part-2)#
		4	Mathematics (Part-2)#
		5	Others
BC	Grade	08	Grade 8
		09	Grade 9
		10	Grade 10
		11	Grade 11
		12	Grade 12
DE	Chapter	XY	Chapter XY
F	Explanations	1/0	Present or absent
G	Examples	1/0	Present or absent
H	Rigour of presentation	0	not present (not stated)
		1	weak (incorrectly or inadequately stated)
		2	moderate (correctly or adequately stated with no justifications/ evidence/ proof)
		3	weak evidence (stated correctly but with inadequate justifications/ evidence/ proof)
		4	good evidence (stated correctly with adequate justifications/ evidence/ proof)

The above 23 units of analysis are related to each other and the interlinkages are important for the student to grasp the entire conceptual space of vectors. We have specifically captured the way the interlinkages between these units are presented in the textbooks. A 23 x 23 symmetric matrix (see figures 2A.2(a), 2A.2(b)) with the same conceptual units as row and column heads captures the links between the concepts (529 concept-concept links: CCLs) along the rows and columns. There are certain links between each of these topics which are identified to be relevant CCLs (255 out of 529); e.g. links between the trigonometric ratios, unit vectors, the unit circle and the rectangular components, etc. The others (274) where a direct link was not meaningful, were considered irrelevant CCLs and are denoted by the hyphenated cells; e.g. links between the trigonometric ratios and the vector definition. the triangle law and the applications cases such as a mass on an inclined plane, etc. Each of these 255 relevant CCLs was coded using an 8-character (ABCDEFGH) coding system. We used this coding scheme (Table 2A.2) to capture the location (textbook and the chapter, ABCDE), mode (explanations and problem-solving FG) and rigour (H) of content-treatment.

- *Location*: The first character (A) refers to the subject (mathematics, physics, etc.) a CCL is dealt with. The second and third characters together (BC) denote the grade-level of that particular textbook. The fourth and fifth characters together (DE) denote the chapter number presenting that link.

- *Mode*: The sixth character (F) captures the presence or absence of the link in the form of explanations (including descriptions or discussion in the main body of the chapter). The seventh character (G) captures the presence or absence of the link in the form of problem-solving either as worked-out examples or exercises.

- *Rigour*: The eighth character (H) denotes the rigour of presentation in either of the modes rated on a scale of 5 (0-4): 0-not present (not stated), 1-weak (incorrectly or inadequately stated), 2-moderate (correctly or adequately stated with no justifications/ evidence/ proof), 3-weak evidence (stated correctly but with inadequate justifications/ evidence/ proof), 4-good evidence (stated correctly with adequate justifications/ evidence/ proof).

	ABCDEF	Direction (Vector Definition)	Magnitude (Vector definition)	Rectangular Components (Resolution)	Non-rectangular Components (Resolution)	Unit Vectors	Triangle Law	Polygon Law	Parallelogram law	Rotation of Frame of Reference	Resolved Forces	Resultant Forces	Inclined Plane	Properties of Angles and lines	Trigonometric Ratios	Unit Circle	Heights and Distances	3D components	Polar Coordinates	Geometric Interpretation of Scalar Product	Algebraic Interpretation of Scalar Product	Geometric Interpretation of vector Product	Algebraic Interpretation of vector Product	Algebraic addition(using rect comps)
Direction (Vector Definition)	11104113	8	-	11104113	11104111	11104112	11104113	11104000	11104113	11105012	11104011	11102101	-	-	-	21103000	-	41210111	-	11106101	11106112	11107112	11107000	11104101
Magnitude (Vector definition)	11104113	-	8	11104113	11104111	11104112	11104000	11104000	11104113	11105012	11104011	11102101	-	-	-	21103000	-	41210111	-	-	-	11107112	11107111	11104101
Rectangular Components (Resolution)	11104113	11104113	11104113	8	11104111	11104112	-	-	11104101	11105112	11104012	11105112	11105112	-	21008000	21103000	-	41210111	21105000	11106111	11106112	11107111	11107111	11104111
Non-rectangular Components (Resolu	11104111	11104111	11104111	11104111	8	11102000	11104000	11104000	11104000	-	-	11105010	11105000	-	-	-	-	-	-	-	-	-	-	11104000
Unit Vectors	11104112	11104112	11104112	11104112	11102000	8	-	-	-	11104012	11104012	11105000	-	-	-	21103000	-	41210112	21105000	-	11106101	11107111	11107111	11104111
Triangle Law	11104113	11104113	11104000	-	11104000	-	8	11104000	11102102	-	-	11104102	-	-	-	21103000	-	-	-	-	-	-	-	11104000
Polygon Law	11104000	11104000	11104000	-	11104000	-	11104000	8	11104000	-	-	11104000	-	-	-	-	-	-	-	-	-	-	-	11104000
Parallelogram law	11104113	11104113	11104113	11104101	11104000	-	11102102	11104000	8	-	-	11104013	-	11104112	21008000	21103000	-	41210000	-	-	-	-	-	11104112
Rotation of Frame of Reference	11105012	11105012	11105012	11105112	-	11104012	-	-	-	8	11104011	11105011	11104011	11104011	-	21103000	-	41210000	21105000	-	-	-	-	-
Resolved Forces	11105112	11104011	11104011	11104012	-	11104012	-	-	-	11104011	8	11105011	11104011	-	21008000	21103000	-	-	-	-	-	-	-	11105012
Resultant Forces	11105113	11102101	11102101	11105112	11105010	11105000	11104102	11104000	11104013	11105011	11105011	8	11105011	-	21008000	21103000	-	-	-	-	-	-	-	11105012
Inclined Plane	11105112	-	-	11105112	11105000	-	-	-	-	11104011	11104011	11105011	8	11104000	-	-	-	-	-	-	-	-	-	-
Properties of Angles and lines	20906112	-	-	-	-	-	-	-	11104112	11104011	-	-	11104000	8	-	-	21008012	-	21105101	-	-	-	-	-
Trigonometric Ratios	21008113	-	-	21008000	-	-	-	-	21008000	-	21008000	21008000	-	-	8	21103112	21008113	41210000	21105101	11106101	-	11107111	-	11105111
Unit Circle	21103102	21103000	21103000	21103000	-	21103000	21103000	-	21103000	21103000	21103000	21103000	-	-	21103112	8	-	41210000	21105101	-	-	-	-	11104000
Heights and Distances	21008113	-	-	-	-	-	-	-	-	-	-	-	-	21008012	21008113	-	8	-	-	-	-	-	-	-
3D components	41210111	41210111	41210111	41210111	-	41210112	-	-	41210000	41210000	-	-	-	-	41210000	41210000	-	8	21105000	-	11106111	-	11107111	11104101
Polar Coordinates	21105111	-	-	21105000	-	21105000	-	-	-	21105000	-	-	-	21105101	21105101	21105101	-	21105000	8	11102111	11106000	11107000	-	11104000
Geometric Interpretation of Scalar Proc	11106112	11106101	-	11106111	-	-	-	-	-	-	-	-	-	-	11106101	-	-	-	11102111	8	11106000	-	-	-
Algebraic Interpretation of Scalar Produ	11106112	11106112	-	11106112	-	11106101	-	-	-	-	-	-	-	-	-	-	-	11106111	11106000	11106000	8	-	-	-
Geometric Interpretation of vector Proc	11107112	11107112	11107112	11107111	-	11107111	-	-	-	-	-	-	-	-	11107111	-	-	-	11107000	-	-	8	11107000	-
Algebraic Interpretation of vector Prod	11107112	11107000	11107111	11107111	-	11107111	-	-	-	-	-	-	-	-	-	-	-	11107111	-	-	-	11107000	8	-
Algebraic addition(using rect comps)	11104102	11104101	11104101	11104111	11104000	11104111	11104000	11104000	11104112	-	11105012	11105012	-	-	11105111	11104000	-	11104101	11104000	-	-	-	-	8

Figure 2A.2: (a) left- Coding of the concept-concept links in (NCERT)

	ABCDEF	Direction (Vector Definition)	Magnitude (Vector definition)	Rectangular Components (Resolution)	Non-rectangular Components (Resolution)	Unit Vectors	Triangle Law	Polygon Law	Parallelogram law	Rotation of Frame of Reference	Resolved Forces	Resultant Forces	Inclined Plane	Properties of Angles and lines	Trigonometric Ratios	Unit Circle	Heights and Distances	3D components	Polar Coordinates	Geometric Interpretation of Scalar Product	Algebraic Interpretation of Scalar Product	Geometric Interpretation of vector Product	Algebraic Interpretation of vector Product	Algebraic addition(using rect comps)
Direction (Vector Definition)	11102112	8	-	11102101	11102000	11102001	11102102	11102103	11102112	11102101	11104011	11102101	-	-	-	21102000	-	21108000	-	11102011	11102011	11102101	11102000	11102011
Magnitude (Vector definition)	11102112	-	8	11102001	11102000	11102001	11102000	11102000	11102102	11102101	11104011	11102101	-	-	-	21102000	-	21108000	-	-	-	11102000	11102111	11102011
Rectangular Components (Resolution)	11102102	11102101	11102001	8	11102000	11102111	-	-	11102001	11104011	11104012	11105012	11104012	-	21004000	21102000	-	21108101	21203111	11102102	11102111	11102111	11102111	11102112
Non-rectangular Components (Resolut	11102102	11102000	11102000	11102000	8	11102000	11102000	11102000	11102000	-	-	11105010	11104011	-	-	-	-	-	-	-	-	-	-	11102000
Unit Vectors	11102011	11102001	11102001	11102111	11102000	8	-	-	-	11102000	11104000	11105000	-	-	-	21102000	-	21108101	21203101	-	11102111	11102111	11102111	11102101
Triangle Law	11102102	11102102	11102000	-	11102000	-	8	11102104	11102102	-	-	11102102	-	-	-	21102000	-	-	-	-	-	-	-	11102000
Polygon Law	11102103	11102103	11102000	-	11102000	-	11102104	8	11102000	-	-	11102000	-	-	-	-	-	-	-	-	-	-	-	11102000
Parallelogram law	11102112	11102112	11102102	11102001	11102000	-	11102102	11102000	8	-	-	11102112	-	11102101	21004011	21102000	-	21108000	-	-	-	-	-	11102102
Rotation of Frame of Reference	11102101	11102101	11102101	11104011	-	11102000	-	-	-	8	11104011	11105011	11104011	11104011	-	21102000	-	21108000	21203000	-	-	-	-	-
Resolved Forces	11104011	11104011	11104011	11104012	-	11104000	-	-	-	11104011	8	11105011	11104011	-	21004000	21102000	-	-	-	-	-	-	-	11104011
Resultant Forces	11104011	11102101	11102101	11105012	11105010	11105000	11102102	11102000	11102112	11105011	11105011	8	11105011	-	21004000	21102000	-	-	-	-	-	-	-	11105011
Inclined Plane	11104012	-	-	11104012	11104011	-	-	-	-	11104011	11104011	11105011	8	11104000	-	-	-	-	-	-	-	-	-	-
Properties of Angles and lines	20803112	-	-	-	-	-	-	-	11102101	11104011	-	-	11104000	8	-	-	21004112	-	21203000	-	-	-	-	-
Trigonometric Ratios	21004113	-	-	21004000	-	-	-	-	21004011	-	21004000	21004000	-	-	8	21102111	21004112	21206101	21203101	11102111	-	11102111	-	11102101
Unit Circle	21102101	21102000	21102000	21102000	-	21102000	21102000	-	21102000	21102000	21102000	21102000	-	-	21102111	8	-	21206000	21102101	-	-	-	-	11102000
Heights and Distances	21004112	-	-	-	-	-	-	-	-	-	-	-	-	21004112	21004112	-	8	-	-	-	-	-	-	-
3D components	21206101	21108000	21108000	21108101	-	21108101	-	-	21108000	21108000	-	-	-	-	21206101	21206000	-	8	21206111	-	11102112	-	1102111	11102101
Polar Coordinates	21203112	-	-	21203111	-	21203101	-	-	-	21203000	-	-	-	21203000	21203101	21102101	-	21206111	8	11102111	11102000	11102111	-	11102000
Geometric Interpretation of Scalar Proc	11102102	11102011	-	11102102	-	-	-	-	-	-	-	-	-	-	11102111	-	-	-	11102111	8	11102101	-	-	-
Algebraic Interpretation of Scalar Produ	11102111	11102011	-	11102111	-	11102111	-	-	-	-	-	-	-	-	-	-	-	11102112	11102000	11102101	8	-	-	-
Geometric Interpretation of vector Proc	21108101	11102101	11102000	11102111	-	11102111	-	-	-	-	-	-	-	-	11102111	-	-	-	11102111	-	-	8	11102101	-
Algebraic Interpretation of vector Produ	11102112	11102000	11102111	11102111	-	11102111	-	-	-	-	-	-	-	-	-	-	-	1102111	-	-	-	11102101	8	-
Algebraic addition(using rect comps)	11102012	11102011	11102011	11102112	11102000	11102101	11102000	11102000	11102102	-	11104011	11105011	-	-	11102101	11102000	-	11102101	11102000	-	-	-	-	8

Figure 2A.2: (b) right–Coding of the concept–concept links in (MH)
Note: Each cell of the chart corresponds to a CCL and is coloured based on the rigour of the presentation (last character, H. 0 to 4 mapped to green to red colour–scale)

The coding exercise involved creating a chart for each curriculum separately (figure 2A.2(a): NCERT; figure 2A.2(b): MH state board). All the ratings in the coding exercise are carried out by the researcher; and hence is prone to a subjective judgment of the researcher. Efforts are taken to make it as objective as possible (especially for the rigour rating using the guidelines stated in the bracket) and the entire analysis is vetted and validated through a in-house open seminar with subject-matter experts, a few of whom had been on textbook writing committees earlier; however, it can never be without a touch of subjectivity. These codes were then used to capture content treatment by each of the curricula using two kinds of analysis, namely: (1) Link analysis (2) Flow analysis

1.2 *Link Analysis*

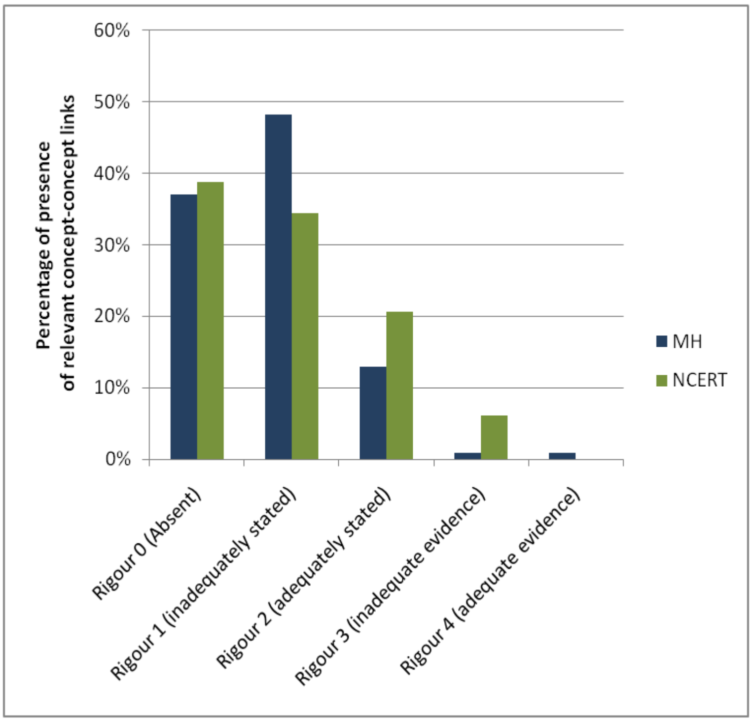


Figure 2A.3: Rigour level proportions of various unit links (NCERT and MH).

This analysis looks for patterns in the mode and rigour of treatment of the topic of vectors. Each CCL has certain significance in building the large coherent model in the student and requires a particular way of treatment in the textbook, depending on the nature of the CCL. The analysis reveals the larger patterns in the manner in which the textbooks actually treat each of these CCLs. Certain consistent sets of patterns in the nature of the topics treated in different modes and the rigour of that treatment emerged. These patterns, as discussed later in the chapter along with other findings, appear to be very closely linked to the affordances of the paper-based textbooks.

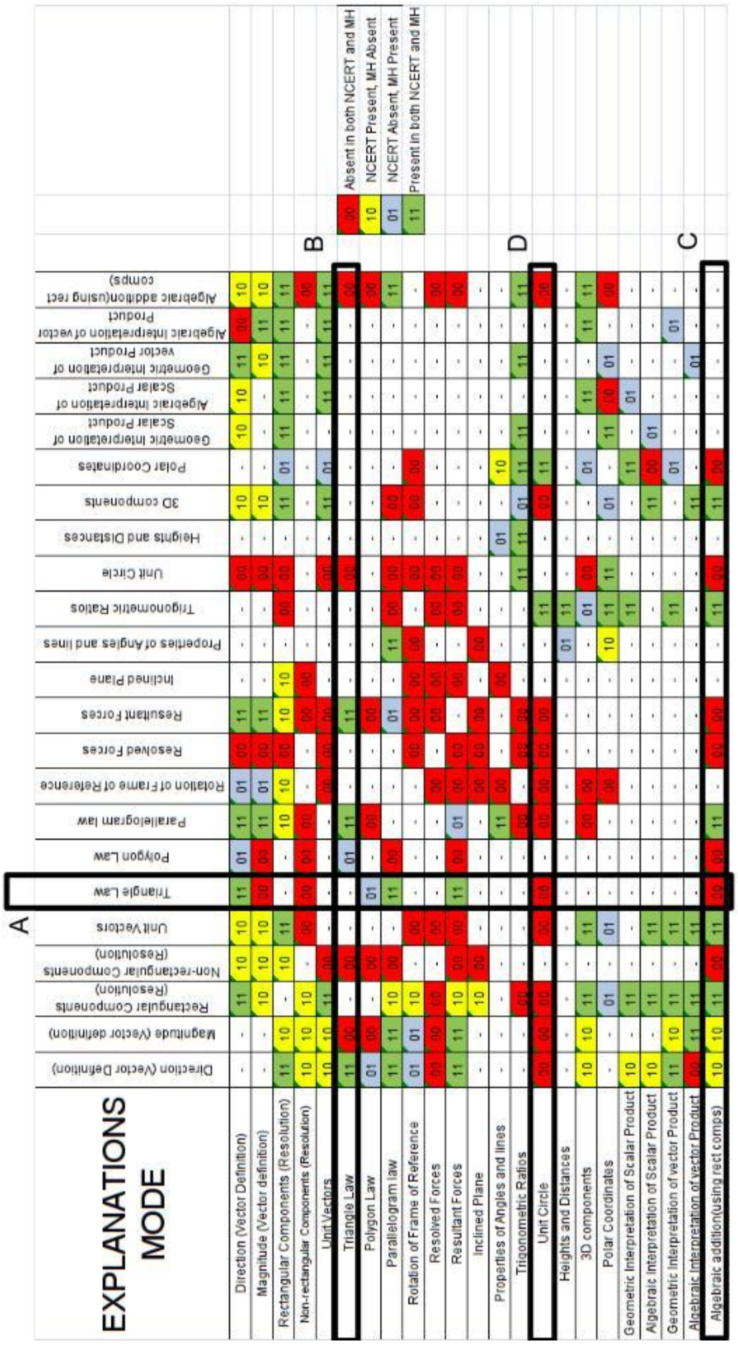


Figure 2A.4: (a) Content-treatment in explanations mode (XY in each cell=$F_{NCERT}F_{MH}$).

Before we get into the modes of treatment, the rigour provides a very broad picture of the coverage of all the relevant CCLs by the textbooks. A quick look at the proportions of topics presented at various levels (in figure 2A.3) indicate that about 40% of the relevant interconnections between units are not even mentioned (rigour = 0) and another 30-50% are presented only weakly (rigour-1) in both the curricula (figure 2A.3).

Further, the modes of content-treatment (explanations "F" and problem-solving "G") present an interesting pattern in both the curricula. Reflecting on the possible reasons for this treatment gives us a sense of the feasibility of using the paper-based textbook as a medium in treating a particular CCL in a particular way.

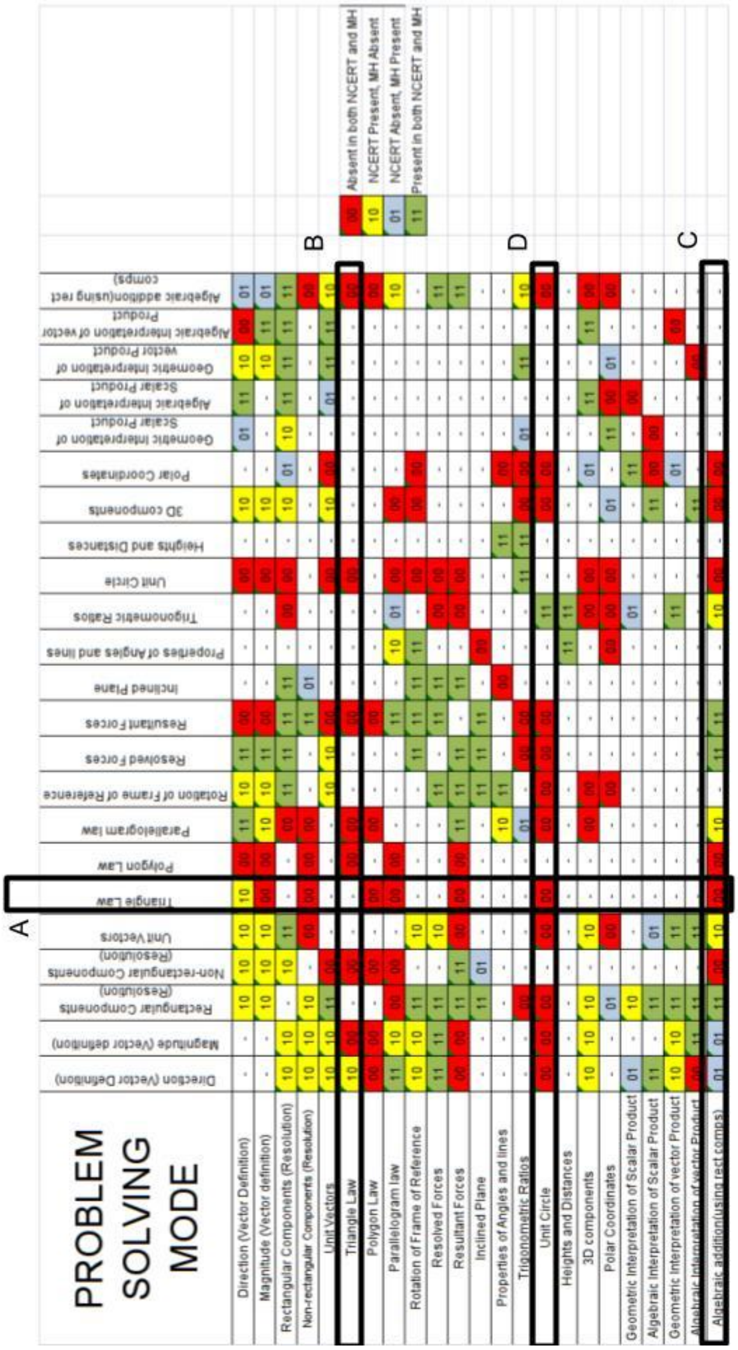


Figure 2A.4: (b) Content-treatment in problem-solving mode (XY in each cell=$G_{NCERT}G_{MH}$).

Charts in figures 2A.4a and 2A.4b are obtained by concatenating the corresponding characters in NCERT and MH curricula (respective F's and G's). The colour coding scheme is also captured in the figures. The red cells (00) correspond to the absence and green cells (11) correspond to the presence of a CCL in that particular mode (explanation or problem-solving) in both the curricula. The textbooks that we analysed present the geometrical methods of adding vectors while introducing the idea of vector addition in explanation-mode (green rows/columns corresponding to triangle/polygon law in top-left figure 2A.4a highlighted as A, B), but not in problem-solving mode (red rows/columns corresponding to triangle/polygon law in top-left of figure 2A.4b highlighted as A, B). Once the resolution of vectors into rectangular components is introduced, the addition of vectors is mostly performed using rectangular components. Further, the algebraic mode of adding vectors is not linked with the geometric modes (triangle or parallelogram law) in the mode of example problems.

There are a large number of units which are absent in the explanations-mode in both the curriculum (red cells-'00' in figure 2A.4(a)); a significant chunk of these untreated/absent CCLs are those related to

1. the non-rectangular components and addition using triangle and parallelogram laws (the red cluster in the top left corner of the matrix in figure 2A.4(a))
2. the applications of the addition and resolution of vectors in understanding the motion in-plane (red cluster in the centre of the matrix in figure 2A.4(a): resolving forces, finding resultant forces, rotating a frame of reference) and
3. the connections with the unit circle.

Interestingly the units related to these applications of resolution and additions of vectors in mechanics are dealt with using the example problems (green cluster in the centre of the matrix in figure 2A.4(b)). Although the statements of laws of motion are dealt with in the explanatory sections, the applications using the vectors are actually presented in the process of solving the example or exercise problems.

Similar to the case with explanations, the topic of the unit circle is not dealt with in the mode of example problems as well.

Both the curricula present a wide range of CCLs in the explanatory sections (green cells-'11'). Among them, NCERT appears to focus more on strengthening links between the fundamental units like the definition of the vector, the addition and the resolution concepts (yellow cells-'10' in the top-left corner) in both explanations and problem-solving modes.

1.3 Flow Analysis

Grade	9		10		11										12							
						Ph Ch 3									Mt Ch 10							
Subject/ Chapter	Math	Phys	Math	Phys	Math	Sq 1	Sq 2	Sq 3	Sq 4	Sq 5	Sq 6	Ph C5	Ph C6	Ph C7	Sq 1	Sq 2	Sq 3	Sq 4	Sq 5	Sq 6	Sq 7	Phys
Properties of Angles and lines	2																					
Trigonometric Ratios			3																			
Heights and Distances			3																			
Unit Circle					2																	
Direction (Vector Definition)						3									1							
Magnitude (Vector definition)						3									2							
Triangle Law							3									2						
Polygon Law							0															
Parallelogram law								3									2					
Rectangular Components (Resolution)										3								3				
Unit Vectors								2									2					
Non-rectangular Components (Resolution)									1													
Algebraic addition(using rect comps)											2											
Rotation of Frame of Reference												2						2				
Resolved Forces												2										
Resultant Forces												3										
Inclined Plane												2										
3D components															1							
Polar Coordinates					1																	
Geometric Interpretation of Scalar Product													2									
Algebraic Interpretation of Scalar Product													2						3			
Geometric Interpretation of vector Product														2						2		
Algebraic Interpretation of vector Product														2							3	
Applications of Scalar Product													2								1	
Application of Vector Product														2							2	2

Figure 2A.5a: Trajectory in NCERT Board

The flow analysis is a way of qualifying a curriculum's alignment with a given progression of the topic. This representation captures the order in which the topics progress and are spread across textbooks in each of the curricula. The more ordered and smooth the progression of topics is, the easier it is to coherently integrate all the topics for the teachers and the students; this, thus, indicates the level of additional effort that the teachers and students need to put in integrating the concepts smoothly and coherently. These patterns can also hint at certain affordances of the paper-based medium in content-treatment.

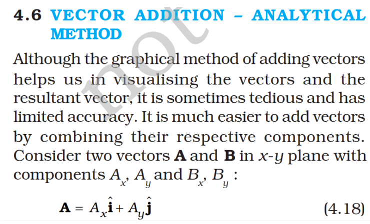
We have reorganised the 23 units along with the applications of scalar and vector products (for completion), into an order which we think would be a meaningful way of presenting them; they progress from direct prerequisites, basic-definitions, addition and applications, products and applications of vectors. These were validated by an in-house physics education researcher, who was not part of the research team. A chart is made using the first 5 characters (ABCDE) to indicate the grade-level and the subject of the textbook and the chapter number. The characters in the cells filled in the charts (figures 2A.5a and 2A.5b) are the rigour of the topics covered. An agreement with our trajectory will reflect as a straight unbroken diagonal from top left to bottom right in the chart (in figures 2A.5a and 2A.5b).

Grade	8	9		10		11																											12	
Subject/ Chapter						Ph Ch 2											Ph Ch4			Mt Ch 8														
	Math	Math	Phys	Math	Phys	Sq 1	Sq 2	Sq 3	Sq 4	Sq 5	Sq 6	Sq 7	Sq 8	Sq 9	Sq 10	Eg.	Sq 1	Sq 2	Math Ch 2	Sq 1	Sq 2	Sq 3	Sq 4	Sq 5	Sq 6	Sq 7	Sq 8	Sq 9	Sq 10	Sq 11	Sq 12	Sq 13	Math	Phys
Properties of Angles and lines	2																																	
Trigonometric Ratios				3																														
Heights and Distances				2																														
Unit Circle																			1															
Direction (Vector Definition)						2														1														
Magnitude (Vector definition)						2														2														
Triangle Law							2														2													
Polygon Law								3															3											
Parallelogram law									2													2												
Rectangular Components (Resolution)											2																							
Unit Vectors											1													2										
Non-rectangular Components (Resolution)										2																								
Algebraic addition(using rect comps)																2									2									
Rotation of Frame of Reference												1																						
Resolved Forces																		1												2				
Resultant Forces																	1												1					
Inclined Plane																		1																
3D components																																	1	
Polar Coordinates																																	2	
Geometric Interpretation of Scalar Product													2													1								
Algebraic Interpretation of Scalar Product														1												2								
Geometric Interpretation of vector Product																												2						
Algebraic Interpretation of vector Product															1												2							
Applications of Scalar Product																	2														1			
Application of Vector Product																		2														1		2

Figure 2A.5b: Flow of content treatment in MH Board

In NCERT (figure 2A.5(a)), the necessary concepts are covered across grades and textbooks. Here the basic mathematical concepts were covered in math textbooks of grade 9 and 10. Though the topic of vectors fundamentally is mathematical, it is introduced first in physics (grade 11) and then in mathematics (grade 12). Within physics, the concepts follow roughly the trajectory as indicated by lesser deviations from the straight diagonal. A similar chart made for MH board (figure 2A.5(b)) shows that the initial concepts like the angles in parallel lines till trigonometric ratios are dealt with in mathematics in lower grades of 8, 9 and 10. Vectors are introduced in grade 11 Physics (Chapter 2 onwards) and are applied in various later chapters, in which we see a break/ deviation from our trajectory. As with NCERT, these are dealt with in the math textbook only later in chapter 8. In both the curricula, one can notice a flow in the mathematics curricula almost parallel to the corresponding physics curricula. Overall, this analysis gives a picture of the spread of the related topic across about 8 textbooks, which needs to be integrated to be able to coherently understand the topic.

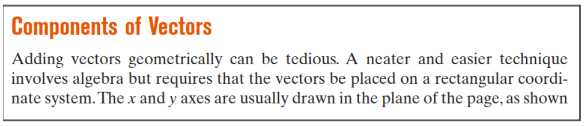
1.4 *Excerpts related to the treatment of geometric and algebraic modes*



4.6 VECTOR ADDITION – ANALYTICAL METHOD

Although the graphical method of adding vectors helps us in visualising the vectors and the resultant vector, it is sometimes tedious and has limited accuracy. It is much easier to add vectors by combining their respective components. Consider two vectors **A** and **B** in *x*-*y* plane with components A_x, A_y and B_x, B_y :

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} \quad (4.18)$$



Components of Vectors

Adding vectors geometrically can be tedious. A neater and easier technique involves algebra but requires that the vectors be placed on a rectangular coordinate system. The *x* and *y* axes are usually drawn in the plane of the page, as shown

Figure 2A.6: Excerpts from (left) NCERT physics textbook and (right) 'Fundamentals of Physics' by Resnick Halliday (Halliday et al., 2013)

Besides the above coding based analysis, we found some interesting excerpts consistently across the textbooks when introducing the topic of rectangular components: the algebraic mode of vectors. The textbooks explicitly privilege the algebraic methods over the geometric methods. For example, see excerpts from two textbooks in figure 2A.6. The grade 11 physics textbook by the NCERT states: "*Although the graphical (geometrical) method of adding vectors helps us in visualising the vectors and the resultant vector, it is sometimes tedious and has limited accuracy. It is much easier to add vectors by combining their respective components.*" Another popular book, widely referred to by Indian students (Halliday et al., 2013), states — "*Adding vectors geometrically can be tedious. A neater and easier technique involves algebra but requires that the vectors be placed on a rectangular coordinate system.*" Both the books explicitly undermine the geometric method of adding vectors to introduce rectangular components, by ascribing tedium and inaccuracy, and thenceforth use only algebraic addition (as also indicated in the coding exercise of NCERT books)[13].

2 Interaction with Teachers

For more indicators of the role of the paper-based medium in relation to the difficulties faced by students with the topic of vectors as manifested on ground (in the classrooms), we interacted with the teachers. Along with the observations from the textbook analysis, this would give us a sense of how the textbooks are used and how the medium affords and limits the teachers

[13] Though there are instances in this latter book, involving geometric methods, like the example problem in p44, (sample 3.01), they are still limited compared to those involving algebraic methods.

while teaching the topic. We interacted with 4 (1 math and 3 physics) grade-11 teachers of varied teaching experiences to capture if and how the issues similar to those found in the textbook analysis manifest in the classroom practices. The interviews were semi-structured surrounding the curriculum, the gaps and problems they found when teaching the topic of vectors and the issues identified in the textbook analysis. The discussion was guided by the following questions.

1. If they find any difficulties in teaching these concepts related to vectors?
2. If and how the difficulties identified are addressed during the teaching?
3. What kind of conceptual issues related to this topic do they find students facing?
4. What kind of connections do they find and make between the mathematical entities and the physics concepts during the teaching sessions on these concepts?

The written material and the recordings during the interview were used for further analysis. The interviews were quickly analysed for some key aspects that they identified with the topic of vectors specifically and other related difficulties in general. Key inputs from the teacher interactions varied from the conceptual to procedural level narratives depending on their teaching experience. For example, all the teachers noted the difficulty that students face in resolution and addition of vectors, and finding products (especially cross products). But the reasons cited by the teachers varied from - procedural aspects of not being able to solve a determinant to find cross product or, confusions with using trigonometric ratios while resolving vectors or, using the formulae of the dot and cross products - to conceptual aspects of students not understanding the notion of direction of a vector, and problems with the geometrical addition in a triangle giving the resultant vector. Teachers have also noted the issues around the applications of vectors in rotational dynamics and electricity and magnetism, and confusions on resolving vectors in rotated frames of reference (like that of forces on a body on an inclined plane). Also, other mathematical ideas like instantaneous quantities and prerequisite concepts of properties of angles (in a set of parallel lines) are mentioned as possible problem areas that these teachers notice among the students.

Besides these, teachers mentioned the problems that they face when handling chapters on waves and oscillations and their limitations to draw diagrams multiple times to demonstrate changes in geometry. In this connection, a teacher says -

> *"... is a constraint in visualising, the way it is presented. Because it is verbal, no? So, from those words only you have to understand. And even if you put it in the form of a picture, it is in 2D. It is not moving in time. So it is difficult for them. Like the superposition of waves. What exactly is happening? What are stationary waves? There are students who have asked - "ma'am, in the stationary waves, does the wire split into two?" [Link]*

When probed for the physics teachers' engagement with the required mathematics, the teachers informed that they do have some discussion on the relevant mathematical concepts and also have discussions with the peers teaching mathematics. But they reported that this interaction with mathematical perspectives is not a regular practice, due to a very tight timeline for completion of syllabus. Moreover, teachers note that students with engagements like extra-tutorial classes to excel in the competitive entrance tests like CET and IIT-JEE[14] are highly over-loaded with little time that they can afford for productive, holistic and intense engagements with these topics at a conceptual level.

3 Findings

From the patterns found in the above two analyses, 3 broad themes emerge, which are useful in organising the empirical work from the SCIARM framework.

3.1 Lack of geometric manipulation

The textbooks that we analysed present the geometrical methods of adding vectors while introducing the idea of vector addition, but not in problem-solving. Once the resolution of vectors into rectangular components is introduced, the addition of vectors is almost exclusively performed using rectangular components. Further, the textbooks, as indicated by the excerpts related to rectangular components, explicitly privilege the algebraic methods over the geometric methods. And, there is no attempt made to establish the equivalence between the algebraic mode

[14] CET and IIT-JEE are some common entrance tests for students to get admissions into universities or colleges for their engineering degrees, which is a popularly aspired path.

of adding vectors and the geometric modes (triangle or parallelogram law) either by explanation or by problem-solving. The topics like the unit circle, which could have been effective in making this connection, are covered neither in explanations nor in problem-solving. This leaves students with little scope to realise that the addition and resolution are reverse processes and appreciate how the unit vectors capture the directional aspect of vectors.

The interaction with the teachers also identifies similar issues with geometric components (relation between addition resolution, directional aspect of vectors, and the cross products). Further, the cases raised during discussion like the superposition of waves highlights this limitation of the paper-based medium — a lack of dynamics and possibilities of manipulating pictures (and geometric entities) which have spatial nature.

3.2 Serial ordering

There are no scaffolds interlinking all prerequisites like the geometry of lines, angles and triangles, trigonometric ratios to the topic of vectors, in the textbooks we analysed. For example, the unit circle could have been an effective scaffold to establish the equivalence of polar and cartesian forms (connecting the geometric and algebraic representations of a vector), but it is presented elsewhere in the mathematics curriculum, and is not linked to vectors. Further, trigonometry itself is a topic which is considered difficult among the students and an application of it in the new context of resolution in vectors could be even more intimidating to the students.

Similarly, the necessary scaffolds and linkages between the units are absent as is evident from the fact that almost 40-50% of the relevant unit links are not brought out in the textbooks, as seen in the figure 2A.3. For example, the algebra-geometry connection is essential for the students to imagine and understand the spatial significance of the algebraic manipulations they perform using the rectangular components. The fact, as evident from the flow analysis, that all these units related to the vectors are covered across different grades along with improper scaffolds makes it an uphill task for students to develop a coherent understanding. In higher secondary level (grade 11-12 in India) which is a different stage of schooling, a lot of topics similar to vectors (like the usage of integral and differential calculus in physics) are introduced with a fragmented and incoherent presentation with very limited scaffolds.

Also, the interaction with the teachers indicates the difficulty in having a coherent approach in collaboration with the mathematics curriculum and the teachers, due to the practical constraints. This converges with the evidence from the disintegrated content flow (seen in flow analysis) where the medium is constrained and presents content in a modular way, making it difficult for the teachers to present related topics spread across about 8 textbooks in an integrated manner.

3.3 Opaque problem-solving

Often the operations and approaches used during problem-solving remain less transparent for the students, without explicit explanations. Applications of the operations of addition and resolution of vectors can be found in mechanics with physical quantities like force, velocity, momentum, to analyse the motion of a body using physical principles like Newton's laws or the conservation laws. After applying these principles, representations like free-body diagrams are made, which can be further manipulated using vector operations. Here, explanations or any kind of heuristics for application of vectors are missing in the textbooks as evident from the explanatory modes of presentation. For example, in the case of a mass on an inclined plane, the student is expected to grasp necessary patterns or templates, by solving number of problems but without any explicit explanations, to make decisions about the choice of the mass for which the free body diagram is to be made, to make suitable free body diagrams and, and to choose a frame of reference to resolve the forces etc. A similar opacity of the procedures reflects in the teachers' remarks on the difficulties that students have in making sense of the determinant method of finding vector products and the usage of trigonometric ratio in resolution among others.

4 Chapter-discussion

We have seen, until now, the details of the text book analysis and teacher interactions, and the findings broadly organised into 3 themes. Here, in line with the SCIARM framework. we shall interpret and discuss these limitations further to connect with the larger research questions of the thesis.

With respect to the first theme, we argue that the tedium and a lack of accuracy is not an inherent limitation of the geometric method of addition, as claimed by the textbooks, but could be stemming from the limited affordances of the paper-based medium to perform geometric manipulations (using paper-based medium, geometric tools like pencil, scale, compass, etc). And it is manifested in textbooks as the limitations in the treatment of geometric aspects of vectors. This limitation of the medium is also acknowledged by Dray and Manogue (2006) when they make a similar claim that the difficulty in offering geometric proofs (which would require performing certain geometric manipulation) in the textbooks could be partly because of the "difficulty in translating them into words on the printed page". Similar limitations of the textbooks in treating geometry, though not explicitly ascribed to the static paper-based medium, can be found elsewhere as well. Fuys & Geddes (1984) note that geometry related material in most textbooks limits students' level of thinking to an elementary level[15] (referring to van Hiele's (1959) levels). As a related point, the overall proportion of geometry to algebra in the curricula is reported to be less (Harel, 1987; Jones, 2002; Ramadas, 2009).

From the SCIARM framework, it is interesting to note that in the regular teaching-learning practices, the representations are mostly static. Despite this limitation, teachers, sometimes described as performing artists (Rives, 1979), try and make the representations dynamic by enacting using gestures and meticulously planned board work, as captured by elaborate Japanese lesson studies (Doig & Groves, 2011; Fernandez & Yoshida, 2012). However, these dynamic classroom practices are just scaffolds for a static medium. Content and the formulations are presented as finished products along with "problem-solving based on algorithmic recitations", leaving no scope for variations (Stinner, 1992). From the constitutivity hypothesis, and in turn the corollary and the SCIARM framework, this limitation could be a strong constraint arresting the sensorimotor interactions, which in turn could limit the cognition of the students.

[15] Van Hiele's categorisation of geometric reasoning starts from grasping superficial apparent/ visual features to gradually moving towards abstract and formal levels of thinking with mathematical rigor.

Related to the second finding from the SCIARM framework, it is practically impossible for textbooks, which are written in a modular and linear way as sections and chapters, to present or connect every related unit at the given point of time. It would be daunting for a physics textbook writer to also revisit all necessary prerequisites every time. The paper-based textbook ends up presenting them in smaller chunks and modules in some order spread across math and science/physics textbooks across grades, as can be found in our flow analysis. This will naturally make it difficult to present the conceptual interlinks in a coherent manner. These limitations in the conceptual interlinks are acknowledged by the work of Usiskin (2018), when comparing paper-based and e-textbooks. A similar set of limitations in covering content in mathematics and science textbooks was described elsewhere as "a mile wide and an inch deep" (Schmidt et al., 2007).

A possible way to interpret our third finding of limited explanations in problem-solving approaches, from the SCIARM framework is that: in the paper-based medium, there is a significant additional workload and hence hardly any incentive for the students to apply and explore various models and try out the other possible ways to solve and appreciate the rationale behind the particular choices made. In relation, Simon (1980, p. 92) notes that — *"textbooks (in general) are much more explicit in enunciating the laws of mathematics or of nature than in saying anything about when these laws may be useful in solving problems"*. Dreyfus (1999) and Lithner (2003) note that there is a lack of 'instructions or indications' in the textbooks, to support the desired quality of reasoning in these contexts. Elsewhere, Stinner (1992) characterises this by noting that "textbook- centred science teaching mostly delves into the logical plane (mathematical- algorithmic- factual) with only occasional excursions to what we shall refer to as the evidential plane (experiential- experimental- intuitive)." Lack of qualitative explanation in comparison to quantitative treatment was also found in textbooks in topics like acceleration (Dall'Alba et al., 1993) making it difficult for students. Oversimplifications in the content are found in other textbooks also (Forjan & Sliško, 2014).

In conclusion, the analysis of the textbooks using the case of the vectors helps us in identifying 3 key patterns in the way content is treated by virtue of them being paper-based, also supported by the interactions with the teachers. These are manifested in various levels in the way the content is treated ranging from the kind of topics that are presented and the mode and the rigour with which they are presented, and also the order in which all the topics are presented. These also are reflected in the teacher interactions about their execution of the topic and the difficulties that they find. And there are similar acknowledgments across literature analysing textbooks, all hinting at the limitations of the medium, though not explicitly linking to the affordances of the static paper-based medium. This analysis, along with the above acknowledgments, thus provides us with a good portrait of the existing medium shaping the curriculum and teaching-learning practices.

Publications from parts of this chapter

Karnam, D.P., Mashood, K. K., & Sule, A. (2020). Do student difficulties with vectors emerge partly from the limitations of static textbook media? *European Journal of Physics*, 41(3), 035703. https://doi.org/10.1088/1361-6404/ab782e

Karnam, D. P., & Sule, A. (2018). Vectors in Higher Secondary School Textbooks. In S. Ladage & S. Narvekar (Eds.), *Proceedings of epiSTEME 7 — International Conference to Review Research on Science, Technology and Mathematics Education* (pp. 159–167). Cinnamon Teal.

Chauhan, P., Joel, A.J., Kurup, P., Srinivasan, P.S., & Karnam, D.P. (2019). Experiences of teaching Vectors in Indian pre-university classrooms: An account by Teachers. *In Proceedings of the Inaugural Conference of the Mathematics Teachers' Association* - India. (pp 126-127) Mumbai: HBCSE.

2B

Study-2: Existing Students' Conceptual Reasoning Behaviour

The objective of the chapter: To capture the existing *conceptual reasoning behaviour (CRB)* in the students.

Methodology: Probing students through written test and follow up interviews

Key findings: Patterns in students' CRB organised broadly into 3 themes.

- Reasoning approaches with an algebraic dominance
- Interconnections between conceptual units
- Approaches in Problem-solving

In line with the SCIARM framework, there is an interesting semblance in these patterns, which provides evidence for a tentative relationship between the students' difficulties (with the topic vectors in our case) and the affordances of the medium.

This chapter characterises the students' conceptual reasoning behaviour[16] (aspects of student behaviour with utterances like writings, articulations and gesticulations while reasoning and problem-solving) in the case of vectors. Conceptual reasoning behaviour is one of the important indicators of students' cognitive processes in the context of reasoning or problem-solving in a given topic. This chapter reports the findings from a systematic probing of a wide range of students through instruments of written tests and interviews evoking their behaviour.

1 Methodology

1.1 *Materials and protocol*

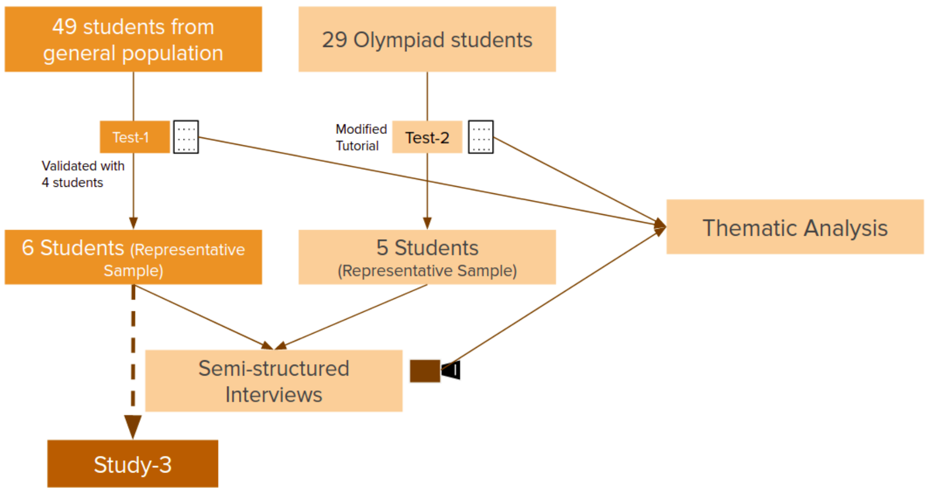


Figure 2B.1a: A schematic representing the methodology of the Study-2

See figure 2B.1a. We subjected two groups of students (groups 1 and 2) using a protocol of a written test followed by a semi-structured interview (the tests to the two groups were different but related to vectors). Our first sample was a group of students (N=49) who passed grade-11 (16-18 years of age) from a typical urban school in India, which we call typical students (TS). They were administered a test (sample questionnaire in Appendix-2B.1) consisting of a set of problems, which examined their understanding of basic vector concepts like addition, resolution into components as well as their application to problems in mechanics. This test was validated by administering to a group of 4 students at a similar stage of learning the topic of vectors. Out of these 49 TSs, a smaller representative sample of 6 students was selected for a follow-up interview.

[16] For a detailed description of CRB See section 3.1 in Ch 1B

In addition to the above student group, we studied another group (N=29) of students who had finished grades-10/11, which we call the Olympiad Students (OS). These students were shortlisted through a highly competitive national-level selection test to participate in a national level selection and training camp leading to representing India in the International Olympiad on Astronomy and Astrophysics (IOAA). They may be considered as a nationally drawn sample representative of the academic high-achievers. The rationale for including them in the study was to capture common patterns in CRB among students with widely different scholastic abilities. This group was given a scaffolded questionnaire related to the derivation of Lagrangian Points L4 and L5 for a 3 body system (adapted from the derivation presented in Stern, 2004) as a written test (Appendix-2B.2). A representative sample of 5 students from this group was also interviewed.

During the test, all the students were encouraged to also write down if and why they were confused by any particular question, being as specific as they can. This allowed us to access their methods of solving problems, reasoning approaches, and conceptual inter-relations that they make. These written scripts of the students formed a set of data for analysis.

1.2 Details of the Interview

Further, semi-structured interviews lasting 30-45 mins were conducted individually with each of the 11 (6TSs+5OSs) students around their test responses. They explained their reasoning and were allowed to correct their test responses if needed. Besides the test responses, some prompts were used, which required them to explain:

- The process of resolution of vectors (using trigonometric ratios in right triangles)
- The equivalence between adding the vectors geometrically and algebraically (using rectangular components)
- The advantages of using rectangular components for adding vectors
- *The components of components paradox:* Related to the motion of a mass on an inclined plane (figure 2B.1b), the normal force can be resolved along and perpendicular to the ground as $N\cos\theta$ and $N\sin\theta$. Say, we then resolve $N\cos\theta$ again along and perpendicular to N, and the force along N may be incremented by $N\cos^2\theta$ (a component of $N\cos\theta$ along

N). This process can be repeated numerous times. We asked the students if they find it paradoxical.

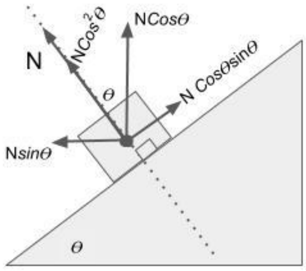


Figure 2B.1b: Repetitive resolution leading to the components of components paradox

Not all of these prompts were used in every interview. The use of a prompt depended upon the context. These prompts provided openings to ask many intermediate questions, which helped in capturing in detail the students' CRB. The interviews were video recorded.

2 Analysis and Findings

Students' written responses, video recordings and written material generated during their interviews were subjected to a thematic analysis. The focus of our analysis was to find recurring patterns in students' CRB associated with vectors. The videos were analysed after meshing them with corresponding written material, thus generating coherent episodes. The diagrams, equations and gestures employed by the students while answering the questions were carefully examined. The episodes identified from the video and written data were iteratively organised into themes, from which certain recurrent patterns emerged. Three broad themes emerged while categorising these episodes of behaviour.

- Reasoning approaches with algebraic dominance
- Limitations in interconnections between conceptual aspects
- Limitations in Problem-solving approaches

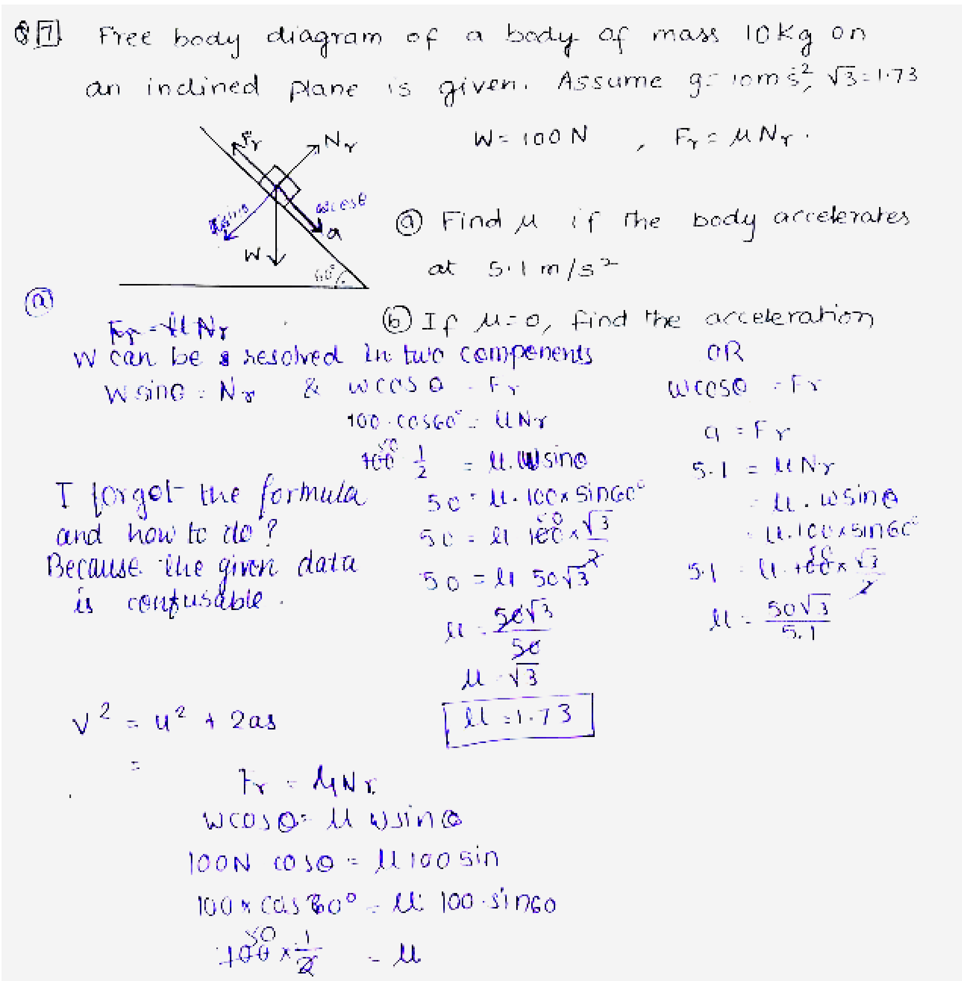
Each of the reported episodes may not have been observed in the case of every student, but collectively, the episodes cover the entire range of patterns found among the students.

2.1 *Theme-1: Reasoning approaches with an Algebraic Dominance*

4 indicators of students' reasoning patterns were categorised under this theme.

Indicator-1: Reliance on memorised-formulae and algebraic manipulations

For questions that required explanations with reasons (e.g. questions like Q5.d in figure 2B.4 for TSs, and Q10 in figure 2B.3 for OSs) 49 TSs and 29 OSs wrote mostly algebraic expressions (formulae) or performed algebraic manipulations. Even in the cases where geometric reasoning was essential or easier, a dominant reliance on algebraic modes was observed. To questions that had the potential to elicit geometric reasoning (e.g. Q5.d in figure 2B.4), students like TS2 used algebraic manipulations (incorrectly). Similarly, the responses of OS2 and OS4 to a slightly more complex problem were based on direct algebraic substitution (figure 2B.3). It is worth noting that this problem could have been solved geometrically, using similar triangles formed by the force vectors and ρ (scaled) vectors (Stern, 2004).



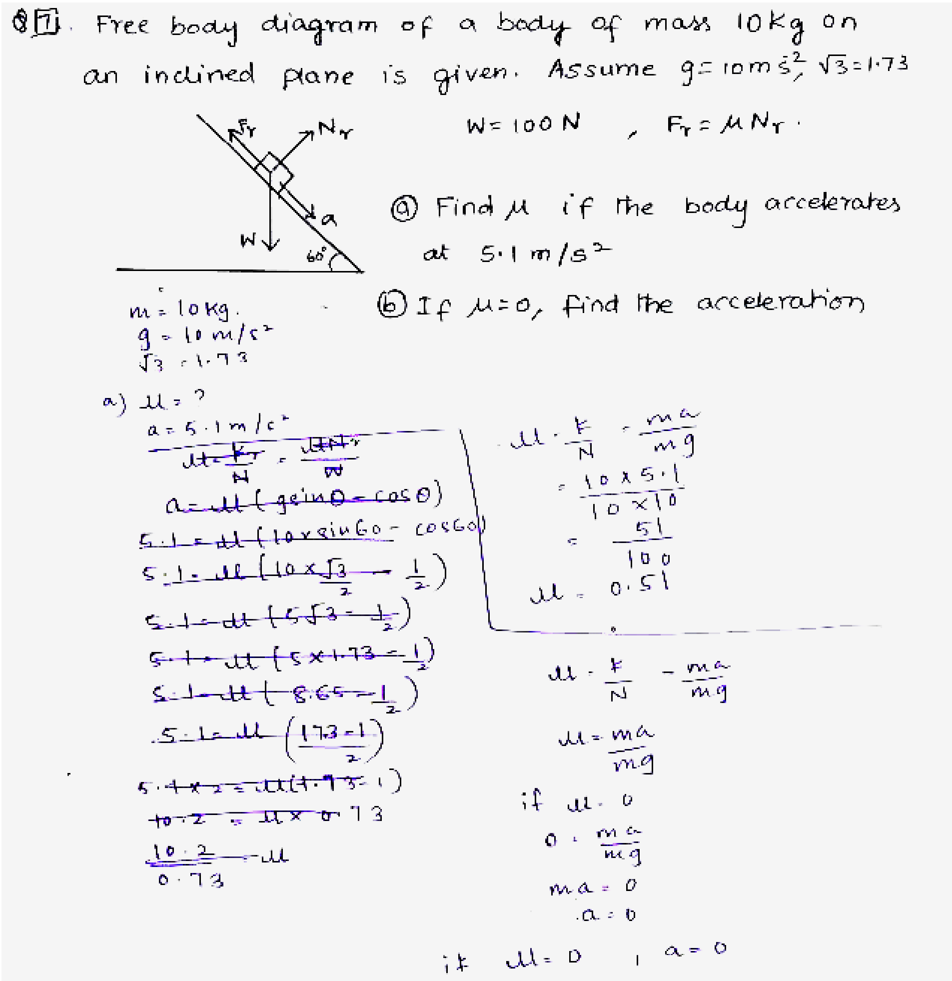


Figure 2B.2: Responses by TS1(top) and TS3(bottom) to Q7 in the test.

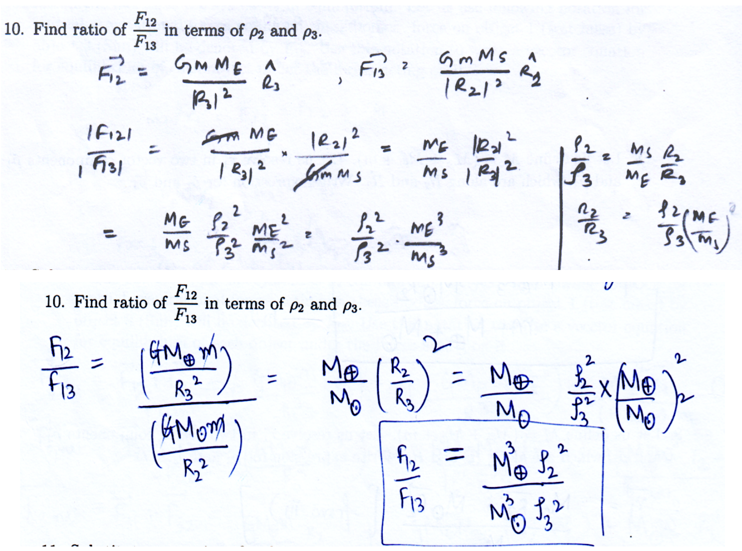


Figure 2B.3: Response by OS4 (left) and OS2 (right) to Q10 in the test.

Answering a problem in mechanics that required applying vectors, (figure 2B.2), TS3, without drawing any free body diagrams and force equations, wrote an expression for coefficient of friction–μ directly from the textbook.

TS1 drew some arrowheads indicating free-body diagram and wrote equations based on them; but then deleted some of them and eventually appeared to return to memorised formula, as reflected in the statement - "I forgot the formula…". Statements of this kind, indicating memorised formulae, were used by most students across questions.

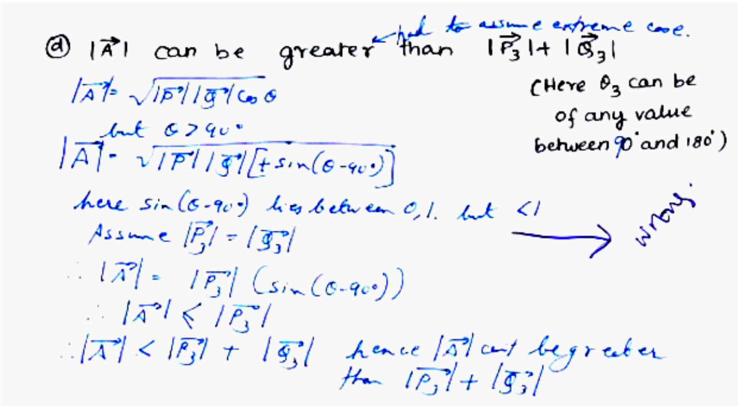


Figure 2B.4: Response by TS2 to Q5.d in the test.

Indicator-2: Treating vectors as scalars (ignoring directional aspects)

Consistent with the literature (reviewed in Ch-1B), we observed students treating vectors akin to scalars, ignoring the directionality. For example, many students wrote $|\vec{A}|+|\vec{B}|=|\vec{C}|$ as A+B=C and confused it with the vector equation $\vec{A} + \vec{B} = \vec{C}$ (reflected in TS responses to Q5). Usage of phrases like "… *basically, we are adding (a) term on either side*" and "… *in vectors, we can add, multiply, or divide by constants*" by OS4 were common. Most students (like OS3) were comfortable with addition using rectangular components (adding like terms) akin to the algorithm used in scalar algebra. They often ignored the unit vectors and added rectangular components of vectors like scalars.

Indicator-3: Preference for algebraic explanations during interviews

We observed that even during interviews, algebraic reasoning dominated students' reasoning-behaviour. For example, when asked to explain the addition of vectors, OS5 proceeded to derive the expression of the magnitude of the resultants using triangle law and mostly ignored the directional aspects. In another related case, to prove the equivalence of addition using rectangular components (algebraic method) and the triangle law (geometric method), all students tried to establish an algebraic derivation (employing cosine rule) leading from one method to the other. They could have explained the equivalence using simple geometric constructions and manipulations, which none of them tried. Only a few students (e.g. OS5) could appreciate this possibility when pointed out later. These cases indicate a preference for algebra-based explanations and solutions among both groups. When probed for indicators of understanding of geometric addition of vectors, there were no satisfactory responses from 4 out of 6 TSs.

Indicator-4: Underlying algebraic influences in the talk

Using algebra does not necessitate the absence of a geometric understanding of vectors. But we found deeper evidence in students' usage of terms, particularly in the interviews. OS1 used the terms 'resultant' for 'product' and 'rms' for 'resultant' (probably because root-mean-square and cosine rule of resultant have visibly similar algebraic expressions with '√', the square-root symbol). A similar case of conflating dot product with the x component (cosθ in the expressions) and with the expression of cosine rule (as by TS2 in figure 2B.4) was observed. A lack of precision in categories and labels can be granted to novices. Nonetheless, the interesting and subtle aspect here is that of the consistent underlying algebraic aspect (cosθ, '√') across the cases. Further, students frequently used verbalisations of the algebraic expressions as actual definitions. For example, OS3 defined dot product as the "*sum of the product of corresponding components.*" Dot products and cross products were defined simply as *abcosθ* or *absinθ*. OS5 defined the centre of mass as "*summation of position vectors of the* i^{th} *particle multiplied by the mass of the* i^{th} *particle, divided by (the) total mass.*" The dominance of algebraic reasoning in students' imagination is clear from episodes like OS1 gesturing a '√' symbol in the air when discussing resultant, instead of gesturing geometric aspects related to the triangle law. This indicates that the meaning attached to the vector concept is mostly algebraic.

2.2 *Theme-2: Limitations in interconnections between conceptual units*

Discussions with both TSs and OSs exposed numerous weak spots in their understanding and fragile interconnections between conceptual units.

Addition

Students, especially TSs, were not able to understand the geometric addition of vectors as anything beyond drawing triangles and parallelograms, often with improper directions (figure 2B.5). Notice that the resultant drawn is $\vec{A} - \vec{B}$ and not $\vec{A} + \vec{B}$ as was asked. Largely, these responses were in confirmation with the findings of Nguyen and Meltzer[17] (2003). In several cases, they wrote improper algebraic expressions similar to the cosine rule (like in figure 2B.4). This could be caused by an imperfect recollection of memorized formulae. Despite their inaccuracies, when probed for any indicators of what the addition of vectors actually means, either geometrically or physically, there were no satisfactory responses from 4 of the 6 TSs. The OSs too were not clear about whether different methods of adding vectors would lead to the same answer. Some of them were under the impression that each law of vector addition (triangle and parallelogram laws) gives different answers.

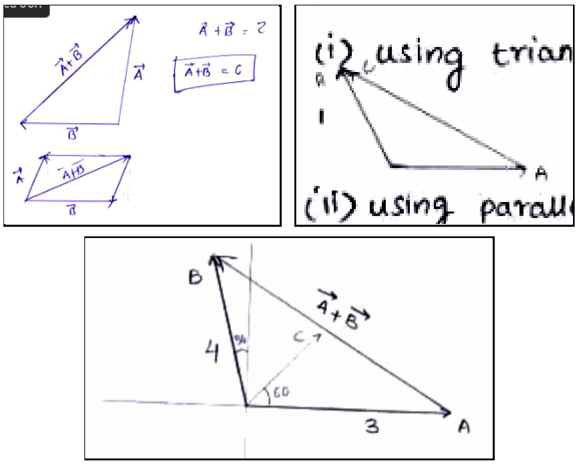


Figure 2B.5: TS1 TS3 and TS5 addition with triangle law

Another related conceptual issue was in establishing the equivalence between the addition using rectangular components (algebraic) and triangle or parallelogram laws (graphical/geometric). Even the OSs struggled to reconcile this, and as described in indicator-3 earlier, there was a significant reliance on algebraic methods to establish that.

Resolution and trigonometric ratios

When asked to explain the process of resolution 4 out of 6 TSs gave the expressions $r\cos\theta$ and $r\sin\theta$, but could not derive them using the correctly

[17] See similar figures in Ch-1B Section 2.1 (figure 1B.1).

constructed right triangles and trigonometric ratios. One of the six could draw the right triangles and explain coherently to some extent. Among OSs, though many were aware of the right triangle and the derivation, students like OS4 made incorrect and inconsistent constructions by drawing a longer vector and took the projection halfway (figure 2B.6-left). The TS's had a general difficulty in appreciating the relationship between the resolution and the addition of vectors. The OSs had a sense that resolution and addition are reverse actions.

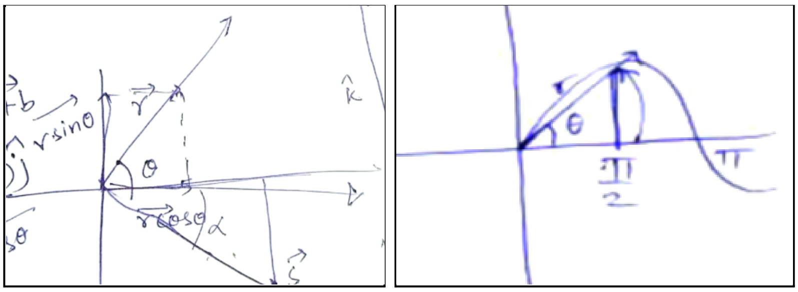


Figure 2B.6: Students showing the connection between resolution and trigonometric ratios

The notion of trigonometric ratios is introduced in a different context and, as seen in chapter 2A, these are not linked and effective topics like unit circle are not presented in the context of vectors by the textbooks. And when students are probed about these ratios, they drew and defined them reasonably accurately in right triangles; but they (especially TSs) struggled with the application of these ratios in the case of vector resolution. Interestingly, two TSs drew the sinusoidal curve showing a variation of $sin\theta$, and one of them tried to construct a right triangle within the sinusoidal curve (figure 2B.6-right), and then was clueless.

<u>Importance to the rectangular components</u>

When vectors are resolved in mutually orthogonal components like rectangular components, the addition along a given direction becomes similar to scalars. We noticed difficulties among students with understanding the general notion of a unit vector, their role in adding vectors and the basis of addition using $\hat{i}\,\hat{j}\,\hat{k}$ components. All the students equated components with only rectangular components (x and y components). That, a vector can be resolved into non-rectangular components also, is crucial to internalising the fact that addition and resolution are reverse processes. Moreover, 10 of the 11 students could not explain the significance of using rectangular components for addition; only

OS2 could explain it explicitly. Three others had some sense of the significance by suggesting that it simplifies algebraic addition, but all of them gave related but incoherent justifications for the significance, like the ones below.

- OS4 cites, the right triangle being amenable to Pythagoras theorem,
- Gets simpler at θ=90° as *cosθ* becomes 0, making the dot product 0 or, as per OS5, simplifying the expression for the resultant (cosine rule).
- OS3 wonders what would happen if they were not orthogonal, say at θ=120°, and then says, *it is not possible, they must be at θ=90°*. He says, "if they are not orthogonal they would affect each other". Though this was in right direction, he could not build this reasoning completely to establish how adding vectors using $\hat{i}\,\hat{j}\,\hat{k}$ components become easier.

Each of these students was correct in bits and pieces, but none of them had a complete and coherent picture. The OSs had a naive sense of comfort and familiarity with using rectangular components and they could not get their head around non-rectangular axes and coordinate axes. They could not prove that when the components are rectangular, there are no components of one along the other and magnitudes of vectors in the same direction could be added like scalars. This is also linked to their inability to fathom clearly the equivalence of addition geometrically and algebraically.

Components of components

This paradox, described earlier, was not explained by any of the TSs. Among OSs, two students (OS4 and OS2) could explain it after a long discussion. The fact that the resultant is replaced by the rectangular components once you resolve them is at the root of this paradox. Resolving this paradox requires an understanding of the process of resolution and the relation between the components and the initial unresolved vector.

Products of vectors

Students' understanding of vector products was another indicator of the incompleteness of their conceptual interlinks (models). One may superficially view it as merely a deficiency in mathematical skills, as seen from the students' struggle with managing the $\hat{i}\,\hat{j}\,\hat{k}$components when calculating the dot products or the determinant of the matrix when finding the cross product. However, it pertains more to the imagination of vector products and scalar products. Interaction with the students confirms their struggle in assimilating this new form of products. They defined the product simply using expressions of $abcos\theta$ and $absin\theta$. They came up with simplistic explanations like "*use a scalar product if the final answer needs to be a scalar (like work done)*" and "*a vector product if the product needs to have a direction (like a torque)*". This explanation immediately breaks when we question them about how the magnitudes of these products are consistent with the required quantities.

2.3 Theme-3: Limitations in Problem-solving approaches

There are also certain patterns in the students' problem-solving practices besides the indicators of algebraic dominance.

Superficial and Procedural description

When asked to explain the addition of vectors, some students (like OS4) explained the procedure of drawing the triangle or the parallelogram. Even after persistent probing, they could not articulate the addition with any reference to its geometrical or physical significance. There were students like OS1, OS3 and OS2 who could give a physical meaning — "*arriving at a vector which has the same effect as the individual vectors acting together*". However, this was a singular case, as the same students when discussing dot products or cross products, failed to provide physical interpretations or explanations beyond procedure. This indicates that the students currently don't hold a coherent conceptual understanding of these geometrical laws beyond the procedures of drawing them.

Attempts to map to standard algorithms of problem-solving

In problems related to mass on an inclined plane (for TS), students tended to rely on the standard algorithm of drawing free-body diagrams and resolving and writing equations (figure 2B.2). This problem was a simple yet a non-standard one. We could see attempts by students to follow the standard steps (see the kind of vector resolutions attempted, the structures of the equation written like directly replicating some formulae). Students preferred to rely on this approach even at the cost of losing mathematical coherence, and eventually gave up.

3 Chapter Discussion

The patterns found in the rich descriptions of the conceptual reasoning behaviour in students in all three themes are tightly interrelated. Many of these behavioural indicators confirm evidence not just related to but beyond the topic of vectors reported in the literature (reviewed in sections of Ch-1B).

It is acknowledged that transforming and reasoning using spatial and geometric entities (like diagrams) is central to reasoning and imagining in mathematics (Dörfler, 2005) and sciences in general (Ramadas, 2009). However, the first theme shows the dominant and superficial reasoning among the students lacking ability of geometric reasoning. This is in agreement with numerous studies in literature that report students being more comfortable with $\hat{i}\,\hat{j}\,\hat{k}$ form of vectors than with the geometric form (as reviewed in Ch-1B). Similar behaviour in students is reported in mathematics education research on other topics like cartesian geometry, where there doesn't seem to be any depth beyond superficial algebraic manipulations in students' reasoning (e.g. Knuth, 2000). This becomes far more compelling in cases like vectors and cartesian geometry, which have a significant geometrical aspect along with the algebraic ones. Connecting this to student difficulties, Atiyah (2002) notes that when we start doing algebraic manipulations, we have a tendency to "stop thinking geometrically and about the meaning".

As per the SCIARM framework, a lack of geometric manipulation while practicing, could limit the student's ability of visuospatial transformation of models while reasoning. One may always attribute the observed tendency of algebraic approaches to the assessment system with questions more amenable to algebraic manipulation. Even if we concede that this tendency is due to the paper-based format of the written assessments, students should not have any limitations to reason and explain geometrically, when speaking or gesturing in the interviews. But that is not the case, as evident in this study where even during interviews and when the situation is more amenable to geometric reasoning.

Further, the indicators-3,4 in theme-1 strengthen the rationale of the study to look at the affordances of instruments. These subtler indicators reveal stronger influences on cognitive aspects (gestures, perceptual nature of symbolic reasoning). For example, Indicator-4 of algebra underlying imprecision of vocabulary brings out a subtle yet strong influence algebra has on students' reasoning. These indicators of dependence on algebra reveal a subtle yet deep influence algebra has on students' reasoning (and cognition).

Indicator-1 in theme-1, as well as the behaviour in theme-3, confirm with widely reported conceptual behaviour of a tendency to use algebraic expressions and memorise the formulae (Lithner, 2003), often without a coherent understanding of them (Schoenfeld, 1985). Indicator-2 of adding vectors comfortably with rectangular components similar to scalars is in line with the literature (Cuoco et al., 1996). Extending the above 2 indicators of habituation to certain ways of thinking and reasoning, we see in indicator-3 a tendency to close a question with an algebraic expression. This is very similar to that among younger students, when transiting from numerical arithmetic to symbolic algebra, struggling to cognitively reconcile a final answer being symbolic (an algebraic expression) and not a number (Chalouh & Herscovics, 1984).

Further, the evidence in theme-3 indicates physics problem-solving seen as following certain standard algorithms without an appreciation of the methods and the role in modelling the physical system. This reflects superficial algorithmic reasoning with neither a deeper sense nor coherence. According to Fischbein and Barash (1993), the algorithmic approach creates conflicting mental models, which leads to poor understanding. Reasoning coherent schemes or models have been argued to be important for effective problem-solving (Chinnappan, 1998). The evidence in theme-2 indicates numerous incoherent conceptual linkages agreeing with these general findings in the literature.

The bias towards algebra in their reasoning-behaviour is not problematic as long as geometric understanding is also available, and both are strongly integrated. However, literature reports students' struggle with geometric aspects of vectors and poor geometry-algebra integration. These reports, along with the evidence across the themes from our study, point towards a clear pattern in student reasoning-behaviour. Deeper investigations into the reasons behind the pattern are required, as student difficulties cannot be systematically addressed without such an analysis.

4 Conclusion

Further, the corollary (addressing the research question) that the interactive affordances of representational medium shapes the learners' STEM cognition, gets strengthened by the evidence in the two chapters 2A and 2B. There is an interesting semblance that emerges in the patterns in the existing medium affordances (as shown in ch-2A) and the existing students' CRB (evident in this chapter). See figure 2B.7.

Students' CRB	Existing media affordances
Dominant algebra based reasoning	Limited geometric manipulation
Rote learning of algorithms	Opaque dynamics during problem solving
Incomplete and incoherent models of vectors	Modular and serial ordering

Figure 2B.7: Semblance in the patterns in the existing students' CRB and the existing media affordances

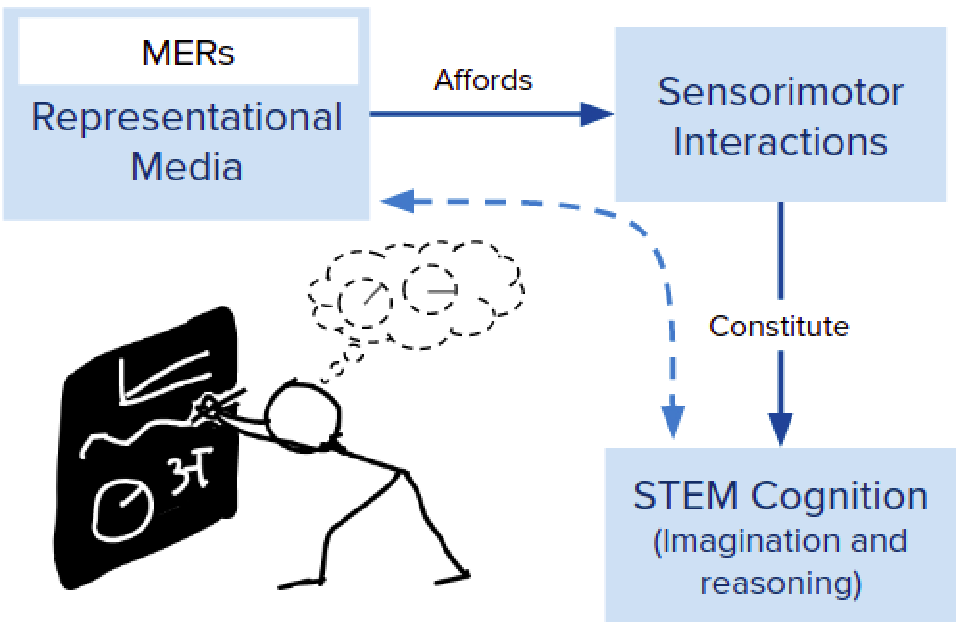


Figure 2B.8: SCIARM framework- Strengthening of the corollary relating interactive affordances of representational media and the students' STEM cognition, indicated by the dotted lines

The semblance in these patterns indicates that student's existing STEM cognition (as reflected in their behaviour) partly be emerging from the limited interactive affordances of existing representational medium (static paper-based). This strengthens the corollary (indicated by one dotted line in figure 2B.8). Now the question is, does students' CRB change when interactive affordances of the representational medium change? The next chapters of execution-part are towards extending this evidence further, through some careful changes in the interactive affordances of the representational media. These results are discussed again in greater detail in ch-3A.

Summary Table

*Table-2B.1: Conceptual gaps in the textbooks and related conceptual behaviour of students, deriving from the limitations of the textbook media using the framework developed in the next chapter. *Refer to the CL/SCA section introduced in figure 2D.2 (in Ch-2D).*

Limitations of paper-based medium	Missing SCAs/CLs* in the Textbooks(from the textbook analysis)	Specific conceptual behaviour in the students (from the student and teacher interactions)	General tendencies among the students (from the student and teacher interactions)
Lack of geometric manipulation	• Triangle law of addition not applied beyond definition and theoretical description (SCA-1, CL-1,2,3) • Parallelogram law of addition not applied as a dynamic geometric method beyond the formula (SCA-2, CL 4,5,6) • No scaffolds for resolution and addition using resolved rectangular components (SCA3 CL 8,9)	• Students don't understand the geometric addition of vectors and follow rote procedures. They are more comfortable with adding vectors just algebraically. • The direction of vectors ignored in Triangle Law of addition • Establishing equivalence of addition methods (triangle and parallelogram and using ijk components) • Teachers report students struggling with the notion of direction and understanding unit vectors while adding vectors algebraically.	• Students express ○ High tendency to use rectangular components. ○ Preference for algebraic approaches while explaining or solving (even when reasoning using geometric methods is best) ○ Superficial and meaningless usage of algebra ○ Algebra of scalars as a habit ○ Gesturing the algebraic manipulations and structures (like a square root in the expression for the resultant vector) ○ Written scripts filled with formulae and algebraic manipulations ○ Definitions as statements with verbalized algebra • Teachers noted that it is very difficult for them to draw diagrams numerous times on the board to show the processes (both physical and mathematical) dynamically
Serial Ordering	• No scaffolding for the use of trigonometric ratios for the expressions of resolved components explained. (SCA3 CL8,9) • No link between the	• Students rote learn the expressions and can't explain the process of arriving at the rcosθ and rsinθ expressions. • Resolution and addition	• Difficulty in integrating the knowledge of trigonometric ratios and geometry of circles, and triangle, while resolving vectors • Difficulty in identifying the

	addition and resolution as inverse operations (SCA4 CL 12,13)	not understood as related operations	ease that rectangular components bring to the algebra of vectors
Opaque Problem Solving	• No explanations provided in solving problems in linear motion of objects in a plane for (SCA5 CL-14,15,16) ○ choosing the masses of interest ○ drawing free body diagrams ○ choosing the frames of reference • No explicit explanations of how vector addition and the resolution are used in the process of problem-solving (SCA5 CL-14,15,16)	• Students rely on the memorised formulae and standardised algorithms to solve problems. • When given non-standard problems, they try to fit the problem to suit the standard algorithm or formulae, rather than trying to use the first principles (which they lack)	• Students understand science as algorithmic problem solving rather than an endeavour based on model-based reasoning. • This understanding of the nature of science alters their perspective towards it.

Publications from parts of this chapter

Karnam, DP., Mashood, K. K., & Sule, A. (2020). Do student difficulties with vectors emerge partly from the limitations of static textbook media? *European Journal of Physics*, 41(3), 035703. https://doi.org/10.1088/1361-6404/ab782e

2C

Design-1: Touchy Feely Vectors-1

Objective of the chapter: To report the design of a new media system called Touchy-Feely Vectors, which address the limitations of the paper-based medium in the previous chapter making the abstract symbolic system tangible.

Design principles:

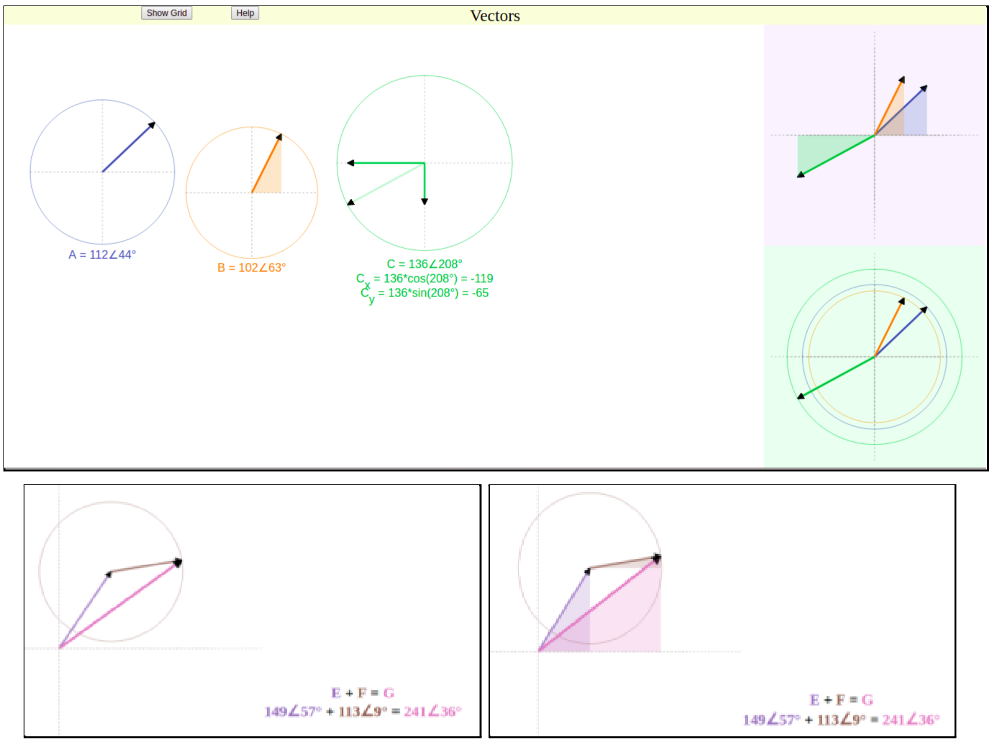
- Dynamic representations with real-time integration
- Manipulation of the representations as entities
- Unit circle as an integrating model

Key ideas: The limitations identified in the earlier chapters are systematically compensated using the affordances of the new media into a manipulable system called Touchy-Feely Vectors

This chapter describes the design of the first version of the computer-based interactive system, called Touchy Feely Vectors - 1 (TFV-1) to address the limitations that were identified in the paper-based textbooks (Ch-2A). This system which supports resolution and addition of vectors was based on a javascript-based platform built with an engineer from IIT Roorkee, who worked as a project staff in "LSR Group" of HBCSE. The system is OS independent and can run on any web browser both on a normal computer or laptop (using keyboard and mouse) and on a touch interface. However, in the study-4, which used this design, we used the keyboard and mouse-based interface.

1 Description of the features

TFV-1 allows students to create, manipulate and operate (add and resolve) vectors on a browser, using a mouse-based interface. The screen has broadly three panels (See figure 2C.1a). The central panel is a larger panel, where vectors can be created and operated on. The side panels show the right triangle projected on the x-axis and the circles of all the vectors on the screen.



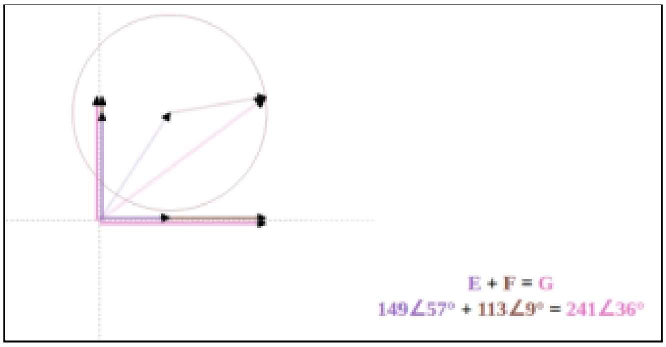


Figure 2C.1: (a-d from top to bottom) (a) Creating a vector, changing its direction and magnitude, and resolving it into rectangular components. (b) Addition of two vectors. (c) Right Triangles superimposed. (d) Rectangular Components getting added. Touchy-Feely Vectors-1 (bit.ly/tfv-1); scan the QR code to interact with the simulation.

In the central panel, a left-click on the screen creates a vector (figure 2C.1a). A vector is represented by a conventional arrow (like a ray) within a circle and faint coordinate axes; this is a spatial/geometric representation for the vector. Further, this has a tag with the name of the vector (a capital alphabet) and numerical representation in $r\angle\theta°$ form: r stands for the magnitude and θ° for the angle made by the vector with the positive x-axis in degrees; this tag is an algebraic representation for the vector. The vector can be moved around the screen by a right-click and drag interaction. The entire circle, coordinate axes and the algebraic tag move along with the vector in the screen. The direction of each vector can be manipulated by a left-click and drag on the line of the vector; the magnitude can be changed by a left-click and drag anywhere inside the circle other than around the line.

The vector can be resolved by a double click (displaying the right triangle form, vector B in figure 2C.1a) followed by a single click (leading to a resolved form, vector C in figure 2C.1a) anywhere inside the circle. The x and y components emerge from the two orthogonal sides of the right triangle and move to the respective axes through a short animation. The related changes in algebraic equations are displayed simultaneously, capturing the relation between the direction and magnitude ($r\angle\theta°$) and the rectangular components ($r\cos\theta°$, $r\sin\theta°$). The initial vector becomes faint and is available for manipulation of the direction and magnitude. The corresponding changes in the x and y components can be seen in real time both geometrically in the figure and algebraically in the equations in the tag. Moving back to the initial state of the vector just requires reversing the steps (single click to right triangle state, followed by a double click). A vector in any form can be removed from the panel by a left-click on the vector while holding the shift key.

Next, the system also supports the addition of vectors. Clicking two vectors one after the other while pressing the control key (Ctrl) adds the two vectors using the triangle law of vector addition (figure 2C.1b). The vectors move to the left half of the central panel and arrange themselves as the two sides of the triangle and the third side appears as the resultant of the vector. The coordinate axes now correspond to the resultant vector. The axes and the circles of the initial two vectors are removed; now, only arrowheads represent the vectors geometrically. The tags are also replaced by an addition equations in $r\angle\theta°$ form in the right half of the panel. The direction and magnitude of the active vector (one with the circle in Figure 2C.1b) can be manipulated using the same controls as for normal manipulation. When this vector is manipulated, result the third side of the triangle, i.e., the resultant vector changes. The other vector is made active for manipulation by a right-click within the circle of the first vector and vice versa. The changes in the magnitude and direction of the component vectors, along with those of the resultant, reflect in the equations in real-time. This allows students to see geometrically the process of addition, and thus could help address some of the student reasoning and imagination difficulties related to the addition of vectors.

Further, algebraic addition with rectangular components can be performed by the same series of steps used for resolution earlier (figure 2C.1c). A double click provides the right triangle form (figure 2C.1c) and a single-click later provides the rectangular components of all the three vectors: the two initial vectors and the resultant vector (figure 2C.1d). The x and y components of the vectors are indicated by a set of three juxtaposed arrows along the axes corresponding to the respective components of the three vectors. As the coordinate axes visible correspond only to that of the resultant vector, the algebraic addition becomes apparent. In figure 2C.1d, the x-components of the two vectors add up to give the x component of the resultant vector and y-components also add up similarly. A summary of the key interactions are presented in the table 2C.1 below.

Table 2C.1: The Interactions used and the effects on TFV-1

Effect of the interaction	Interaction (Action [visualisation]) in TFV-1
Create a vector	Left-click on the central panel
Activate/deactivate the vector for changing magnitude and direction	NA (A vector is always active for changing direction and magnitude)
Change the direction of the vector	Left-click + drag on the line of the arrowhead
Change the magnitude of the vector	Left-click + drag on the line of the arrowhead
Move the vector around in the screen	Right-click+drag anywhere in the circle
Delete a vector	Shift + Right-click
Add two vectors	Ctrl + Left-click on two vectors one after other
Change the direction and magnitude of the vectors in addition mode	Make the vector active for manipulation [as indicated by the circle around the arrowhead] To switch control of manipulating the vector, right-click within the circle [the shift of the circle indicates this]. Once the circle appears the interactions are similar to a normal vector.
Parallelogram Addition	NA
Resolve a vector	This is a 2-step procedure (Double-click + single-click). 1. Double-click anywhere inside the circle [A right triangle appears] 2. Then, single left-click anywhere inside the circle [Animation shows the x and y components emerging from the right triangle and the equations appear]
Manipulate in resolution mode	Same as earlier on the faintly visible initial vector
Move back to the initial vector	Reverse the Double-click + single-click interactions [Animation also reverses]
Resolve the resultant vector (to see the algebraic addition)	Same as those for resolving the vector. [x and y components of the 3 vectors emerge from the 3 respective right triangles]
Move back to the initial vector (unresolving) in addition mode	Reverse the Double-click + single-click interactions [Animation also reverses]
Finish Addition	Click on the done addition button on the top. [Resultant vector remains. The initial vectors disappear]

To scaffold practice, we designed a series of tasks that smoothly extended the textbook representations. The tasks required students to explore all the possibilities of manipulation, such as creating and manipulating vectors, resolving them, adding them geometrically using triangle law of vectors, and finding sets of vectors that have a given resultant. More details about these can be found in the chapter 2D, where we describe the implementation of the TFV-1 as part of the laboratory study.

2 Key principles guiding the design

2.1 *Dynamic representations with real-time integration*

All the representations used are dynamic in nature. They change in time. The geometric representation of the vector — in the form of the arrow head — is dynamic and mobile. When resolved, the components are also dynamic; they change with the changes in the initial vector. When two vectors are added, the resultant vector is also dynamic; it changes with the changes in the initial two vectors, and when the vectors are resolved. In all the above conditions, the corresponding algebraic representations also change. Other important features about these dynamics are the interlinking of the changes in real-time; geometric and algebraic representations are interconnected in real-time. A change in the direction and magnitude of the vector in its geometric representation reflects in realtime in the changes to the algebraic representations in tags, changes to the resultant vector when the vectors are added, and changes to the x and y components when resolving the vector and adding resolved vectors.

Another set of dynamics that is embedded in the design is in the form of the animated transitions. When the vectors are added, an animation dynamically reveals the vectors aligning themselves and the resultant vector emerging in a particular spatial and temporal order of events. When a vector is resolved, an animation dynamically reveals the emergence of x and y components from the right triangle and they align with the coordinate axes. When the resultant vector is resolved, a similar animation again dynamically reveals the x and y components of initial vectors emerging from respective right triangles and aligning themselves in a particular order. All these dynamics help unravel certain conceptual aspects underlying the interconnections between the algebraic and geometric modes of vectors. Such an interconnected dynamic system is one of the central strengths of the TFV system.

2.2 *Manipulation of the representations as entities*

The second key principle behind the design of TFV was to allow manipulation of the geometric entities. We ensured active agency in making the manipulations instead of passive visualisations. Most of the dynamics described above were triggered by a set of coherent interactions especially on the geometric representations of the vectors. The direction and magnitude were two key manipulable parameters across the system. Further, the principle of making the representation itself tangible, as in the case of 'Touchcounts' for numbers (Sinclair & de Freitas, 2014), was implemented; the geometric representation of the vector is made like an entity which can be moved around, incorporating a kind of concreteness to it; these representations become amenable for senosri-motor actions, thus triggering the affordances for interaction. Given the mouse and keyboard based interactions, we tried to start with a coherent set of interaction protocols described in the table above. The circle around the arrow representation of the vector has played a very important role as a spatial entity in binding these interactions together; for example, the interactions to manipulate the direction and magnitude of the vector, or to resolve or add are tightly connected to the circle. The interactions are deliberately confined to the geometric representations, as the paper-based medium already does a good job in allowing the manipulations of the algebraic representations, and we intend our system to augment but not replace the existing medium.

2.3 *Unit circle as an integrating model*

The circle around the vector arrow, besides supporting the interactions, had a very strong conceptual basis. The transition from polar (r,θ) to the rectangular coordinates (x,y) is central to integrating the geometric and algebraic representations of the vectors. The process of resolution involving the trigonometric ratios, and the related process of x and y components of initial vectors linearly adding up to give the respective components of the resultant vector remain as opaque conceptual aspects for most of the students. Students appear to confuse this particular application of trigonometric ratios among others in physics and mathematics. Unit circle is said to be central to bridge this particular connection in the context of complex numbers (Panaoura et al., 2006; Smith, 2016). A combination of the unit circle and the right triangle is thus effectively introduced and used in the system all through to support students integrating various conceptual aspects ranging from the particular application of trigonometric ratios, and geometry of circle and right triangle. The unit circle construction, which binds various conceptual aspects along with the interaction support, plays a pivotal role in the design of the TFV system.

2.4 *Other similar applications*

As reviewed in the literature in ch-1A, this aspect of dynamics is taken advantage of in many applications like dynamic geometric environments. Even for the topic of vectors, there are two popular applications on PhEt simulations (Perkins et al., 2006) and Geogebra (e.g. Brzezinski, 2016; International GeoGebra Institute, 2002). However for our purposes, especially given the nature of the research question concerning the interactive affordances, we chose to develop the interface in-house for better control on the design and also adoption in the Indian classroom conditions. We were deeply inspired by similar new media interfaces like TouchCounts (Sinclair & Heyd-Metzuyanim, 2014), which tries to make numbers tangible, in the same way we attempt to do with vectors. Further, these are informed by 4E cognition (constitutivity hypothesis) and hence more amenable to testing the corollary based on SCIARM framework and addressing the research question. Further, our approach is based on identifying key gaps in the existing medium and carefully addressing the limitations, as outlined in the next sections.

3 How the design compensates for the limitations of textbook media

Chapter 2A identified three main limitations generated by the static nature of textbooks (Limited geometric manipulation, Serial ordering, Opaque problem solving). We discuss below how TFV-1 systematically compensates for these limitations. Table 2C.2 provides a summary.

3.1 *Limited Geometric Manipulation*

A central design feature of TFV-1 is making the vector manipulable as a geometrical entity. For example, a vector's magnitude and direction can be directly altered geometrically. Addition of vectors using the triangle law (geometrically) can now be performed, and the resultant is now easily and actively explorable, for many combinations of vectors. Further, the effect of the changes in magnitude and direction can be directly seen in the rectangular components. These together take advantage of the spatial dynamics afforded by the computational media, in allowing manipulation of geometric aspects of vectors with real-time effects, which are otherwise not supported by the paper-based medium easily.

3.2 Serial ordering

A major issue with the serial ordering created by static media is the lack of integration between earlier and later concepts. One way the TFV-1 design addresses this problem is through the representation of a circle (the unit circle) connected to the vector, which provides a good way to integrate vectors, the geometry of triangles, as well as the geometry of circle and trigonometry. The short animation unfolds the underlying dynamic process of rectangular components emerging from the right triangle. This provides students an active component to manipulate in their imagination. The relations between magnitudes and directions, as well as the rectangular components, are more coherent and tightly modelled by a unit circle, unlike in the case of an equation-based derivation, which is more opaque. As the manipulations also change equation values and capture the reverse nature of the addition and resolution process, the links between geometric and algebraic representations of vectors could also be strengthened.

3.3 Opaque problem solutions

Since the TFV-1 system does not present problems yet, this feature has not been explicitly addressed in the design. However, some elements to address this issue are part of the design. For instance, the changes in geometric manipulation are interlinked with changes in the numerical entities and equations (algebraic entities). This allows algebraic entities such as equations to be understood as active entities ((Majumdar et al., 2014) presents a related design that makes equations enactable), helping students alter their view of algebra as static strings of symbols. The static symbol view is an artefact of static media, and it encourages a focus on equation solving based on moving components around. This becomes a meaningless mechanical procedure when the connection to equations and geometry is not understood, encouraging algorithmic modes of reasoning and problem-solving. By making geometry an active component, the system allows students to develop a spatial understanding of vectors, which is a precursor to model-based reasoning. The dynamic real-time interactions and explorations provide critical support to develop spatial imagination and also explore new geometry-algebra states. This process allows developing a sense of mathematical modelling.

Summary table

The table below summarises how the TFV-1 design compensates for the three limitations of the paper-based medium that are identified after the textbook analysis in the ch-2A.

Table 2C.2: Design principles emerging from the limitations and the missing SCAs/CLs identified in the textbook analysis

Limitations of paper-based medium	Missing SCAs/CLs in the Textbooks (from the textbook analysis)	Design Principle addressing the limitation	How the design could address the missing SCAs/CLs
Lack of geometric manipulation	• Triangle law of addition not applied beyond definition and theoretical description (SCA-1, CL-1,2,3) • Parallelogram law of addition not applied as dynamic geometric method beyond the formula (SCA-2, CL 4,5,6) • No scaffolds for resolution and addition using resolved rectangular components (SCA3 CL 8,9)	• Dynamic And Tangible Representations ○ Make the geometrical entities tangible, dynamic and manipulable and linked to algebraic entities in real-time. ○ Unpacking the underlying processes of addition and resolution dynamically. • Interactions for synthesis of concepts ○ Active agency in making the manipulations instead of passive visualisations ○ Make the interactions for manipulation consistent with the conceptual links ○ Circle as an interaction entity/interface	• It allows ○ adding vectors using triangle law of vector addition emphasizing on the directions of vectors ○ dynamically changing the component vectors to see the effect on the resultant • Currently, there is no support for parallelogram version of vector addition. • Presenting geometrical processes dynamically and allowing manipulations in ○ resolution of vector into rectangular components ○ addition using rectangular components

Serial Ordering	• No scaffolding for the use of trigonometric ratios for the expressions of resolved components explained. (SCA3 CL8,9) • No link between the addition and resolution as inverse operations (SCA4 CL 12,13)	• Dynamics in Resolution integrating circle and right triangle and trigonometric ratios • Addition as a reverse process of resolution (and the notion of non-rectangular components)	• Unpacking processes in resolution operation dynamically by animating the emergence of rectangular components thereby integrating the geometry of triangles and circle and trigonometry. • Interlinking the processes of addition and resolution through reversible interactions.
Opaque Problem Solving	• No explanations provided in solving problems in linear motion of objects in a plane for (SCA5 CL-14,15,16) say, ○ choosing the masses of interest ○ drawing free body diagrams ○ choosing the frames of reference • No explicit explanations of how vector addition and the resolution are used in this process of problem-solving (SCA5 CL-14,15,16)	• Present the related topics in an integrated manner enabling the creation of a coherent model for students. (E.g. the unit circle to integrate trigonometry, geometry and vector resolution and addition)	• The processes of addition and resolution are made transparent and the inter-relations between them could be established using the conceptually consistent interactions. • However, no direct support for applications in mechanics. • Suitable tasks can be developed as illustrated in the tasks table 2D.1.

Publications from parts of this chapter

Karnam, DP., Agrawal, H., Parte, P., Ranjan, S., Borar, P., Kurup, P., Joel, A J., Srinivasan, PS., Suryawanshi, U., Sule, A., & Chandrasekharan, S (2020). Touchy-Feely Vectors: a compensatory design approach to support model-based reasoning in developing country classrooms. Journal of Computer Assisted Learning, 1–29. https://doi.org/10.1111/jcal.12500

Karnam, DP., Agrawal, H., Mishra, D., & Chandrasekharan, S. (2016). Interactive vectors for model-based reasoning. In W. Chen, T. Supnithi, A. F. Mohd Ayub, M. Mavinkurve, T. Kojiri, J.-C. Yang, ... S. Iyer (Eds.), *The Workshop Proceedings of the 24th International Conference on Computers in Education* (pp. 401–406). Mumbai, India: IIT Bombay.

Karnam, DP., Agrawal, H., & Chandrasekharan, S. (2018). "Touchy Feely Vectors" changes students' understanding and modes of reasoning. In J. C. Yang, M. Chang, L.-H. Wong, & M. M. T. Rodrigo (Eds.), *Proceedings of the 26th International Conference on Computers in Education* (pp. 143–152). Philippines.

2D

Study-3: Laboratory Study

The objective of the Chapter: To report the quantitative and qualitative evidence for the effect of this system on the students through a small scale intervention-based laboratory study.

Key Findings:

- Interaction with this system enhances the students' conceptual link strength in ~70% of the cases. And disrupts their existing models.
- Detailed qualitative analysis of the student interviews after the intervention indicates a change in their reasoning approaches and increased epistemic access.

Let us revive the line of argument after Ch-2B, where we reported initial evidence strengthening the corollary based on the SCIARM framework. We found in Ch-2A and Ch-2B that student's existing STEM cognition (as reflected in their behaviour) could partly be emerging from the limited interactive affordances of the existing representational medium (static paper-based). Then we designed a new media interface, TFV-1 (see Ch-2C) which changed interactive affordances. If the corollary were to be valid, we hypothesise that interactions with the changed representational medium should create a change in the students' STEM cognition. In other words, our design hypothesis is that interaction with TFV-1 would allow students to move closer to a spatial and model-based understanding of vectors, building on their learning based on textbooks. To test this hypothesis, we conducted the study reported in this chapter.

1 Sample and methodology

We interacted with grade-11 students (n=49) from a city-based English medium junior college (pre-university level institution for grades 11 and 12). The student population at this particular junior college had achieved average to above-average marks in grade 10 to gain admission. Some of the students studied in the vernacular medium till grade 10. The students had finished their academic year (grade 11) and had learned and worked with vectors all through the year. These were the same 6 typical students involved in study-2 (in ch-2B) who continued to participate in this study.

1.1 Materials (samples included in appendix):

Pre-test and post-test questionnaires:

The pre-test had 9 questions related to prerequisites (basics of vectors, trigonometric ratios, the geometry of lines), vector knowledge (addition using triangle and parallelogram laws of addition, and rectangular components; resolution and components), applications (in mechanics), and products (direct dot and cross products). This is the same test that these 6 students were administered as part of study-2 (See sample student sheet in appendix 2B.1 for both pretest and the post-test). Questions required drawing and writing, as well as answering true/false and multiple-choice questions. The post-test had a similar, but different, set of questions covering vector knowledge and applications. We encouraged students to express the reasons for their responses, and why they found a question difficult.

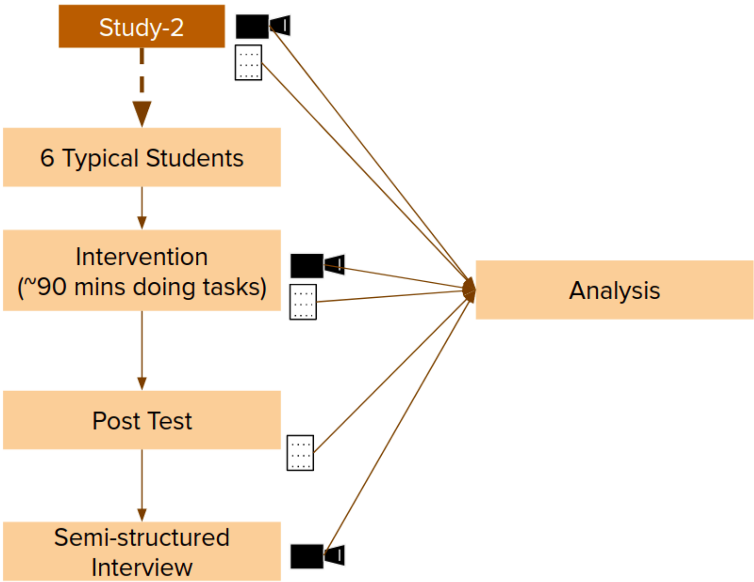


Figure 2D.1: The protocol of the study-3

<u>*The intervention tasks:*</u>

The tasks were designed to make students interactively explore various conceptual features supported by TFV-1. They were designed to address the conceptual (Table 2D.1) and media (Table 2D.2) limitations identified by the textbook analysis and teacher interviews (Ch-2A).

Table 2D.1: The tasks given to students during the intervention, and how they could help students develop various conceptual links (CLs).

Tasks given to students	Rationale
1. Please make a vector of a. Magnitude 130 and an angle of 60 degrees. 130∠60° b. Magnitude 100 and an angle of 270 degrees. 100∠270° c. Please play around with the magnitude and direction of the vectors for a while.	This was an introductory task. This introduces the TFV-1 controls to create and manipulate the vector as a geometrical entity. It was expected that the presence of the circle would evoke some conceptual priors related to angle and rotation.
2. Please break the above vectors into components.	This introduces the process of resolution and links the algebraic and geometric representations of a vector. The dynamics behind the process of resolution is revealed, as well as the emergence of corresponding equations based on trigonometric ratios. This task was expected to help interconnect the relevant geometry of triangle, circle and trigonometry, into a coherent model to reason with. These tasks thus address the limitations of serial ordering and opaque problem solutions, and
3. Please make a vector with a. x component 50 and y component 80 b. x component 50 and y component 100 c. x component 70 and y component 100	

d. x component 50 and y component -80	particularly the SCA-3 (CLs 8,9) related to rectangular components.
4. Please add the given two vectors. And observe the changes by changing the magnitudes and directions of both the vectors.	This open-ended task introduces addition using triangle law of vector addition. This specifically addresses SCA-1 (CLs 1,2,3).
5. Create two vectors (50i) + (90j) and (70i)+ (-60j). Add these two vectors; i,j are the unit vectors.	This addresses addition using resolved components. This is a scaffolded task, requiring students to make vectors with given components (repeating task 3 above) and then performing addition. This provides scope for students to explore the inter-relations between the processes of addition and resolution, thus addressing SCA-4 (CLs 12, 13)
6. Please create two vectors in such a way that the resultant of the two is 120∠60°.	This extends task 4 and provides an opportunity to explore the nature of vector addition using the triangle law of addition. An understanding of this process could support model-based reasoning while solving problems. This open-ended task addresses specifically SCA-1 (CLs 1,2,3) and in turn SCA-5 (CLs-14,15,16).
7. Please create a vector in such a way that it along with 100∠140° results in 80∠60°.	This is a close-ended task but requires extensive exploration of the system using the triangle law of addition. This process would also integrate prior knowledge in relation to the addition of vectors, and thus help build a model of the addition of vectors. This task thus addresses the limitation of serial ordering and opaque problem solutions, as well as SCA-1 (CLs 1, 2, 3).
8. Please create two vectors in such a way that the resultant of the two is 120∠60°. Create as many examples as possible.	This open-ended task requires extensive exploration of the system, using the triangle law of addition. This process would help integrate prior knowledge in relation to the addition of vectors and thus build a coherent model of the addition of vectors. This task thus addresses the limitation of serial ordering and opaque problem solutions besides SCA-1 (CLs 1, 2, 3) and in turn SCA-5 (CLs-14,15,16).
9. Please create two vectors in such a way that the resultant of the two is -80i+110j. Create as many examples as possible.	This open-ended task extends task 8, as it requires connecting addition using the geometrical method with addition using rectangular components. This task provides scope for interesting ways of integrating algebraic and geometric addition, thus addressing SCA-3,4 (CLs 8,9,12,13) and SCA-1 (CLs- 1,2,3).
10. Please create two perpendicular vectors in such a way that the resultant of the two is 120∠60°. Create as many possibilities as you can for the above task.	Same as task 8, but provides additional constraints. This task could establish the inverse relationship between addition and resolution, especially as rectangular components. This task thus strengthens SCA-3,4 (CLs 8,9,12,13) and

	SCA-1 (CLs- 1,2,3) and in turn SCA-5 (CL-14,15,16).

Table 2D.2:. How the intervention tasks address the limitations of textbooks

Limitations of paper-based medium	Missing SCA/CLs in the Textbooks (from the textbook analysis)	Tasks addressing these limitations and SCAs/CLs
Lack of geometric manipulation	• Triangle law of addition not applied beyond definition and theoretical description (SCA-1, CL-1,2,3) • Parallelogram law of addition not applied as a dynamic geometric method beyond the formula (SCA-2, CL 4,5,6) • Presentation of resolution and resolved rectangular components for addition (SCA3 CL 8,9)	• All the tasks required students to manipulate geometric entities on the screen. Especially Tasks 4, 6, 8, 9 and 10 are completely open-ended and would provide exploratory contexts to manipulate the geometric entities. (See Table-S3) • Students arrive at methods of solving the problems on their own. The vector model is progressively applied and refined during interactions.
Serial Ordering	• No scaffolding for the use of trigonometric ratios for the expressions of resolved components explained. (SCA3 CLs 8,9) • No link between the addition and resolution as reverse operations (SCA4 CLs 12,13)	• A range of tasks – manipulation-based, exploratory, closed-ended (Tasks 1, 3, 5, 7) and open-ended (not single solution Tasks 8, 9, 10) – covering all the concepts related to the conversion of the vector from geometric to algebraic denotation, resolution, and addition using triangle and rectangular components.
Opaque Problem Solving	• No explanations provided in solving problems in linear motion of objects in a plane for (SCA5 CLs -14,15,16) ○ choosing the masses of interest ○ drawing free body diagrams ○ choosing the frames of reference • No explicit explanations of how vector addition and resolution are used in the process of problem-solving (SCA5 CL-14,15,16)	• Students arrive at methods of solving the problems on their own. The vector model is progressively applied and refined during interactions. • All the open-ended tasks (such as 4,6,8,9, and 10) were designed to push students to look for models, beyond the blind application of formulae. This, in turn, could help in applying vectors during problem-solving in mechanics.

Protocol

Students (n=49) were first administered the written pre-test. After a week, a representative subgroup (n=14), covering the entire range of test performance, was invited for an interview, based on their responses on pre-requisites and the reasons in the responses. Only 8 students attended this session and they were interviewed in detail, based on their pre-test responses, and rich characterization of their CRB was developed (reported in study-2). After 4 to 5 days, 6 students (who could continue to attend the rest of the study) attended the individual intervention sessions, which involved performing tasks (in table 2D.1) on the TFV-1 system for about 70-90 minutes. While doing the tasks, the students were also provided with paper and pen to work out anything when needed in finishing the tasks. Student actions were recorded in a variety of ways, including video, written scripts (rough work), screen capture, and eye-tracking (Tobii X2-60). After a week, these 6 students were individually administered the post-test and interviewed grounded in their responses immediately, to capture their CRB.

2 Data analysis

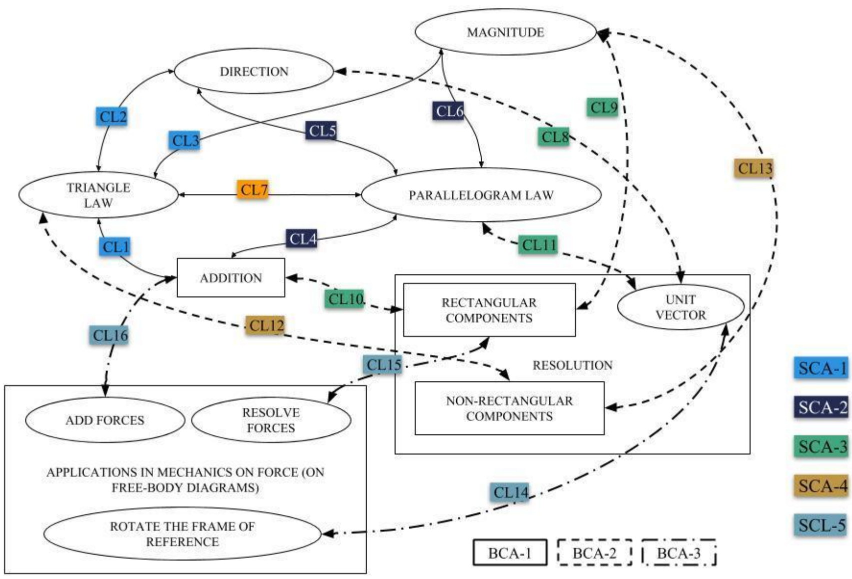


Figure 2D.2: A map showing the conceptual links and concept areas. Also, see Table in Appendix 2D.1

The pretest and the pretest interviews formed a part of the data analysed and interpreted in chapter 2B. This chapter extends that analysis with that of the remaining data: the intervention and the post-test data.

The pre and post-test results of the 6 students who attended the intervention were analysed in detail, to identify the nature of changes in their conceptual behaviour. For analysis, the topics of interest[18] presented were categorised into three Broad Concept Areas (BCAs, represented by the dots/bold links in figure 2D.2). These were then further classified into 5 sub-concept areas (SCAs, represented as 5 coloured boxes labelling the links; SCA1 - Triangle Law; SCA2 - Parallelogram Law, SCA3 - Rectangular components, SCA4 - Non-rectangular components, SCA5 - Application in the context of forces). The concept areas were constituted by 16 links between concepts (CLs). CL7 with only one CL was not considered as a separate SCA, and not used in this analysis.

Student responses were independently rated by 3 raters. Appendix 2D.2 provides a snapshot of the rating sheet of each student. Ratings were given to all relevant CL-question pairs. Only those CLs which could be expressed in response to a particular question in pre or post-test formed relevant CL-question pairs (all CLs could not be expressed in each question). A CL-question pair was deemed irrelevant if found irrelevant by at least 2 raters. A 5-point rating scale for conceptual strengths was developed for each of these concept links (1 = no indication, 5 = strong indication; see rubric in the Table in Appendix 2D3). This structure follows studies examining levels of conceptual understanding, based on the analysis of reasons and judgments in test responses (Besterfield-Sacre & Gerchak, 2004; Niemi, 1996). The scale does not measure the correctness, but rates for the conceptual clarity of that particular concept link, as expressed in the response to a given question.

The mode (statistical) of the 3 ratings by raters was taken as the final score for each CL-question pair. For cases where all the 3 ratings varied, a consensus of at least 2 raters was arrived at through a discussion, thus ensuring that at least 2 raters agree upon every rating. These ratings (out of 5 points) were then converted into percentages, denoting the CL strengths for each student (See Appendix 2D.2 for the rating sheet of each student). This analysis provides a comprehensive picture of change in each student's understanding, in the form of CL-strengths, pre and post-interaction with TFV-1.

[18] Listed in the Table 2A.1 in Ch-2A while analysing the textbooks

3 Findings

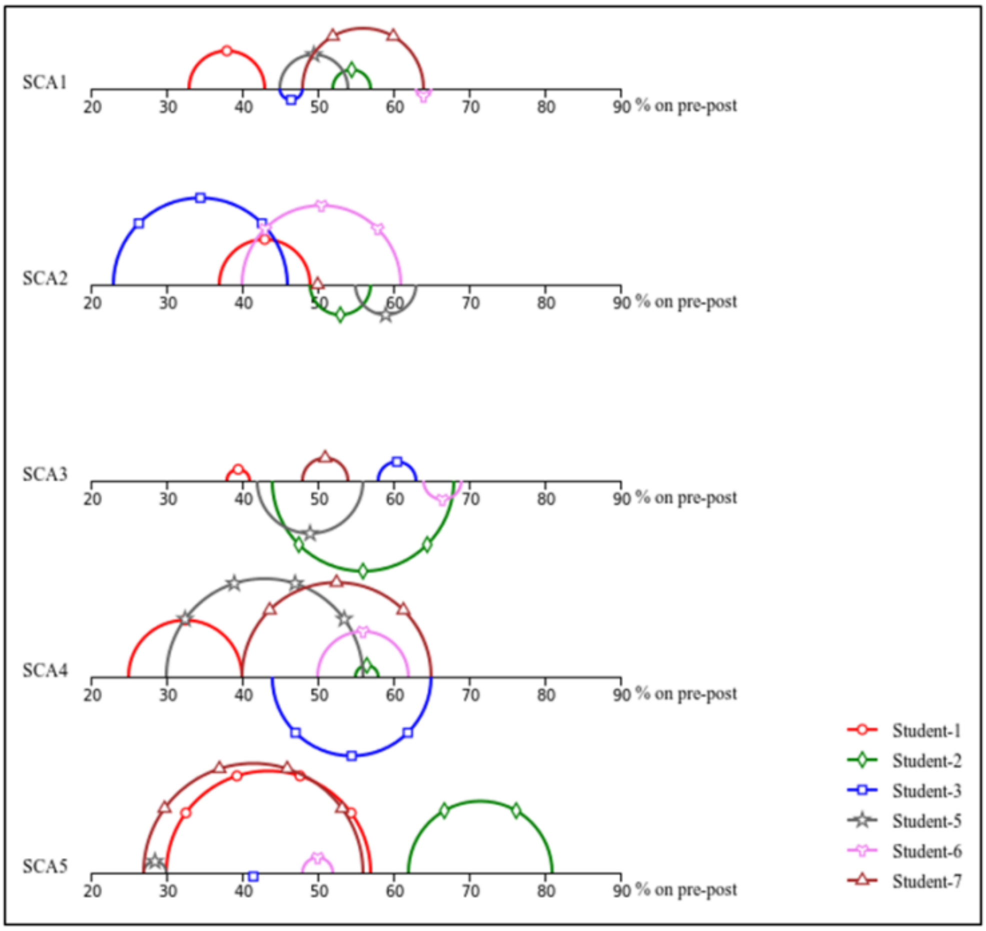


Figure 2D.3. Arcs: Trajectories of change in strengths across SCAs of 6 students

For each student, the strengths of the 15 CLs were aggregated into strengths of 5 SCAs. Strengths at the SCA level gave more meaningful patterns than those at CL, which were too detailed. Figure 2D.3 shows all the 6 students' CL strength changes in 5 SCAs (5 horizontal strength axes). Each arc corresponds to the transition from pretest to post-test of one student. The size of the arc captures the magnitude of growth/fall in the strength of the links, in that particular SCA of that particular student. An arc above the strength axis denotes a growth from pretest to post-test and an arc below the axis denotes a fall.

Students' SCA strengths changed in the following ways. At least 3 students showed growth in each of the SCAs, with more than 3 students improving in SCA1, SCA4 and SCA5. 4 students improved in SCA1 (triangle law). This is not surprising, as the addition in TFV-1 is based on triangle law. Even the 2 drops were only about 2-3% (smaller arcs) and are as good as no change. The parallelogram law (SAC2) is not very strongly expressed in TFV-1, yet 3 students improved in this. Surprisingly, only 3 students improved in the rectangular component (SCA3), even though rectangular components were a central part of the system. Five students showed growth in SCA4 (non-rectangular components), which is an under-explored topic in textbooks. All three (S2, S5, S6) students, with weakened conceptual understanding in SCA3, showed growth in SCA4. In SCA5 (applying vectors and vector operations in the context of forces), all the students showed improvement. Most students whose performance dropped (7 out of 9 drops) had pre-test strengths around 50-60% and higher pre-test strengths than those who showed growth. A possible explanation for this could be that the system disrupted their existing conceptual models (possibly rote-learnt, as indicated in pre-test interviews), which were adequate to meet their current academic requirements.

Overall, this data suggests that interacting with TFV-1 compensated for students' understanding of vectors in two ways: 1) improving the understanding of addition using triangle law and non-rectangular components, and 2) disrupting their understanding of rectangular components. The case studies in the next section provide more insights into these quantitative observations.

3.1 Case Studies

Besides the above analysis of the changed conceptual link strengths of the students, case-studies of two of the six students were undertaken to develop a richer sense of the way the interaction with TFV-1 has affected the students conceptual reasoning behaviour. To do this, we examined two students for any change in the nature of responses between pre and post-test conditions (both written test and interview data) in connection to their interactions with the TFV. These two were chosen as they had contrasting interaction patterns, and interesting patterns of trajectories from pre to post. Also, they covered a wider range of scholastic ability (beyond a minimum threshold for meaningful insights).

Student S2

S2's teachers found him promising, and he was confident in the pretest interview saying things to the effect that the pretest was very easy. Our later interactions to capture his CRB revealed that his understanding was limited. In the intervention, when others tried to perform the tasks using the affordances of the TFV-1 itself, he relied significantly on using paper and pencil, to restructure the problem task, apply algebraic methods, and arrive at a solution or an estimate of the solution. He then implemented it on the TFV-1 system. For example, when creating the vector 50î+80ĵ (by manipulating the direction and magnitude of the vector on TFV-1), he made a rough estimate of the resulting magnitude using formulae [magnitude=$\sqrt{x^2 + y^2}$, angle=$\tan^{-1}$(y/x)] and then fine-tuned it to arrive at the target and implemented on TFV-1. When trying to create two vectors whose resultant is 120∠60° (Task 6 in Table 2D.1), he simply resolved 120∠60° into rectangular components using the algebraic method and then added the two on TFV-1. Later, the tasks got complicated (with constraints like the multiple sets of vectors or multiple orthogonal sets of vectors resulting in 120∠60°), and not easily solvable using just algebraic estimations. Only after trying hard to find ways in which he could use some formulae to arrive at a rough estimate of the required vectors, he started exploring the affordances of TFV-1. For instance, in the case of a set of orthogonal vectors resulting in 120∠60° (Task 10 in Table 2D.1), he made intelligent assumptions like the initial vectors to be of equal magnitude and then estimated their magnitude as shown in figure 2D.4(e). In short, most of his interactions in accomplishing the tasks were *conceptually-guided interaction*. He described his problem-solving and reasoning using pen and paper as fundamental, associating it with theory. The interaction with TFV-1 was considered an experiment, where he could try things out and observe them. These are interesting strategies, hinting at conceptual clarity in addition and resolution of vectors. However, this strategy does not cohere well with the confusions in the responses in the pre-test, indicating limitations in his conceptual links.

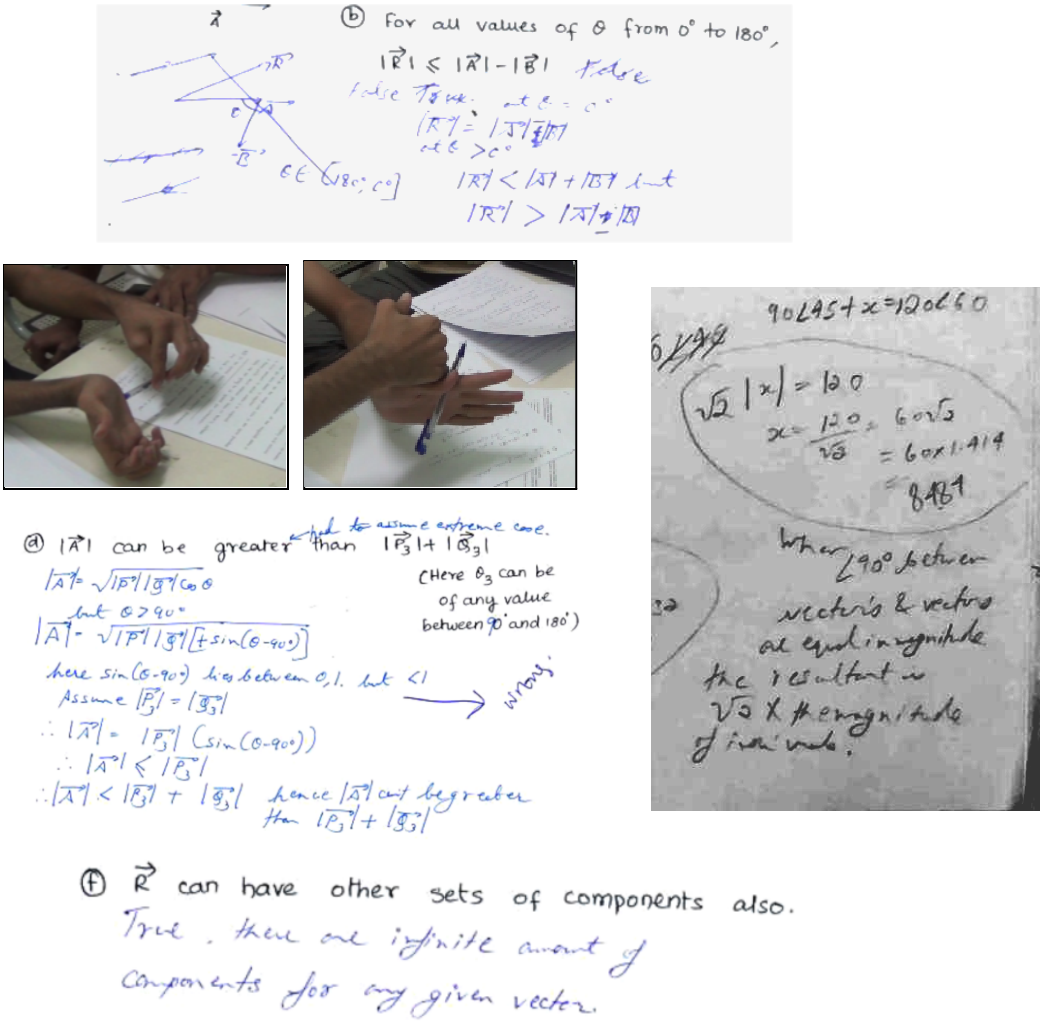


Figure 2D.4: (a) S2's post-test response (b,c) S2 using gestures in the post-test interview (d) algebraic estimations in the pretest (e) algebraic estimation during the intervention (f) A response by S2 in post-test

During the pretest, his responses were very tightly connected to the algebraic expressions, and his reasoning during the pretest interview was also based on manipulations done on algebraic expressions. For example, to a question in figure 2D.4(d) about the relation between the magnitude of resultant of two vectors and the sum of magnitudes of the two vectors (|A| and |P3|+|Q3|, when A=P3+Q3; question 5d in pretest in appendix 2B.1), he tried writing an expression for the resultant (though incorrectly), substitutes angles, and performs algebraic manipulations to arrive at a conclusion. His reasoning is completely based on algebraic expressions. During the interview about the question, he corrected the expression, but the reasoning was still based on algebraic expressions. Also, the discussions in the interview indicate that – though he has breadth in his conceptual repertoire, his understanding seems to break when attending to the details, as in the case of the question Pr.5d in figure .2D.4(d).

His responses in the post-test and the interview indicated a change in approach. To a similar question, he drew diagrams (see figure 2D.4(a), and considered the extreme conditions, and used both equations and diagrams to reason with (though he was not able to arrive at any conclusive response). Further, during the interview, he used gestures indicating dynamic manipulation of the two vectors in space, changing the angle (see figures 2D.4(b,c)). To the question (figure 2D.4(f)), he arrived at a sudden realisation of infinite possible sets of vectors to a given resultant, while he was explaining using gestures. He says- "*...in my mind, I just... I drew two axes and then I shifted the axes like this* (gesturing and rotating his hands in figures 2D.4(b,c)) *... perpendicular, perpendicular, perpendicular... oh!! There can be many!!*" Later, he explicitly noted that the interaction with the system helped him in such a realisation. Further, these gestures indicate the emergence of abstract science concepts (Roth & Lawless, 2002) and imagining spatially (Hegarty et al., 2005).

Student S5

Another student, S5 was meticulous, with all responses neatly and clearly presented in the answer sheets. Her responses in the pretest showed a good understanding of the prerequisites; however, the interviews revealed that her understanding was limited.

S5 took a lot of time to get comfortable with the interactions during the intervention session, and could not complete all the tasks within the time available, though she could explore all the features of the system by doing some extra tasks. Her interactions were not much conceptually-guided and she relied entirely on the affordances and visual cues of TFV-1 to accomplish the tasks. We call them *perceptually-guided interactions*. For example, when creating $50\hat{i}+80\hat{j}$, she always ended up controlling just one of the parameters — magnitude or direction — and never arrived at the required target. After a demonstration of how both the direction and magnitude could be changed, without discussing the general principle, she could proceed smoothly. She did not use paper and pencil medium other than to note answers. Even during the tasks to create a set of vectors that resulted in a given target vector, she kept manipulating the vectors, until she stumbled upon the target vector. After the interactions, she said with some excitement -

> *"the tool helped in understanding addition of vectors... hmm... components (meant rectangular components), like the way it is shown here (while drawing on the paper). Firstly the vectors, which were inside a circle... Earlier I was not used to how the components come. Like it is shown here (pointing to the system), the lines appear and then this line moves here, forming the y component. Resolution of vectors... umm... angle... I liked this very much."* Further, she notes *"As earlier I could not draw things out on my own for addition of vectors and all... angles and others I feel are now easier..."*

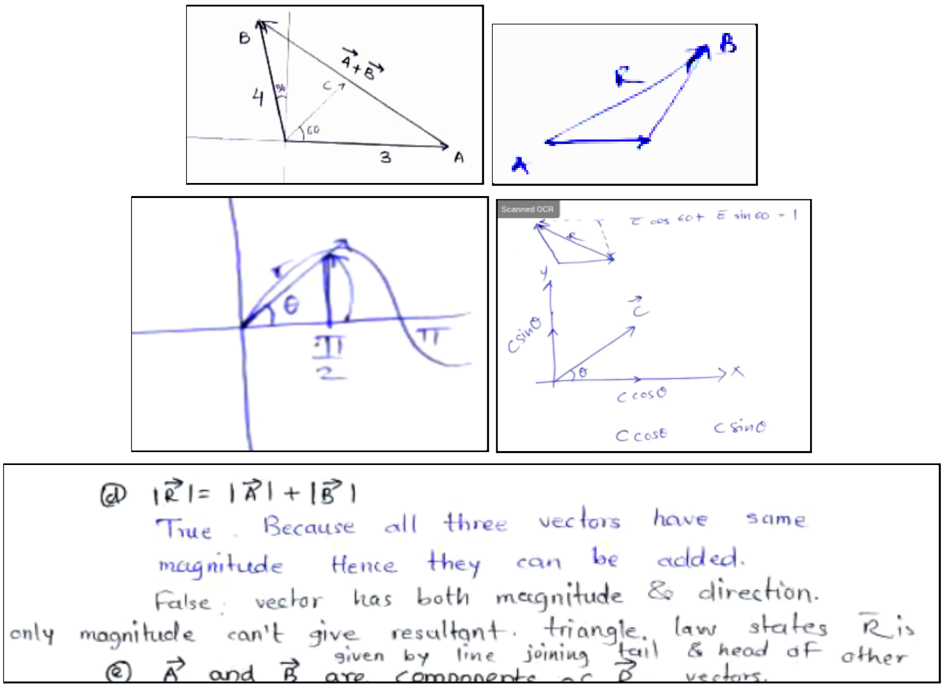


Figure 2D.5: (a,b) S5's response for triangle law (c) S5's drawing (d) S5's writings during the pretest interview (e) S5's response in the post-test (blue) and during the post-test interview (black)

During the pretest and the follow-up interview, it was clear that her conceptual connections were not strong, and her mental models were incomplete. For example, when discussing the triangle law of vector addition, she was anchored to the surface level similarity of the shapes of triangles, and her drawings replicated the same obtuse-angled triangle, without focusing on the directions or the order of vectors (figures 2D.5(a,b)). For rectangular components, she could draw the x and y projections label them as $rcos\theta$ and $rsin\theta$, but could not derive them from right triangle based constructions. When asked to show how the trigonometric ratios were used here, she drew a sinusoidal curve and tried to inscribe a right triangle in it as if in a unit circle (figure 2D.5(c)). While writing the relationship between a vector and its rectangular components, she wrote *Ccos60°+Csin60°=1*, instead of *Ccos60°î+Csin60°ĵ=C* (C being the magnitude of vector C), perhaps confusing with the trigonometric identity $cos^2\theta+sin^2\theta=1$. All these indicate incoherent interlinks and models where she mixes up concepts and references.

Post-test responses showed growths in SCA1 (triangle law 45%→54%), a drop in SCA3 (rectangular components 56%→42%), and a growth in SCA4 (components and addition 30%→56%). Many of her written statements were as inconsistent as in the pretest. However, she gave some conclusive answers in the interviews. For example, in the posttest interview, she rectified one of her answers indicating a realisation of the significance of the direction of a vector (figure 2D.5(e)). She said she understood the problems when done on the system (TFV-1), but not in the post-test.

This case shows small advances in the understanding of vectors, though overall progress is mixed. She had difficulty transferring knowledge from TFV-1 to a paper-based situation. Overall, the pieces did not come together into a completely coherent model yet. But the excitement she had after interacting with TFV-1 indicates a self-perceived sense of the emergence of coherence, but not completely articulated or reflected in the paper-based post-test.

4 Discussion of cases

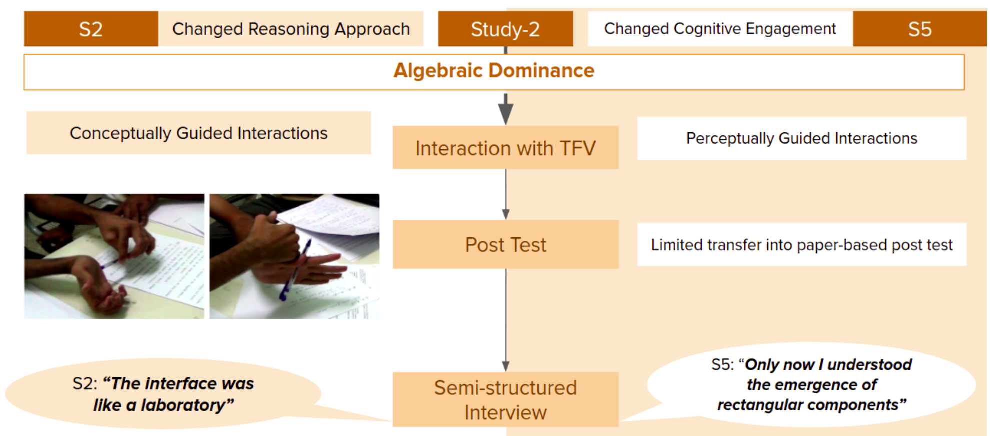


Figure 2D.6: Summary of the two cases

The cases of S2 and S5 show how the interactive vector system could change students' conceptual understanding to different degrees.

S2, who used formulae and algebraic expression extensively in the pre-test, started to reason and imagine geometrically (indicated by his use of diagrams in responses, and the use of gestures during the post-test interview). The affordances of TFV-1 for geometric manipulations and conceptual synthesis, absent in static paper-based media, allowed him to develop coherence in his models, and reason using transformations on the models (Ramadas, 2009). The drops in SCA3 & SCA4 (related to rectangular and non-rectangular components) indicate that the system also disrupted his existing models of vectors which were incomplete — as evident from pretest interviews — yet could have been productive in handling regular school assessments (as indicated by his teachers finding him promising). His comment that working with TFV was like experimentally proving theoretical aspects indicates a self-perceived sense of tangibility of the geometric representations. The combination of the use of algebra-based estimations and conceptually-guided interactions during the intervention and episodes of gestures indicating spatial/ geometric reasoning provide indicators of better geometry-algebra integration and triggering of imagination. More interaction with the system would possibly help him settle some of these persisting confusions. Overall, there is a changed reasoning approach in S2 (See figure 2D.6-left).

Further, it is interesting to note that S2 used algebraic equations for most of the tasks even during the intervention (when geometric entities were available) and his interactions were conceptually-guided. This indicates the strength of the habits with existing methods and models, as also evident from the reluctance to use the new media until the tasks became challenging enough. This indicates that the learning effect of the new computational media is not a clear and quick replacement of existing practices. It is a slower process, where parts of older practices are first used in relation to the new media initially, and the newer practices are used only for cases where older practices don't exist, or where they are of not much help.

Unlike S2's case, the case of S5 conclusively shows, based on her pre-post test strength analysis as well as the interview responses, a growth in her understanding of addition using triangle law and components of vectors (SCA-1 and 4 respectively). Even with SCA-3 (related to the rectangular components), she explicitly notes that she can solve the given problem using the system, which indicates an enabling effect of the system. S5 was elated to see the resolution process, which allowed her to see explicitly the underlying dynamic process while arriving at *rcosθ* and *rsinθ* and hence a better cognitive engagement and *epistemic access*. However, her post-test responses indicate that she struggled to transfer this understanding to text representations in the paper-based post-test. The geometrical transformations allowed her to see how the mathematical system works (unpacking the multiple states), and this, as Simon (1996) notes, could lead to an understanding different from the current one based on static book-based representations. As S5 indicates, this is an exciting shift. Overall, there is a changed cognitive engagement in S5 (See figure 2D.6-right).

See summary table in Appendix 2D.4. Concluding the findings of this study, one can say that the interaction with the interface offered the students novel experiences with the geometrical representations. This is reflected in the strengthened conceptual links for all the students and disrupted models for good performers. And from the case studies, we could see a changed reasoning behaviour among traditional good performers (S2) and better epistemic access to concepts that were opaque hitherto among average performers (S5). This is only initial evidence and there are limitations in the transfer to the paper-based assessment; yet, these indicators of specific effects promise that sustained interaction with the careful designs such as TFV — using the affordances of the digital medium of dynamic real-time interaction with the representations — could trigger students' imagination and shift their conceptual reasoning behaviour in the desired direction and transfer to better performances in paper-based assessments.

Publications from parts of this chapter

Karnam, DP., Agrawal, H., Parte, P., Ranjan, S., Borar, P., Kurup, P., Joel, A J., Srinivasan, PS., Suryawanshi, U., Sule, A., & Chandrasekharan, S (2020). Touchy-Feely Vectors: a compensatory design approach to support model-based reasoning in developing country classrooms. Journal of Computer Assisted Learning, 1–29. https://doi.org/10.1111/jcal.12500

Karnam, DP., Agrawal, H., Sule, A., & Chandrasekharan, S. (2019). Touchy Feely Vectors - Material Experiences of Geometrical Representations of Vectors. In Graven, M., Venkat, H., Essien, A. & Vale, P. (Eds). (2019). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 148). Pretoria, South Africa: PME.

Karnam, DP., Agrawal, H., & Chandrasekharan, S. (2018). "Touchy Feely Vectors" changes students' understanding and modes of reasoning. In J. C. Yang, M.

Chang, L.-H. Wong, & M. M. T. Rodrigo (Eds.), *Proceedings of the 26th International Conference on Computers in Education* (pp. 143–152). Philippines.

Karnam, DP., Borar, P., Agrawal, H., & Chandrasekharan, S. (2018). The Affordable Multitouch Classroom. In *The Future of Learning Conference - Pedagogy, Policy and Technology in a Digital World.* Bangalore: IIMB.

Karnam, DP., Agrawal, H., Mishra, D., & Chandrasekharan, S. (2016). Interactive vectors for model-based reasoning. In W. Chen, T. Supnithi, A. F. Mohd Ayub, M. Mavinkurve, T. Kojiri, J.-C. Yang, ... S. Iyer (Eds.), *The Workshop Proceedings of the 24th International Conference on Computers in Education* (pp. 401–406). Mumbai, India: IIT Bombay.

2E

Design-2: Touchy Feely Vectors-2

The objective of the chapter: To report the evolution of TFV-1 into the second version, TFV-2

- Evolving the interactions from TFV-1
- Addressing the issues encountered by the teachers
- For better adaptability in the classrooms

Design principles:

- Epistemic Access (similar to TFV-1)
- Teacher Practice
- Physical Access

Key ideas: Co-designing with the teachers could help in ensuring smoother integration to existing classroom practices.

1 Broad Design requirements

TFV-1 showed that an interactive concept design could help address the conceptual issues students encounter when they encounter vectors based on static media. The feedback from the students interacting with the TFV-1 indicated that - the conceptual aspects and the geometric affordances were very effective and useful, but the interactions were complex. TFV-1 is computer-based, but most Indian schools have large classrooms (with ~60 students) and limited access to computers. Also, teaching with the TFV-1 requires teachers to move to computers, and thus change their existing classroom practices, which are currently adapted to textbooks and large classrooms. The design thus needed to be rethought, from an access point of view, to address these three issues (large classrooms, lack of computers, text-based teacher practice). This chapter reports the redesign and the key principles and constraints guiding and shaping the redesign.

We chose to address these problems by moving to a smartphone-based design, for three reasons. One, the use of smartphones is becoming more common even in rural areas, and we might assume that most high school science teachers either have a smartphone or can have access to one. Secondly, smartphones provide better avenues for integration with current teacher-practice, as they can be used in large classes, and do not require moving to a lab. Finally, smartphones can address large class-sizes by supporting a flipped classroom format, where, if students also have smartphones, they may interact with the system at home and discuss and solve problems in class. This format is possible because there are now enough devices in rural areas to support brief borrowing for interactive learning, in case a student's immediate family does not own a smartphone yet.

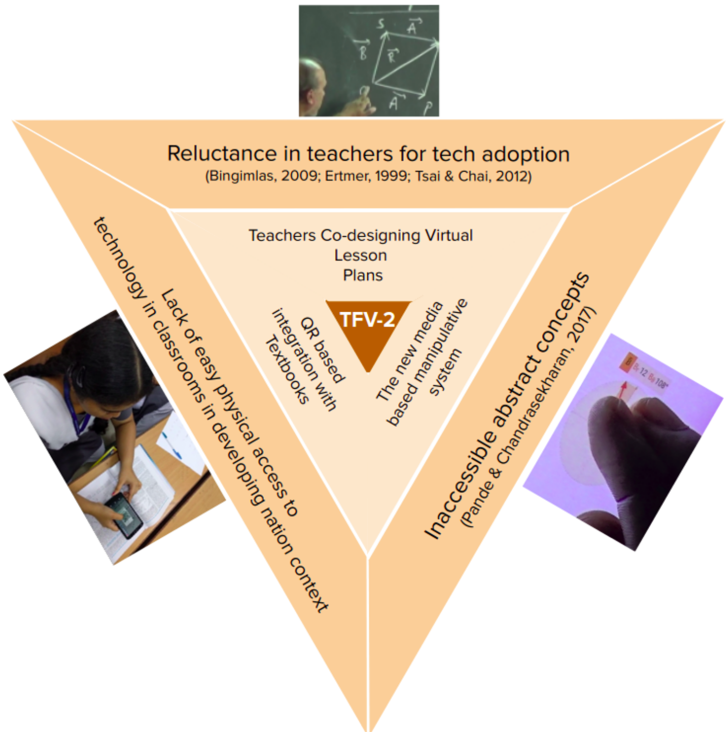


Figure 2E.1:. The triangle capturing the key considerations in the redesign of TFV-2

To address the problem of ingrained teacher practice, we adopted a two-part strategy. We first integrated the TFV system with the textbook using QR codes, so that teachers and students could easily move between the two media formats. Secondly, we co-designed lesson plans with teachers, as this process would allow them to integrate the simulation better with their existing teaching practices.

This iteration also allowed us to rethink the interactive concept design, from the point of view of recent theories that consider concepts to be constituted through interaction. We worked with an interaction designer to redesign all aspects of the interaction, to better fit the concept to student manipulations on the interface.

See figure 2E.1. Together, these design elements allow augmenting all textbooks, to meet the following design requirements: (1) support student imagination of dynamics embedded in mathematics and science formalisms, (2) change teacher practice, to support learning based on such dynamic understanding and collaboration, and (3) provide access to this type of dynamic understanding of formalisms to all classrooms in the developing world. The next section describes the refinement of interactions in TFV-1 with conceptually meaningful interactions of the new system

(TFV-2); this addresses the design requirement (1). And later, we discuss the compensatory design approach, where towards better adoption in the existing Indian classrooms; this addresses the design requirements (2) and (3).

2 TFV-2 extending TFV-1 with conceptually meaningful interactions: towards better epistemic access

2.1 Description of TFV-2

We begin with a description of the TFV-2 and then discuss the changes made from TFV-1 in detail. The structure of the system in terms of the vector related concepts it caters to has remained the same from the first version, TFV-1. TFV-1 used interactions based on arbitrary mouse clicks and button presses. The new design replaced these with conceptually meaningful embodied interactions on a touch-based interface. This provided more agency to students, in controlling the operations on the interface. The javascript-based system allows learners and teachers to create, manipulate, add and resolve vectors on any smartphone screen, using multi-touch gestures. The system evolved over multiple iterations, over a period of two years. As different requirements emerged, the design was revised in an integrated fashion, such that the same design requirements are met by multiple design elements. The design is available for interaction at this link [https://lsr_lab.gitlab.io/vectors/].

In TFV-2, a vector is created using a two-finger gesture (figure 2E.2(b,c)) (double consecutive taps). The created vector is displayed using a unit circle representation, which works as an interface element as well as a pedagogical element (addressing the lack of this representation in textbooks). For all vectors, the algebraic notation is shown alongside in a box, and this algebraic notation changes with every manipulation. Once created, the vector can be made active by tapping at the centre of the circle (figure 2E.2(d)). The magnitude of an active vector can be increased or decreased by touching the arrowhead (becomes highlighted) and pulling the vector away or towards the centre (figure 2E.2(e)). Appropriate handles have been introduced in the representations to give feedback on active and passive states of vector manipulation. The direction can be changed by touching the vector line (becomes highlighted) and rotating in the desired direction, with the angle increasing from 0° from the positive x-axis (figure 2E.2(f)). The vector can also be deleted by long-pressing at the centre of the circle and dragging towards the trash bin icon.

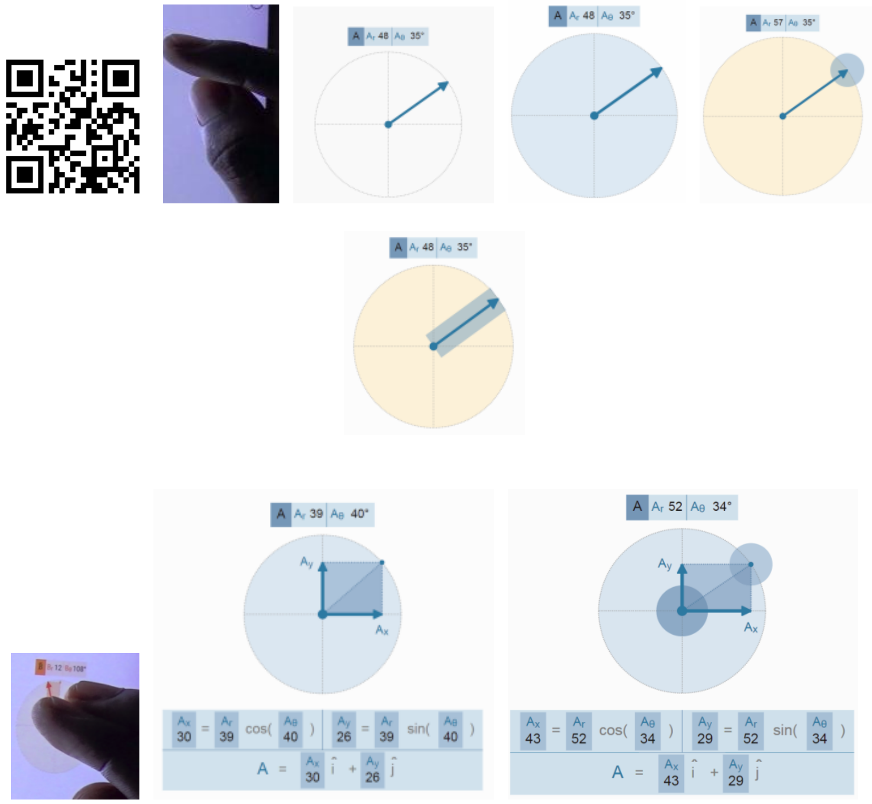


Figure 2E.2. (left to right)(a) QR code to access the TFV-2; (b,c) Gesture for creating a vector (double consecutive tap) and the created vector; (d,e,f))An active vector, changing magnitude, direction; (g,h,i)Pinching away gesture for resolution and gesture for reversing the resolution of a vector.

A vector can be resolved by touching on either side of the vector in its active state (as if holding a physical arrow), and swiping the fingers away from each other (see figure 2E.2(g); pinching away gesture, similar to popular gesture for zooming in). This pinching away gesture for resolution fits with the conceptual understanding of splitting the vector, with respect to the given coordinate axes. This gesture starts an animation that shows how the vector is resolved into components. The resolved vector is indicated using a dotted line, which can be changed (magnitude and direction) to see how the rectangular components are affected by these changes (figure 2E.2(h)). These visual aids detail out the mathematical operations and act as scaffolds for the imagination. The process of resolution can be reversed by touching the head and tail of the vector (see figure 2E.2(i); highlighted using two faint circles). The same interaction is used later for the addition of two vectors, which conceptually integrates the resolution and addition, and ensures consistency, both in concept and interactions terms.

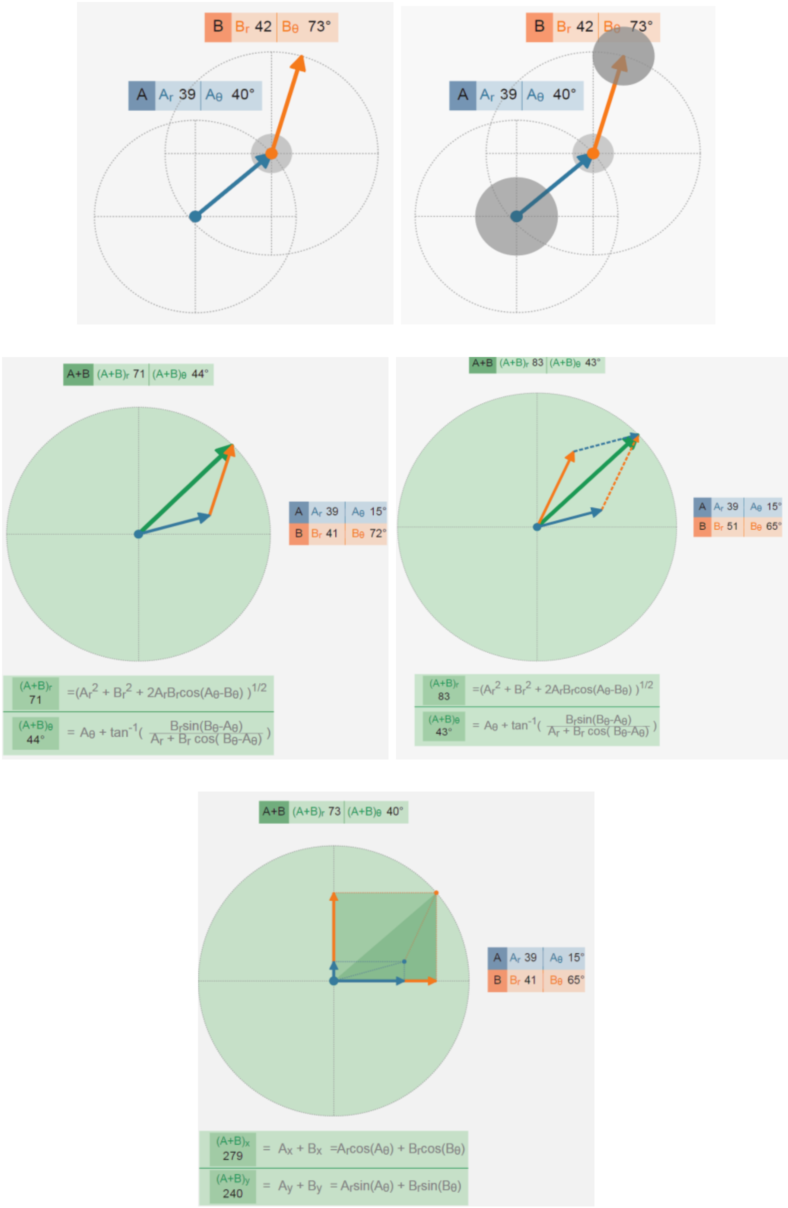


Figure 2E.3. (a,b) Gesture for adding two vectors (top); (c) addition and manipulation in triangle law and, (d) parallelogram law. (e) Pinching away the resultant leads to addition using rectangular components.

Two vectors can be added (figures 2E.3(a,b) using the triangle law by moving the vectors such that the head of one vector touches the tail of the other, and then touching the two remaining points using the thumb and index finger (double consecutive tap on highlighted two points). The vectors add by completing the third side of the triangle, giving the resultant (figure 2E.3(c)). The original vectors can be changed (magnitude, direction) to see how the added vector changes in relation to these changes. Long pressing the centre of the added vector displays the parallelogram law implementation of the addition operation, shown using a parallel translation of vectors (figure 2E.3(d)). The vectors are manipulable in this mode as well. The addition process can also be performed using the resolved rectangular components of the vectors. This is done using the same resolution gesture (pinching away) on the resultant vector in the triangle law mode. An animation displays how the rectangular components of the initial vectors add up to give the resultant. Changing the direction and magnitude of the initial vectors (denoted here by dotted lines) results in changes in the resultant (figure 2E.3(e)). This allows visualising the variations in the rectangular components and the resultant during manipulations. This feature interconnects the graphical (triangle and parallelogram law) method of adding vectors and the algebraic method (using the rectangular components).

Note that these operations just extend the descriptions in the textbook, making the dynamic behaviour of the vector explicit and available for imagination. The visual language in the interface also builds over the representation style in the textbook, to retain the familiarity and maintain consistency. Given the design requirement we discussed (only 'seeding' the imagination), the system is scoped narrowly and does not seek to provide problem-solving opportunities using on-screen manipulation. However, the system is robust enough to support advanced problem-solving if needed. Students can generate and manipulate vectors on screen, and use these operations and the resulting states to lower working memory load while solving complex problems.

2.2 *Key Changes from TFV-1*

The key changes in TFV-2 as is already evident is the shift to the touch-based interface to be suitable to an easily accessible smart-phone interface. Constrained by space on a touch interface, the side panels were removed. A heat map of eye-tracking data also indicated that the side panels were not very much used by the students and hence probably redundant. Further, in the process of redesign, all the interactions were refined as summarised in the table 2E.1 below. The refinement of the interaction tried to make a finer connection between the visualisations on the screen and the interactions and making those interactions conceptually meaningful.

Firstly, a special focus was made on making the interactions simple and intuitive. A lot of visual elements like faint circles or bars at the point of taps were used to let the user be aware of the action performed. This quick visual feedback made the user immediately aware if an action is being performed or not, which was one of the limitations as per the feedback from the students. The action and perception here can be directly linked, consistent with the principles of 4E cognition. The operations like resolution and addition can not be directly connected and are mediated by conceptual aspects (as apparent in the model described in ch-2A). So, attempts were made to systematically connect the actions performed causing suitable visualisations to make underlying conceptual processes tangible. A characterisation of the learning problem to students could be the opaqueness of these underlying processes between the produced-actions and their effects. Here we tried to link the intended conceptual meaning with the embodied meaning that the concrete actions already have.

Conceptually meaningful interactions

The interactions were thus simplified and made consistent with an intention to enhance the concreteness of the representations and meaningfulness of the actions. There are basically two sets of interactions: (1) for manipulating the vector (moving, changing directions and magnitude) and (2) performing operations on vectors (creating, adding and resolving). The interactions for moving the vector and for manipulating the vector — holding and dragging the concrete elements of arrow representation *viz* circle, line or tip with immediate visual feedback — are consistently used in all conditions and modes and enhance the physicality/concreteness of the geometric representation of the vector. The interaction for creating a vector (also a resultant vector after positioning the initial vectors suitably) is the double-consecutive tap. The interaction for resolving the vector (also resultant vector after adding) is the hold and pinch-away gesture. These are also consistent all through and are also linked to the usual actions one would do if the vector were to be an actual concrete object. These interactions designed such that their embodied meanings are linked to the underlying conceptual aspects; for example, the pinching away action resolving the vector — with an embodied meaning of splitting a concrete narrow object — is linked to the meaning of resolution of a vector: splitting the vector along the coordinate axes. These interactions could thus help in closing the distance between the actions performed and the underlying conceptual significance of the actions thus reducing their opacity.

Table 2E.1: The Interactions used and the effects on TFV-1

Effect of the interaction	Interaction (Action + [visualisation]) in TFV-1	Interaction (Action + [visualisation]) inTFV-2
Create a vector	Left-click on the central panel [Vector with arrow representation with a circle and coordinate axes appears]	Double-consecutive tap on the screen [Visual cues provided by the small circles where tapped; vector with arrow representation with a circle and coordinate axes appears]
Activate/deactivate the vector for changing magnitude and direction	NA (A vector is always active for changing direction and magnitude)	Tap at the centre of the circle [The circle gets faintly coloured]
Change the direction of the vector	Left-click + drag on the line of the arrowhead	Tap and drag on the line of the arrowhead [A bar appears around the line]
Change the magnitude of the vector	Left-click + drag on the line of the arrowhead	Tap and drag on the tip of the arrowhead [A faint grey circle appears at the tip of the vector]
Move the vector around in the screen	Right-click+drag anywhere in the circle	Tap and drag the deactivated vector
Delete a vector	Shift + Right-click	Tap+ Hold at the centre of the circle and drag towards a cross mark [A cross mark popping up and growing in size when dragged]
Add two vectors	Ctrl + Left-click on two vectors one after other	1. Ensure the vectors are in the deactivated mode 2. Place the tail of the second vector on the head of the first vector by dragging it. [Visually they snap with a small faint grey circle appear at the contact] 3. Create a third vector (which is the resultant of the two) by double-consecutive taps on the tail of the first vector followed by the head of the second vector. [A short animation showing the emergence of the resultant vector as the third side completing the triangle]

Change the direction and magnitude of the vectors in addition mode	Make the vector active for manipulation [as indicated by the circle around the arrowhead] To switch control of manipulating the vector, right-click within the circle [the shift of the circle indicates this]. Once the circle appears the interactions are similar to a normal vector.	The interactions are similar to manipulating a normal vector.
Parallelogram Addition	NA	Long tap on the centre of the circle. And tap again to get back to triangle law mode.
Resolving a vector	This is a 2-step procedure (Double-click + single-click). 1. Double-click anywhere inside the circle [A right triangle appears] 2. Then, single left-click anywhere inside the circle [Animation shows the x and y components emerging from the right triangle and the equations appear]	This is a 2-step (Hold + pinch away) procedure. 1. When the vector is active, hold the line of the vector with two fingers, as if it is a real rod-like object. [Two right triangles appear as long as we hold the vector]. 2. Pinch away holding the vector (similar to a conventional zoom out gesture used on touch interfaces). [Animation shows the x and y components emerging from the right triangle and the equations appear]
Manipulating in resolution mode	Same as earlier on the faintly visible initial vector	Same as earlier on the faintly visible initial vector
Moving back to the initial vector	Reverse the Double-click + single-click interactions [Animation also reverses]	Double consecutive tap on the faint vector [Animation of the x and y components merging into the initial vector]
Resolving the resultant vector (to see the algebraic addition)	Same as those for resolving the vector. [x and y components of the 3 vectors emerge from the 3 respective right triangles]	Hold and pinch away the resultant vector. [x and y components of the 3 vectors emerge from the 3 respective right triangles]
Moving back to the initial vector (unresolving) in addition mode	Reverse the Double-click + single-click interactions [Animation also reverses]	Double consecutive tap on the faint resultant vector [Animation of the x and y components merging into the initial vector]
Finishing Addition	Click on the done addition button on the top. [Resultant vector remains. The initial vectors disappear]	Tap+ Hold at the centre of the circle and drag towards a tick mark [A tick mark popping up and growing in size when dragged. Resultant vector remains. The initial vectors disappear]

3 Compensatory design approach: towards better adaptability of TFV-2 to the existing Indian classrooms

3.1 Teacher Practice

Towards ensuring smooth adoption of the system in the classroom activities, we approached teachers to introduce an initial prototype of this dynamic vector system in classrooms. However, the teachers requested that the researchers teach the class using the system. This was surprising, and interviews with the teachers to understand this request revealed that they used the descriptions in the textbook in three ways: 1) as a way to organise their thinking about vectors; 2) as a way to sequence the progression of the class over time, and 3) as a way to organise the interactions in the classroom. These three elements together constitute a 'lesson plan' (O'Neill, 1982) for an individual class. This finding, and the rich literature indicating that teacher adoption and usage of digital resources in the classroom has not been smooth (Bingimlas, 2009; Ertmer, 1999; Tsai & Chai, 2012), suggested a further design requirement: the system needs to be integrated with teachers' textbook-based lesson plans. Without this, TFV-2 would not be widely used in classrooms.

The TFV-2 was an open-exploration system and teachers found it difficult to incorporate it into a timed classroom session. To address this integration with teachers' current textbook-based practice, particularly their thinking about vectors and sequencing of the lessons, we decided to design tasks that could be planned for organised classroom activities. We conducted a lesson planning workshop with four teachers (the methodology and protocol of this workshop are described in chapter-2F). Based on this discussion, we developed 'virtual lesson plans' (VLP), which consisted of manipulation tasks in the TFV system that smoothly extended teachers' existing textbook-based lesson plans. The lesson plans we developed were as follows:

1. A first module related to types of vectors, which teachers usually introduce immediately after the definition of vector quantities. Tasks were designed to allow students to make various kinds of vector pairs (equal, opposite, negative), and simultaneously get familiar with the controls (for creating, manipulating and deleting vectors). This is the introductory session (~35-40 minutes).

2. The second module involved addition using triangle law, which is the next topic in the textbook. Here tasks were designed to allow students to add and manipulate the vectors in the triangle law mode. They could also verify commutativity and associativity of vector addition. A challenging task here is creating as many vectors as possible to arrive at a given resultant vector. This involves about 2 sessions (~ 70-80 minutes)
3. The next topic teachers usually deal with (following the sequence in the textbook) is addition using parallelogram law. Tasks were designed to repeat the exploration in the parallelogram mode, as well as the creation of vector sets to arrive at a given resultant vector (reiterating the nature of vector addition). After the teacher derives the expressions for the magnitude and direction of the resultant, students could verify the magnitudes at specific cases of vectors at 0°, 90° and 180°, by recreating them to confirm the algebraic expressions. For those groups who finish the above tasks, there were tasks extending the problems in their textbooks. This was planned to involve about 1.5 sessions (~50-60 minutes)
4. The last topic in the textbook sequence is resolution. After the teacher explains resolution, they demonstrate the use of the system for the same purpose, showing how trigonometric ratios are involved in the operation. Tasks were designed to allow students to create vectors of given x and y components, just by manipulating the magnitude and direction. In this process, the interlinks between r and θ with respect to x and y are built. Then the teacher elaborates on the need of unit vectors, and how this helps in adding using rectangular components. The teacher does a quick demonstration followed by student manipulation of the addition process using the rectangular components. This was planned to involve about 2 sessions, including closing remarks (~70-80 minutes).

3.2 *Physical Access*

To integrate these lesson plans better with teachers' textbook-based classroom practice, we developed a QR-code based system that created an 'augmented textbook'. Each lesson plan above was linked to a QR code label, and this label was printed and attached next to a textbook figure (see appendix 2E.1). The figure thus became manipulable, as scanning the QR codes took students to the related lesson plan module, which had tasks that allowed students to manipulate and understand in dynamic terms the teacher's discussion of the figure. The QR-code based system smoothly connects the static textbook figures to manipulable and dynamic vectors, such that just one or two instances of manipulation based on these lesson plans provides students with the ability to understand and imagine the vector in dynamic terms. This understanding, in turn, gives them the ability to build on the textbook representation, and thus develop the confidence to attempt problems given in the textbook and in exams. Since the level of interaction required for this understanding is minimal, it could be provided by just one smartphone in the classroom (the teacher's), or the village. This feature thus addresses the access design requirement as well.

To provide wider access to the imagination possibilities provided by the TFV system, the QR codes linked to lesson plans were compiled into a table, with QR codes in one column, and the related figures and textbook page numbers in another. This document was then made available for download on our group website (also attached in the appendix 2E.1). Anyone could now download the QR codes and attach them to the relevant pages in the textbook, to make their textbook augmented with the TFV system. This feature allows millions of textbooks in developing countries to be augmented in an affordable way, to support the imagination of dynamics that are required to understand vectors. Note also that this is a highly scalable system, where every module of every science and mathematics (and possibly other) textbook could be made dynamic and simulatable, without requiring drastic changes in textbook design or teacher training.

Thus, the QR code-based system ensures a quick and seamless integration of the dynamic vector system with the existing flow of the classroom practice and reduces the effort needed for teachers to adapt to the system and steer classrooms in new ways of doing and learning. Such VLPs embedded in the textbooks through QR codes (Uluyol & Agca, 2012) can augment the figures and tasks in the textbook, as well as teacher thinking and practices based on them. This makes the introduction of new artefacts into the classrooms smoother and less abrupt, as the system builds on already existing static artefacts (textbooks) and compensates for their limitations, instead of seeking to replace them, which often is also a costly affair in developing conditions like in India.

4 Chapter Discussion

TFV-2's carefully refined interactions with meaningfully linked actions and visualisations seeks to trigger imagination among the students. We seek to enhance the effects found with TFV-1, due to the possibility of smoother and more embodied interactions afforded by the touch interfaces. We ensured that the underlying conceptual elements of the design remaining intact as in TFV-1; the changes introduced in interactions are only in addition to already existing features of real-time dynamic integration of algebraic and geometric representations of vectors, manipulability of geometric entities, and integration of conceptual and interaction aspects by the unit-circle construction.

Further, this iteration of design paid special focus on the adaptability to the classroom conditions in India. As outlined in Ch-1A and Ch-1B, the scale and resource constraints of a developing nation like India cannot afford to replace the existing infrastructure (the artefact of textbooks). These textbooks play an anchoring role in shaping the entire ecosystem ranging from the teaching-learning activities in classrooms to the assessments related activities and teacher training and professional development in the education system. In such a context, it is imperative that for better adoption, TFV-2 builds on this infrastructure. Also, this iteration of redesign involved the teachers and created virtual lesson plans, thus enabling them to smoothly integrate this into their practices as well.

Thus, the TFV-2 design as discussed in the above two key enhancements (1) refined interaction and (2) embedding in the textbook could meet the design requirements for better epistemic access and adaptability and physical access within the existing system.

Publications from parts of this chapter

Karnam, DP., Agrawal, H., Parte, P., Ranjan, S., Borar, P., Kurup, P., Joel, A J., Srinivasan, PS., Suryawanshi, U., Sule, A., & Chandrasekharan, S (2020). Touchy-Feely Vectors: a compensatory design approach to support model-based reasoning in developing country classrooms. Journal of Computer Assisted Learning, 1–29. https://doi.org/10.1111/jcal.12500

Karnam, DP., Agrawal, H., Parte, P., Ranjan, S., Sule, A., & Chandrasekharan, S. (2019). Touchy Feely affordances of digital technology for embodied interactions can enhance 'epistemic access' In M. Chang, R. Rajendran, Kinshuk, S. Murthy, & V. Karnat (Eds.), Proceedings of the 10th IEEE International Conference on Technology For Education (T4E) 2019. (pp. 114-121). Goa, India.

Karnam, DP., Agrawal, H., Borar, P., & Chandrasekharan, S. (2019). The Affordable Touchy Feely Classroom: Textbooks embedded with Manipulable Vectors and Lesson Plans augment imagination, extend teaching-learning practices. In Lund, K., Niccolai, G., Lavoué, E., Hmelo-Silver, C., Gweon, G., and Baker, M. (Eds.). *Proceedings of 13th International Conference on Computer Supported Collaborative Learning (CSCL) 2019,* Volume 1. (pp. 488-495) Lyon, France: International Society of the Learning Sciences.

Karnam, DP., Agrawal, H., Sule, A., & Chandrasekharan, S. (2019). Need to explore affordances of technology for better learning and teaching interface designs. *In The Future of Learning Conference - Learning 4.0: Connecting the Dots and Reaching the Unreached.* Bangalore: IIMB.

*Karnam, DP., Agrawal, H., Borar, P., & Chandrasekharan, S. (2018). Touchy Feely Vectors: exploring how embodied interactions based on new computational media can help learn complex math concepts. In *Proceedings of the 5th ERME Topic Conference MEDA*, 2018. Copenhagen: University of Copenhagen, 313-314.

Karnam, DP., Borar, P., Agrawal, H., & Chandrasekharan, S. (2018). The Affordable Multitouch Classroom. In *The Future of Learning Conference - Pedagogy, Policy and Technology in a Digital World.* Bangalore: IIMB.

Borar, P., Karnam, DP., Agrawal, H., & Chandrasekharan, S. (2017). Augmenting the Textbook for Enaction: Designing Media for Participatory Learning in Classrooms. In R. Bernhaupt, G. Dalvi, A. Joshi, D. K. Balkrishan, J. O'Neill, & M. Winckler (Eds.), Human-Computer Interaction – INTERACT 2017 (Vol. 10516, pp. 336–339). Cham: Springer International Publishing.

2F

Study-4: The Classroom Study

The objective of the study: Extending the promising findings in the study-3, this study reports a larger scale implementation of systems of this kind..

Key Findings:

- TFV-2 changes reasoning behaviour in good performers and enhances cognitive engagement in average-performers

In the laboratory study, we did a detailed analysis for the effects the interaction with TFV-1 had on students' conceptual understanding as well as conceptual reasoning behaviour. This was a more focussed intervention in the laboratory conditions. This along with the study-1 and study-2 provided some evidence in support of the corollary based on the SCIARM framework.

With the intent of examining the effects of the changed representational medium on the field, the redesign leading to TFV-2 was embedded in the textbooks with virtual lesson plans (VLPs) specially codesigned with the teachers. In the present study, this revised system was deployed in the field for real-world testing. The teachers taught their actual vector lessons using the system in their classrooms as per their academic calendars. We report the controlled classroom study using TFV-2 for specific effects of this system on students' reasoning behaviour and cognitive engagement in comparison to classrooms with conventional teaching. We have confined our analysis to their conceptual reasoning behavior, as guided by our examination of the STEM cognition with changed interactions with the representations afforded by the representational medium. Also this serves as an illustration of adoption of the TFV-2 in the Indian school system.

1 Methodological details

This study was to test whether the changed representational media in the classroom changes students' STEM cognition (reflected in their CRB),

1.1 Sample

We did a classroom study (grade 11), with three experimental classrooms where the integrated system was deployed, and three control classrooms where the standard textbook teaching method was used. We worked with 5 physics teachers (T1-T5), in 3 junior colleges (SC1-SC3) (pre-university level high schools) in Mumbai. All the teachers were trained postgraduates in Physics and had about 20 years of experience teaching at this level. Each school had multiple classrooms (divisions/sections) of grade 11. We observed two grade 11 classrooms (Control group - CG of 135 students, and Experimental group - EG 131 students) in each college (a total of 6 classrooms). In CG, the teachers taught using the conventional method. In

EG, the teacher used tablets with the TFV-2 deployed. See Table 2F.1 for more details of the number of students.

School (type, Curriculum)	Classroom	Teacher	No. of students
SC1 (Private, MH)	EG1	T1	63
	CG1	T2	53
SC2 (Government, MH)	EG2	T3	34
	CG2	T3 & T5*	41
SC3 (Government, NCERT)	EG3	T4	34
	CG3	T5 & T3*	41

*Table 2F.1: Codes for colleges and classrooms, and corresponding teachers. NCERT - national, and MH - a provincial (Maharashtra) textbook. * T3 & T5 taught in SC2 and SC3 due to inter-school teacher transfers (could not be controlled).*

1.2 *Procedure (Teacher lesson plans, control and experimental groups and post-test)*

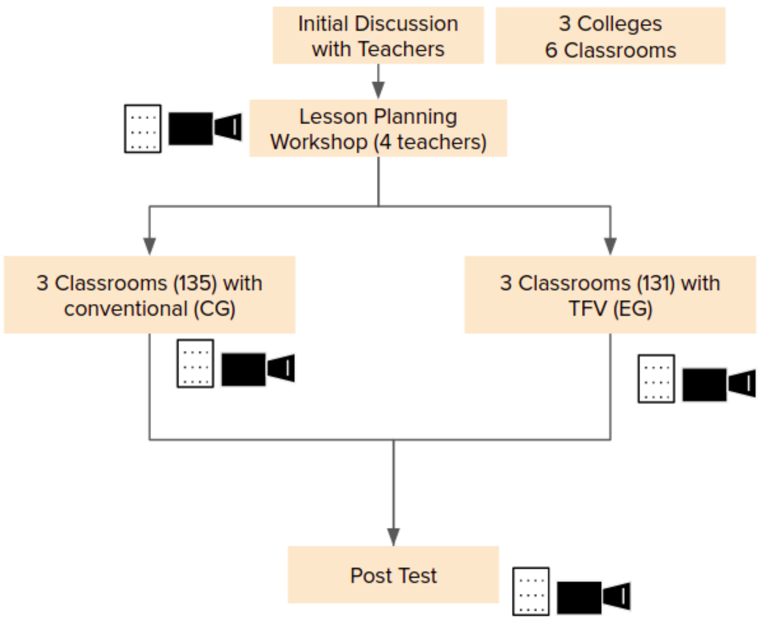


Figure 2F.1. The flow of the study and the sample.

See figure-2F.1 for the flow of the study. A brief introduction to the research study and the TFV-2 system was given to the respective college managements and the teachers, to get their consent for the study. To get some initial familiarity with the system, the teachers were requested to do some exploration of a few tasks (not mandatory) on TFV-1. All the teachers except T5 attended a Lesson Planning Workshop (LPW) for ~3 hours. In the workshop, the teachers based on their already existing lesson plans together discussed and arrived at a virtual lesson plan (VLP)[19] with tasks on

[19] Described in CH-2E

TFV-2, to be used in their teaching. We video recorded all the classroom sessions (both in CG and EG for 5-8 teaching sessions of 35/40 minutes each in different colleges). In CG classrooms, the teacher was given no inputs by the researchers.

In EG classrooms, as per the VLP, a 1-2 page worksheet with predetermined tasks (developed during the LPW) was prepared, depending on the topics the teacher planned to teach in the class (one worksheet per student). About 25-30 6” Android tablets (SWIPE™), with the necessary files and QR reading applications pre-installed, were carried to the EG classrooms by the researchers for every session. The classroom was split into groups of 2-3 students (depending on the total strength) and each group was given one tablet. The tasks are available over the web and can be accessed directly. But the colleges didn’t have reliable internet connectivity, so the necessary files were copied to each tablet, and they were accessed locally using QR codes generated for the file paths. In the EG session, the teacher led the class, discussing the day’s topics. The students in the groups interacted with the tablet-based TFV-2, as directed by the teachers and facilitated by the researchers (figure 2F.2). They also completed related worksheets. The researchers’ roles were 1) logistics related to the tablets, technical support (for the teacher as well as the students), 2) recording the classroom proceedings. After about 1-3 weeks, students in all the 6 classrooms were given a questionnaire (sample questions in figure 2F.3; entire questionnaire in appendix 2F.1), which included questions that sought to capture whether the system was effective in helping students integrate the geometric and algebraic understanding of vectors, and whether their imagination of the vector operations improved through the interaction.

To further assess possible variations in the effect of TFV interaction, particularly with students' academic abilities, we collected each student’s average science and math scores for the Grade 10 board exam, which is a milestone in the Indian schooling system. The grade-10 scores of only 234 students (119 control and 115 experimental) could be collected from the school records (averaged math and science scores). This score provides a summary picture of the overall academic ability of each student.

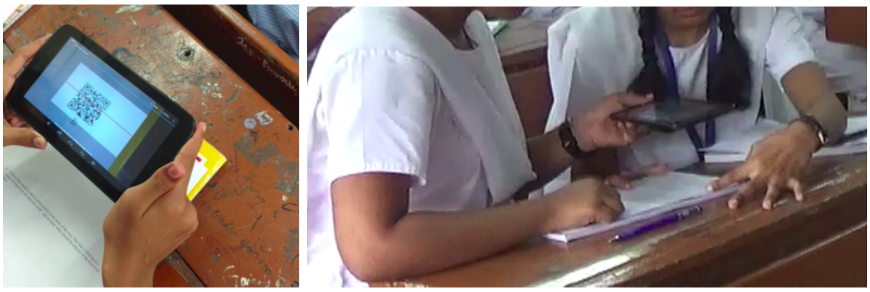


Figure 2F.2: Students scanning the QR code in an EG Classroom

Choose the correct answer (multiple answers allowed) and state the reason. (Q-6, Q-7 and Q-8)

Q-6. $\overline{P}$ and $\overline{Q}$ are two vectors at an angle of 45° with each other at their co-initial points are tails, and $|\overline{P}+\overline{Q}|$ (magnitude of their resultant ($\overline{P}+\overline{Q}$)) is 5 units. If magnitudes of $\overline{P}$ and $\overline{Q}$ are fixed and the angle between them is increased to 120°, which of the following could be the value of $|\overline{P}+\overline{Q}|$?

a) 3 units b) 5 units c) 8 units d) 9 units e) 2 units

Reason:

Q-7. If $\overline{P}$ and $\overline{Q}$ are two **collinear** vectors, and their resultant is $\overline{R}$.
(**magnitudes** are represented by small letters p,q and r respectively)

1. If p = q , r = 0
   a) Never b) Sometimes Possible c) Always true
   Reason:

Figure 2F.3: Sample questions used in the test (snapshot from the test)

2 Analysis and Findings

2.1 *Analysis Framework (Coding Scheme)*

The analysis framework used in the pilot study was developed further, incorporating the findings from the case studies in Ch-2D (Karnam et al., 2018), to better capture changes in students' reasoning approaches, as reflected in their written scripts. We first developed a rating scheme using a small sample, where 3 raters iteratively coded the material, until at least 2 out of 3 raters converged on identical ratings[20]. This rating scheme (details below) was then finalised, and the answer scripts were equally distributed among all the raters, ensuring equal distribution of students from all the 6 classrooms to each rater. We analysed some selected questions (Q6, Q7 with 4 sub-questions, Q8 with 4 sub-questions; see appendix 2F.2 for details) rated for the geometry-algebra integration.

For each student, indicators of geometric and algebraic reasoning were captured, by counting the number of responses with (predominantly)

[20] Two raters rated the same set of students for all the 3 relevant questions, and finally the scores were tallied.

geometric aspects and algebraic aspects. Apart from diagrams and equations as markers of geometric and algebraic reasoning respectively, keywords were identified in 2 of the 3 questions, as students used verbal statements as well in their responses (when answering the 4 sub-questions for each question). These keywords were identified by each rater, after rating ~30 answer sheets. Examples of geometric keywords that emerged were ‘collinear’, ‘lying on the same plane’, ‘parallel’, ‘opposite directions’, qualitative action words like ‘make/form an angle’, ‘rotate’, etc. Examples of algebraic keywords were ‘solve’, ‘value’, ‘formula’, and quantitative action words such as ‘subtract’, ‘substitute’, ‘calculate’, or the verbalisation of algebraic entities like ‘square root’, ‘equals to’ etc. See fig-S2 for rating scheme.

This scheme allowed a student's answer to each question to be analysed based on the reasoning in 4 modes — using geometric entities (diagrams, geometric keywords) and algebraic entities (equations, algebraic keywords). Instead of the total number of diagrams, we counted the number of questions where at least one diagram was used (for example, if a student drew two diagrams for one question, we counted it as just one instance). This is because even one diagram is a sufficient marker for geometric reasoning in a question. The same procedure was adopted for keywords, as well as the markers corresponding to algebraic reasoning. Thus a final geometric score was a count of instances of geometric reasoning out of a total of 17 possible instances [2 questions (Q7, Q8) with 4 sub-questions each and every subquestion rated with 2 types of markers (keywords and diagrams), contributing to (2 x 2 x 4) 16 possible instances, and 1 question (Q6) with just diagrams]. The algebraic score of each student was obtained in a similar manner. This rating exercise gives each student (in the control and experimental groups) a score indicating geometrical reasoning and another indicating algebraic reasoning. These scores were used to examine the effect of TFV-2 in the experimental group.

2.2 *Findings*

The above data was analysed, and interpreted as effects of the lessons with TFV, on two aspects of students' reasoning behaviour with vectors: (1) reasoning approaches with geometry-algebra integration, and (2) cognitive engagement with the content. Furthermore, we analysed the patterns using two kinds of slicing of the data: (1) With all the 266 students (without incorporating grade-10 scores), and slicing them school-wise, and (2) With 234 students (with grade-10 scores), and slicing them bin wise (lower bin: below 85% as average-performers; upper bin: upper 85% good-performers). Interesting patterns related to each of these effects, on students of different academic abilities, were found. The following subsections elaborate on the analysis, report the patterns in the effects and interpretation of each of these effects.

Effect on geometry-algebra integration

We sought to understand the effect of TFV interactions in helping students integrate algebraic and geometric reasoning. This was done by examining the differences in the distribution of geometric and algebraic scores between the groups. To make sense of these differences, we used 3 visualisations. A quick histogram of both these scores indicated that they are not normal distributions, and we needed to compare 2 bivariate samples. We employed kernel density estimation (KDE), a non-parametric method found useful to analyze smaller sample sizes (Anderson et al., 1994; Gretton et al., 2012; Li, 1996; Ramdas et al., 2017; Silverman, 2018). Kernel density estimation is not widely used in education research but is used in multivariate nonparametric testing in genetics (Saunders et al., 2017), and in chemical engineering (Liang, 2008) among others. For each comparison between the control and the experimental groups, we present two plots (see figures 2F.4 and 2F.5). Each geometry-algebra (x-y) plot captures the following analyses.

- Joint probability distribution functions (j-PDFs) of the algebra and geometry score were obtained by smoothing the discrete data points using the Gaussian kernel. The resulting joint probability distribution functions (figures 2F.4 and 2F.5) are normalised distributions for each group, and capture the probability of finding a student with a given algebra and geometry score (the shade of the colour). These j-PDFs (kernel density plots; KDE) allow us to jointly see the overall patterns in the distribution of algebraic and geometric reasoning, and hence the distribution of the geometry-algebra integration. In the plots, darker regions (higher probability of students being in that region) correspond to the mode of the distribution. Any expansion towards the x-axis indicates an improved tendency to reason using geometric entities. Expansion along the y-axis indicates algebraic tendency. Expansion along the diagonal away from the origin indicates a better geometry-algebra integration, which is a desirable effect. A faint diagonal split provides a visual indication of this effect.

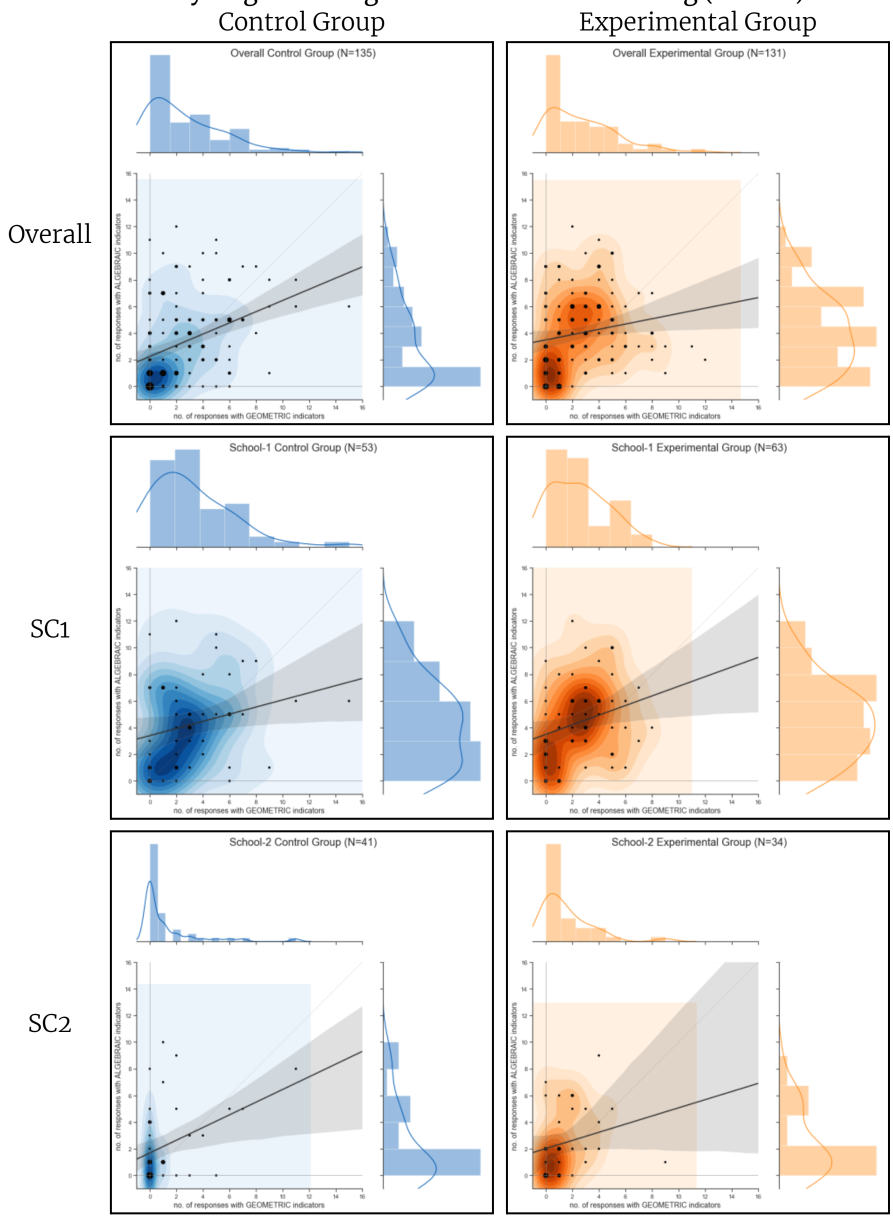
- The KDEs and histograms of algebra and geometry scores are individually plotted along the axes, and can be seen as projections of the j-PDF on the respective axes.
- The scatter plot is overlaid on the j-PDF, with the size of the marker indicating the frequency of students with a particular geometry and algebra score. As can be seen, the data is discrete, as the scatters take only integral values. The higher frequency markers can be seen to overlap with the darker regions of the j-PDF.

Establishing the difference between two KDEs is an ongoing research problem in statistics, and various approaches have been indicated in the literature, such as Maximum Mean Discrepancy (Alba Fernández et al., 2008; Anderson et al., 1994; Gretton et al., 2012), Distance-based tests (Sejdinovic et al., 2013), Kolmogorov–Smirnov (KS) Test (Pavia, 2015) and local significant differences for KDE (Duong, 2013). To measure the significance of the difference visually seen in the j-PDFs, we employed the 2-sample KS test, which is fairly well understood and sufficient for our purposes. The KS-test quantifies the difference between two continuous PDFss, and is suitable for the goodness of fit with KDE (Pavia, 2015). In our case, the KS tests of comparison are done for distributions of algebra and geometry scores individually, and for the total score (algebra + geometry scores) of CG and EG (Tables 2F.2, 2F.3). It provides the D statistic (ranging 0-1), which gives the effect size, and p indicates the 2 tailed p-values with 95% confidence.

From the joint probability distributions for all schools (figure 2F.4, row-1), we see the EG pattern bulging away from the origin, creating a second mode, and spreading out of the distribution along both axes. This indicates that, overall, interaction with TFV has helped an increased proportion of students (increased probability for a student) to reason with better geometry-algebra integration.

A school-wise comparison is more meaningful, given the direct comparability between the groups. In SC1 (Figure 2F.4 row-2), the overall distribution does not seem to be much affected. However, a clear development of the second and higher mode (darkest) away from the origin is seen in EG, which is a shift in the desired direction. In SC2 (Figure 2F.4 row-3), the CG appears to be highly reliant on algebraic reasoning, and EG has a very slight effect in the desired direction.

Geometry-Algebra Integration: School-wise slicing (N=266)



SC3

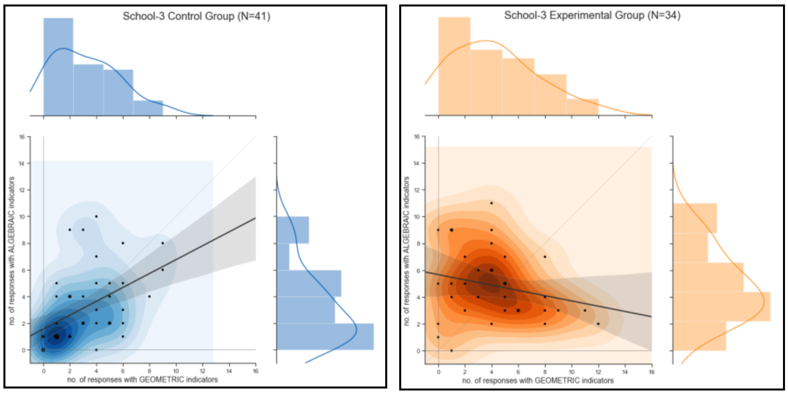


Figure 2F.4: School-wise slicing: Joint Probability Distribution plots of students geometry vs algebra performance in the 3 schools (row-1 overall; rows-2,3,4: Schools-1,2,3 respectively).

In SC3 (Figure 2F.4 row-4), the mode clearly shifts diagonally away from the origin significantly for the EG, indicating improved geometry-algebra integration. Also, expansion along the geometric axes indicates enhanced geometric reasoning. The KS tests confirm the visual interpretation, with the difference being significant overall, and for SC3. The intervention seems to have not impacted the reasoning modes of some students in SC1 and almost the entire SC2.

Fig. No. 2F.	Without Grade-10	N(students)		Alg+Geo		Algebra		Geometry	
		C	E	D	*P*	D	*P*	D	*P*
4.row-1	Total	135	131	0.139	*0.14*	0.201	*0.007**	0.059	*0.968*
4.row-2	SC1	53	63	0.105	*0.889*	0.111	*0.846*	0.131	*0.674*
4.row-3	SC2	41	34	0.144	*0.798*	0.197	*0.417*	0.232	*0.234*
4.row-4	SC3	41	34	0.364	*0.010**	0.360	*0.012**	0.149	*0.768*

*Table 2F.2: Table showing the KS-test results for the above j-PDFs. D statistic (ranging 0-1), which gives the effect size and p indicates the 2 tailed p-value with 95% confidence. *p<0.05*

Given these interesting patterns in the school-wise data, we looked at effects on students with different academic abilities (Bin-wise slicing). We first sliced the sample into good (above 85%) and average (below 85%) performing groups. These samples were further examined using the jPDF. Figure 2F.5 shows the patterns in the distributions. The overall bin-wise pattern (for 234 students) almost matches with the overall school-wise (for all 266 students) (Compare row-1 in Figure 2F.4, Figure 2F.5). An interesting pattern was the differences in the distribution in the above 85% bin in comparison to the below 85% bin. In the above 85% bin (3rd row Figure 2F.5) the mode in EG appears to shift very much away from the origin, in comparison to the CG. The distribution also spreads along the geometric axis. Whereas in the below 85% bin, the pattern does not appear to change much, except for a slight widening of the distribution. This shift, as indicated by the KS test results, again hints at an increasing algebraic representation, but contributing to better geometry-algebra integration in the upper bin (>= 85% grade 10 score). This integration does not appear to be significantly affected in the lower bin.

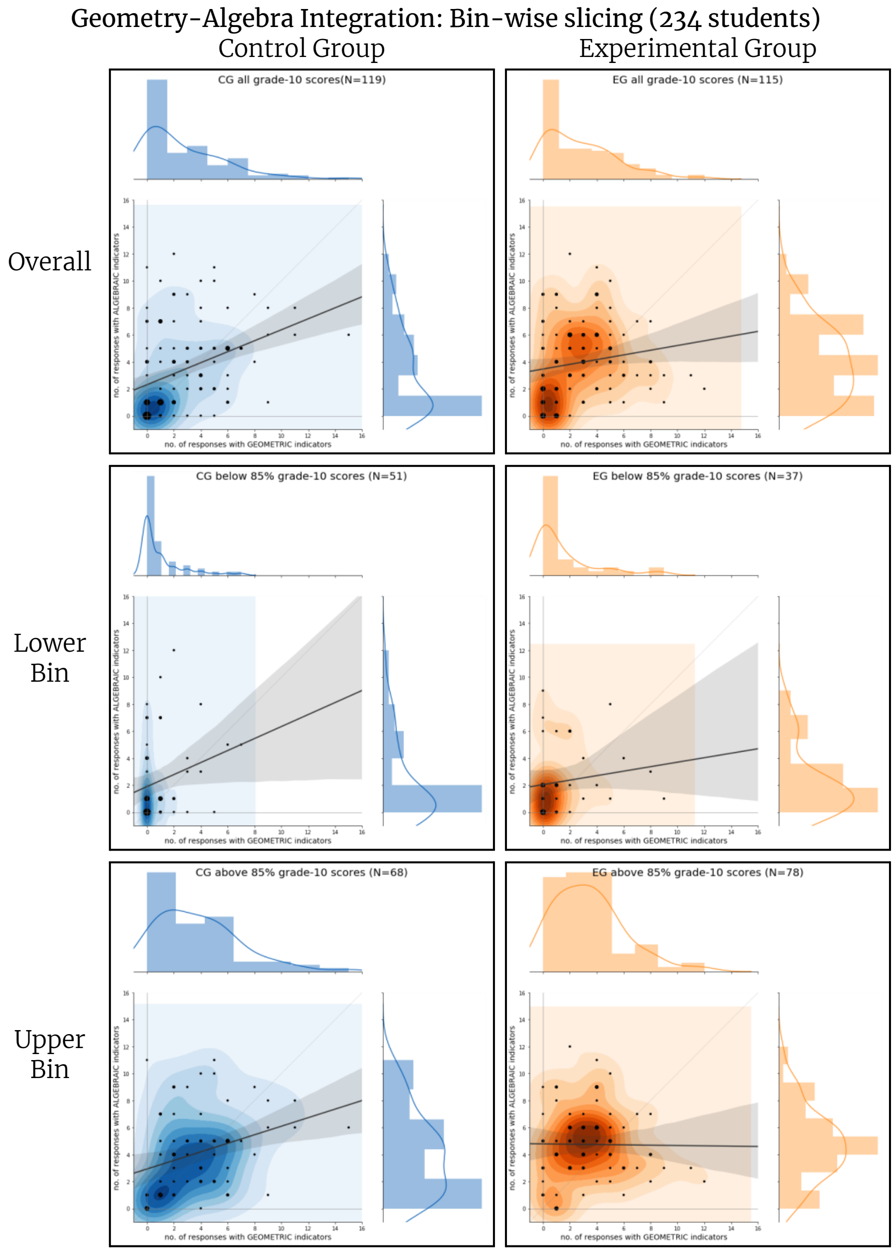


Figure 2F.5: Bin-wise slicing: Joint Probability Distribution plots of students geometry vs algebra performance in students with <85% (lower bin Row-2) and >85% (Upper bin Row-3) grade-10 scores (row-1 overall)

Fig. No. 2F.	With Grade-10	N(students)		Alg+Geo		Algebra		Geometry	
		C	E	D	*P*	D	*P*	D	*P*
5.row-1	Total	119	115	0.141	*0.180*	0.202	*0.014**	0.079	*0.844*
5.row-2	Lower Bin	51	37	0.127	*0.86*	0.187	*0.398*	0.89	*0.99*

5.row-3	Upper Bin	68	78	0.108	*0.76*	0.202	*0.08*	0.09	*0.85*

*Table 2F.3: Table showing the KS-test results for the above j-PDFs. *p<0.05*

Furthermore, it is interesting to note that in both the slicings of data, the difference in the algebra-geometry integration is contributed significantly by algebra, not geometry. Based on the evidence from the written-scripts, we can claim an overall better geometry-algebra integration in EG: visually both in algebra and geometry from the plots (Figures 2F.4, 2F.5) and statistically significant only in algebra (from Tables 2F.2, 2F.3). This indicates that geometry-algebra integration could happen through many complex trajectories. Establishing these trajectories requires detailed investigations of student interactions with TFV (see the discussion in ch-3A), which could be a possible future pursuit.

Effect on Cognitive engagement

We captured student cognitive engagement with the concepts using two measures: the proportion of students who responded using any of the modes (text, equations, drawings or keywords) to (1) all-the-3 questions and (2) at-least-one question. These simple markers, measuring whether the students respond to questions, are interpreted as markers of students' engagement with the content. Their attempting a question is an indicator of students attaching some meaning to the content, not necessarily a correct meaning, and hence does not directly correlate with conceptual understanding. This is thus similar to the markers of academic engagement such as completion of tasks in homework, participation and attendance (Appleton et al., 2006, 2008; Christenson et al., 2012; Lamborn et al., 1992).

Cognitive Engagement: School-wise Slicing (N=266)

Overall

School-wise Slicing

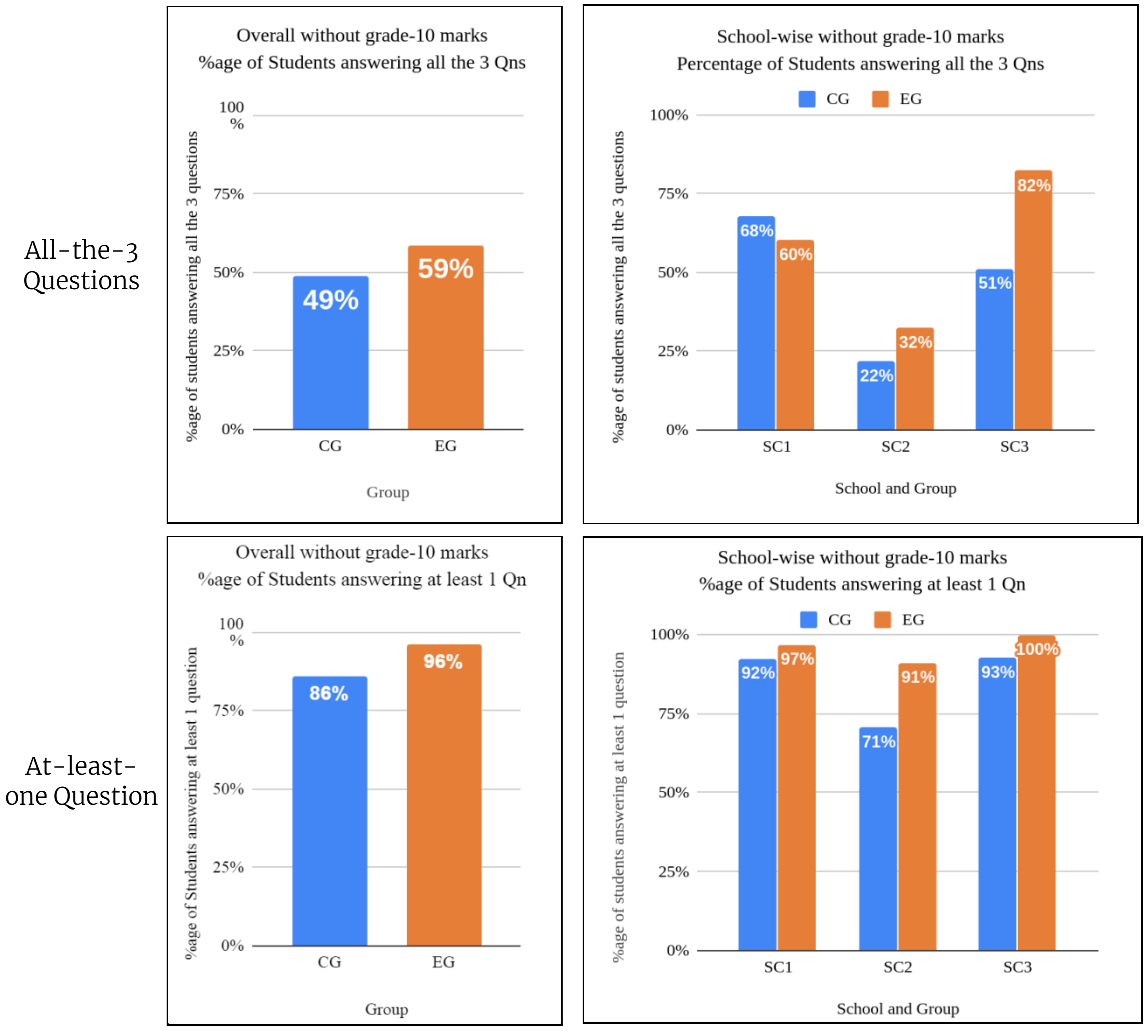


Figure 2F.6: The proportion of students responding to (top row) all-the-3 questions, (bottom row) at-least-one question in the control and experimental group. Left figure: Proportion combining all the schools. Right figure: Proportions in individual schools.

And similar to the geometry-algebra integration case, we do this analysis by slicing the data school-wise (N=266) and bin-wise (N=234). Figure 2F.6 shows the comparison with overall and school-wise slicing. This was repeated (figure 2F.7) for those students who had grade-10 scores slicing them into two bins viz., above 85% (high performing) and below 85% (average performing). These patterns are tested for significance using the odds ratio test (see tables 2F.4, 2F.5). An odds ratio r indicates that it is r times more likely that an individual who completes the tests has used TFV-2 i.e. she belongs to EG. An r greater than 1 is desirable. Greater the value of r, greater is the likelihood of TFV-2 being effective. The statistical significance of this is provided by the Z statistic and the p-value at 95% confidence interval (CI).

A greater proportion of students in EG have better cognitive engagement in all categories in both measures, except for SC1 in the all-the-3 case (see figure 2F.6). In agreement with this, the odds ratios are greater than 1 in all cases except in SC1 of the all-the-3 case (see table 2F.4). These effects are statistically significant in the all-the-3 questions case for SC3, and the at-least-one-question case SC2.

	N(students)		All 3 Questions								
	C	E	C	E	C%	E%	Diff	Odds Ratio	Z-Stat	P-value	CI
Overall	135	131	66	77	48.9%	58.8%	9.9%	1.4907	1.615	*0.1064*	0.9181 to 2.4204
SC1	53	63	36	38	67.9%	60.3%	-7.6%	0.7178	0.848	*0.3965*	0.3335 to 1.5447
SC2	41	34	9	11	22.0%	32.4%	10.4%	1.7005	1.009	*0.3129*	0.6064 to 4.7684
SC3	41	34	21	28	51.2%	82.4%	31.1%	4.4444	2.723	*0.0065**	1.5191 to 13.0031
Without Grade-10 Scores School-wise Analysis			At least one question								
			C	E	C%	E%	Diff	Odds Ratio	Z-Stat	*P-value*	CI
Overall			116	126	85.9%	96.2%	10.3%	4.1276	2.732	*0.0063**	1.4930 to 11.4113
SC1			49	61	92.5%	96.8%	4.4%	2.4898	1.028	*0.3038*	0.4376 to 14.1648
SC2			29	31	70.7%	91.2%	20.4%	4.2759	2.09	*0.0366**	1.0944 to 16.7053
SC3			38	34	92.7%	100.0%	7.3%	6.2727	1.2	*0.2301*	0.3127 to 125.8218

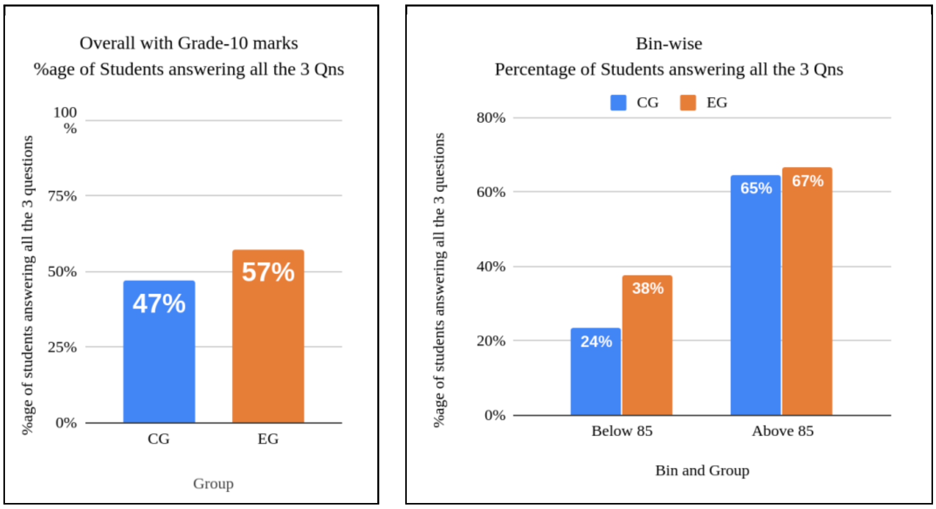
*Table 2F.4: School-wise Slicing: The odds ratio-test for the school-wise analysis (without using grade-10 scores). *p<0.05*

Cognitive Engagement: Bin-wise slicing (N=234)

Overall

Bin-wise Slicing

All-the-3 Questions



At-least-one Question

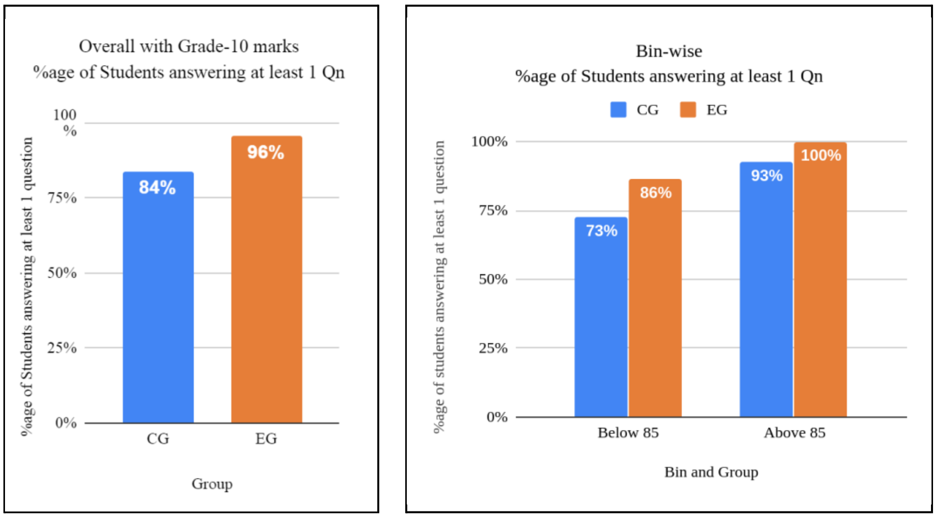


Figure 2F.7: The proportion of students responding to all-the-3 questions (top row); at-least-one question (bottom row) in CG and EG students whose grade-10 scores were available. Left figures: Proportion combining both the bins. Right figures: Proportions in individual bins.

In the bin-wise slicing (N=234 students with grade 10 scores) as well, we see an increased engagement in EG in all categories in both measures. Further, an interesting pattern of greater impact on cognitive engagement was found in the lower bin than in the upper bin. For the all-the-3 case (see figure 2F.7 and Table 2F.5), the lower bin has a greater ~14% difference as compared to the upper bin with a smaller ~2% difference. This is also similar in the at-least-one case with ~14% and ~7% differences in lower and upper bins respectively.

This indicates that lessons with TFV improved cognitive engagement more in average performers than in high performers. This is reasonable, because one may safely say that high performers are already cognitively better engaged with the content (as also reflected in taller bars in upper bins than lower-bins in both CG and EG, figure 2F.7).

This effect is in clear contrast to the geometry-algebra integration seen earlier. TFV improved the geometry-algebra integration, and influenced the reasoning approaches, more in good-performers than in average-performers. However, it increased cognitive engagement more in average-performers than in good-performers. Overall, these results indicate that learning with TFV changed reasoning approaches (esp. geometry-algebra integration), but with no big effect on cognitive engagement in high-performers. In the average-performers, TFV only enhanced cognitive engagement, but could not make any effect on the reasoning approaches. Both of these are meaningful shifts in the desired direction.

	N(Students)		All 3 Questions								
	C	E	C	E	C%	E%	Diff	Odds Ratio	Z-Stat	*P-value*	CI
Overall	119	115	56	66	47.1%	57.4%	10.3%	1.5153	1.579	*0.1144*	0.9046 to 2.5384
Lower Bin	51	37	12	14	23.5%	37.8%	14.3%	1.9783	1.442	*0.1494*	0.7826 to 5.0009
Upper Bin	68	78	44	52	64.7%	66.7%	2.0%	1.0909	0.249	*0.8033*	0.5500 to 2.1638
With grade-10 Scores			At least 1 question								
Bin Analysis			C	E	C%	E%	Diff	Odds Ratio	Z-Stat	*P-value*	CI
Overall			100	110	84.0%	95.7%	11.6%	4.18	2.744	*0.0061**	1.5047 to 11.6116
Lower Bin			37	32	72.5%	86.5%	13.9%	2.4216	1.54	*0.1235*	0.7858 to 7.4623
Upper Bin			63	78	92.6%	100.0%	7.4%	13.5984	1.756	*0.0792*	0.7379 to 250.6117

*Table 2F5: The odds ratio-test results for Bin-wise Slicing: *p<0.05*

2.3 *Qualitative episodes of reasoning behaviour*

Further there were episodes even during the test, where the students in experimental groups traced their actions on paper indicating geometric reasoning, as in the figure 2F.8.

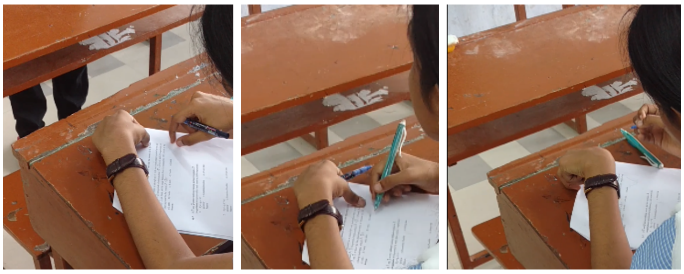


Figure 2F.8: A student moving hands above the paper indicating geometric reasoning in the post-test (See Supplementary material for video) left- finger movements, middle - raising the head during her work with pencil in hand, right- consolidated reasoning and getting ready with pen in hand

The episode of a student from EG2 is related to Q6, which involves changing the angle and estimating the effect on the resultant vector. Here the student starts with moving her fingers on the paper indicating the directions of the given vectors (which are at an acute angle), and then moves fingers as if she was writing numbers (making some numerical reasoning). Then she picks up a pencil (still in the air), and moves it tracing the vectors with a greater angle, and pauses and raises her head for a moment. Then she makes the initial set of movements of acute angle again, and traces the longer diagonal (diagonal represents the resultant). Then repeats the tracing of vectors in obtuse angle and a shorter diagonal and she makes arc-like movements indicating an obtuse angle, indicating a dynamic change of angle this time. Then immediately, she picks her pen, pauses for a moment, attempts to begin to write, slightly picks up the paper and changes her posture and responds to the question. These set of movements appear to be more confident as she consolidates her reasoning.

3 Chapter Discussion

Overall, this chapter presents interesting evidence in the effects of different interactions with the representations. Extending the initial promise found in the results of laboratory study reported in chapter 2D, with interactions with TFV-2, we find that the geometry-algebra integration and changed reasoning approaches are seen in good performers and enhanced epistemic access among average performers. This can be interpreted as the interactions with the TFV-2 enabling the average performers to begin finding the representations meaningful. Irrespective of the correctness of the responses, one can see that a greater proportion of average performers have responded to the question in EG than in CG. There is no difference in this engagement among the good performers; a possible explanation for this is that these students are already able to engage with the content and hence the interaction with TFV-2 does not impact their responding to the questions (more than 60% of good performers could answer all the questions as compared to below 40% among average performers in both CG and EG see figure 2F.7). Among these good performing students, the interaction with TFV-2 worked at a deeper level and helped them with better geometry-algebra integration as reflected by the greater use of equations and diagrams by students in EG than by those in CG.

However these effects are of varying levels of significance in different groups. For example, though all the schools indicate effects in the desirable direction, the patterns in the effects itself are interesting. SC2 has diminished effects in comparison to the other 2 schools in terms of the geometry-algebra integration. SC1 has diminished effects in the proportion of students who responded to all-the-3 questions. Finer aspects of the distribution and the reasons for this difference requires analysing possible factors such as the students scholastic performance and social-economic background in general. Each of these are interesting research questions worth investigating, but are not directly within the scope of our research questions.

Besides the above effects on the students' STEM cognition, the very execution of the study stands as an effective illustration of the adoption of TFV-2 into teaching learning activities in the classroom. These are unravelled in greater detail in the chapter-2G.

Publications from parts of this chapter

Karnam, DP., Agrawal, H., Parte, P., Ranjan, S., Borar, P., Kurup, P., Joel, A J., Srinivasan, PS., Suryawanshi, U., Sule, A., & Chandrasekharan, S (2020). Touchy-Feely Vectors: a compensatory design approach to support model-based reasoning in developing country classrooms. Journal of Computer Assisted Learning, 1–29. https://doi.org/10.1111/jcal.12500

Karnam, DP., Agrawal, H., Parte, P., Ranjan, S., Sule, A., & Chandrasekharan, S. (2019). Touchy Feely affordances of digital technology for embodied interactions can enhance 'epistemic access' In M. Chang, R. Rajendran, Kinshuk, S. Murthy, & V. Karnat (Eds.), Proceedings of the 10th IEEE International Conference on Technology For Education (T4E) 2019. (pp. 114-121). Goa, India.

Karnam, DP., Agrawal, H., Borar, P., & Chandrasekharan, S. (2019). The Affordable Touchy Feely Classroom: Textbooks embedded with Manipulable Vectors and Lesson Plans augment imagination, extend teaching-learning practices. In Lund, K., Niccolai, G., Lavoué, E., Hmelo-Silver, C., Gweon, G., and Baker, M. (Eds.). *Proceedings of 13th International Conference on Computer Supported Collaborative Learning (CSCL) 2019,* Volume 1. (pp. 488-495) Lyon, France: International Society of the Learning Sciences.

Karnam, DP., Agrawal, H., Sule, A., & Chandrasekharan, S. (2019). Need to explore affordances of technology for better learning and teaching interface designs. *In The Future of Learning Conference - Learning 4.0: Connecting the Dots and Reaching the Unreached.* Bangalore: IIMB.

Karnam, DP., Borar, P., Agrawal, H., & Chandrasekharan, S. (2018). The Affordable Multitouch Classroom. In *The Future of Learning Conference - Pedagogy, Policy and Technology in a Digital World.* Bangalore: IIMB.

2G

Touchy-Feely classrooms: Effect on Teaching-learning practices

The objective of the study: To outline the broad differences in the teaching-learning practices in the classrooms with and without the TFV-2 system. This is an illustrative report of the practice changes.

Key Findings:

- Teachers adopted the system into their practice, but are still largely grounded in the textbooks.
- The larger organisation and flow of knowledge in the classrooms was more decentralised and distributed in the EG classrooms.

In chapter 2A, we had a brief elaboration on teachers and the problems they face in enacting the lessons. In this chapter, we discuss qualitative changes, both in student learning (particularly imagination) and teacher practice when TFV-2 is introduced into the classroom setting. The specific pedagogical effects of the system, particularly in the way it helps augment the imagination and affect students' STEM cognition, can only be understood after extensive analysis of the post-test worksheets, videos of the classroom, and post-test interviews. This is a report of the changes illustrating the adoption towards addressing the research question 2.

The main data sources for the current analysis were video recordings, which were analysed to understand the differences in classroom dynamics (CG and EG), interactions between teachers and the students, and interactions with various artefacts (textbook, blackboard, notebooks and worksheets, the tablet). For this, the video recordings were analyzed using two lenses – differences in learning, particularly imagination, and differences in teacher/teaching-related practices. Representative episodes of the classroom practices discussed below are captured in the video provided as supplementary material.

1 Learning practices

The main differences in the learning process, particularly related to imagination, between the CG and EG classrooms were in terms of the use of gestures, trajectories, collaboration and flow of learning.

1.1 Gestures

As one of our key design objectives was to help students imagine the vector in geometric terms, we examined the gestures students made in the two classrooms, as these indicate geometric thinking. In the CG classroom, students were passive spectators most of the time, except for a few episodes where they took note of what was on the blackboard. Activity, when rarely present, was limited to nodding in agreement or tracing pens over the diagrams seen on the board (figure 2G.1 left). Gestures of this kind suggest a very limited engagement with the geometric content of vectors. The teacher's enaction of the textbook figures, which is the sole source for triggering the imagination of dynamic operations, is not resonated by the students.

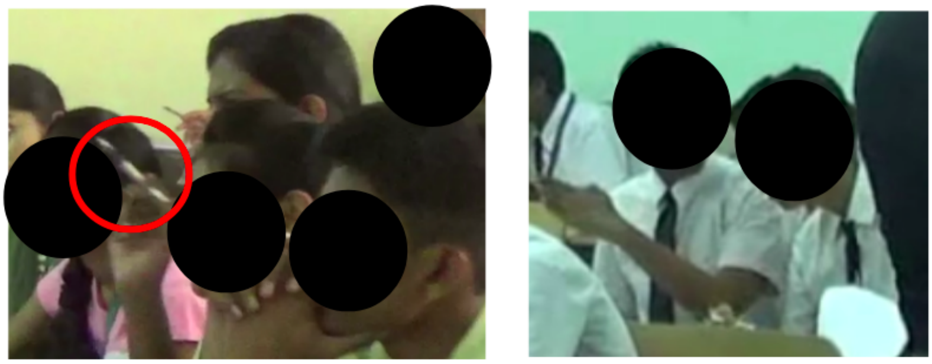


Figure 2G.1: Student in CG using gestures to remotely access the content on the board (left).A student in EG using gestures to explain to the peer in the context of activity on TFV (right)

In the EG classrooms, there was more active engagement with the geometric representations. This was reflected in more meaningful and sustained gestures, along with discussions, which indicate conceptually deeper engagement with geometric operations (figure 2G.1-right). These gestures were usually in conversation, where one student was actively trying to explain his imagination to another. The gestures indicate the possibility of students resonating with the teacher's writings (equations) or drawings (diagrams).

1.2 *Multiple trajectories*

In the CG classroom, the learning opportunities were only when the teachers enacted a concept, and most of the activity in the classroom involved taking notes, which followed the teacher's description of the topic and solutions to problems.

In the EG classroom, the exploratory and open-ended nature of tasks led to a surprising number of problem-solving trajectories. The diversity of approaches indicate the high imagination and learning potential provided by the TFV system. To illustrate, in one of the tasks, students were asked to create two vectors, whose resultant is a given vector with magnitude 60 and a direction of 40°. The following approaches were seen.

- *Trial and error* (the most common approach, where students manipulate the magnitude and direction of vectors randomly). Some students found some patterns of change as they kept interacting.
- *Estimating the rectangular components along x and y axes and creating them as the two vectors.* However, these students struggled when

they were asked to create another set of vectors. Here the students explicitly applied the interconnections between resolution into rectangular components and addition.

- *Creating two vectors with magnitude 30 at 20°* from the x-axis and then adding to find resultant of magnitude 60 at 20° instead of at 40°. This combination provides the opportunity for students to realise that magnitudes and directions (represented as angles) cannot be algebraically added directly.
- *Creating the vector of magnitude 60 at 40° from the x-axis*, and arguing that the other vector is a zero vector. This particular group of students tried out this task for a long time using the trial and error method and eventually applied the idea of zero vector quite intelligently.

1.3 Collaboration

In the CG classroom, there is very limited scope for students to engage in discussions with peers. The only conversations recorded between students involved asking for an extra pen or pencil, asking neighbours to show some part of his/her notes for clarification, or giggles with playing (disinterested in the content of the lesson).

Figure 2G.2: Students in EGs collaborating within their designated groups as well as beyond their groups

In the EG classroom, the very nature of the lesson plan involved making groups of 2 or 3 students, who worked together on the tablet. This inherently made the class collaborative. In addition, there were many episodes (similar to figure 2G.2) where students naturally started interacting across groups. The nature of collaboration was grounded in the tasks they were performing on the tablets. Collaboration is not necessarily an indicator of an effective classroom, as there could be a lot of social and cognitive loafing (O'Donnell & O'Kelly, 1994), which could have happened in the EG classroom as well. However, as the students were involved in the collaborative activity using the TFV system, the teacher could monitor student progress, to start discussions.

1.4 *Flow of learning*

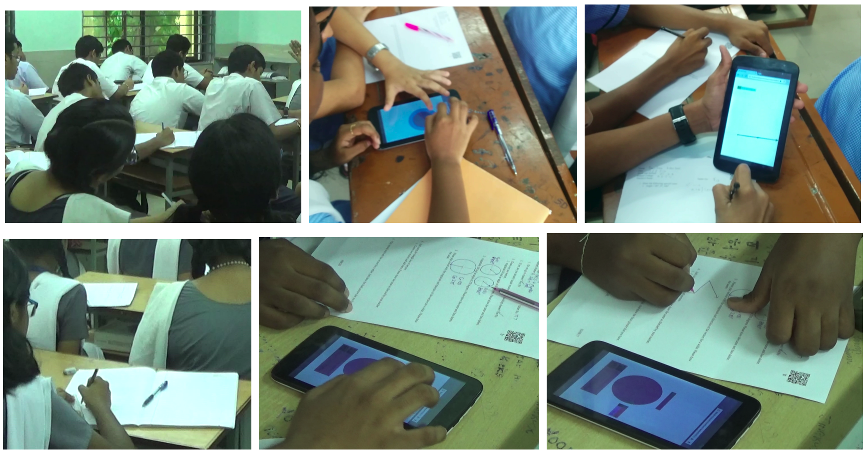


Figure 2G.3:. Students in a CG taking notes in their notebooks (left) Students in EG interacting with the tablet (middle) and taking note of it in the worksheet (right)

In CG classrooms, most of the time was spent on the teacher delivering content in the lecturing mode. The teacher thus had full control over the flow of knowledge and learning in the classroom. There was very little scope for students to intervene in the flow and build their own knowledge. Further, the teacher decided when students could take notes, when to listen to him/her, which book was to be taken, who should respond to questions etc. This authoritative stance is central to conventional classrooms, and this is required to some extent for smooth classroom management. However, this power structure often makes students intimidated, and they thus seldom interact with the teacher. Students tend to overcome this fear using chorus responses. The teacher wielded further control by appreciating (saying words like very good, interesting) or dismissing (ignoring or not attending to) student responses.

EG students actively participated in the enaction process along with the teacher, and this transferred agency to the students, who had more power in directing the flow of learning than in a normal classroom. This was reflected in the free interactions among students, the change in the nature of teacher-student interactions, and also students' emotional connections with the tasks. The students were active participants with the teacher, and their interaction with the content was active and meaningful (see figure

2G.3). Based on these interactions, along with questions and discussions with the teacher, they controlled the flow of learning, and hence constructed their own knowledge. The teacher was no longer the sole controller of the flow of knowledge. Students generated knowledge during their interactions with their peers as well, and knowledge flows horizontally through collaborative work. Moreover, as suggested by the teachers, if tasks could be designed for students to do at home (thus flipping the classroom), the flow of learning could become even more student-centred.

1.5 Other observations

The CG classroom was very calm, and there were never any emotionally charged moments. In contrast, the EG classroom had a charged feeling, and there were numerous moments of excitement and disappointment.

There were multiple instances where EG students asked for similar systems for other topics in the curriculum. Students were very excited to see the tablets on the first day. By day 3 and 4, they started engaging with the TFV system in a more serious way and they were not just excited due to the presence of the tablet.

2 Teacher/Teaching practices

The introduction of the TFV system into the classrooms was not fully and easily embraced by the teachers, though there were changes that indicated possible integration eventually. Here we discuss some of the central differences (and lack of changes) captured in the video data. We start with a discussion of practice elements resistant to change.

2.1 Resistant teacher practice

Three out of five EG teachers (T1, T2, T5) used the same diagram for teaching parallelogram law of vector addition, where the angle is less acute, even though the vectors used in triangle law, shown simultaneously on the board, were quite different (See figure 2G.4). This drawing closely follows the one in the textbook and indicates the deep and subtle influence of the textbooks, where the teachers are conditioned by the textbook representation to such an extent that the diagrams seep into their practice without explicit awareness. While the TFV system has changed the way teachers present the content (as discussed in later subsections), there were such subtle aspects that were still driven by the textbook representations. We present this result first to indicate how deeply the textbook is integrated into teaching practice.

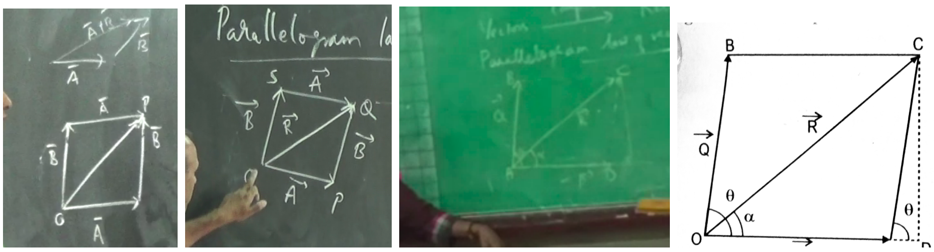


Figure 2G.4. Teachers making parallelograms very similar to the one given in their textbook (right) with almost 90°angle while teaching parallelogram law of vector addition.

This conditioning by textbooks extended to the way content was presented. In the EG classroom, where teachers had the possibility of widening the scope of discussions (like emphasizing and discussing the patterns of changes in the algebraic expressions through geometric manipulations, or the nature of the x and y components being interconnected by the circle). Such topics were not discussed, and the scope of the class was still within the limits of the topics presented in the textbook. This reaffirms the conditioning role played by textbooks in setting classroom practices.

In the CG classroom, the teaching narrative usually took the form of a lecture, where the teacher introduced the definition or the statement of any law, explained it using some examples, and then drew diagrams. The teacher used some gestures to enact the diagram. The students took written notes based on these practice elements. When some extra time was available before the end of the session, the teachers solved some numerical problems. The flow of the lesson was very linear and monotonic. In the EG classroom, the narrative was similar in the beginning (introduction to the definitions or statements, making diagrams). It then changed to describe the tasks the students needed to do using the system. Interestingly, there was a tendency for both CG and EG teachers to follow a set question and answer template for discussion (especially when stating the definitions or the laws of additions), as this helps students in answering exam questions. In later discussions, T1, who has taught in an EG class, noted that despite the teachers being given the flexibility to use new modes of teaching, there is a lot of emphasis on examination results and time limitations. This emphasis comes from the way the education system in India is structured around exams, which restricts the possible narratives in the classroom.

2.2 *Practice elements that changed*

<u>*Teacher utterances*</u>

A central change was the nature of teacher talk in EG classrooms, from mere description of static diagrams (like one arrow and another arrow forming two sides of a triangle, and the third side being the resultant, in triangle law of vector addition), to a more dynamic enaction, where teachers described the addition process. For instance, the teacher used the gestures in the TFV system (figure 2G.5 right) to explain the process of adding two vectors saying "now you know how to take vectors, just bring this vector (2nd vector) and attach here (head of 1st vector), then you will have to press your thumb (at the tail of 1st vector) and another finger here (head of 2nd vector), you will get the resultant".

Teachers in both CG and EG tended to use many gestures and bodily actions when explaining content using diagrams. However, there were differences in the gestures used by CG and EG teachers. In CG, teachers tended to use iconic gestures (like drawing an arrow in the air or using an opened palm to represent a vector: figure 2G.5 left) and described a static diagram: one arrow and another arrow forming two sides of a triangle, and the third side being the resultant.

Figure 2G.5: (Left) Teacher in CG using iconic gestures to show a vector, which is still a diagram. (Right) Teacher in EG using a picking action, reflecting that the vector is a touchable entity.

In the EG classroom, teachers used similar gestures as a CG teacher during lecturing. However, they also incorporated gestures used in the TFV system. For example, T4 and T3 used gestures influenced by the system when explaining the triangle law of vector addition (see quote above). These gestures signify a change in the way teachers thought about vector operations, based on interaction with the system.

Teacher Physical Movement

In the CG classroom, the teacher stayed near the blackboard (as seen in many figures). As the TFV system allowed all EG students to enact vector operations, the teacher did not need to demonstrate the enaction in the class, and she thus had a lot of time available to move around in the class and engage in discussions with the students. The role of the TFV system in generating this behaviour is clear from the contrasting behaviour of T3 (the only teacher present in both EG and CG) in the CG and EG classrooms. The discussions involved clarifying conceptual doubts and questions. This allowed the teacher to also get real-time feedback about students' understanding of vector concepts, and initiate discussions with individual groups (figure 2G.6 right and centre) or the entire classroom (figure 2G.6 left), thus enriching the collective learning experience for the students.

Figure 2G.6: (Left and Center) Teachers engaging in discussion with student groups in EG. (Right) Teacher showing a group's work on the tablet to the entire classroom (a potential space for initiating a discussion)

Teacher questions

Usually, teachers ask questions to check for understanding (Rosenshine, 1983), or when students are not attentive. In the CG classroom, the monologue of the teacher was often interspersed by questions, where the teacher explained a concept or a problem and asked one of the students to stand or come to the board and solve or answer a similar problem (See figure 2G.7). There were some such episodes in the CG classroom, where the teacher called a student to perform subtraction of two vectors, or derivation of expression of parallelogram law by simple substitution, or to solve a numerical problem similar to the one the teacher had just solved. Irrespective of the purpose of asking the question, they were close-ended questions with a unique answer.

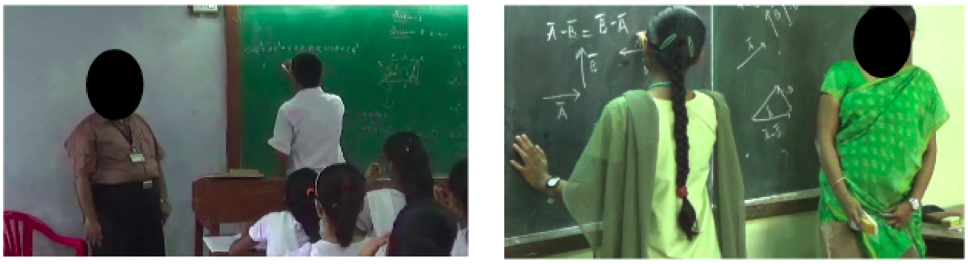


Figure 2G.7: Teachers, in CG, calling students to the board to answer or solve some questions.

In contrast, in the EG classroom, the teacher spoke to the entire classroom only in the beginning. The rest of the time was spent answering questions from students. The nature of the questions was open-ended (with no single answer) and exploratory, following the exploratory nature of TFV tasks.

Other changes

Besides the above changes, a few other episodes indicate significant changes in teacher thinking and practice, induced by the TFV system.

- As teachers started discussing possible lesson plans during the lesson planning workshop (LPW), they immediately suggested that we could design tasks for students to do at home (as they get more time to engage with the TFV system) and then discuss their findings and responses in the classroom next day, effectively flipping the classroom.
- After one of the lessons, T3 said - "besides helping the students in visualising and understanding the concepts, it has clarified and made imagination easier for topics for me too". This shows how systems like TFV could also help teachers understand complex concepts better, and also help develop content knowledge as well as pedagogical content knowledge (Shulman, 1986), particularly given the acute shortage of good teacher trainers and programs in the developing world.
- Teachers, like the students, appreciated TFV and have repeatedly requested extensions to the TFV system, particularly to include 3D vectors, products of vectors, as well as waves. This is because it is difficult to draw repeatedly to explain these formal systems using the blackboard.

- A physics teacher who was not part of the study sat through an entire classroom session, to see how the system was being used by students

3 Chapter Discussion

We could see a transformation of the classroom space for the students and situations and practices for their learning emerged in them smoothly. Among teachers, we see they could adapt the technological intervention smoothly, but the changes in their practices were far subtler, as we see indicators of them still anchored to textbooks. It is here, a key insight emerges about the central role textbooks play in directing teacher practice, and how building on the textbook could lead to rapid changes in teaching practices, particularly in terms of the way teachers direct/control the flow of knowledge in the classroom. The augmentation of textbooks could also help in swiftly upgrading teacher thinking and pedagogical knowledge of science and mathematics concepts, which is a critical issue in the developing world. Such upgradation of pedagogical content knowledge (Van Driel et al., 1998) could be better facilitated by QR augmentations that support flipping the classroom, as this feature would raise the number of potential questions from students, and in turn, motivate teachers to come prepared to answer these questions.

Overall, the report of the changed classroom practices in this chapter, along with the evidence for changes in student CRB in ch-2F, illustrates possibilities of effective adoption of the TFV system provided by the teacher-driven QR-based augmentation of textbooks. These, thus offer a new approach to address key resource limitations in developing world classrooms, in terms of both technology access and training costs. The QR-based approach also provides a way for educational research institutions to curate good learning resources (such as videos and problem banks), and include these periodically as annotations to current textbooks, without waiting for the lengthy and bureaucratic process involved in revising the textbook.

Publications from parts of this chapter

Karnam, DP., Agrawal, H., Borar, P., & Chandrasekharan, S. (2019). The Affordable Touchy Feely Classroom: Textbooks embedded with Manipulable Vectors and Lesson Plans augment imagination, extend teaching-learning practices. In

Lund, K., Niccolai, G., Lavoué, E., Hmelo-Silver, C., Gweon, G., and Baker, M. (Eds.). *Proceedings of 13th International Conference on Computer Supported Collaborative Learning (CSCL) 2019,* Volume 1. (pp. 488-495) Lyon, France: International Society of the Learning Sciences.

Karnam, DP., Borar, P., Agrawal, H., & Chandrasekharan, S. (2018). The Affordable Multitouch Classroom. In *The Future of Learning Conference - Pedagogy, Policy and Technology in a Digital World.* Bangalore: IIMB.

PART-3

CONCLUSION

3A

Discussion of Findings

The objective of the chapter: To summarise, interpret and discuss the findings, and identify the limitations

Key findings:

- Interactions with medium changes students' CRB (reasoning approaches and epistemic access)
- Compensatory media design approach — smoothly augmenting textbooks, instead of replacing them — with teacher-co-designed virtual lesson plans change classroom practices

Limitations:

- Limited analysis of interaction processes, and hence weak in establishing a causal relation between STEM cognition and affordances of the representational medium
- Confined to the MERs of vectors, but could not extend to modelling scenarios
- Methodological pluralism addressing the limitations of statistical rigour
- Experiences meaningful, but limited to the TFV system, and not grounded in other real-world experiences.
- Limited consideration of rural and remote regions physical access

In this chapter, we begin with a summary of the key findings from each study. Then we interpret these findings in addressing the research problems identified in part-1 (Ch-1A,1B). Lastly, we discuss these specifically from a vector learning problem standpoint and then, for some wider insights, related to the medium and learning in connection to the literature. Then we highlight some of the limitations of the study.

1 A summary of the findings

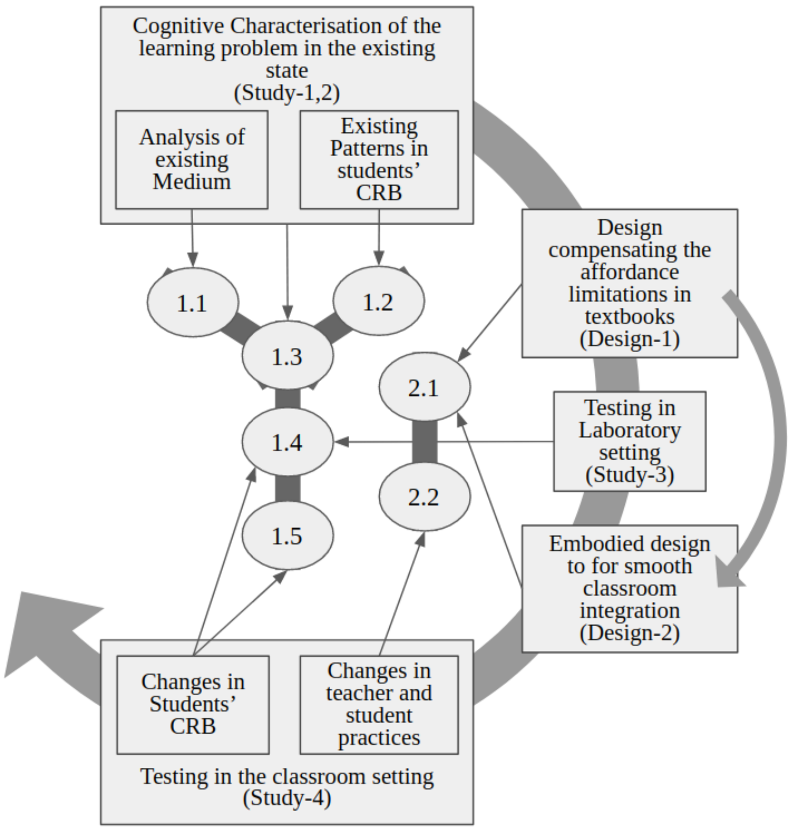


Figure 3A.1. The flow of the studies and their mapping to the research questions

1.1 *Study-1: Textbooks manifest the limitations of the static paper-based medium*

This study addressed the sub-questions 1.1 and 1.3 towards characterizing the existing representational medium, specifically the way the topic of vectors is treated. This study was a detailed analysis of textbooks and supporting evidence from interactions with the teachers. Three limitations of the paper-based medium manifest in the way the textbooks treat the topic of vectors and the teachers' practice.

- *Lack of geometric manipulation:* The medium of paper does not easily allow performing geometric manipulations. Textbooks across curricula attribute this limitation of tediousness and inaccuracy to geometric methods and justify the usage of algebraic methods of adding vectors. Further from the treatment patterns, we note that the addition of vectors is done mostly using algebraic methods. Geometric methods are merely stated as laws in the explanatory section and there is no scope in problem-solving contexts where geometrical methods are used. We ascribe these patterns emerging from the static nature of the paper-based medium and it's limitations to afford manipulations of geometrical entities accurately and easily; manipulating geometrical entities accurately in paper-based medium requires meticulous and tedious usage of geometrical instruments like a compass, ruler etc. Supporting evidence, for this limitation of geometric manipulation of the existing media in classrooms, emerges from teachers when they cite difficulties in drawing multiple diagrams like in the case of superposition of waves. To compensate for this limitation, they try to use gestures and other artefacts at hand but often find it difficult to trigger imagination in students.
- *Serial ordering:* The medium of paper does not allow presenting content in a cohesive and integrated manner. Textbooks across the curricula present the topics related to understanding vectors like the geometry of lines, triangles and circles, trigonometry etc., in multiple grades and textbooks. Textbooks have to present all of them in a serial order. We ascribe these patterns to the limitation of the paper-based medium to present content in an integrated manner; the paper-based nature of the medium has limited affordances to dynamically present content from multiple topics together in a meaningfully integrated manner. Supporting evidence, for this limitation to present content in an integrated manner, emerges from teachers when they cite their inability to integrate them in collaboration with teachers for other subjects and other grades, which is often not logistically possible. To compensate for this, they try to quickly revise the required content, but it becomes

impractical to spend sufficient time to ensure students integrate the content meaningfully.

- *Opaque problem solving:* Textbooks across the curricula when solving problems of applying vectors tend to not explicate the rationale of a particular approach or underlying heuristics; for instance, the mass chosen to draw free body diagrams or the frame of reference chosen to resolve the vector are not explicated. This will make problem-solving opaque for students. We ascribe this to the limited affordances of the paper-based medium to try multiple solutions approaches quickly and easily. Supporting evidence emerges, for this limitation, when the teachers repeatedly narrate anecdotes highlighting students rote learning formulae, or resorting to applying memorised problem-solving algorithms etc.

Some of these limitations are already articulated in various contexts in the literature as discussed in detail in the chapter discussion (Ch-2A).

1.2 Study-2: Students' CRB has a dominant algebraic mode of reasoning.

This study addressed the sub-questions 1.2 and 1.3. This study probed two groups of students (typical students and olympiad students) through a questionnaire and follow up interview. We did a thematic analysis of data from written scripts meshed along with data in the form of episodes from the interviews. Together, these presented indicators of some consistent patterns of CRB in both groups of students. These patterns in students' CRB are organised broadly into 3 themes.

- *Reasoning approaches with an Algebraic Dominance:* This theme emerges from a set of indicators that hint at underlying algebraic modes of reasoning in the students. These include tendencies of dominant reliance on memorised formulae and algebraic manipulations; treating and operating on vectors as scalars, ignoring the directionality of vectors; preference for algebraic explanations even when geometric reasoning is relevant, easier and simpler; many indicators of algebraic modes in the nature of vocabulary (conflating terminology based on algebraic similarities) and the gestures (gesturing structural aspects of algebraic expressions) they use in the conversations.

- *Interconnections between conceptual units:* In connection with the conceptual aspects related to vectors, we see many indicators of fragile or incoherent mental models that students held. Some of those included: limited understanding of geometric modes of addition beyond drawing simple diagrams (often ignoring the directions of vectors) and not realising the equivalence of various methods of adding (a sense of realisation that all of them lead to the same resultant); limitations in their understanding of resolution as a process and a proper application of trigonometric ratios (and related issues); limitations in understanding the significance of using rectangular components; other minor issues evoked from components of components paradox and products of vectors.

- *Approaches in Problem-solving:* In connection to the indicators related to problem solving, we see certain behavioural patterns emerging consistently. These include a superficial and procedural level engagement with the problems, and a tendency to look for standard algorithms while solving problems.

Evidence for many of these behavioural patterns can be already found in various contexts in the literature especially in math education, as discussed in detail in the chapter discussion (Ch-2B).

1.3 Study-3: When systematically compensated by design-1 students CRB changes.

This study addressed sub-question 1.4 on whether a change in the interactive affordances of representational medium changes the students' CRB. This study involved interaction with 6 students through pre and post-tests followed by interviews and an intervening interaction with the Design-1, the Touchy-Feely Vectors -1 (TFV-1) through a series of carefully designed tasks. The analysis involved: a detailed rating of students written scripts for changes in the strengths of conceptual links based on an analysis framework; and case studies of two students with different nature of interactions and scholastic abilities. The key findings of this study are:

- Interaction with this TFV-1 enhances the students' conceptual link strength in ~70% of the cases in all the students. And it also disrupted their existing models like in the case of the student S2.

- Detailed cases of two students using their interview and the interaction data indicate a change in their reasoning approaches and increased epistemic access.
    - The first case student S2, a scholastically better case, relied significantly on the paper and pencil based numerical calculations and estimations with some conceptual basis during the interaction with the system; his interactions were *conceptually guided.* We see an increased strength of conceptual links across many SCAs and a drop in the case rectangular components; this as revealed from the interview was due to certain confusions and hence the drop in the post-test. However, during the interview, we find more coherence in his thinking. We see a shift in his reasoning approaches from the pretest to the post-test in the form of usage of geometric entities like vector diagrams to reason about a similar kind of question and in the forms of episodes of using gestures to explain dynamically the variations in the resultant with varying the directions of initial vectors. This is a very important change in his reasoning approach from that at the pre-test stage, where he only used algebraic expressions and substitutions. All these together indicate triggering of imagination and providing epistemic access through novel tangible experiences with the vectors (note his comments of TFV as experimental space to test theoretical calculations of vectors), enabled by the interactions with TFV-1.
    - The second case student S5, a scholastically average case, relied significantly on the perceptual features of the TFV-1 system with no interaction with paper-based medium (other than to just note down the final answer); her interactions were *perceptually guided.* Though she could not complete all the tasks in the intervention, she managed to explore all the affordances of the system through a period of about 90 minutes. We see an increased strength of conceptual links across all the SCAs in her case. During the interview, though, we find that her understanding was still fragile and incoherent, there is a self-perceived clarity in her talking about the concepts. There

are indicators of enhanced epistemic access to resolution and addition from her comments to the effect that earlier she could not make sense of the process of resolution and the underlying trigonometric ratios, but the interaction helped her see the same, and a similar realisation with addition and the emphasis on directions; there is an accompanying excitement of understanding these in her comments. She notes that she could understand and address some questions on the system, but not on a paper-based test indicating a limited transfer of skills from one medium to another. This thus improved cognitive engagement with the content providing epistemic access to the abstractions, but this has not transferred to her reasoning while working with paper-based tests.

Together the above findings gave promising evidence that affordances of TFV-1 helped change reasoning approaches and enhanced epistemic access in the students, by providing novel experiences with formal structures.

1.4 Study-4: Finer effects on students of different scholastic abilities

This study addressed the sub-questions - 1.4 and 1.5 on if a change in the interactive affordances of representational medium changes the students' CRB and capturing the effects in actual classroom implementation. This involved a large-scale (266 students in 3 schools each with a control and an experimental classroom) implementation of the Design-2 TFV-2. We worked along with the teachers in developing virtual lesson plans. 3 teachers executed these lesson plans to teach the students a part of the chapter on vectors in their regular academic calendar; 3 other classrooms were taught in conventional methods. After the lessons for 1-2 weeks, both the groups of classrooms were administered a test. The scripts were then analysed for the patterns of usage of representations (algebraic and geometric) while reasoning to respond to the test-items; we used their previous grade math and science scores to capture their general scholastic abilities. Using these data, we analysed geometry-algebra integration using KDEs and cognitive engagement (responding to the questions) slicing the data based on schools and students' scholastic abilities. Some of the key findings from the analysis include:

- TFV-2 changes reasoning behaviour in good performers and enhances cognitive engagement in average-performers. (reported in Ch-2F)
    - We could see that there is a shift in the proportions of students who could use both algebraic and geometric modes in all the schools, except in one of them. Also from slicing the data based on scholastic abilities: the reasoning approaches are improved with better geometry-algebra integration among good-performers and there was no visible change in the average-performers.
    - We could see that there is a change in the proportion of students who could respond to all the questions overall. From further slicing the data, we could see that this rise was more pronounced in the case of average performers than good performers; that is, interaction with the system could give a greater boost to average performers than good performers towards epistemically accessing the abstract content. There is a raise, but not a very significant one among good performers, who already can cognitively engage with the content.

This evidence along with some qualitative episodes, which show students using dynamic geometric reasoning even when they are answering on a paper-based test, strengthens the possibility of TFV-2 triggering imagination among the students (by enhancing cognitive engagement/ epistemic access and by shifting their reasoning towards better geometry-algebra integration).

1.5 Design and Practice changes:

The 2 design iterations of TFV as well as qualitative observations of changes in teaching and learning practices in classrooms address the research question 2 on the way a systematic design of media-intervention looks like in a developing nation context. These two design iterations show that:

- The interactive affordance of computational media can be used systematically to address the limitations of the paper-based medium (identified in chapter 2A). In the case of the vectors, this involves

    - Presenting the geometric and algebraic representations of vectors in an integrated manner (with dynamic and real-time integration).
    - Allowing conceptually meaningful interactions with these representations.
- Compensatory nature (augmenting the textbooks) and co-designing of the tasks with the teachers could enable a smooth integration. This involved
    - Co-designing virtual lesson plans along with the teachers smoothly integrating with their already existing teacher practices based on textbooks.
    - Embedding these lesson plans in the textbooks using QR codes.

Qualitative findings from the classroom implementation as part of the study-4 has shown changes in the teaching and learning practices yet a smooth adaptation (reported in Ch-2G). These include

- Teachers adopted the system into their practice but their practices are still mostly grounded in the textbooks.
- The larger organisation and flow of knowledge in the classrooms was more decentralised and distributed in the classrooms with the TFV-2.

2 Interpretation

In this section, we interpret the above findings and integrate these interpretations towards addressing the research problems (listed in Ch-1A and 1B).

2.1 RQ1: Do the interactive affordances of representational mediums in STEM shape learners' STEM cognition?

The findings from studies 1,2,3 and 4 together strengthen the corollary that interactive affordances of the representational media affect students' STEM cognition (displayed as CRB). From studies 1 and 2, a very interesting semblance between the patterns of existing (paper-based) — medium manifesting as patterns in textbooks' treatment of the topic of vectors — and patterns in the students' existing STEM cognition — the themes of indicators in the students' CRB.

- Lack of geometric manipulations leading to too much reliance on algebra-based reasoning without proper understanding and corresponding imagination,
- Improper scaffolding and interlinking between the units in the textbooks leading to their fragile conceptual understanding without a coherent model
- Lack of explanations in application problems leading to students following fixed algorithmic procedures even in certain novel problems;

See the left blocks of figure 3A.2. The apparent semblance and the fundamental nature of the patterns in textbooks and student conceptual behaviour point to a possible meaningful link from the SCIARM framework. This provides initial evidence for the corollary of the constitutivity hypothesis and extends the SCIARM framework. Student's existing STEM cognition (as reflected in their behaviour in study-2) could partly be emerging from the limited interactive affordances of the existing representational medium (static paper-based). This strengthens the corollary as indicated by one dotted link in figure 3A3.

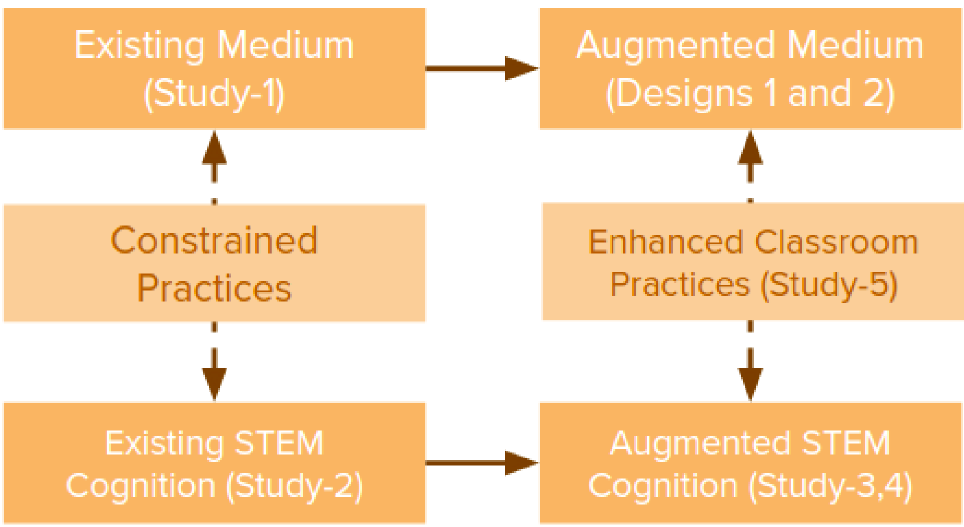


Figure 3A.2. The interpretation of the results from the studies in connection with the corollary of the SCIARM framework. (left) Existing System with the paper-based medium. (right) An augmented system with TFV

This further is strengthened by the evidence from studies 3 and 4 (right blocks of Figure 3A2), where a carefully designed interface compensates for the limitations of the paper-based medium, allowing interactions with the geometric representations, presenting related concepts in an integrated manner and transparent manner. We see here that students after interacting with the interface have stronger conceptual links. Further, the detailed case studies as well as the large scale study with 266 students reveals certain nuances of the effects on students of different scholastic abilities. Good performers, like student S2 in study-3, report a changed reasoning approach; this is evident from the usage of gestures by S2 as well as the spread of the contours in the KDEs in study-4. And average performers, like student S5 in study 3, do not show any visible markers of changed reasoning approach; this is evident from the analysis of interactions as well as interviews of the student S5, and the limited difference in KDEs between EG and CG. However the average performers seem to have gained better epistemic access, as evident from greater cognitive engagement among the students; student S5 made comments to the effect that she for the first time could access the underlying processes of addition and resolution, and the students in the lower bin (<85% category) showed a greater difference between CG and EG groups than those in the upper bin. As briefly discussed earlier this makes sense because the good performers (upper bin students) can already cognitively engage with the content.

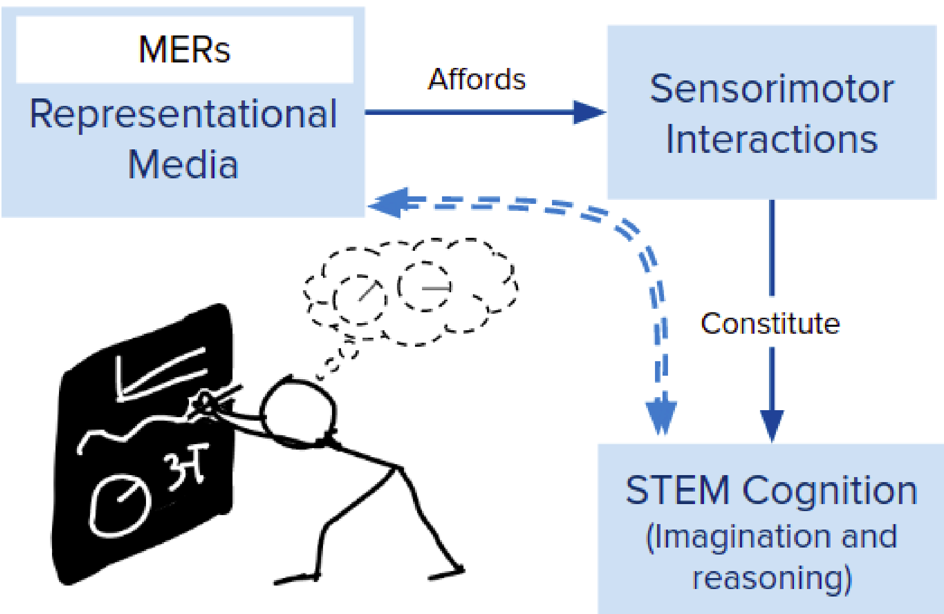


Figure 3A.3. Strengthening of the corollary captured by the SCIARM framework (indicated by the 2 dotted links).

These studies together provided three strands of data, which, when considered together, present converging evidence that limitations of the paper-based medium significantly contribute to student difficulties while learning vectors. First, the limitations of the paper-based medium are reflected in the way vector concepts are presented by textbooks. Second, besides the conceptual gaps related to this textbook presentation, students also display reasoning patterns that are driven by the limitations of the paper-based medium, such as the dominance of algorithmic thinking and rote learning and reasoning based entirely on algebra. Third, after interacting with a digital media system with specific compensatory features that were designed to overcome the media limitations of textbooks, students shift from these methods to more coherent and integrated reasoning approaches (See Figure 3A.2 right). This shows that the interaction with the representations afforded by the new media, could trigger imagination (an ability to reason with models) as well as enhance epistemic access to the abstract concepts and models.

Overall, these 4 studies together provide promising evidence strengthening the corollary (see figure 3A.3) that the interactive affordances of representational media — both existing paper-based and the new media-based TFV — do shape the students' STEM cognition — students' reasoning behaviour in both cases. The evidence thus in turn confirms the constitutivity hypothesis.

2.2 *RQ2- What does a systematic design of media-intervention look like in a developing nation context?*

The entire design iterations informed by the findings of studies 1 and 2 (existing medium and existing students CRB) and compensating for the limitations of the paper-based medium and implementation through studies 3 and 4 after refining the design to suit teacher practice and enhance physical access illustrates how a media-intervention can be conceptualised and implemented in the developing nation context. We will briefly gather the design aspects and how these have been effectively integrated with the classroom practices.

Epistemic access (compensatory)

The first iteration of the design systematically addresses the limitations of the existing static paper-based medium. The interactions of the TFV-2 are refined with more conceptually meaningful ones. The evidence addressing

RQ1 shows how these novel affordances of interaction with the representations allowed by the computational media can allow students to epistemically access the content, and with good performers, we could see the further triggering of imagination.

Teacher practice (codesign):

From the interactions with the teachers, we realised that the teachers are used to and are more comfortable with the textbooks as instructional artefacts. To smoothly merge with the existing teacher practice, we worked along with them, and without much change to their existing trajectories of teaching, through a workshop co-designed virtual lesson plan. This included a specific set of tasks on the TFV-2 system which the students would do in the classroom in place of their regular paper-based exercises.

Physical access (augmentation):

Furthering the interactive affordances of the TFV system, we have implemented them on smartphones and developed QR codes for the specific tasks as per the virtual lesson plan developed by the teachers. These QR codes thus embedded the TFV system in the textbooks and could very smoothly augment them. The ideal situation for the applicability of TFV is where every student has access to a touch-enabled device for some part of the day, a scenario towards which Indian society is moving. However, in a remote school even if one smartphone is available with the teacher, and with latching on to the existing textbook-based infrastructure, it becomes accessible to the students at the least to visualise and get novel experience with the geometric representations[21].

Compensatory augmentation using virtual lesson plans codesigned by the teachers allowed us to smoothly implement the medium in the classroom practices (3 classrooms for about 1-2 weeks in each). From the differences in the teaching-learning practices found as part of the study-5 (reported in

[21] Of course, here all the students will not have the vibrant sensory experience, but it is still better than not having any. Let us analyse this using another similar example- a microscope. The scenario is similar to a high-school classroom with just one microscope. It could be similar to how seeing an onion peel or a moving organism under a single microscope in the class helps learners. Though this does not give access to the whole knowledge about microorganisms, this can still trigger a paradigm shift in the way students imagine them. The ideal situation is where every student has access to a microscope, which agencies like the Prakash Labs are attempting with Foldscopes.

Ch-2G), we see that introduction of the medium could change the organisation of the classroom interactions as well as a more decentralised control of the flow of knowledge. Together this implementation serves as a good model illustrating computational medium-integration, especially in developing nation context.

3 Discussion

In this section, we will discuss these results with respect to the SCIARM framework and other literature we reviewed earlier. As per the corollary captured by the SCIARM framework, we expect that the cognition of an agent changes with the changed interactions with the representations inscribed in the representational medium. And the literature review of media in learning warns of taking a technocentric approach towards the medium and ignoring the method, or implementational aspects which could be influencing the learning. Furthering the same line of thought, one may question that it is the eventual usage of (or method of using) the artefacts — either the textbook or the TFV system — and not the treatment or presentation of the content in these artefacts that could influence the learning. To address this, we use the SCIARM framework to argue that the usage (the method) and the medium are again tightly connected and thus can influence students' cognition and learning.

Discussing the influences on students' reasoning-behaviour in a typical classroom, Dreyfus (1999) notes - "*College students do not usually read mathematics research papers, or see research mathematicians in action. But they do listen to lectures and participate in exercise sessions; they see and experience the talk and actions by their teachers; they read textbooks; they hand in assignments and tests, and they consider the grader's remarks when they receive them back; their mathematical behaviour is shaped, consciously or subconsciously, by these influences*". In India and other developing country contexts, paper-and-pencil is still the dominant medium of interaction, and teaching and learning practices are shaped by the static nature of this medium. For complex and dynamic formal systems (Moreno-Armella et al., 2008) like vectors, geometric manipulations are difficult to imagine and tedious to execute, given the static nature of the representations in the textbook medium. This limitation could thus constrain teachers' actions, and hence students’ actions and their imaginations. (See constrained practices in the existing system in the left of Figure 3A.2).

In Indian classrooms like that of Samantha’s, textbooks have very high institutional authority (prescribed by the state authority), and their indispensable nature anchors all teaching-learning practices towards writing, such as instruction (chalkboard), classwork and homework (using notebooks), and assessments (written examinations). Even if an atypical teacher puts extra effort and creates lecture notes and exercises (rare in Indian classrooms), these would be strongly connected with the patterns used in textbooks to discuss content. For instance, many teachers we interacted with made similar figures (figure 3A.4, while teaching parallelogram law), which mimicked the one used in the textbook (Karnam et al., 2019). This gives a hint of the extent of conditioning by the textbooks. Interestingly, when teachers started using a digital media system we have developed (Touchy-Feely Vectors), their drawings and gestures changed to the ones used in the application (Karnam et al., 2019). This is evident from the changed teaching and learning practices in the presence of TFV-2 (See right blocks in figure 3A2). Given these patterns, it is reasonable to conjecture that the way static media shapes the treatment of content in textbooks leads to a general static-media driven ecosystem in Indian classrooms.

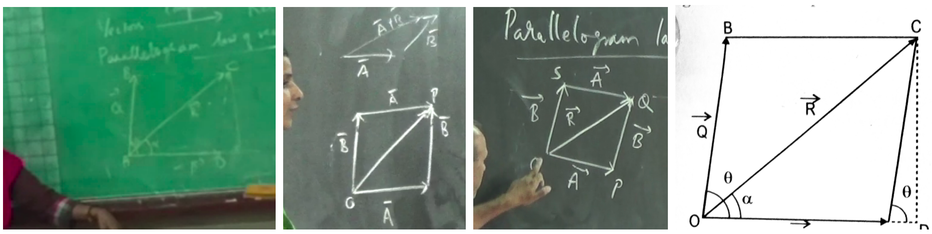


Figure 3A.4: Multiple teachers making parallelograms very similar to the one given in their textbook (right) with almost 90°angle while teaching parallelogram law of vector addition (Karnam et al., 2019)

The limitation of the media in allowing geometric manipulations requires teachers to 'act out' the dynamics of formal structures such as vectors. Teachers are highly constrained in triggering in students the imagination of the dynamics of geometric aspects of vectors, as this can be done only through gestures, words and enaction (Chauhan et al., 2019). Given these constraints, students’ actions (geometric manipulations), and the resulting actions in the imagination, are highly restricted. These restrictions could reflect in their reasoning habits, as observed in their behaviour (also see ‘habits of mind’ used by mathematicians (Cuoco et al., 1996)). Related to this point, Atiyah (2002), a celebrated mathematician, notes that when we start doing algebraic manipulations, we have a tendency to “stop thinking geometrically and about the meaning”. Given students’ lack of experience in imagining geometric processes, the role of static media in constraining cognition may actually be wider, adversely affecting not only student learning and understanding of vectors (among other geometry-related topics) but also their wider visuospatial abilities and reasoning (2002). These broadly explain the plight of students like Samantha struggling to meaningfully attempt problems.

Now, the introduction of a different medium allows new kinds of actions in both teachers and students and new experiences for the students with the representations, through carefully planned tasks or lessons. This is evident from the qualitative and quantitative findings in both studies 3 and 4. Changed teaching and learning practices in the classrooms (See right blocks in figure 3A2) indicate among the students who were only reasoning using algebraic forms initially, a change in this CRB towards using geometry to imagine vectors, was observed when they started using the Touchy-Feely Vector system as seen in the studies 3 and 4 (Karnam et al., 2019, 2018). This suggests new digital media can orient students towards geometric reasoning, and the earlier reasoning pattern is based partly on the textbook media’s treatment of vectors.

Based on these patterns, as well as the potential role played by representational media in shaping cognition, elaborated by the SciARM framework, it is reasonable to conjecture that the limitations of the static paper-based medium could be one of the root causes of student difficulties with vectors, and novel interactive affordances of computational medium, when designed and implemented through careful considerations of larger contexts, is a potential intervention approach to address these student difficulties. The compensatory nature of the thesis’s stance does not undermine the importance of algebraic reasoning or the advantages of the paper-based medium in supporting this reasoning pattern. And particularly with the topic of vectors, the thesis augmented paper-based medium with interactive affordances of computational medium and just stresses on the already acknowledged need for beginner students to start with geometric aspects (Dorier & Sierpinska, 2002; Roberts et al., 2008), as it helps trigger imagination processes based on formal models. Similar to these findings, weaker students are reported to be benefited by using enactive methods with representational modes (Armstrong, 1972).

The thesis, in the process, further meets the first three of the four needs identified from the literature review in ch-1A completely (need for medium+method approach, need to shift from the notion of mode of delivery, need for a changed notion of knowing, and need to pay attention to processes of learning). The thesis accounts for both method and medium as discussed in the previous paragraphs, the very framework demands to look at medium beyond just as a mode of delivery, and is in alignment with the changing notions of knowing as informed by 4E models. And in terms of attending to the processes in learning, we have a limited analysis of the interactions, but this gets compensated for to some extent by the grounding of the design considering the adaptability issues and documenting the changed teaching-learning practices with the changed-medium.

4 Limitations

Though there is enough promise in the evidence from multiple qualitative and quantitative data points there are some limitations in the studies as well as the design which one needs to pay attention to when generalising the conclusions of this thesis.

- The surprising finding that algebra contributed to the growth in geometry-algebra integration (study-4), as well as the conceptual disruptions shown by the pilot study (study-3), indicate that learning of vectors can occur through many complex trajectories, and we need to attend to these processes. Our study has not explored these underlying processes, but only the overall cognitive effect of representational media on students' cognition, brought about by new sensorimotor interactions. Establishing the underlying mechanisms that lead up to this change requires controlled laboratory experiments, examining cognitive processes. The factors involved in these changes are not just the interactive design features, but a whole set of factors triggered by the novel affordances of TFV. Technology features are thus one component of a holistic learning experience, which is shaped by social interactions, including pedagogy, as discussed in Karnam et al. (2019).

- The claims of the study are based on a pluralistic methodology, combining content analysis, qualitative analysis, design and quantitative analysis. The study's results are thus not fully evaluable from standard frameworks based on quantitative analysis. We consider this a strength of the study, as recent developments in educational technology have also led to shifts in study methods, where researchers combine different approaches (methodological pluralism) and develop thick descriptions to develop insight into technology-mediated cognitive processes. There is thus a shift away from the exclusive focus on the statistical significance of effects (Hew & Brush, 2007; Papert, 1988; Reiser, 1994; Ullmer, 1994). The coherence of our quantitative data (with larger patterns), particularly findings of the textbook analysis, students' CRB, the design and design-based changes in CRB, as well as the case studies from the pilot study (Karnam et al., 2018), provide wide-ranging support for our claims, across methods. This convergence compensates for the limited statistical significance of some findings in the field study.
- *Evidence confined to the topic of vectors:* The evidence gathered in this thesis is in the case of the topic of vectors. Though we have provided justifications on how this topic is suitable for the required investigation, there is a limitation in the exploration of the modelling aspects in this thesis; usage of the vectors as a representational system to model phenomena just in Samantha's case was not explored. Though we had some related questions in the tests used, these are not analysed in detail, and the design of the system was focused more on providing novel interactions with the geometric representations of vectors but not extending it to the model an actual physical phenomenon. Though this does not seriously affect the strength of the claims made by the thesis, as it is now, it would have been more elegant if the modelling aspect could also be illustrated. We hope to explore these in the future extensions of this investigation.

- Though TFV provides some meaningful context to learning of vectors, its scope is limited to enriching experiences with vector abstractions within classrooms. TFV does not provide contexts of physical applications of vectors, say to model forces etc. It is confined to creating a world of vectors, where the tangible objects (the arrows and others on screen) obey vector laws, and hence through interaction, concrete ways of learning abstract mathematical ideas become possible. It is also confined to triggering imagination and providing meaning through islands of mathematical experiences created in classrooms using the interface. As this is still disconnected with the real-world — e.g. there are no connections to the action one performs on TFV with any relation or effect in the real world like forces, etc., — these experiences are still limited. The real-world contextualisation needs to be and could be done by the teachers. Once there is such a learning experience of vectors, by which an imagination is triggered in the learners, and the students gain epistemic access to the abstract modelling capabilities, the teachers can use this structure, and extend these to physical contexts. There are complex pedagogical and learning issues in making this connection — where learners use the mathematical tools (like vectors) to model the physical world — as well, which is beyond the scope of the thesis, but need to be examined in future. TFV could definitely play some role in this process.
- In the TFV design, the technological aspects and the affordances for geometrical manipulability are entangled. The emphasis of the thesis is not so much the digital medium as its interactive affordances. Any medium which allows manipulation of geometric representations could have this effect. And the claimed cause of the effect is not the technological affordance, but the geometric affordances of TFV. But it was not possible to both propose this design and disentangle this subtle point in the thesis.
- The reach of TFV, and other similar digital media systems, could be limited in rural and remote parts of developing countries like India, which may not always have physical-access to computing devices. Textbooks are still important artefacts for a majority. However, given the trend of increasing smartphone usage in India, we hope

systems like TFV-2, embedded in textbooks, could foster epistemic-access in motivated children, even if one person/ teacher has a smartphone, until universal and equitable physical-access to open and free computational technology is available[22]. The findings of the study thus generalise in principle to most contexts where static-media use is dominant (Karnam, Mashood, et al., 2020; Karnam & Sule, 2018).

- *Practices focus but are sticking to the curriculum and specific content:* Though the thesis adopts the position that science curriculum should reflect science practice, it examines a specific topic in the curriculum, and the study is thus situated within the traditional content-based paradigm. This is a deliberate choice, based on the acknowledgement that the educational system, particularly in India and other developing nations, is still in a transition from the content-based paradigm. Nevertheless, to focus on practices within this limited framework, our study paid more attention to overall changes in practices with MERs, like reasoning using algebraic and geometric representations, instead of just specific changes in conceptual understanding of, say, resolution, addition etc.

[22] Further, we are fully for an open and inclusive access to all technology (including hardware). Society should be allowed to tinker with software as well as the hardware (as representational media) to undergo a social transformation or a revolution, akin to what open access to technologies such as writing and printing did in previous eras (renaissance, flourishing of modern science and technology, and even emergence of democratic values; (e.g. McLuhan 1964; Postman 2006; Ong 2013)).

3B

Implications

The objective of the chapter: To highlight the relevance and draw key implications of the thesis to various stakeholders of the educational system and research and propose potential future pursuits

Relevance:

- Changing paradigms of cognitive theory
- Changing STEM practice and its characterization
- Policy push for technology adoption in education

Implications:

- Practitioners
    - STEM Education (practice)
    - Design
- Researchers
    - Education technology
    - STEM education (Research)
    - HCI and Cognitive Science
    - Philosophy of science and Society
- Policy

The thesis has a lot of relevance to some contemporary circumstances and broad implications to many stakeholders of the educational system, as it could provide insights into the role of technology and medium in the process of learning. Though there are limitations as elaborated in the last chapter, the thesis, we believe, clears the ground and provides good starting points to trigger deeper investigations in future. This chapter outlines our reflections and implications for each of the stakeholders inviting them for more explorations.

1 The relevance of the research

1.1 Changing affordances of computational media

Unlike traditional models of cognition (Boden, 1988; Newell & Simon, 1976), 4E models require paying attention to the affordances of the artefacts we interact with. The advent of digital technology representational media has been changing this interaction for the last few decades. There is a trend towards increasing access to such technologies as well, thanks to the internet and mobile technologies, especially in countries like India (Young, 2016). Some of the already popular technologies include embodied controllers, augmented reality and virtual reality, which are mostly used in entertainment like gaming etc. These transitions are not merely new technological advancements, but they alter the fundamental modes of interaction between the human body, the world, physical artefacts and the symbolic world with MERs. The boundaries of body, world and knowledge are increasingly blurred through these affordances of novel interfaces. These are of course in addition to the rising computational capabilities of advanced technologies like machine learning and AI, and progress made in the fields like robotics and the internet of things.

Given these novel experiences with representations, as well as the new 4E cognitive models to understand and advance these experiences, we might be at the cusp of an epistemological revolution — a revolution of what and how humankind knows — given the fundamental changes to the affordances of the objects in the environment we interact with. Some of these trends are already identified and reflected on by cognitive scientists like Clark (2001), in the context of changes that ICT and mobile technologies brought to our lives, by erasing the constraints of space and time in interacting with information. But the changes that we are now walking into could be even more fundamental. And in this context, the thesis could be seen as highly urgent and relevant in carefully understanding the nature of the effects these representational media have in our lives as they reshape our cognitive capacities and the terrains of cognitive explorations, particularly for the future of knowledge, STEM learning and education.

1.2 *Consequent changes in the terrain of STEM*

Computational models and simulation models are increasingly becoming a norm in contemporary STEM. Unlike the conventional models, computational models are developed using data fitting, or machine learning kind of mechanisms, and the nature of cognitive roles of the STEM practitioners is also fast changing (Chandrasekharan & Nersessian, 2015, 2017; Dowek, 2015; Humphreys, 2002). Consistent with the SCIARM framework, the usage of the computational medium as a representational medium provides entirely new kinds of affordances, and hence the emergence of new kinds of models and new modes of reasoning and STEM cognition (e.g. Winsberg, 2010); for example, computer simulations could merge experimenting and modelling (Lenhard, 2007). Testifying the criticality of this issue, many philosophers and practitioners have been debating about the legitimacy of these new forms of knowledge and the changing epistemic values (e.g. Pylyshyn, 1978): for example, debates on the legitimacy of computational proofs among mathematicians (e.g. four-color theorem Gonthier, 2008). These transitions in the STEM practices are living examples of the defining role representational media play in shaping STEM cognition to an extent or redefining what is STEM.

These reflect in the gaining momentum of computational thinking among educational researchers as well. Systematic educational interventions to support the shift to the cognitive augmentation ('external imagination') role of digital media are still in exploratory stages (Lakshmi et al., 2016; Majumdar et al., 2014; Shayan et al., 2015; Sinclair & de Freitas, 2014). Aspects of this shift reflected in the efforts to promote computational thinking (CT) under three broad overlapping domains:

1) Learning programming and software concepts (Kazimoglu et al., 2012; Repenning et al., 2014; Wang, 2011; Weiguo, 2015)

2) Learning existing material in the curricula, particularly in physics and mathematics, using computational simulations and related classroom activities (Araujo et al., 2008; De Jong et al., 2014; Perkins et al., 2006; Van Joolingen & De Jong, 2003; White, 1993)

3) Learning new scientific concepts generated by computational modeling (such as self-organization, emergence and non-linear causality), typically using agent-based models (Gu & Blackmore, 2015; Sengupta et al., 2013; Stieff & Wilensky, 2003; Wilensky, 1999).

We in this thesis, sensitive to the existing practices in the educational system, followed the second category of CT studies above, using the case of a particular modelling topic of vectors from the existing curriculum. With careful reflections on the underlying cognitive mechanisms and the larger context and scheme of things, we tried to show how the design of learning interfaces can become less alienating and adaptable into the existing system.

1.3 *Policy push for Technology adoption in the schools (including developing nations)*

Technology, like in other walks of life, can potentially revolutionise the way we access knowledge, and in turn, teach and learn within our educational systems. The acknowledgement of this potential of digital technology, to assist universalising education by widening access to learning (Unesco, 2000, p. 15), is reflected in the emphasis on technology adoption in educational policies across the world (Kozma, 2008; Kozma & Vota, 2014; Pelgrum & Law, 2003; Unesco, 2000, p. 21) including India (Bajwa, 2003; NCERT, 2005, p. 49,92). Traditional classrooms have been in conventional instructional modes for far too long (e.g. Kaput & Roschelle, 2013). This policy push could be a force multiplier, and it has gained newfound justification and traction by the Covid-19 pandemic. But there is a lurking danger that such a policy push could lead to a thoughtless and blind adoption of digital technology, especially given the limitations in our understanding of the role of media in learning (Kozma & Vota, 2014). This can prove costly for developing nations with scarce resources. This research, being carefully grounded in cognitive and social processes of learning, and embedded within the systemic constraints like in India, could provide a very illustrative case of new media-based design for learning, and develops useful insights to make informed choices and make smoother the ongoing technology integration in education.

2 Implications for Practitioners

2.1 STEM Education Practitioners: the teachers

- Teachers teaching vectors have some direct implications from the thesis. They can directly benefit from the analysis of the learning problem of vectors, and the use of the TFV system through the virtual lesson plans developed and customised as downloadable QR codes.
- The thesis also can illustrate and inspire more teachers with concrete models of engaging with designers and researchers through co-design processes creating virtual lesson plans. This collaboration is inevitable for the future-education, as teachers are the only stakeholders who can appreciate and bring out the nuances involved in pedagogy, when in practice.
- In general, the thesis could provide some productive insights to teachers about technological intervention and probably also some conceptual tools to talk, reflect and critically engage with various technological solutions.

2.2 Designers

The design has been at the heart of the entire execution of the thesis. And we at various stages had to work with designers and think and be like one. The thesis has some important insights for the designers interested in the problem of learning, and those interested in general in the human-computer interaction (HCI) space. A designer may view the thesis as a rigorously executed user study; and that is what, we believe, education research also requires to be, as also reflected by the emergence of numerous design-based research studies. But our elaborated inspirations from cognitive sciences, nature of scientific practices and the explicit attempt to ground in the contexts of the educational system could trigger new perspectives in a designer. We shall elaborate on some of them in some detail at two levels: allowing epistemic access, and system adaptability.

Embodied interactions for Epistemic access

The now widely-accepted theoretical model of embodied cognition, and the parallel development of embodied interaction systems, offer new and exciting avenues for designers of learning media. The interaction design illustrated in this thesis (esp of TFV-2) provides two key design constructs that can be built to develop such new media systems:

1. *concept-laden interactions:* TFV-1, like many other existing new media designs like PhET, Geogebra, relied a lot on the visualisations to address the conceptual issues (like using animations for resolution, opaque triangles and circles appearing to trigger certain conceptual aspects etc); the gestures were mostly a combination of clicks and key presses — so is the case with other learning systems which mostly use sliders and virtual knobs, or fields to key in text, which remain the same when operated using keyboard + mouse/ touchpad or a device with a touch interface. In TFV-2 we reflected more on the gestures used as well: the way the action performed and the visualisations perceived could be conceptually linked. We tried to find ways of reducing the semantic distance between these two — the actions and the visualisations representing the concepts — to be able to enhance the epistemic access and trigger the imagination in a grounded manner. As conceptual aspects were already well-considered in designing the visualisations, we designed actions that are less arbitrary and well connected to the conceptual aspects. We tried to overlap the embodied meanings of the actions as much as possible with the conceptual meaning so that students can directly latch on to something to start their imagination and progressively refine it further. These form conceptually-grounded action-perception (or gesture visualisation) pairs as conceptually meaningful interactions, which we call *concept-laden interactions*. For example, we used the pinching away gesture to resolve a vector; pinching away action on a long object resembles splitting it like a stick, and has an embodied meaning of splitting; this overlaps with the conceptual meaning of

the resolution as splitting a vector into its rectangular components; this along with suitable visual cues on the screen makes it a conceptually-meaningful interaction. This is something we could use to enhance the tangible experience of the representations and simultaneously enhance the access to the abstract meaning.

2. *Tangible concepts*: Another interesting feature that we would want to discuss within the epistemic access section is the use of the circle (a Unit circle). Across the interface in both versions, we used a circle along with the arrowhead of the vector. It proved a very useful feature both from conceptual aspects as well as interactional aspects. Conceptually, a circle is central to the inter-relation between the geometric and algebraic representations; note that geometry-algebra integration is a key part of the learning problem with vectors. And in terms of the interactions, the circle helped a lot in binding the gestures as well as the visualisations when manipulating the vectors' magnitude and direction; in fact, TFV-1 used the circle itself as the object on which action is performed (to change the magnitude, you had to click anywhere inside the circle). Thus, in the TFV system, we see the unit circle was like a central machine binding both conceptual and interaction aspects in a meaningful manner. Such entities with dual ontological nature (affording concrete interactions as well as abstract conceptualisations) could be a very useful feature in any learning design to enhance epistemic access for beginners. We call these as *tangible-concepts*.

There are interesting insights related to the kind of gestures used, and the visualisations produced, and the conceptual significance of these two. We would want to have a disclaimer that this design was made for providing epistemic access or trigger imagination to a beginner, and may not prove very useful for someone who already has a sound understanding of the topic.

Adaptability (Physical access + Practice)

This aspect of the design is at the heart of the second research question, which tries to illustrate a design approach implemented in developing nation context. Addressing this would involve certain aspects about the nature of the research used in this thesis, which we shall discuss at length in the implication to educational technology researchers. Here, we shall confine to some of the reflections purely from the design point of view. Also, we want to remind the reader that only after the pilot run of the first iteration we have designed for wider adaptability; so all the below discussions in connection to TFV-2. Designing something for wider adaptability requires a deeper understanding of the existing state of affairs, a glimpse of which is available from studies 1 and 2.

- *Choosing technology:* As we designed this on a javascript, this can be run on any simple browser as an HTML file. this is a significant enhancer of physical access because of the following reasons:
    - this is platform-independent — can operate on any touch interface (smartphone or tablet or touch laptop) of any operating system (android, windows or iOS);
    - This is supported on a smartphone; this is particularly helpful in developing nation context, where a large number of users are embracing smartphones outpacing the users of computers; this can reasonably address the infrastructural and maintenance constraints with a laptop or a computer
    - This does not require any additional installation; this is an HTML file, and can be directly accessed on a web browser, and does not require installation of a separate application.

  These are to ensure generic accessibility of the technology among the users and could have some specific lessons for designers to choose the suitable technologies.
- *Replace vs augment towards a compensatory design approach:* The next feature is related to the existing infrastructure, and a design approach to augment and not replace the existing infrastructure. This is particularly relevant to the developing nation contexts from an economical perspective as well. Textbooks have been a very effective and affordable artefact in the system for many years. A strong grounding in the cognitive and social processes (as indicated by the literature review in ch-1B) through frameworks like SCIARM, allows informed choices with acknowledgement of merits of the existing system, and avoid fanciful and blanket replacement of existing artefacts. With TFV, we can see that the design carefully

chooses to compensate for only those limitations of the existing medium and later uses QR codes to augment them. This way, users who are already using a particular artefact do not need to adapt to a new artefact with the existing base demolished but can smoothly assimilate the new artefact, augmenting the existing one.

The above points of technical ease and latching on the already pervasive textbooks could ensure wider physical access to TFV.

- *Teacher Practice:* Another aspect of adoption is related to the important stakeholders in the educational system - the teachers. In our design, we involved the teachers in developing virtual lesson plans (involving a series of tasks with TFV-2 embedded in their already existing lesson plans) which were embedded in the textbooks using QR codes. This is a very important aspect that literature also suggests addressing technology adoption by the educational system (Tsai & Chai, 2012). This codesigning could have enabled the teachers, who were initially hesitant to use it in their lessons, to implement it in their classroom teaching. Further, the involvement of teachers also especially in the designing of the computational media allows in creating more meaningful affordances and tasks tapping into their pedagogical content knowledge (PCK) (Van Driel et al., 1998) and also technological PCK (TPACK) (e.g. Archambault & Barnett, 2010).

Together, we reflected on and discussed some broad implications for designers who are interested in developing technological interfaces for better learning and adaptability. Also, this gives enough promise to focus on the interactive affordances of the medium as a potential interventional approach for designers of learning technologies. Besides these insights, generic designers would also be interested in the discussion related to Human-computer interaction and Cognitive science in the later sections.

3 Implications to Researchers

3.1 STEM Educators

Education researchers are forebearers of any new change in the educational system. And they need to proactively engage with important stakeholders like teachers in facilitating the shift. They need to be the channels communicating and translating the shifts seen in the cognitive models the STEM practices, into STEM education. For example, we see that there is a shift in STEM practices towards computational models, and education researchers must be at the forefront in engaging with both philosophers, practitioners and educators of STEM.

- Firstly, the thesis provides enough evidence to pay careful attention to the bodily interactions students have, and specifically the media-usage to represent abstractions using MERs in the classrooms. In more short or medium terms, the thesis can have more implications specifically to math and physics education researchers to interpret student learning problems. This illustrates the application of novel cognitive paradigms into the interpretation of the learning problems, beyond the conventional description of misconceptions. This provides sufficient justification to revisit the learning problems described across science and math education literature, paying special attention to the interactions with media, and also students' conceptual reasoning behaviour. In this context, this thesis can be broadly belong to the category of research studies which analysed the role of everyday experiences in students understanding of physics (e.g. Reiner et al., 2000) and the process of conceptual change (e.g. Samarapungavan et al., 1996).
- This thesis, being grounded in the '*practices*' image of science[23], carries this baton as acknowledged by standards in NGSS further. Also, by focussing on the reasoning approaches visible in the students' utterances, interpreted as the usage of MERs, we consciously nudged away from the conventional conceptual-change kind of framing of learning. These have implications for the learning objectives and the role of content in the process. More careful thought needs to be put in clarifying these and this requires a collaborative effort by educators, education researchers, philosophers of science and education and other relevant stakeholders. This is more of a longer-term pursuit and educational researchers need to play the role of framing and binding these collaborative pursuits. Further, it is needed to identify the key practices defining STEM (e.g. Lehrer & Schauble, 2015) and

[23] For images of science see (Lehrer & Schauble, 2015)

proactively research and develop novel pedagogical paradigms to facilitate the development of such practices.

- This thesis also has some insights for methodology and analysis. One can draw some insights from the methodological aspects of the thesis, to trigger pursuits towards a practices-focussed STEM. Also, the thesis employs multiple methods for various requirements in the studies and also involves iterations of designing and testing learning media. Thus this illustrates a mixed methodological study, which was recommended to be a productive method especially given the limitations of traditional methods in developing insights in the application of media in learning contexts (Reiser & Dempsey, 2012) and triggering new disciplines like learning sciences. Further in terms of the analysis, again this has introduced novel approaches like the concept link strength analysis (similar to the epistemic network analysis (Csanadi et al., 2018)), the rubrics for representational usage, and rich visualisations and statistical analysis using KDEs (which are not widely used in education research).

3.2 Ed Tech and Media

The thesis is broadly situated in the debates around the role of media in learning, as conceptualised in chapter-1A. We shall see in this section, the implications the findings of the thesis have, specifically to those debates and generally for researchers interested in media for learning and educational technology.

Grounded exploration of interactive affordances

The thesis, as summarized and discussed in the previous chapter (Ch-3A), corroborates the corollary by providing evidence on the interactive affordances of the representational media playing a role in shaping students' STEM cognition (operationalised as CRB). Further, literature broadly indicates technocentricism in the research around the media in education. And it is recommended to ground explorations in cognitive and social processes of learning, informed by relevant theories. In this context, the investigation carried out by us in the thesis is an exploration strongly grounded in recent cognitive theories and informed by larger discussion around media and cognition. This was facilitated by the development of the

SCIARM framework. Consequently, our research questions are not technocentric. All through the thesis, from the operationalisation, the methodologies, the analysis of the data and the interpretation of results, we were not particularly bound to the efficacy of a *particular* intervention design towards understanding a *particular* concept, but the efficacy of a feature of any representational medium on a particular behaviour (usage of MERs).

We analysed the existing media in the classrooms for its interactive affordances for students and teachers. We examined students' utterances (CRB) to capture patterns in their reasoning approaches and the possible relationship between the interactive affordances of the representational medium and their reasoning approaches. We then carefully compensated for only those limitations, and tested through two iterations: one with a small sample with deeper analysis and thicker description, and the other with a larger sample getting into nuances of the impact. In all the analysis, we were less interested in the conceptual understanding of the particular topic of vectors than in the larger usage of MERs with a close connection with the affordances of the representational medium. The learning problem of vectors was also operationalised more in terms of MER usage rather than a conceptual understanding of specific concepts. The specificity of the thesis is about the cognitive role of a particular affordance (of interaction) of the representational medium.

Compensating, not replacing paper-based medium: Mixed media.

The thesis adopts a wider perspective towards technology embracing even non-digital technologies under the umbrella; the review of media studies and studies by those like Vygotsky, allow us to start this trajectory of technology from any simple physical tool, and speech as the first representational medium, slowly transitioning to media affording writing, printing, and now typing and a lot more. This widened perspective towards technology provides a realistic, non-inflated and grounded viewpoint towards digital technology, and opens up realistic possibilities of framing mixed media futures (similar to the coexisting media of speech and writing) rather than an overestimated digital technology push.

In relation, the thesis highlights certain limitations of the paper-based medium to allow manipulations with geometric entities and hence possible limitations in triggering imagination and reasoning abilities. However, positions such as Harrell (2008), argue that pencil-paper, combined with strategies like argument mapping, is sufficient for developing such critical thinking. This claim is true at one level, as many students have learned vectors using pencil and paper, and gone on to become successful STEM practitioners; this is also consistent with our earlier admission that a good method without all interactive affordances also could lead to better learning outcomes. But to widen the epistemic access to a lot of students, careful design and integration of digital media, as illustrated by the thesis is very important. TFV, as evident in study-4, enhances epistemic access, as it specifically compensates and integrates with paper-based medium, with a central focus on enriching the interactions with the MERs. Eventually when focussing on learning, any technology, either paper-based, digital or any other, which rides on the fundamentally embodied learning faculties of the learner, as a cognitive being, could be employed.

Seeding vs replacing imagination: cognitive augmentation

Another aspect that is widely raised in relation to the digital medium usage is as follows. It could be argued that the student effort involved with the paper-based medium is higher (compared to TFV), and this effort is a good feature provided by the text media, particularly when supported by teacher guidance, as this effort could possibly lead to better imagination abilities. Given this view, it is possible to argue for the following design principle: making all possible manipulations available through digital media is not good for learning, as this feature may lead to students not imagining, and becoming dependent on the system, using it in ways similar to a calculator. Our interviews and pre-test results question this design rationale, as without manipulation it was difficult for students to start, or even identify, the imagination process that is needed. Taking the above critique into account, the challenge is to develop a compensatory system that just seeds (but not offload) the imagination, particularly in students who find it difficult to imagine the geometric operations, when using just paper media.

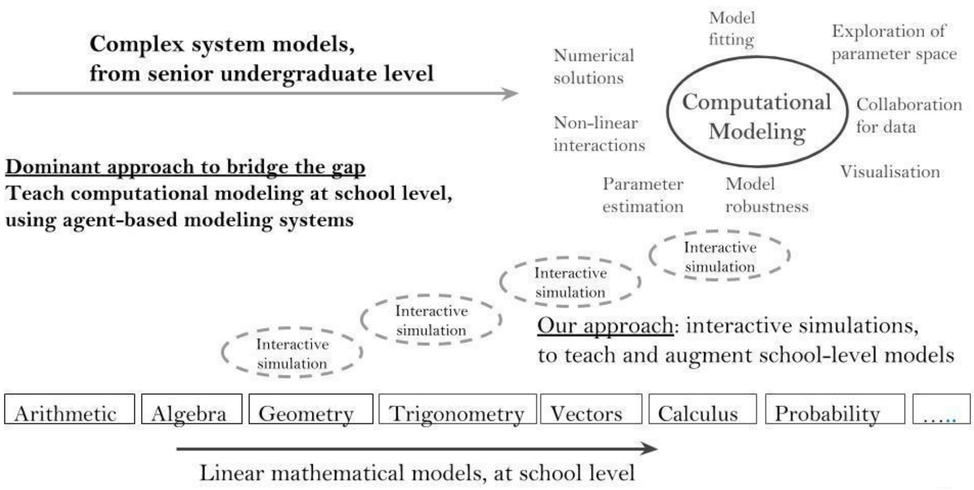


Figure 3B.1: Contrasting our approach, of interactive simulations grounded in existing curricula (linear mathematical model at school) gradually transitioning towards computational modeling (complex system models at senior UG level), with dominant approaches to promote computational modeling like agent-based modeling systems

However, in the larger scheme of affairs (viewing beyond the current curricular requirements), a system that only allows such seeding may not be ideal as well, as systems that allow many manipulations could be used to address more complex problems (say, involving many vectors), compared to what textbook-based reasoning can support. Such complex problems, based on interactive simulations, could also pave the way towards a better understanding of computational modelling and its practice, and computational thinking in general (fig-3B.1), which changing STEM terrain is demanding as mentioned in the earlier sections on the relevance of the research.

This cognitive analysis, examining the connection between media and imagination, shows that designing educational technology requires managing a complex cognitive trade-off – between Seeding vs. Replacing imagination on the one hand, and Computational vs. Derivational (or analytical) approaches to problem-solving on the other. This work provides preliminary evidence for media triggering imagination reflected in students' CRB; and illustrates the way, careful digital media designs for cognitive augmentation can help to shift students' reasoning toward computational thinking, being grounded in existing curricula.

Conclusion

The current narrative in the EdTech industry appears to favour a radical restructuring of the entire educational ecosystem, replacing existing practices with purely digital technology and its practices. Though such a shift may be needed eventually, particularly to support abilities like computational thinking, the thesis illustrates how a smoother, rather than disruptive, transition of the educational system towards such approaches is possible. This approach requires educational technology researchers to develop deeper and more meaningful engagement with the stakeholders in the educational system, and develop theoretically grounded designs and studies that examine the cognitive augmentation potential of different media. We discuss below some detailed implications for educational technology practitioners and researchers.

- Not every learning problem needs digital technology and digital learning tools may not be needed for each and every topic. Some learning objectives, such as the understanding of the area-topic, can be better addressed perhaps using a combination of various representational media and physical artefacts (Rahaman et al., 2017).
- Every medium has its place and relevance, and careful examination and weighing of the affordances of different media is needed before advocating the adoption of particular technologies. Critically, this analysis requires understanding the cognitive facilitation each media could provide.
- Laboratory explorations of prototype systems can be technocentric, but such prototypes need to be systematically integrated with the existing practice for applications of technology to social enterprises such as education to succeed. This requires shifting to a systems approach to design once the prototype stabilises, to ground the technology in larger social processes related to education, starting from classrooms.

3.3 HCI and Cognitive Science

Given the broader grounding in the cognitive theories, our thesis can also have some important implications for the 4E models themselves. 4E models are gaining wider acceptance due to their amenability of being biologically and evolutionarily grounded (due to which for e.g. we do not require different explanatory paradigms for human cognition and animal cognition); however, evidence for abstract cognition like in STEM is emerging only recently, most of which are reviewed in ch-1A towards developing SCIARM. In such a context, the affirmative evidence, though weak from a purely cognitive science point of view, provides a good trigger for future investigations. We outline the nature of some of those future explorations, given the evidence from the thesis.

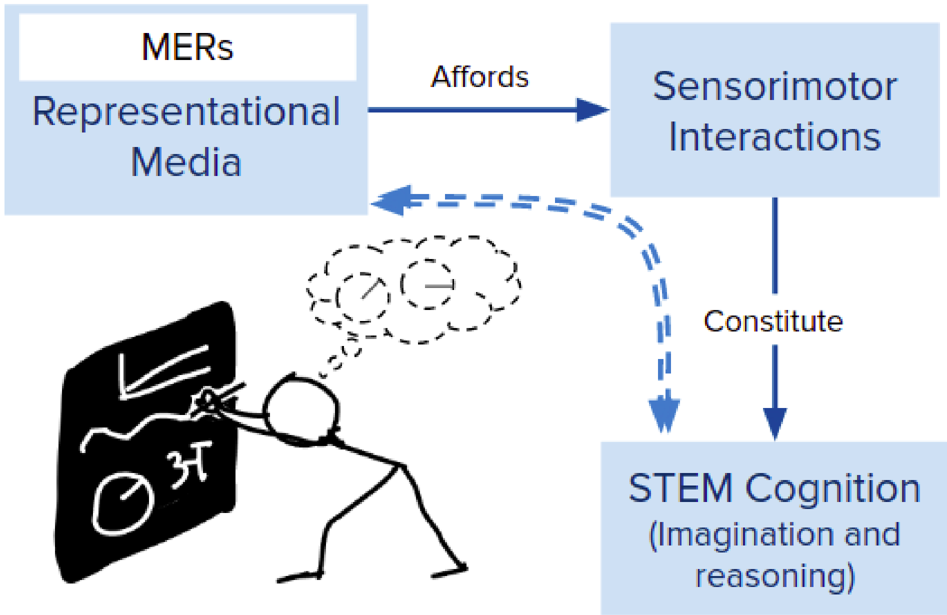


Figure 3B2. Horizontal arm relating technology and interactions, vertical arm relating interactions and cognition

The thesis provides promising evidence of affordances of objects of interaction shaping cognition in epistemic domains like STEM (at the knowledge level activities). This interrelation is a triad of artefacts (including technology), interactions (afforded by them) and cognition (constituted by these interactions). So, keeping the focus on each arm of the two relations leads to two directions of future investigations.

- *Human-computer interaction:* The first arm (figure 3B.2) is the relation between the artefacts and the interaction afforded by the body-environment (here body-artefact system). There is a lot of interest among the designers and technology-enthusiasts in this space of research. Our research at a conceptualisation level could trigger developing useful frameworks grounded in cognitive theories, in conducting explorations grounded in HCI explorations especially related to human abstract activities. These could provide new ways of framing and interpreting the already existing results and framing future explorations in the field. For example, the explorations of technologies like VR and AR could benefit from the insights in the thesis.

- *Cognitive Science:* The second arm (figure 3B.2) is the relation between the interactions and cognition. The evidence in the thesis, by strengthening the SCIARM framework can in turn strengthen the constitutivity hypothesis, especially in abstract human activities like STEM. Such explorations in the domains dealing with abstractions like STEM are pretty recent and require more concerted research. Primarily being a STEM education thesis, our claims, without rigorous and controlled experimentation, are weak with respect to the standards of cognitive science research. However, this can help in triggering future pursuits with carefully controlled experiments studying this constitutive relation between interactions and the imagination; such efforts require multi-disciplinary teams and are slowly emerging in research communities like math education. Eventually, these can provide stronger evidence for the 4E models of human cognition. Overall, our thesis can be seen to start with cognitive theories and at the end contribute back to the discipline.

4 Implications to Policy

- Educational Technology policy in developing countries currently lacks a clear direction on the way to promote computational thinking, while also simultaneously addressing access issues and technology adoption in classrooms. This thesis illustrates a new approach to address this complex problem (figure 3B1), particularly in developing countries. This model could be extended to develop new ways to address other such complex policy problems, such as the integration of data science and sustainability into the curriculum.

- It is well-known that technology systems change organisational structures and institutions (Shah, 2020). It is currently unclear how embodied digital media technologies will change institutional structures, and how such structures need to adapt to support new technologies. The changes in classroom practice enabled by TFV provides an initial guideline for developing policy frameworks to address this complex issue.

- Tech policy (widening access for more Samanthas and moderation/ regulation of usage): The policymakers advocating for the educational system to embrace technology must pay careful attention to the implications provided to the educational researchers as well. As the interface with practitioners, commercial interests as researchers, they need to moderate the adoption of technology by taking informed decisions, grounded with the best current understanding. They should encourage the exploration of mixed media applications as well. They should consider taking a holistic approach by seeking to augment/ complement but not replace the existing infrastructure, as illustrated by the thesis.
- Curricular policy (from learning science and practices shift): Another aspect related to the curricular policies is adapting quickly to the changing trends. The content-based focus of the instruction as well as assessments need to change towards practice-based approaches. The curricula could be built around a skeleton based on desired STEM practices, rather than just content. And there is the immense potential of digital technology in enabling this, using all its affordances as reviewed in the ch-1B. For example, learning analytics and AI-based approaches can be widely used in analysing the processes (of students' activities), flipped classrooms can be imagined where students' real-world experiences and other epistemological resources are meaningfully integrated into their activities.
- Other systematic changes (assessments, teacher research collaboration etc): Policy also needs to create platforms and nurture collaboration of teachers and researchers for socially-meaningful and practically implementable insights to emerge. This collaboration as illustrated in this research can be particularly productive for both teacher and researcher communities. The current TPD mode of interaction could be reorganised towards a more collaborative mode, through a series of programs of platforms for such interactions.

3C

Conclusion

The thesis broadly provided some insight into the reasoning patterns widely visible in STEM learners. It speculates based on the field theories of cognition, and the constitutive nature of human cognition, that this could be partly stemming from the limitations of the media we interact with. The thesis using detailed studies, provides evidence for the possibility of the students' learning and the STEM cognition be shaped by the representational media they interact with. Though the evidence does not establish a causal relation between the **representational medium** and the STEM **cognition**, given the contitutivity hypothesis and the SCIARM framework, it could be a reasonable starting point for a lot of future research.

Some broad connections

In relation to the role of representational media on cognition, there is a very interesting anthropological narrative emerging from the media studies, which is consistent with the findings of the thesis. These concern the abstractions that human cultures hold. Abstract cognitive aspects of our societies, like ideas and knowledge, have always been found to be moulded by the representations, which are in turn shaped by the affordances provided by the representational medium available. Various studies have traced the progress of human thought, values and social organisation as following the progress of the technologies to represent, and a very close, yet subtle, relation between them has become conspicuous.

The transitions from oral to written media present an interesting link to the changes in the oral to written cultures (Ong, 2013). For example, knowledge is memorised, as an oral medium (e.g. speech) can't leave an external trace unlike written text; people who memorise are considered storehouses of knowledge and hence are revered more, similar to the reverence shown to the written text in written cultures. In written cultures, abstract expressions became possible with the system of stringing concrete arbitrary symbols/ characters (alphabets) to encode a sound, which could mean something (Logan, 1986). These representations now etched in the representational medium became concrete objects of interaction; this concreteness becomes very clear from some early representational mediums like clay tablets and quipu knots[24]. The beginnings of science, geometry, logic and rational philosophy — the age of reason and rationality — became possible due to "analytical management of knowledge" (in Ong's words) afforded by the written text (McLuhan & Logan, 1977; Postman, 2006). And more compact syntax of manipulation of formalisms emerged with algebra around the time of the invention of the printing press (Kaput & Roschelle, 2013), widening the access to knowledge. Access to these representational media to wider populations is closely tied with the shaping of the politics (e.g. Innis, 2008; McLuhan, 1964; Postman, 2006) and economics (e.g. Swetz, 1987) of the era, through wider access to knowledge and ideas. Similar acknowledgements can be found by endowing material status to signs, simulations and code (Baudrillard, 2016) shaping the economics and politics in the modern era.

All the above studies in the space of media, society and culture hint at the fundamental role the representational media (as technologies to represent) had in shaping the abstractions of the societies of the era, which in turn reflected in the values and the way we organised ourselves in the past. These thus give some evidence from the past of a possible relation between the representational media and cognition. Here in conclusion, we would want to position the thesis as a small addition to these larger sets of reflections on human cultural evolution.

---

[24]For a fascinating account of early representational mediums like Sumerian clay tablets to quipu knots (knots on coloured cords) of Incas, see Harari (2014, pp. 138–142); these mediums were primarily used to record trade transactions and other bureaucratic exchanges.

Conclusion

Let us come back to the question, I was wondering about in the preface, Why does a system which can learn to ride a bike or even learn language robustly, fail to learn to solve the calculus or vector algebraic problem?

With the above evidence it could fundamentally reframe the issue of the learning problem of millions of students like Samantha, as a design problem. The problem is with the design of our educational system, all the way down the concrete infrastructure we occupy (as Foucoult would love to see) and we interact with (as Gibson, and the entire embodied cognition camp would love to).

As an afterthought, it makes absolute sense that we as biological beings, who have evolved to live interacting with the dynamic world in real-time, face problems with engaging and making meaning of static entities (representations as concrete entities are static in paper-based medium). As the dominant medium used does not afford smooth and dynamic interactions, our fundamental learning faculties are not put to use in the schools. A completely different training of the cognitive system is needed to be able to pay attention to and engage with the static representations. Some of us, who lend ourselves or succumb to this training, who can overcome their natural biological learning faculties to engage with real-time dynamic worlds, end up performing well in the formal education system. This perspective leads to radically different perspectives to behaviour like rote-learning that is very rampant in school education. Of course, this is a provocative stance, but the thesis firmly grounded in the evidence aims to provoke this line of thinking among a lot of us researchers and educators, to radically reimagine how we upbring or enculture our off-springs.

From oral to written and now to digital media, humanity has gone through many transitions in the way knowledge is represented. In the transition from oral to written cultures, human civilisation witnessed tectonic shifts, in knowledge systems and their resulting social orders. We are currently witnessing how digital technology is reshaping our world -- the way we travel, transact, and do science -- at a breathtaking pace. STEM education is a crucial cultural activity, which will also change soon, dramatically, with the introduction of digital media. This thesis provides initial directions towards both initiating and managing this critical and drastic change. It also could help us understand the cognitive realities of millions of children like Samantha and advance their learning using novel perspectives towards representational media. Meaningful collaboration between stakeholders like teachers, an instance of which is illustrated by this thesis work, is the way ahead to reach, and manage, this radically different future.

BIBLIOGRAPHY

Abrahamson, D., & Sánchez-García, R. (2016). Learning Is Moving in New Ways: The Ecological Dynamics of Mathematics Education. *Journal of the Learning Sciences*, *25*(2), 203–239.

Aguirre, J. M. (1988a). Student preconceptions about vector kinematics. *Physics Teacher*, *26*(4), 212–216.

Aguirre, J. M. (1988b). Students' preconceptions about the independent characteristics of orthogonal component velocities. *AIP Conference Proceedings*, *173*, 235–240.

Aguirre, J. M., & Erickson, G. (1984). Students' conceptions about the vector characteristics of three physics concepts. *Journal of Research in Science Teaching*, *21*(5), 439–457.

Aguirre, J. M., & Rankin, G. (1989). College Students' Conceptions about Vector Kinematics. *Physics Education*, *24*(5), 290–294.

Ainsworth, S. (2008). The educational value of multiple-representations when learning complex scientific concepts. In *Visualization: Theory and practice in science education* (pp. 191–208). Springer.

Alba Fernández, V., Gamero, J., & Muñoz García, J. (2008). A test for the two-sample problem based on empirical characteristic functions. *Computational Statistics & Data Analysis*, *52*(7), 3730–3748.

Anderson, N. H., Hall, P., & Titterington, D. M. (1994). Two-Sample Test Statistics for Measuring Discrepancies Between Two Multivariate Probability Density Functions Using Kernel-Based Density Estimates. *Journal of Multivariate Analysis*, *50*(1), 41–54.

Andres, M., Di Luca, S., & Pesenti, M. (2008). Finger counting: The missing tool? *The Behavioral and Brain Sciences*, *31*(6), 642–643.

Appleton, J. J., Christenson, S. L., & Furlong, M. J. (2008). Student engagement with school: Critical conceptual and methodological issues of the construct. *Psychology in the Schools*, *45*(5), 369–386.

Appleton, J. J., Christenson, S. L., Kim, D., & Reschly, A. L. (2006). Measuring cognitive and psychological engagement: Validation of the Student Engagement Instrument. *Journal of School Psychology*, *44*(5), 427–445.

Araujo, I. S., Veit, E. A., & Moreira, M. A. (2008). Physics students' performance using computational modelling activities to improve kinematics graphs interpretation. *Computers & Education*, *50*(4), 1128–1140.

Archambault, L. M., & Barnett, J. H. (2010). Revisiting technological pedagogical content knowledge: Exploring the TPACK framework. *Computers & Education*, *55*(4), 1656–1662.

Armstrong, J. R. (1972). Representational Modes as They Interact with Cognitive Development and Mathematical Concept Acquisition of the Retarded to Promote New Mathematical Learning. *Journal for Research in Mathematics Education*, *3*(1), 43–50.

Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practises in Cabri environments. *Zentralblatt Für Didaktik Der.* https://link.springer.com/article/10.1007/BF02655708

Atiyah, M. (2002). Mathematics in the 20th century. *Bulletin of the London Mathematical Society*, *34*(1), 1–15.

Bajwa, G. S. (2003). ICT policy in India in the era of liberalization: Its impact and consequences. *Global Built Environment Review*, *3*(2), 49–61.

Bakas, C., & Mikropoulos, T. (2003). Design of virtual environments for the comprehension of planetary phenomena based on students' ideas. *International Journal of Science Education*, *25*(8), 949–967.

Banerjee, A. V., Cole, S., Duflo, E., & Linden, L. (2007). Remedying Education: Evidence from Two Randomized Experiments in India. *The Quarterly Journal of Economics*, *122*(3), 1235–1264.
Barniol, P., & Zavala, G. (2014). Test of understanding of vectors: A reliable multiple-choice vector concept test. *Physical Review Special Topics - Physics Education Research*, *10*(1), 010121.
Baudrillard, J. (2016). *Symbolic Exchange and Death*. SAGE.
Bauman, R. P. (1992). Physics that textbook writers usually get wrong: III. Forces and vectors. *Physics Teacher*, *30*(7), 402–407.
Besterfield-Sacre, M., & Gerchak, J. (2004). Scoring concept maps: An integrated rubric for assessing engineering education. *Journal of Engineering Education*, *93*(2), 105–115.
Bhattacharya, S., Tiwari, K. K., Mohammad, T., Singh, D., Gautam, R. P., & Rani, S. (2018). *Educational Statistics at a Glance*. Department of school education & literacy, MHRD, GOI, New Delhi.
Bingimlas, K. A. (2009). Barriers to the successful integration of ICT in teaching and learning environments: A review of the literature. *Eurasia Journal of Mathematics, Science & Technology Education*, *5*(3), 235–245.
Boden, M. A. (1988). *Computer Models of Mind: Computational Approaches in Theoretical Psychology*. Cambridge University Press.
Bollen, L., van Kampen, P., Baily, C., Kelly, M., & De Cock, M. (2017). Student difficulties regarding symbolic and graphical representations of vector fields. *Physical Review Physics Education Research*, *13*(2), 020109.
Brzezinski, T. (2016, March 29). *Vector Operations*. Geogebra.org. https://www.geogebra.org/m/tKGYbk7J
Byers, P. (2010). Investigating Trigonometric Representations in the Transition to College Mathematics. *College Quarterly*, *13*(2).
Calder, N., Brown, T., Hanley, U., & Darby, S. (2006). Forming conjectures within a spreadsheet environment. *Mathematics Education Research Journal*, *18*(3), 100–116.
Carlson, D. (1993). Teaching Linear Algebra: Must the Fog Always Roll in? *The College Mathematics Journal*, *24*(1), 29–40.
Carrillo, P. E., Onofa, M., & Ponce, J. (2011). *Information Technology and Student Achievement: Evidence from a Randomized Experiment in Ecuador*. https://doi.org/10.2139/ssrn.1818756
Chalouh, L., & Herscovics, N. (1984). From letter representing a hidden quantity to letter representing an unknown quantity. *Proceedings of PME-NA-VI, Madison, Wisconsin*, 71–76.
Chandrasekharan, S., & Nersessian, N. J. (2015). Building Cognition: The Construction of Computational Representations for Scientific Discovery. *Cognitive Science*, *39*(8), 1727–1763.
Chandrasekharan, S., & Nersessian, N. J. (2017). Rethinking correspondence: how the process of constructing models leads to discoveries and transfer in the bioengineering sciences. *Synthese*. https://doi.org/10.1007/s11229-017-1463-3
Chauhan, P., Joel, A. J., Kurup, P., Srinivasan, P. S., & Karnam, D. P. (2019). Experiences of teaching Vectors in Indian pre-university classrooms: An account by Teachers. *In Proceedings of the Inaugural Conference of the Mathematics Teachers' Association - India*, 126–127.
Chemero, A. (2003). An outline of a theory of affordances. *Ecological Psychology: A Publication of the International Society for Ecological Psychology*, *15*(2), 181–195.
Chen, F., Hu, Z., Zhao, X., Wang, R., Yang, Z., Wang, X., & Tang, X. (2006). Neural correlates of serial abacus mental calculation in children: a functional MRI study. *Neuroscience Letters*, *403*(1-2), 46–51.

Cheung, A. C. K., & Slavin, R. E. (2013). The effectiveness of educational technology applications for enhancing mathematics achievement in K-12 classrooms: A meta-analysis. *Educational Research Review*, *9*, 88–113.
Chinnappan, M. (1998). Schemas and mental models in geometry problem solving. *Educational Studies in Mathematics*, *36*(3), 201–217.
Chisolm, E. (2012). Geometric Algebra. In *arXiv [math-ph]*. arXiv. http://arxiv.org/abs/1205.5935
Christenson, S. L., Reschly, A. L., & Wylie, C. (Eds.). (2012). *Handbook of Research on Student Engagement*. Springer, Boston, MA.
Clark, A. (2001). Natural-Born Cyborgs? *Cognitive Technology: Instruments of Mind*, 17–24.
Clark, A., & Chalmers, D. (1998). The Extended Mind. *Analysis*, *58*(1), 7–19.
Clark, R. E. (1983). Reconsidering research on learning from media. *Review of Educational Research*, *53*(4), 445–459.
Clark, R. E. (1994). Media will never influence learning. *Educational Technology Research and Development*, *42*(2), 21–29.
Clark, R. E., & Feldon, D. F. (2005). Five common but questionable principles of multimedia learning. In *The Cambridge Handbook of Multimedia Learning* (Vol. 6, pp. 97–115). Cambridge University Press.
Clark-Wilson, A. (2010). Emergent pedagogies and the changing role of the teacher in the TI-Nspire Navigator-networked mathematics classroom. *ZDM: The International Journal on Mathematics Education*, *42*(7), 747–761.
Clement, J. (1982). Students' preconceptions in introductory mechanics. *American Journal of Physics*, *50*(1), 66–71.
Clements, D. (1985). Research on Logo in education: Is the turtle slow but steady, or not even in the race? *Computers in the Schools*, *2*(2-3), 55–71.
Clow, D. (2013). MOOCs and the Funnel of Participation. *Proceedings of the Third International Conference on Learning Analytics and Knowledge*, 185–189.
Csanadi, A., Eagan, B., Kollar, I., Shaffer, D. W., & Fischer, F. (2018). When coding-and-counting is not enough: using epistemic network analysis (ENA) to analyze verbal data in CSCL research. *International Journal of Computer-Supported Collaborative Learning*, *13*(4), 419–438.
Cuban, L., & Kirkpatrick, H. (1998). Computers Make Kids Smarter--Right? *TECHNOS*, *7*(2), 26–31.
Cuoco, A., Paul Goldenberg, E., & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curricula. *The Journal of Mathematical Behavior*, *15*(4), 375–402.
Dall'Alba, G., Walsh, E., Bowden, J., Martin, E., Masters, G., Ramsden, P., & Stephanou, A. (1993). Textbook treatments and students' understanding of acceleration. *Journal of Research in Science Teaching*, *30*(7), 621–635.
Daston, L., & Galison, P. (2007). *Objectivity*. Zone Books.
de Freitas, E. (2016). Material encounters and media events: what kind of mathematics can a body do? *Educational Studies in Mathematics*, *91*(2), 185–202.
de Freitas, E., & Sinclair, N. (2012). Diagram, gesture, agency: theorizing embodiment in the mathematics classroom. *Educational Studies in Mathematics*, *80*(1-2), 133–152.
De Jong, T., Sotiriou, S., & Gillet, D. (2014). Innovations in STEM education: the Go-Lab federation of online labs. *Smart Learning Environments*, *1*(1), 3.
Doig, B., & Groves, S. (2011). Japanese lesson study: Teacher professional development through communities of inquiry. *Mathematics Teacher Education and Development*, *13*(1), 77–93.
Domahs, F., Moeller, K., Huber, S., Willmes, K., & Nuerk, H.-C. (2010). Embodied numerosity: implicit hand-based representations influence symbolic number

processing across cultures. *Cognition*, *116*(2), 251–266.
Donevska-Todorova, A. (2018). Fostering Students' Competencies in Linear Algebra with Digital Resources. In S. Stewart, C. Andrews-Larson, A. Berman, & M. Zandieh (Eds.), *Challenges and Strategies in Teaching Linear Algebra* (pp. 261–276). Springer International Publishing.
Dörfler, W. (2005). Diagrammatic Thinking. In M. H. G. Hoffmann, J. Lenhard, & F. Seeger (Eds.), *Activity and Sign: Grounding Mathematics Education* (pp. 57–66). Springer US.
Dorier, J.-L. (1998). The role of formalism in the teaching of the theory of vector spaces. *Linear Algebra and Its Applications*, *275*, 141–160.
Dorier, J.-L., Robert, A., Robinet, J., & Rogalski, M. (2000). On a research programme concerning the teaching and learning of linear algebra in the first-year of a French science university. *International Journal of Mathematical Education in Science and Technology*, *31*(1), 27–35.
Dorier, J.-L., & Sierpinska, A. (2002). Research into the Teaching and Learning of Linear Algebra. In D. Holton, M. Artigue, U. Kirchgräber, J. Hillel, M. Niss, & A. Schoenfeld (Eds.), *The Teaching and Learning of Mathematics at University Level: An ICMI Study* (pp. 255–273). Springer Netherlands.
Dowek, G. (2015). *Computation, Proof, Machine: Mathematics Enters a New Age*. Cambridge University Press.
Dray, T., & Manogue, C. A. (1999). The vector calculus gap: mathematics≠ physics. *Problems, Resources, and Issues in Mathematics Undergraduate Studies*, *9*(1), 21–28.
Dray, T., & Manogue, C. A. (2006). The geometry of the dot and cross products. *Journal of Online Mathematics and Its Applications*, *6*, 1–13.
Dreyfus, T. (1999). Why Johnny can't prove. In D. Tirosh (Ed.), *Forms of Mathematical Knowledge* (pp. 85–109). Springer.
Dreyfus, T., Hillel, J., & Sierpinska, A. (1998). Cabri-based linear algebra: Transformations. In I. Schwank (Ed.), *Proceedings of the First Conference of the European Society for Research in Mathematics Education* (pp. 209–221). Forschungsinstitut fuer Mathematikdidaktik.
Drier, H. S. (2001). Teaching and Learning Mathematics With Interactive Spreadsheets. *School Science and Mathematics*, *101*(4), 170–179.
Dubinsky, E. (1997). Some thoughts on a first course in linear algebra at the college level. *MAA NOTES*, 85–106.
Duffy, T. M., & Jonassen, D. H. (2013). *Constructivism and the technology of instruction: A conversation*. Routledge.
Duong, T. (2013). Local significant differences from nonparametric two-sample tests. *Journal of Nonparametric Statistics*, *25*(3), 635–645.
Entwistle, T., Gentile, D., Graf, E., Hulme, S., Schoch, J., Strassenburg, A., Swartz, C., Chiaverina, C., Clark, R. B., Durkin, T., Gavenda, D., Peterson, F., Robertson, C., & Sears, R. (1999). Survey of high-school physics texts. *Physics Teacher*, *37*, 284–296.
Ertmer, P. A. (1999). Addressing first- and second-order barriers to change: Strategies for technology integration. *Educational Technology Research and Development: ETR & D*, *47*(4), 47–61.
Fabry, D. L., & Higgs, J. R. (1997). Barriers to the Effective Use of Technology in Education: Current Status. *Journal of Educational Computing Research*, *17*(4), 385–395.
Falcade, R., Laborde, C., & Mariotti, M. A. (2007). Approaching functions: Cabri tools as instruments of semiotic mediation. *Educational Studies in Mathematics*, *66*(3), 317–333.
Fernandez, C., & Yoshida, M. (2012). *Lesson study: A Japanese approach to improving*

mathematics teaching and learning. Routledge.
Fischbein, E., & Barash, A. (1993). Algorithmic models and their misuse in solving algebraic problems. In I. Hirabayashi, N. Nohda, K. Shigematsu, & F.-L. Lin (Eds.), *Proceedings of the 17th PME Conference* (Vol. 1, pp. 162–172).
Flores, S., Kanim, S. E., & Kautz, C. H. (2004). Student use of vectors in introductory mechanics. *American Journal of Physics*, 72(4), 460–468.
Forjan, M., & Sliško, J. (2014). Simplifications and idealizations in high school physics in Mechanics: a study of Slovenian curriculum and textbooks. *European Journal of Physics Education*, *5*(3), 20–31.
Frisch, M., Heydekorn, J., & Dachselt, R. (2010). Diagram Editing on Interactive Displays Using Multi-touch and Pen Gestures. *Diagrammatic Representation and Inference*, 182–196.
Fuys, D., & Geddes, D. (1984). *An Investigation of Van Hiele Levels of Thinking in Geometry among Sixth and Ninth Graders: Research Findings and Implications*. 681–711.
Gabert, S. L. (2001). *Phase world of water: a case study of a virtual reality world developed to investigate the relative efficiency and efficacy of a bird's eye view exploration and a head-up-display exploration*. University of Washington.
Geith, C., & Vignare, K. (2008). Access to Education with Online Learning and Open Educational Resources: Can They Close the Gap? *Journal of Asynchronous Learning Networks*, *12*(1), 105–126.
Gibson, J. J. (1977). The theory of affordances. In R. Shaw & J. Bransford (Eds.), *Perceiving, acting, and knowing. Toward an ecological Psychology* (pp. 67–82). Lawrence Erlbaum Associates.
Gierre, R. N. (1988). *Explaining science*. Univ. Chicago Press, Chicago.
Glenberg, A. M. (2010). Embodiment as a unifying perspective for psychology. *Wiley Interdisciplinary Reviews. Cognitive Science*, *1*(4), 586–596.
Glenberg, A. M., & Kaschak, M. P. (2002). Grounding language in action. *Psychonomic Bulletin & Review*, *9*(3), 558–565.
Godfrey-Smith, P. (2006). The strategy of model-based science. *Biology and Philosophy*, *21*(5), 725–740.
Gonthier, G. (2008). Formal proof--the four-color theorem. *Notices of the AMS*, *55*(11), 1382–1393.
Greeno, J. G. (1994). Gibson's affordances. *Psychological Review*, *101*(2), 336–342.
Gretton, A., Borgwardt, K. M., Rasch, M. J., Schölkopf, B., & Smola, A. (2012). A Kernel Two-Sample Test. *Journal of Machine Learning Research: JMLR*, *13*(Mar), 723–773.
Gur, H. (2009). Trigonometry Learning. *New Horizons in Education*, *57*(1), 67–80.
Gu, X., & Blackmore, K. L. (2015). A systematic review of agent-based modelling and simulation applications in the higher education domain. *Higher Education Research & Development*, *34*(5), 883–898.
Halliday, D., Resnick, R., & Walker, J. (2013). *Fundamentals of Physics Extended*. John Wiley & Sons Inc.
Halloun, I. A., & Hestenes, D. (1985). The initial knowledge state of college physics students. *American Journal of Physics*, *53*(11), 1043–1055.
Hanakawa, T., Honda, M., Okada, T., Fukuyama, H., & Shibasaki, H. (2003). Neural correlates underlying mental calculation in abacus experts: a functional magnetic resonance imaging study. *NeuroImage*, *19*(2 Pt 1), 296–307.
Harari, Y. N. (2014). *Sapiens: A brief history of humankind*. Random House.
Harel, G. (1987). Variations in linear algebra content presentations. *For the Learning of Mathematics*, 7(3), 29–32.
Harel, G. (1989). Learning and Teaching Linear Algebra: Difficulties and an Alternative Approach to Visualizing Concepts and Processes. *Focus on Learning*

Problems in Mathematics, *11*, 139–148.
Harel, G. (1990). Using geometric models and vector arithmetic to teach high-school students basic notions in linear algebra. *International Journal of Mathematical Education in Science and Technology*, *21*(3), 387–392.
Harel, G. (1999). Students' understanding of proofs: a historical analysis and implications for the teaching of geometry and linear algebra. *Linear Algebra and Its Applications*, *302*(303), 601–613.
Harrell, M. (2008). No Computer Program Required: Even Pencil-and-Paper Argument Mapping Improves Critical-Thinking Skills. *Teaching Philosophy*, *31*(4), 351–374.
Hawkins, J. M., Thompson, J. R., Wittmann, M. C., Sayre, E. C., & Frank, B. W. (2010). Students' Responses To Different Representations Of A Vector Addition Question. *AIP Conference Proceedings*, *1289*(1), 165–168.
Heckler, A. F., & Scaife, T. M. (2015). Adding and subtracting vectors: The problem with the arrow representation. *Physical Review Special Topics - Physics Education Research*, *11*(1), 010101.
Heft, H. (1989). Affordances and the Body: An Intentional Analysis of Gibson's Ecological Approach to Visual Perception. *Journal for the Theory of Social Behaviour*, *19*(1), 1–30.
Hegarty, M., Mayer, S., Kriz, S., & Keehner, M. (2005). The Role of Gestures in Mental Animation. *Spatial Cognition and Computation*, *5*(4), 333–356.
Hestenes, D. (1988). Universal geometric algebra. *Simon Stevin : A Quarterly Journal of Pure and Applied Mathematics*, *62*(3 – 4).
Hestenes, D. (2010). Modeling Theory for Math and Science Education. In R. Lesh, P. L. Galbraith, C. R. Haines, & A. Hurford (Eds.), *Modeling Students' Mathematical Modeling Competencies* (pp. 13–41). Springer.
Hestenes, D., & Sobczyk, G. (2012). *Clifford algebra to geometric calculus: a unified language for mathematics and physics* (Vol. 5). Springer Science & Business Media.
Hew, K. F., & Brush, T. (2007). Integrating technology into K-12 teaching and learning: current knowledge gaps and recommendations for future research. *Educational Technology Research and Development: ETR & D*, *55*(3), 223–252.
Hew, K. F., Kale, U., & Kim, N. (2007). Past research in instructional technology: Results of a content analysis of empirical studies published in three prominent instructional technology journals from the year 2000 through 2004. *Journal of Educational Computing Research*, *36*(3), 269–300.
Hillel, J. (2000). Modes of Description and the Problem of Representation in Linear Algebra. In J.-L. Dorier (Ed.), *On the Teaching of Linear Algebra* (pp. 191–207). Springer Netherlands.
Hohenwarter, M., & Jones, K. (2007). Ways of linking geometry and algebra, the case of Geogebra. *Proceedings of the British Society for Research into Learning Mathematics*, *27*(3), 126–131.
Hokanson, B., & Hooper, S. (2000). Computers as cognitive media: examining the potential of computers in education. *Computers in Human Behavior*, *16*(5), 537–552.
Holsanova, J. (2014). Reception of multimodality: Applying eye tracking methodology in multimodal research. *Routledge Handbook of Multimodal Analysis*. https://www.researchgate.net/profile/Jana_Holsanova/publication/259501157_Reception_of_multimodality_Applying_eye_tracking_methodology_in_multimodal_research/links/546e31990cf2b5fc17606f88.pdf
Hou, J., Rashid, J., & Lee, K. M. (2017). Cognitive map or medium materiality? Reading on paper and screen. *Computers in Human Behavior*, *67*, 84–94.

Hoyles, C., & Noss, R. (2003). What can digital technologies take from and bring to research in mathematics education? In *Second international handbook of mathematics education* (pp. 323–349). Springer.
Humphreys, P. (2002). Computational Models. *Philosophy of Science*, *69*(S3), S1–S11.
Hutchins, E. (1995a). *Cognition in the Wild*. MIT Press.
Hutchins, E. (1995b). How a Cockpit Remembers Its Speeds. *Cognitive science*, *19*(3), 265–288.
Innis, H. A. (2008). *The Bias of Communication*. University of Toronto Press.
International GeoGebra Institute. (2002). *GeoGebra | Free Math Apps - used by over 100 Million Students & Teachers Worldwide*. GeoGebra. https://www.geogebra.org/
Jonassen, D. H., Howland, J., Moore, J., & Marra, R. M. (2003). *Learning to solve problems with technology: A constructivist perspective*. Pearson.
Jones, K. (2002). Issues in the teaching and learning of geometry. In L. Haggarty (Ed.), *Aspects of teaching secondary mathematics: Perspectives on practice* (pp. 121–139). Routledge.
Jordan, K. (2014). Initial trends in enrolment and completion of massive open online courses. *The International Review of Research in Open and Distributed Learning*, *15*(1), 133–160.
Kalelioglu, F., & Gülbahar, Y. (2014). The Effects of Teaching Programming via Scratch on Problem Solving Skills: A Discussion from Learners' Perspective. *Informatics in Education*, *13*(1), 33–50.
Kaput, J. J. (1986). *Information Technology and Mathematics: Opening New Representational Windows*. http://eric.ed.gov/?id=ED297950
Kaput, J. J., Noss, R., & Hoyles, C. (2002). Developing new notations for a learnable mathematics in the computational era. In *Handbook of international research in mathematics education* (pp. 51–75). Lawrence Erlbaum.
Kaput, J. J., & Roschelle, J. (2013). The Mathematics of Change and Variation from a Millennial Perspective: New Content, New Context. In S. J. Hegedus & J. Roschelle (Eds.), *The SimCalc Vision and Contributions* (pp. 13–26). Springer Netherlands.
Karaca, F., Can, G., & Yildirim, S. (2013). A path model for technology integration into elementary school settings in Turkey. *Computers & Education*, *68*, 353–365.
Karnam, D. P., Agrawal, H., Borar, P., & Chandrasekharan, S. (2019). The Affordable Touchy Feely Classroom: Textbooks Embedded with Manipulable Vectors and Lesson Plans Augment Imagination, Extend Teaching-Learning Practices. In K. Lund, G. Niccolai, Lavoué E Hmelo-Silver, G. Gweon, & M. and Baker (Eds.), *13th International Conference on Computer Supported Collaborative Learning (CSCL) 2019* (pp. 488–495). ENS Lyon.
Karnam, D. P., Agrawal, H., & Chandrasekharan, S. (2018). "Touchy Feely Vectors" changes students' understanding and modes of reasoning. In J. C. Yang, M. Chang, L.-H. Wong, & M. M. T. Rodrigo (Eds.), *Proceedings of the 26th International Conference on Computers in Education* (pp. 143–152).
Kazimoglu, C., Kiernan, M., Bacon, L., & Mackinnon, L. (2012). A Serious Game for Developing Computational Thinking and Learning Introductory Computer Programming. *Procedia - Social and Behavioral Sciences*, 47, 1991–1999.
Kieran, C., & Drijvers, P. (2006). The Co-Emergence of Machine Techniques, Paper-and-Pencil Techniques, and Theoretical Reflection: A Study of Cas use in Secondary School Algebra. *International Journal of Computers for Mathematical Learning*, *11*(2), 205.
Kirkwood, A., & Price, L. (2014). Technology-enhanced learning and teaching in higher education: what is "enhanced" and how do we know? A critical literature review. *Learning, Media and Technology*, *39*(1), 6–36.

Kirsh, D. (2010). Thinking with external representations. *AI & SOCIETY*, *25*(4), 441–454.
Klein, U. (2001). Paper tools in experimental cultures. *Studies in History and Philosophy of Science. Part B. Studies in History and Philosophy of Modern Physics*, *32*(2), 265–302.
Knight, R. D. (1995). The vector knowledge of beginning physics students. *Physics Teacher*, *33*(2), 74–77.
Knuth, E. J. (2000). Student Understanding of the Cartesian Connection: An Exploratory Study. *Journal for Research in Mathematics Education*, *31*(4), 500–507.
Knuuttila, T. (2011). Modelling and representing: An artefactual approach to model-based representation. *Studies in History and Philosophy of Science. Part B. Studies in History and Philosophy of Modern Physics*, 42(2), 262–271.
Knuuttila, T. (2017). Imagination extended and embedded: artifactual versus fictional accounts of models. *Synthese*. https://doi.org/10.1007/s11229-017-1545-2
Knuuttila, T., & Voutilainen, A. (2003). A Parser as an Epistemic Artifact: A Material View on Models. In *Philosophy of Science* (Vol. 70, Issue 5, pp. 1484–1495). https://doi.org/10.1086/377424
Konyalioglu, A. C., Isik, A., Kaplan, A., Hizarci, S., & Durkaya, M. (2011). Visualization approach in teaching process of linear algebra. *Procedia - Social and Behavioral Sciences*, *15*, 4040–4044.
Konyalioglu, S., Konyalioglu, A. C., Ipek, A. S., & Isik, A. (2005). The role of visualization approach on student's conceptual learning. *International Journal for Mathematics Teaching and Learning*, 47, 1–9.
Kozma, R. B. (1994). Will media influence learning? Reframing the debate. *Educational Technology Research and Development*, 42(2), 7–19.
Kozma, R. B. (2008). Comparative Analysis of Policies for ICT in Education. In J. Voogt & G. Knezek (Eds.), *International Handbook of Information Technology in Primary and Secondary Education* (pp. 1083–1096). Springer US.
Kozma, R. B., & Vota, W. S. (2014). ICT in Developing Countries: Policies, Implementation, and Impact. In *Handbook of Research on Educational Communications and Technology* (pp. 885–894). Springer, New York, NY.
Kraemer, K. L., Dedrick, J., & Sharma, P. (2009). One laptop per child: vision vs. reality. *Communications of the ACM*, *52*(6), 66–73.
Kress, G., & Van Leeuwen, T. (2001). Multimodal discourse. *The Modes and Media of Contemporary Communication. (Cappelen, London 2001)*.
Kumar, K. (1988). Origins of India's "textbook Culture." *Comparative Education Review*, *32*(4), 452–464.
Kumpulainen, K., & Kajamaa, A. (2019). From Material Objects to Social Objects: Researching the Material-Dialogic Spaces of Joint Attention in a School-based Makerspace. *CSCL 2019 Proceeding of the 13th International Conference on Computer Supported Collaborative Learning. Volume 1: A Wide Lens: Combining Embodied, Enactive, Extended, and Embedded Learning in Collaborative Settings.(Eds.) Kristine Lund, Gerald Niccolai, Elise Lavoué, Cindy Hmelo-Silver Gahgene Gweon, Michael Baker.*, 352.
Lakoff, G., & Johnson, M. (1980). Metaphors we live by. *Chicago, IL: University of*. https://sci-hub.tw/http://www.cc.gatech.edu/classes/AY2013/cs7601_spring/papers/Lakoff_Johnson.pdf
Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic books.
Lakshmi, T. G., Narayana, S., Prasad, P., Murthy, S., & Chandrasekharan, S. (2016). Geometry-via-Gestures: Design of a gesture based application to teach 3D

Geometry. *Proceedings of the 24th International Conference on Computers in Education*, 180–189.
Lamborn, S., Newmann, F., & Wehlage, G. (1992). The significance and sources of student engagement. In *Student Engagement and Achievement in American Secondary Schools* (pp. 11–39). Teachers College Press.
Landy, D., Allen, C., & Zednik, C. (2014). A perceptual account of symbolic reasoning. *Frontiers in Psychology*, *5*, 275.
Landy, D., & Goldstone, R. L. (2007). How abstract is symbolic thought? *Journal of Experimental Psychology. Learning, Memory, and Cognition*, *33*(4), 720–733.
Latour, B. (1990). Drawing Things Together. In Michael Lynch And (Ed.), *Representation in scientific practice* (Vol. 3, pp. 19–68). MIT Press.
Latour, B., & Woolgar, S. (1979). *Laboratory Life: The Construction of Scientific Facts*. Princeton University Press.
Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. content.taylorfrancis.com.
Lehrer, R., & Schauble, L. (2010). What kind of explanation is a model? In *Instructional explanations in the disciplines* (pp. 9–22). Springer.
Lehrer, R., & Schauble, L. (2015). The Development of Scientific Thinking. In R. M. Lerner (Ed.), *Handbook of Child Psychology and Developmental Science* (pp. 671–714). John Wiley & Sons, Inc.
Lenhard, J. (2007). Computer Simulation: The Cooperation between Experimenting and Modeling. *Philosophy of Science*, 74(2), 176–194.
Liang, J. (2008). Multivariate Statistical Process Monitoring Using Kernel Density Estimation. *Developments in Chemical Engineering and Mineral Processing*, *13*(1-2), 185–192.
Lindgren, R., Tscholl, M., Wang, S., & Johnson, E. (2016). Enhancing learning and engagement through embodied interaction within a mixed reality simulation. *Computers & Education*, *95*, 174–187.
Lin, L., & Atkinson, R. K. (2011). Using animations and visual cueing to support learning of scientific concepts and processes. *Computers & Education*, *56*(3), 650–658.
Li, Q. (1996). Nonparametric testing of closeness between two unknown distribution functions. *Econometric Reviews*, *15*(3), 261–274.
Lithner, J. (2003). Students' mathematical reasoning in university textbook exercises. *Educational Studies in Mathematics*, *52*(1), 29–55.
Liu, D., & Kottegoda, Y. (2019). Disconnect between undergraduates' understanding of the algebraic and geometric aspects of vectors. *European Journal of Physics*, *40*(3), 035702.
Liyanagunawardena, T. R., Adams, A. A., & Williams, S. A. (2013). MOOCs: A systematic study of the published literature 2008-2012. *The International Review of Research in Open and Distributed Learning*, *14*(3), 202–227.
Logan, R. K. (1986). *The alphabet effect*. New York: Morrow.
Madden, S. P., Jones, L. L., & Rahm, J. (2011). The role of multiple representations in the understanding of ideal gas problems. *Chemistry Education Research and Practice*, *12*(3), 283–293.
Magnani, L., Nersessian, N., & Thagard, P. (2012). *Model-based reasoning in scientific discovery*. Springer Science & Business Media.
Majumdar, R., Kothiyal, A., Ranka, A., Pande, P., Murthy, S., Agarwal, H., & Chandrasekharan, S. (2014). The Enactive equation: Exploring How Multiple External Representations are Integrated, Using a Fully Controllable Interface and Eye-Tracking. *2014 IEEE Sixth International Conference on Technology for Education*, 233–240.
Mäki, U. (2009). MISSing the World. Models as Isolations and Credible Surrogate

Systems. *Erkenntnis. An International Journal of Analytic Philosophy*, *70*(1), 29–43.
Martin, T., & Schwartz, D. L. (2005). Physically distributed learning: adapting and reinterpreting physical environments in the development of fraction concepts. *Cognitive Science*, *29*(4), 587–625.
Mason, R. (2005). Media for delivering global education. In *Globalising education: Trends and applications* (pp. 19–38). Routledge.
Masood, M. (2004). *Trends and issues as reflected in traditional educational technology literature: A content analysis*. Indiana University.
Mayer, R. E. (1997). Multimedia learning: Are we asking the right questions? *Educational Psychologist*, *32*(1), 1–19.
McArthur, D., Lewis, M., & Bishary, M. (2005). The roles of artificial intelligence in education: current progress and future prospects. *British Journal of Educational Technology: Journal of the Council for Educational Technology*, *1*(4), 42–80.
McLuhan, M. (1964). The medium is the message. In *Understanding media: The extensions of man* (pp. 23–35). Signet.
McLuhan, M., & Logan, R. K. (1977). ALPHABET, MOTHER OF INVENTION. *Etc.; a Review of General Semantics*, *34*(4), 373–383.
Mikula, B. D., & Heckler, A. F. (2017). Framework and implementation for improving physics essential skills via computer-based practice: Vector math. *Physical Review Physics Education Research*, *13*(1), 010122.
Moreno-Armella, L., Hegedus, S. J., & Kaput, J. J. (2008). From static to dynamic mathematics: historical and representational perspectives. *Educational Studies in Mathematics*, *68*(2), 99–111.
Morrison, M., & Morgan, M. S. (1999). Models as Mediating Instruments. In M. M S Morgan & Morrison (Ed.), *Models as mediators. Perspectives on natural and social science* (pp. 10–37). Cambridge University Press.
Motiwalla, L. F. (2007). Mobile learning: A framework and evaluation. *Computers & Education*, *49*(3), 581–596.
Naik, G., Chitre, C., Bhalla, M., & Rajan, J. (2020). Impact of use of technology on student learning outcomes: Evidence from a large-scale experiment in India. *World Development*, *127*, 104736.
NCERT. (2005). *National Curriculum Framework*. NCERT.
Nersessian, N. (2008). Model-based reasoning in scientific practice. *Teaching Scientific Inquiry*. https://brill.com/downloadpdf/book/edcoll/9789460911453/BP000005.pdf
Newell, A., & Simon, H. A. (1976). Computer Science As Empirical Inquiry: Symbols and Search. *Communications of the ACM*, *19*(3), 113–126.
Ngss, L. S. (2013). *Next Generation Science Standards: For states, by states*. National Academies Press.
Nguyen, N.-L., & Meltzer, D. E. (2003). Initial understanding of vector concepts among students in introductory physics courses. *American Journal of Physics*, *71*(6), 630–638.
Niemi, D. (1996). Assessing Conceptual Understanding in Mathematics: Representations, Problem Solutions, Justifications, and Explanations. *The Journal of Educational Research*, *89*(6), 351–363.
Noë, A. (2004). *Action in Perception*. MIT Press.
Nugroho, D., & Lonsdale, M. (2010). *Evaluation of OLPC programs global : a literature review*. https://research.acer.edu.au/digital_learning/8/
O'Donnell, A. M., & O'Kelly, J. (1994). Learning from peers: Beyond the rhetoric of positive results. *Educational Psychology Review*, *6*(4), 321–349.
O'Donovan-Anderson, M. (1997). *Content and comportment: On embodiment and the epistemic availability of the world* (Karsten Harries And (Ed.)) [Ph.D.]. Yale School.

O'Neill, R. (1982). Why use textbooks? *ELT Journal*, *36*(2), 104–111.
Ong, W. J. (2013). *Orality and Literacy: 30th Anniversary Edition*. Routledge.
Orhun, N. (2004). Students' mistakes and misconceptions on teaching of trigonometry. *Journal of Curriculum Studies*, *32*(6), 797–820.
Ottmar, E., Weitnauer, E., Landy, D., & Goldstone, R. (2015). Graspable Mathematics: Using Perceptual Learning Technology. *Integrating Touch-Enabled and Mobile Devices into Contemporary Mathematics Education*, 24.
Padalkar, S., & Ramadas, J. (2011). Designed and Spontaneous Gestures in Elementary Astronomy Education. *International Journal of Science Education*, *33*(12), 1703–1739.
Panaoura, A., Elia, I., Gagatsis, A., & Giatilis, G.-P. (2006). Geometric and algebraic approaches in the concept of complex numbers. *International Journal of Mathematical Education in Science and Technology*, *37*(6), 681–706.
Pande, P. (2018). *Rethinking Representational Competence: cognitive mechanisms, empirical studies, and the design of a new media intervention* (S. Chandrasekharan (Ed.)) [Unpublished doctoral dissertation]. Tata Institute of Fundamental Research, Mumbai.
Pande, P., & Chandrasekharan, S. (2017). Representational competence: towards a distributed and embodied cognition account. *Studies in Science Education*. https://www.tandfonline.com/doi/abs/10.1080/03057267.2017.1248627
Pande, P., & Chandrasekharan, S. (2017). Representational competence: towards a distributed and embodied cognition account. *Studies in Science Education*, *53*(1), 1–43.
Papert, S. (1980). *Mindstorms: Children, Computers, and Powerful Ideas*. Basic Books, Inc.
Papert, S. (1987). Information Technology and Education: Computer Criticism vs. Technocentric Thinking. *Educational Researcher*, *16*(1), 22–30.
Papert, S. (1988). A critique of technocentrism in thinking about the school of the future. In *Children in the information age* (pp. 3–18). Pergamon Press.
Park, S. H., & Ertmer, P. A. (2008). Examining barriers in technology-enhanced problem-based learning: Using a performance support systems approach. *British Journal of Educational Technology: Journal of the Council for Educational Technology*, *39*(4), 631–643.
Pavia, J. M. (2015). Testing goodness-of-fit with the kernel density estimator: GoFKernel. *Journal of Statistical Software*, *66*(1), 1–27.
Pelgrum, W. J., & Law, N. W. Y. (2003). *ICT in education around the world: Trends, problems and prospects*. UNESCO: International Institute for Educational Planning.
Penuel, W. R., Boscardin, C. K., Masyn, K., & Crawford, V. M. (2007). Teaching with student response systems in elementary and secondary education settings: A survey study. *Educational Technology Research and Development: ETR & D*, *55*(4), 315–346.
Pepper, R. E., Chasteen, S. V., Pollock, S. J., & Perkins, K. K. (2012). Observations on student difficulties with mathematics in upper-division electricity and magnetism. *Physical Review Special Topics - Physics Education Research*, *8*(1), 010111.
Perkins, K., Adams, W., Dubson, M., Finkelstein, N., Reid, S., Wieman, C., & LeMaster, R. (2006). PhET: Interactive Simulations for Teaching and Learning Physics. *Physics Teacher*, *44*(1), 18–23.
Phan, T., McNeil, S. G., & Robin, B. R. (2016). Students' patterns of engagement and course performance in a Massive Open Online Course. *Computers & Education*, *95*, 36–44.
Pickering, A. (1995). *The Mangle of Practice: Time, Agency, and Science*. University of

Chicago Press.
Port, R. F., & Van Gelder, T. (1995). *Mind as Motion: Explorations in the Dynamics of Cognition*. Bradford Books.
Postman, N. (2006). *Amusing ourselves to death: Public discourse in the age of show business*. Penguin.
Presmeg, N. C. (2006). A semiotic view of the role of imagery and inscriptions in mathematics teaching and learning. *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education*, *1*, 19–34.
Pylyshyn, Z. W. (1978). Computational models and empirical constraints. *The Behavioral and Brain Sciences*, *1*(1), 91–99.
Rahaman, J., Agrawal, H., Srivastava, N., & Chandrasekharan, S. (2017). Recombinant Enaction: Manipulatives Generate New Procedures in the Imagination, by Extending and Recombining Action Spaces. *Cognitive Science*, *42*(2), 370–415.
Ramadas, J. (2009). Visual and Spatial Modes in Science Learning. *International Journal of Science Education*, *31*(3), 301–318.
Ramdas, A., Trillos, N. G., & Cuturi, M. (2017). On Wasserstein Two-Sample Testing and Related Families of Nonparametric Tests. *Entropy* , *19*(2), 47.
Reddy, V. K., & Mukherjee, T. (1915). *UGC MHRD e Pathshala*. http://epgp.inflibnet.ac.in/epgpdata/uploads/epgp_content/S000013EN/P001451/M014806/ET/1487326013Paper9,Module12,EText.pdf
Reiner, M., Slotta, J. D., Chi, M. T. H., & Resnick, L. B. (2000). Naive Physics Reasoning: A Commitment to Substance-Based Conceptions. *Cognition and Instruction*, *18*(1), 1–34.
Reiser, R. A. (1994). Clark's Invitation to the Dance: An Instructional Designer's Response. *Educational Technology Research and Development: ETR & D*, *42*(2), 45–48.
Reiser, R. A., & Dempsey, J. V. (2012). *Trends and issues in instructional design and technology*. Pearson.
Repenning, A., Webb, D. C., Brand, C., Gluck, F., Grover, R., Miller, S., Nickerson, H., & Song, M. (2014). Beyond Minecraft: Facilitating Computational Thinking through Modeling and Programming in 3D. *IEEE Computer Graphics and Applications*, *34*(3), 68–71.
Rives, F. C. (1979). The teacher as a performing artist. *Contemporary Educational Psychology*, *51*(1), 7–9.
Roberts, A. L., Sharma, M. D., Sefton, I. M., & Khachan, J. (2008). Differences in two evaluations of answers to a conceptual physics question: a preliminary analysis. *CAL-Laborate International*, *16*(1), 28–38.
Rosenshine, B. (1983). Teaching functions in instructional programs. *The Elementary School Journal*, *83*(4), 335–351.
Ross, S. M., Morrison, G. R., & Lowther, D. L. (2010). Educational technology research past and present: Balancing rigor and relevance to impact school learning. *Contemporary Educational Technology*, *1*(1), 17–35.
Roth, W. M., & Lawless, D. (2002). Scientific investigations, metaphorical gestures, and the emergence of abstract scientific concepts. *Learning and Instruction*, *12*(3), 285–304.
Russell, T. L. (1999). *The no significant difference phenomenon: As reported in 355 research reports, summaries and papers*. North Carolina State University.
Samarapungavan, A., Vosniadou, S., & Brewer, W. F. (1996). Mental models of the earth, sun, and moon: Indian children's cosmologies. *Cognitive Development*, *11*(4), 491–521.
Sancho-Gil, J. M., Rivera-Vargas, P., & Miño-Puigcercós, R. (2019). Moving beyond the predictable failure of Ed-Tech initiatives. *Learning, Media and Technology*,

1–15.
Saunders, G., Fu, G., & Stevens, J. R. (2017). A Bivariate Hypothesis Testing Approach for Mapping the Trait-Influential Gene. *Scientific Reports*, 7(1), 12798.
Schmidt, W. H., McKnight, C. C., & Raizen, S. (2007). *A Splintered Vision: An Investigation of U.S. Science and Mathematics Education*. Springer Science & Business Media.
Schoenfeld, A. H. (1985). *Mathematical Problem Solving*. Academic Press.
Sejdinovic, D., Sriperumbudur, B., Gretton, A., & Fukumizu, K. (2013). Equivalence of distance-based and RKHS-based statistics in hypothesis testing. *Annals of Statistics*, *41*(5), 2263–2291.
Sengupta, P., Kinnebrew, J. S., Basu, S., Biswas, G., & Clark, D. (2013). Integrating computational thinking with K-12 science education using agent-based computation: A theoretical framework. *Education and Information Technologies*, *18*(2), 351–380.
Shaffer, P. S., & McDermott, L. C. (2005). A research-based approach to improving student understanding of the vector nature of kinematical concepts. *American Journal of Physics*, *73*(10), 921–931.
Shah, P. (2020). *Media, cognition and assemblage perspectives on ICT in education: a three-part study in an early-adopter Indian school* [Indian Institute of Management Ahmedabad]. http://vslir.iima.ac.in:8080/jspui/handle/11718/23145
Shayan, S., Abrahamson, D., Bakker, A., Duijzer, A. C. G., & Van der Schaaf, M. F. (2015). The emergence of proportional reasoning from embodied interaction with a tablet application: an eyetracking study. *Proceedings of the 9th International Technology, Education, and Development Conference (INTED 2015)*, 5732.
Shvarts, A. (2018). Joint attention in resolving the ambiguity of different presentations: A dual eye-tracking study of the teaching-learning process. *Signs of Signification*. https://link.springer.com/chapter/10.1007/978-3-319-70287-2_5
Siew, N. M., Geofrey, J., & Lee, B. N. (2016). Students' algebraic thinking and attitudes towards algebra: the effects of game-based learning using Dragonbox 12+ App. *The Electronic Journal of Mathematics & Technology*, *10*(2). http://www.academia.edu/download/46188339/2dragonbox.pdf
Silverman, B. W. (2018). The kernel method for multivariate data. In *Density Estimation for Statistics and Data Analysis* (pp. 75–94). Routledge.
Simon, H. A. (1980). Problem solving and education. In D. T. Tuma & F. Reif (Eds.), *Problem solving and education: Issues in teaching and research.* (pp. 81–96). Lawrence Erlbaum Associates.
Simon, M. A. (1996). Beyond inductive and deductive reasoning: The search for a sense of knowing. *Educational Studies in Mathematics*, *30*(2), 197–210.
Sinclair, N., & de Freitas, E. (2014). The haptic nature of gesture: Rethinking gesture with new multitouch digital technologies. *Gesture*, *14*(3), 351–374.
Sinclair, N., & Heyd-Metzuyanim, E. (2014). Learning number with TouchCounts: The role of emotions and the body in mathematical communication. *Technology, Knowledge and Learning*, *19*(1-2), 81–99.
Sleeman, D., & Brown, J. S. (1982). *Intelligent tutoring systems*. Academic Press.
Smith, E. M. (2016). *Students' Understanding of Complex Numbers in Middle-Division Physics* [ir.library.oregonstate.edu]. https://ir.library.oregonstate.edu/concern/graduate_thesis_or_dissertations/bv73c288n
Steenbergen-Hu, S., & Cooper, H. (2013). A meta-analysis of the effectiveness of intelligent tutoring systems on K--12 students' mathematical learning. *Journal*

of Educational Psychology, *105*(4), 970.
Sterelny, K. (2004). Externalism, epistemic artefacts and the extended mind. In R. Schantz (Ed.), *The Externalist Challenge. New Studies on Cognition and Intentionality* (pp. 239–254). Walter de Gruyter.
Stern, D. P. (2004, September 24). *The Lagrangian Points L4 and L5--alternative derivation*. http://www.phy6.org/stargaze/Slagrng3.htm
Stieff, M., & Wilensky, U. (2003). Connected Chemistry—Incorporating Interactive Simulations into the Chemistry Classroom. *Journal of Science Education and Technology*, *12*(3), 285–302.
Stinner, A. (1992). Science textbooks and science teaching: From logic to evidence. *Science Education*, *76*(1), 1–16.
Suppe, F. (1989). *The Semantic Conception of Theories and Scientific Realism*. University of Illinois Press.
Suppes, P. (1960). A comparison of the meaning and uses of models in mathematics and the empirical sciences. *Synthese*, *12*(2), 287–301.
Swetz, F. J. (1987). Capitalism and Arithmetic: The New Math of the Fifteenth Century. *Open Court, LaSalle, Ill*.
Tall, D. (2000). Technology and Versatile Thinking in Mathematics. *In: Proceedings of the International Conference on Technology in Mathematics Education*, 33–50.
Thelen, E., & Smith, L. B. (1996). *A Dynamic Systems Approach to the Development of Cognition and Action* (Reprint). MIT Press.
Tinio, V. L. (2003). *ICT in Education*. e-ASEAN Task Force. http://unpan1.un.org/intradoc/groups/public/documents/unpan/unpan037270.pdf
Tsai, C.-C., & Chai, C. S. (2012). The "third"--order barrier for technology-integration instruction: Implications for teacher education. *Australasian Journal of Educational Technology*, *28*(6), 1057–1060.
Turvey, M. T. (1992). Affordances and Prospective Control: An Outline of the Ontology. *Ecological Psychology: A Publication of the International Society for Ecological Psychology*, *4*(3), 173–187.
Tversky, B. (2005). Visuospatial reasoning. In R. G. M. Keith J. Holyoak (Ed.), *The Cambridge Handbook of Thinking and Reasoning* (pp. 209–240). Cambridge University Press.
Ullmer, E. J. (1994). Media and learning: Are there two kinds of truth? *Educational Technology Research and Development: ETR & D*, *42*(1), 21–32.
Uluyol, C., & Agca, R. K. (2012). Integrating mobile multimedia into textbooks: 2D barcodes. *Computers & Education*, *59*(4), 1192–1198.
Unesco, E. (2000). The Dakar framework for action: Education for all: Meeting our collective commitments. *Dakar Senegal*, 26–28.
Usharani, D., & Meera, B. N. (2018). Exploration of Students' understanding of vector addition and subtraction. In S. Ladage & S. Narvekar (Eds.), *Proceedings of epiSTEME 7 — International Conference to Review Research on Science, Technology and Mathematics Education* (pp. 374–382). Cinnamon Teal.
Usiskin, Z. (2018). Electronic vs. paper textbook presentations of the various aspects of mathematics. *ZDM: The International Journal on Mathematics Education*, *50*(5), 849–861.
Van Deventer, J., & Wittmann, M. C. (2007). Comparing Student Use of Mathematical and Physical Vector Representations. *AIP Conference Proceedings*, *951*(1), 208–211.
Van Dooren, W., De Bock, D., & Verschaffel, L. (2013). How Students Connect Descriptions of Real-World Situations to Mathematical Models in Different Representational Modes. In G. A. Stillman, G. Kaiser, W. Blum, & J. P. Brown (Eds.), *Teaching Mathematical Modelling: Connecting to Research and Practice* (pp.

385–393). Springer Netherlands.
Van Driel, J. H., Verloop, N., & de Vos, W. (1998). Developing science teachers' pedagogical content knowledge. *Journal of Research in Science Teaching*, *35*(6), 673–695.
van Fraassen, B. C., & Van Fraassen, of P. B. C. (1980). *The Scientific Image*. Clarendon Press.
Van Gelder, T. (1999). Dynamic approaches to cognition. In *The MIT Encyclopedia of Cognitive Sciences.* (pp. 244–246). Cambridge MA: MIT Press.
Van Hiele, P. M. (1959). *The child's thought and geometry*. City University of NewYork.
Van Joolingen, W. R., & De Jong, T. (2003). Simquest. In *Authoring Tools for Advanced Technology Learning Environments* (pp. 1–31). Springer, Dordrecht.
Varela, F., Thompson, E., & Rosch, E. (1991). *The embodied mind: cognitive science and human experience*. Cambridge.
Victor, B. (2011, April). *Kill Math*. http://worrydream.com/KillMath/
Vorms, M. (2011). Representing with imaginary models: Formats matter. *Studies in History and Philosophy of Science. Part B. Studies in History and Philosophy of Modern Physics*, *42*(2), 287–295.
Vosniadou, S. (2002a). Mental Models in Conceptual Development. In L. Magnani & N. J. Nersessian (Eds.), *Model-Based Reasoning: Science, Technology, Values* (pp. 353–368). Springer US.
Vosniadou, S. (2002b). On the Nature of Naïve Physics. In M. Limón & L. Mason (Eds.), *Reconsidering Conceptual Change: Issues in Theory and Practice* (pp. 61–76). Springer Netherlands.
Vosniadou, S., & Brewer, W. F. (1992). Mental models of the earth: A study of conceptual change in childhood. *Cognitive Psychology*, *24*(4), 535–585.
Wai, J., Lubinski, D., & Benbow, C. P. (2009). Spatial ability for STEM domains: Aligning over 50 years of cumulative psychological knowledge solidifies its importance. *Journal of Educational Psychology*, *101*(4), 817.
Wang, L. (2011). Computational thinking and computer fundamental education. *2011 International Conference on Computer Science and Service System (CSSS)*, 1158–1159.
Weiguo, Z. (2015). Computational Thinking Ability Training in College Computer Teaching. *Proceedings of the 2015 International Conference on Social Science and Technology Education*, 817–821.
White, B. Y. (1983). Sources of difficulty in understanding Newtonian dynamics. *Cognitive Science*, *7*(1), 41–65.
White, B. Y. (1993). ThinkerTools: Causal Models, Conceptual Change, and Science Education. *Cognition and Instruction*, *10*(1), 1–100.
Wilensky, U. (1999). *NetLogo*. Center for Connected Learning and Computer-Based Modeling, Northwestern University, Evanston, IL. http://ccl.northwestern.edu/netlogo/
Wilensky, U., & Papert, S. (2010). Restructurations: Reformulations of knowledge disciplines through new representational forms. *Proceedings of Constructionism*, 97–111.
Wilensky, U., & Stroup, W. (1999). Learning Through Participatory Simulations: Network-based Design for Systems Learning in Classrooms. *Proceedings of the 1999 Conference on Computer Support for Collaborative Learning.* http://dl.acm.org/citation.cfm?id=1150240.1150320
Winn, W. (2003). Learning in artificial environments: Embodiment, embeddedness and dynamic adaptation. *Technology, Instruction, Cognition and Learning*, *1*(1), 87–114.
Winsberg, E. (2010). *Science in the Age of Computer Simulation*. University of Chicago Press.

Woody, W. D., Daniel, D. B., & Baker, C. A. (2010). E-books or textbooks: Students prefer textbooks. *Computers & Education*, *55*(3), 945–948.
Woolf, B. P. (2010). *Building Intelligent Interactive Tutors: Student-centered Strategies for Revolutionizing E-learning.* Morgan Kaufmann.
Wu, H. K., & Shah, P. (2004). Exploring visuospatial thinking in chemistry learning. *Science Education*, *88*(3), 465–492.
Wutchana, U., & Emarat, N. (2011). Students' Understanding of Graphical Vector Addition in One and Two Dimensions. *Eurasian Journal of Physics & Chemistry Education*, *3*(2), 102–111.
Xing, W., Chen, X., Stein, J., & Marcinkowski, M. (2016). Temporal prediction of dropouts in MOOCs: Reaching the low hanging fruit through stacking generalization. *Computers in Human Behavior*, *58*, 119–129.
Young, E. &. (2016). *Future of Digital Content Consumption in India.* EY.
Zhang, J. (1997). The nature of external representations in problem solving. *Cognitive Science*, *21*(2), 179–217.

APPENDICES

Appendices

1

Name : **APPENDIX 2B1 (pp 257-276)** KPr p1

School : Date :

Instructions: Test Duration: 90 minutes.

1. Please go through all the questions carefully and answer them in the space provided itself.

2. Please donot use any calculator. All the required data is being provided.

3. Draw and do all the rough work also in the space provided only. Ask for any extra sheets if required.

4. Write answers in detail where it is required.

5. Some questions are of multiple choice type and may have more than one correct answer.

6. Please, write down whatever you know or think suitable. These are not going to be graded for their correctness.

Q1 Look at the following picture which of the following quantities are scalars and vectors?

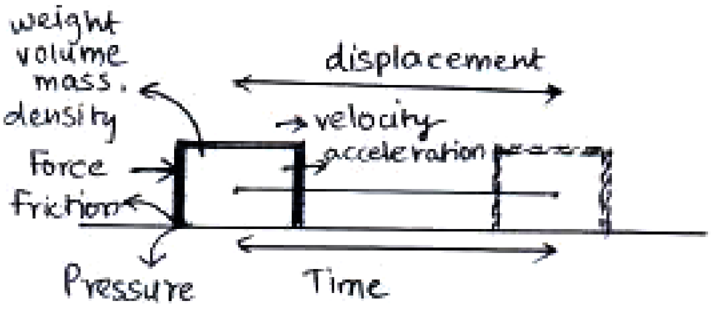
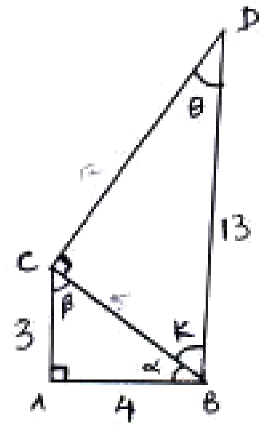


fig. a body in motion

Mass – scalar
Volume – scalar
density – scalar
weight – scalar
force – vector
Friction – vector
Acceleration – vector
velocity – vector
Pressure – scalar
time – scalar
displacement – vector

Q 2

KRrP2



(picture not to scale).

In the adjacent figure, Find

(i) length of AB
(ii) length of BC
(iii) ∠DCB
(iv) length of CD
(v) $\sin(\alpha)$
(vi) $\tan(K)$
(vii) $\cos^2(\beta)+\sin^2(\alpha)$
(viii) $\tan^2(K)-\cot^2(\theta)$

→ i) length of AB = $\boxed{4}$ unit

ii) length of BC =

$= \sqrt{4^2+3^2} = \sqrt{16+9}$

$= \sqrt{25}$

$\boxed{= 5}$ unit

iii) $\angle DCB = \boxed{90^\circ}$

iv) length of CD =

in right ∠ △DCB

$DB^2 = DC^2 + BC^2$

$13^2 = DC^2 + 5^2$

$169 - 25 = DC^2$

$144 = DC^2$

$\boxed{DC = 12}$ unit

v) $\sin\alpha = \frac{AC}{BC}$

$\boxed{\sin\alpha = \frac{3}{5}}$

vi) $\tan(k) = \frac{DC}{BC}$

$\boxed{\tan(k) = \frac{12}{5}}$

vii) $\cos^2(\beta) + \sin^2(\alpha)$

$= \left(\frac{3}{5}\right)^2 + \left(\frac{3}{5}\right)^2$

$= \frac{9}{25} + \frac{9}{25}$

$= \boxed{\frac{18}{25}}$

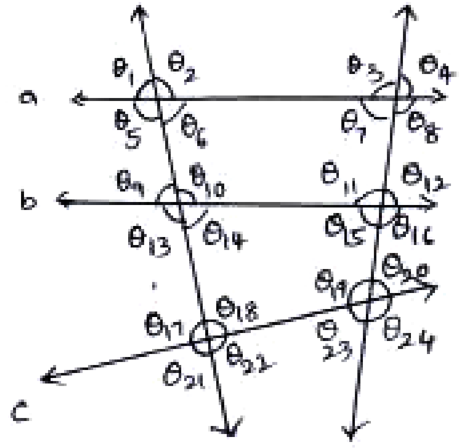
viii) $\tan^2(k) - \cot^2(\theta)$

$= \left(\frac{12}{5}\right)^2 - \left(\frac{12}{5}\right)^2$

$= \frac{144}{25} - \frac{144}{25}$

$= \boxed{0}$

Q 3



In the adjacent figure, lines $a \parallel b$, and all other lines are non-parallel

If $\theta_2 = 110°$, $\theta_3 = 120°$, $\theta_{18} = 100°$,

Find all the other angles.

(Hint: Sum of angles in a quadrilateral = 360°)

① $\theta_1 + \theta_2 = 180°$ — linear pair ∠s
$\Rightarrow \theta_1 = 180 - 110$
$\theta_1 = 70°$

② $\theta_3 + \theta_4 = 180°$ — linear pair
$\theta_4 = 180 - 120$
$\theta_4 = 60°$

③ $\theta_2 = \theta_5$ — opp. ∠s
$\therefore \theta_5 = 110°$

④ $\theta_1 = \theta_6$ — opp. ∠s
$\therefore \theta_6 = 70°$

⑤ $\theta_4 = \theta_7$ — opp ∠
$\therefore \theta_7 = 60°$

⑥ $\theta_1 = \theta_9$ — corresponding ∠'s
$\therefore \theta_9 = 70°$ — $a \parallel b$

⑦ $\theta_2 = \theta_{10}$ — $a \parallel b$
$\therefore \theta_{10} = 110°$

⑧ $\theta_3 = \theta_{11}$ — $a \parallel b$
$\therefore \theta_{11} = 120°$

⑨ $\theta_4 = \theta_{12}$ — $a \parallel b$
$\therefore \theta_{12} = 60°$

⑩ $\theta_{10} = \theta_{13}$
$\therefore \theta_{13} = 110°$

⑪ $\theta_9 = \theta_{14}$
$\therefore \theta_{14} = 70°$

⑫ $\theta_{12} = \theta_{15}$ opp.
$\therefore \theta_{15} = 60°$

⑬ $\theta_{11} = \theta_{16}$ opp.
$\theta_{16} = 120°$

⑭ $\theta_{19} = 130$

⑮ $\theta_{17} = 80°$

⑯ $\theta_{20} = 50°$

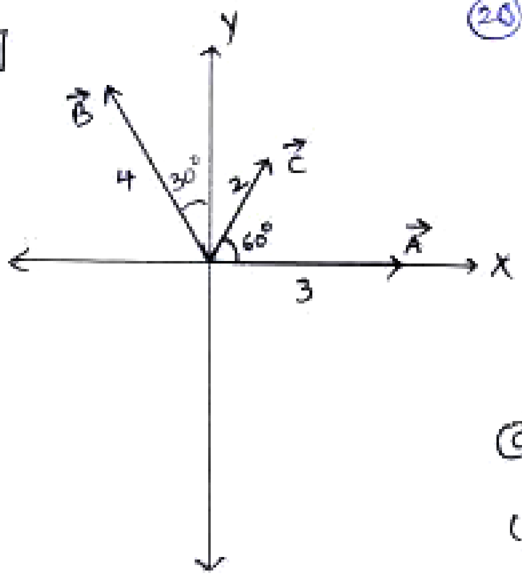
⑰ $\theta_{21} = 100°$

⑱ $\theta_{22} = 80°$

⑲ $\theta_{23} = 50°$

⑳ $\theta_{24} = 130°$

Q 4



In the adjacent figure, $\vec{A}, \vec{B}, \vec{C}$ are three vectors. $\hat{i}$ and $\hat{j}$ are unit vectors along x axis and y axis respectively.

ⓐ Find $\vec{A} + \vec{B}$

(i) using triangle law of addition (just show the geometrical picture)

(ii) using parallelogram law of addition

4

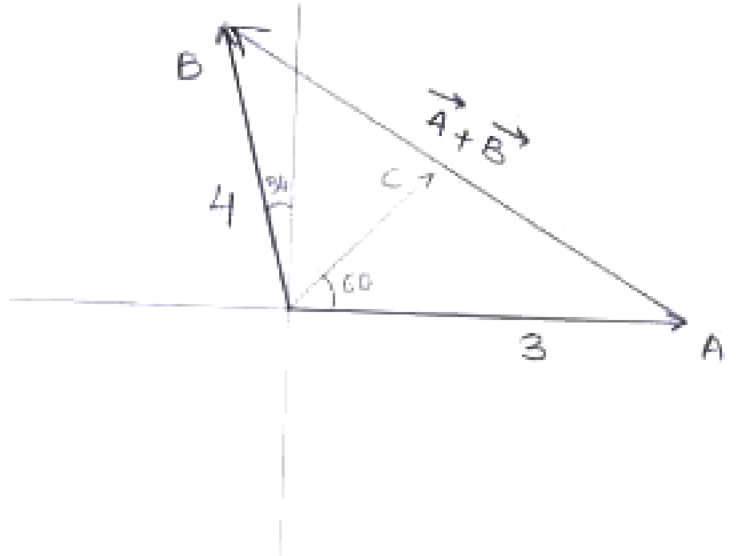
KPrP4

a) $\vec{A} = 3\hat{i}$

$\vec{B} = 4\hat{j}$

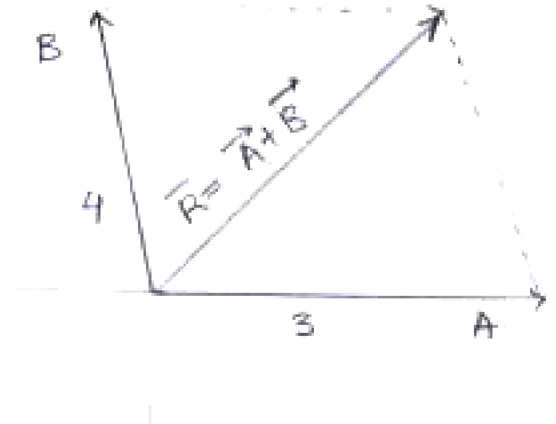
$\vec{A} + \vec{B} = 3\hat{i} + 4\hat{j}$

i)



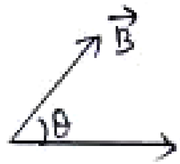
Triangle law of vector addition

ii)



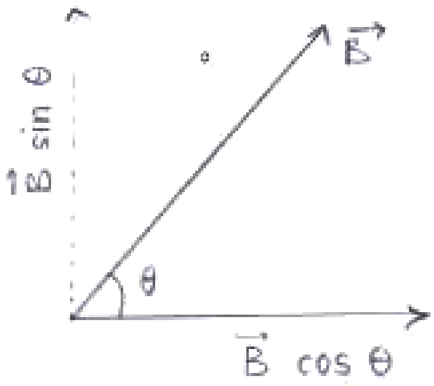
Parallelogram law of vector addition

(b) Break $\vec{B}$ into rectangular components as shown is the below example

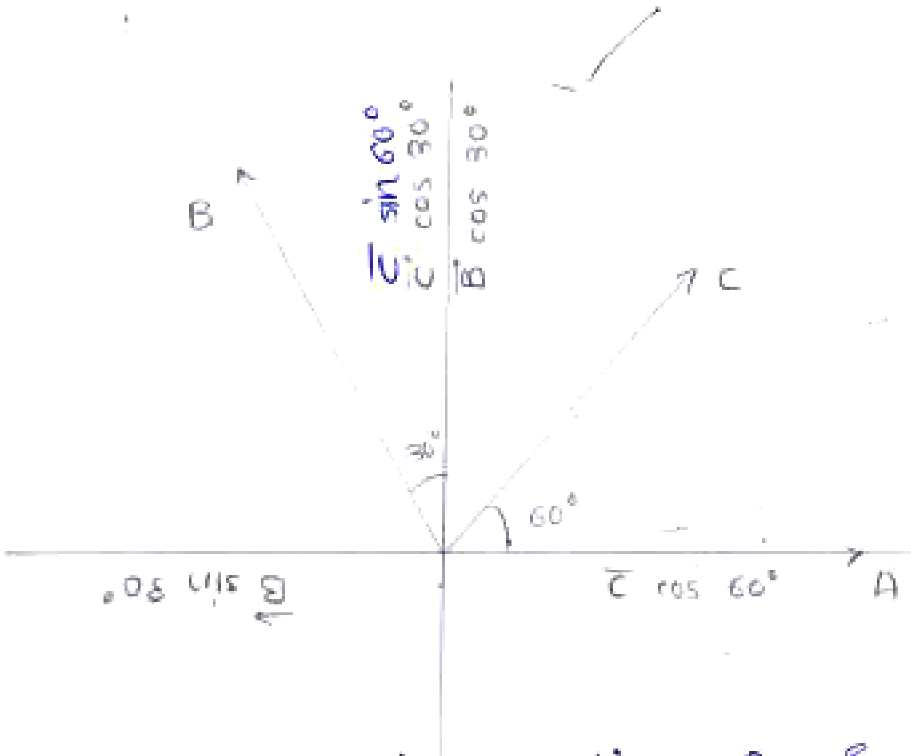


$\vec{B} = |\vec{B}|\cos\theta\,\hat{i} + |\vec{B}|\sin\theta\,\hat{j}$

x Com   y Com



(c) Find $\vec{B}+\vec{C}$ by breaking $\vec{B}$ and $\vec{C}$ into rectangular components



I felt difficult by breaking B & then C & then again their addition

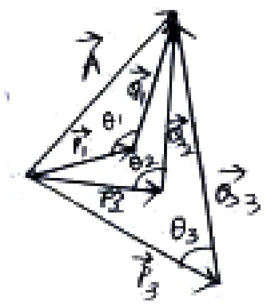
(c) If $\vec{A} = 3\hat{i}$ $\vec{C} = \hat{i} + \sqrt{3}\hat{j}$ Find $\vec{A} + \vec{C}$.

$= \vec{A} + \vec{C}$

$= 3\hat{i} + \hat{i} + \sqrt{3}\,\hat{j}$

$= \boxed{4\hat{i} + \sqrt{3}\,\hat{j}}$

Q 5



In the adjacent figure

$\theta_1 > 90^\circ$ $\theta_2 = 90^\circ$ $\theta_3 < 90^\circ$

State whether the following statements are True or False. Give reasons.

(a) $\vec{P_1} + \vec{Q_1} = \vec{A}$

Yes this statement is true. Because it obeys triangle law of vector addition.

(b) $\vec{P_2}$ and $\vec{Q_2}$ can be called as components of $\vec{A}$.

(c) $\vec{P_1}$ and $\vec{Q_1}$ are components of $\vec{A}$.

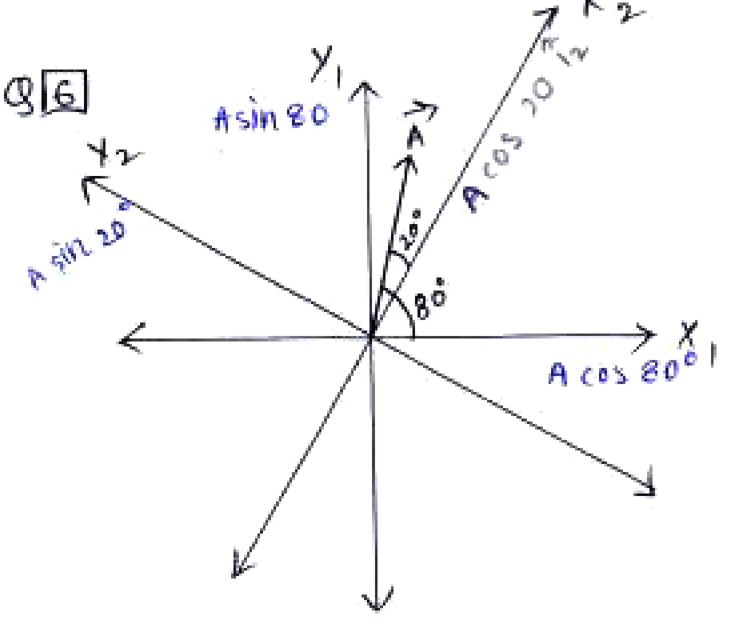
(d) $|\vec{A}|$ can be greater than $|\vec{P_3}| + |\vec{Q_3}|$

(Here θ_3 can be of any value between 0° and 180°)

How can P_1 & Q_1 also P_2, P_3 & Q_2, Q_3 be components they are vectors And if they are components then they have to split also.

KP7 P8

Q 6



In the adjacent figure $X_1 Y_1$ and $X_2 Y_2$ are two different coordinate axes (frames) with same origin.

$\hat{i}_1$ and $\hat{j}_1$ are unit vectors of $X_1 Y_1$

$\hat{i}_2$ and $\hat{j}_2$ are unit vectors of $X_2 Y_2$

(a) Which of the following is/are the correct representation(s) of $\vec{A}$

(i) $|\vec{A}| \sin 20° \hat{i}_1 + |\vec{A}| \cos 20° \hat{j}_1$

(ii) $|\vec{A}| \cos 20° \hat{i}_2 + |\vec{A}| \sin 20° \hat{j}_2$

(iii) $|\vec{A}| \cos 80° \hat{i}_2 + |\vec{A}| \sin 80° \hat{j}_2$

(iv) $|\vec{A}| \sin 80° \hat{i}_2 + |\vec{A}| \sin 80° \hat{j}_2$

(b) Which of the following is/are NOT true?

(i) magnitude of $\vec{A}$ is same in both the frames

(ii) x components of $\vec{A}$ are same in both the frames

(iii) x components of $\vec{A}$ are not same in both the frames

(iv) y components of $\vec{A}$ are not same in both the frames

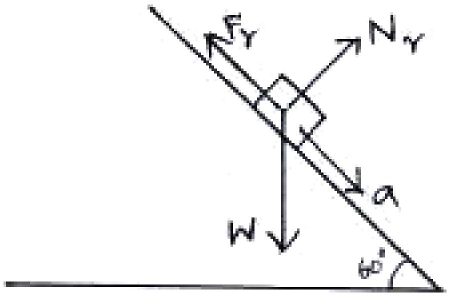
(v) y components are same in both the frames.

a) in part a) one option also should have i.e

$|\vec{A}| \cos 80 \hat{i}_1 + |A|$ ~~cos~~ sin $80 \hat{j}_1$

b) x & y components of different frames should ~~have~~ different

Q 7 Free body diagram of a body of mass 10kg on an inclined plane is given. Assume $g = 10 m s^{-2}$, $\sqrt{3} = 1.73$



$W = 100 N$, $F_r = \mu N_r$.

ⓐ Find μ if the body accelerates at $5.1 m/s^2$

ⓑ If $\mu = 0$, find the acceleration

a) $m = 10 kg$
$W = 100 N$

$F = \mu N$

$\mu = \frac{F}{N} = \frac{F}{mg}$

$= \frac{100}{10 \times 10}$

$\boxed{\mu = 1}$

$\mu = \frac{F}{N}$

$= \frac{ma}{mg}$

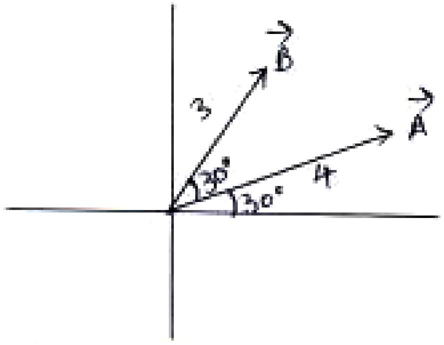
$= \frac{10 \times 5.1}{10 \times 10}$

$\boxed{\mu = 0.51}$

b) About b) part I can't understand when μ become 0 in formula $\frac{F}{N} = \mu$ Whole term become zero if we cross multiply N to μ then it also become 0.

KPrP10

Q [5] Find the dot products of 2 vectors $\vec{A}$ and $\vec{B}$ using two different methods. 1/10



(i) $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos(\theta_{AB})$

(ii) Breaking $\vec{A}$ and $\vec{B}$ as components

$\vec{A} = |\vec{A}|\cos(\theta_A)\hat{i} + |\vec{A}|\sin(\theta_A)\hat{j}$ and

$\vec{B} = |\vec{B}|\cos(\theta_B)\hat{i} + |\vec{B}|\sin(\theta_B)\hat{j}$.

(iii) Compare the two results.

i) $\vec{A}.\vec{B} = |\bar{A}||\bar{B}|\cos\theta$

$= 3 \times 4 \times \cos 30$

$= 12 \times \frac{\sqrt{3}}{2}$

$\boxed{\bar{A}.\bar{B} = 6\sqrt{3}}$

ii) $\vec{A} = |\vec{A}|\cos(\theta_A)\hat{i} + |\bar{A}|\sin(\theta_A)\hat{j}$

$= 4\cos(30)\hat{i} + 4\sin(30)\hat{j}$

$= 4\left(\frac{\sqrt{3}}{2}\right)\hat{i} + \frac{4}{2}\hat{j}$

$= 2\sqrt{3}\hat{i} + 2\hat{j}$

$= 2(\sqrt{3}\hat{i} + \hat{j})$

$\vec{B} = |B|\cos(\theta_B)\hat{i} + |\bar{B}|\sin(\theta_B)\hat{j}$

$= 3\cos(60)\hat{i} + 3\sin(60)\hat{j}$

$= \frac{3}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$

$\vec{A}.\vec{B} = (2\sqrt{3}\hat{i} + 2\hat{j}) \cdot \left(\frac{3}{2}\hat{i} + \frac{3\sqrt{3}}{2})\hat{j}\right)$

$= \left(\frac{2\sqrt{3}\times 3}{2}\hat{i} + 2\times\frac{3\sqrt{3}}{2}\hat{j}\right)$.

$= 3\sqrt{3}\hat{i} + 3\sqrt{3}\hat{j}$

Answer of i) how can compare with answer of ii) because in ii) there are $\hat{i}$, $\hat{j}$ unit vector in i) Ans. there are no such components.

Q9 How are trigonometric ratios and knowledge useful in solving above questions about vectors.

→ components of particular vectors split up into trigo ratios i.e. sin θ & cos θ. When a vector is placed in a co-ordinate axis when it makes an angle with axis it gives its magnitude.

KPrP12

	0	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
cosec	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞
cot	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Name: KPoP1 [1]

School/College:

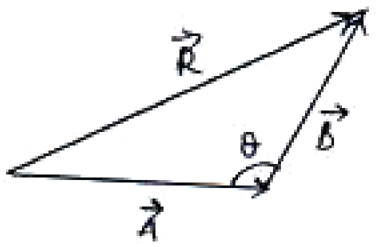
Date:

Instructions:

Test Duration = 40 minutes

1. Please go through all the questions carefully and answer them in the space provided.
2. Please donot use any calculator. All the required data is provided.
3. Draw and do all the rough work also in the space provided. Ask for extra sheets if required.
4. Write answers in detail where it is required
5. Please, write down whatever you know or think is suitable. These are not going to be graded for correctness.

Q [1] In the picture below, state if the following statements are True or False with proper explanation.



(a) The figure follows triangle law of addition - True

(b) For all values of θ from $0°$ to $180°$,

$$|\vec{R}| \leq |\vec{A}| + |\vec{B}|$$

(a) True. Because triangle law of addition states the resultant given by joining tail of 1st vector & head of 2nd vector.

K Po P2 [2]

(c) $\vec{R} = \vec{A} + \vec{B}$

→ ~~True~~. False. Because this expression gives the direction of all the three vectors. These all three vectors are at different direction.

(d) $|\vec{R}| = |\vec{A}| + |\vec{B}|$

True. Because all three vectors have same magnitude Hence they can be added.

False. vector has both magnitude & direction. only magnitude can't give resultant. triangle law states $\bar{R}$ is given by line joining tail & head of other vectors.

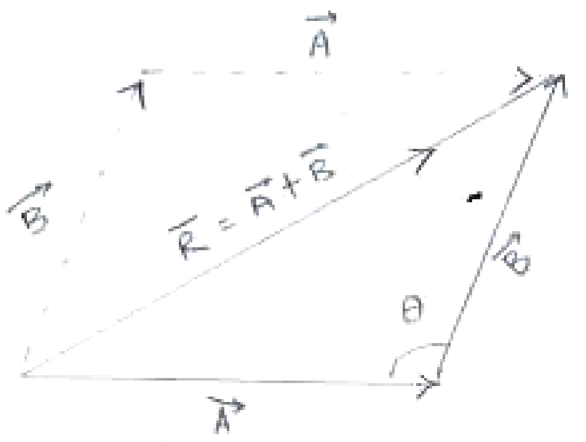
(e) $\vec{A}$ and $\vec{B}$ are components of $\vec{R}$.

True. because there exist angle θ between A & B. As I seen on the tool.

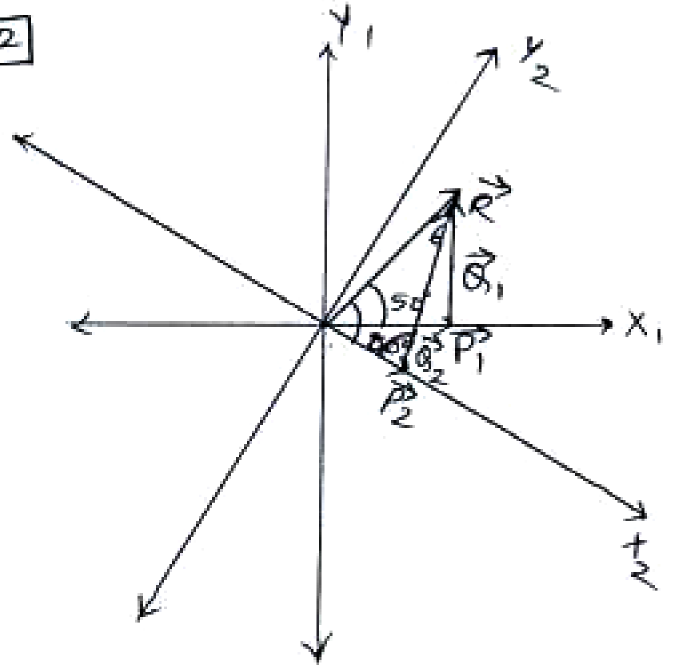
(f) $\vec{R}$ can have other sets of components also.

True. $\bar{R}$ can have components also ranging from angle 180° to 360° i.e. it can have components -ve magnitude.

(g) Draw an equivalent picture of parallelogram law of addition of $\vec{A}$ and $\vec{B}$



Q 2



State True or False with proper explanations.

① $\vec{P}_1 + \vec{Q}_1 = \vec{R}$

True. it obeys triangle law of addition.

② $\vec{P}_2 + \vec{Q}_1 = \vec{R}$

False. because $\overline{P}_2$ is not collinear with $\overline{Q}_1$.

③ If $|\vec{P}_1| = 6\cos 50°$, $|\vec{P}_2| = 6\sin 50°$, $\vec{P}_1 + \vec{P}_2 = \vec{R}$.

last two are false. $|P_2|$ has $= 6 \sin 80$

How can P_2 added to P_1 to give resultant it exceed by P_1 by $30°$.

④ If $\vec{P}_1 = 6\cos 50\,\hat{i}_1$, $\vec{Q}_1 = 6\sin 50\,\hat{j}_1$, $\vec{P}_1 + \vec{Q}_1 = \vec{R}$.

True. because P_1 & Q_1 are components of R on co-ordinate axes $X_1 - Y_1$.

⑤ $|\vec{P}_2| = 6\cos 80°$, if $|\vec{Q}_2| = 6\sin 80°$.

P_2 & Q_2 should be perpendicular to each other as they are components of $\overline{R}$

⑥ if $\vec{P_1} = 6\cos 50\,\hat{\imath}_1$, $\vec{P_1}$ is perpendicular to $\vec{Q_1}$

Data is insufficient if P_1 to Q_1 region is // to axes then it is easy to measure ∠

⑦ if $\vec{P_2}$ and $\vec{Q_2}$ are rectangular components of $\vec{R}$,

$\vec{P_2} = 6\cos 80\,\hat{\imath}_2$

⑧ if $\vec{Q_2} = 6\sin 80^\circ\,\hat{j}_2$ then $\tan 80^\circ = \dfrac{|\vec{P_2}|}{|\vec{Q_2}|}$

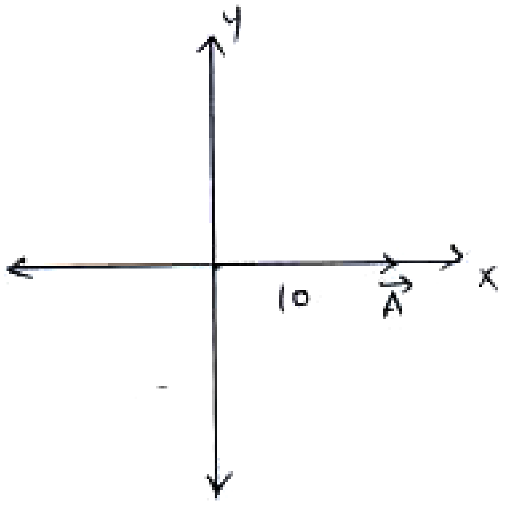
$\overline{P_2} = 6\cos 80^\circ\,\hat{i}$

$\dfrac{\overline{P_2}}{\overline{Q_2}} = \dfrac{6\cos 80\,\hat{i}}{6\sin 80\,\hat{j}}$

$\dfrac{\overline{P_2}}{\overline{Q_2}} = \cot 80^\circ$

It is not true

Q 3



How much should the vector $\vec{A}$ be turned in anti-clockwise direction (↺) so that the x component becomes 5 (half of 10)

(Hint: Refer trigonometric values in the last page)

x component of $\bar{A}$

if we rotate arm in 60° in anticlockwise direction

$A_x = 10 \cos 60°$

$= 10 \frac{1}{2}$

$A_x = 5$ //

It should move by 60° in anticlockwise

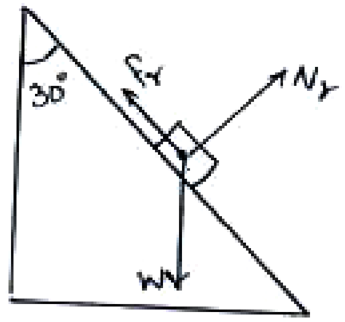
$\cos(-60) = \cos 60$

in anticlockwise dire^n $\theta = 300$

θ can $60°, -60°, 300°$

KPoP6

Q 4



If the body is at rest,

$f_r = \mu_s N_r$, W = 100 Newtons,

(i) What is the value of μ_s.

(ii) Find $\vec{W} + \vec{N_r}$.

① W = 100 N

μ_s = ?

$\frac{F}{N} = \mu_s$

F = 100 N

F = W

N = mg

= 100 × 9.8

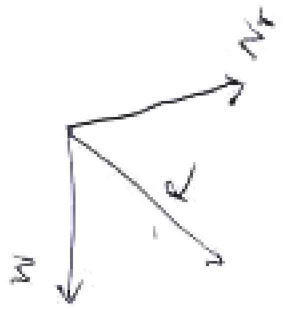
$\mu_s = \frac{F}{mg}$

$= \frac{100}{100 \times 9.8}$

$= \frac{1}{9.8} = \frac{10}{98}$

$= \frac{5}{49}$

μ_s = 0.102



②. W + Nr

= 100 + 100 × 9.8

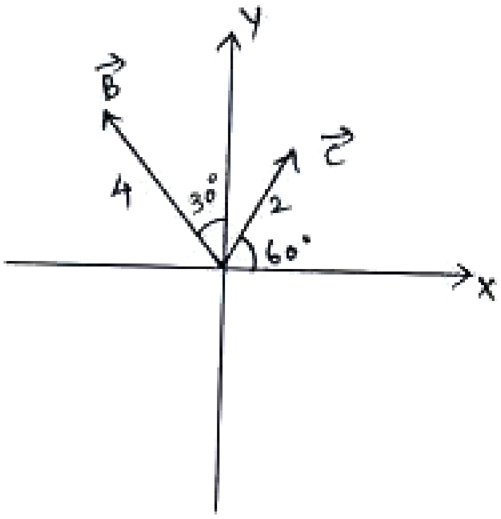
= 100 (1 + 9.8)

= 100 (10.8)

= 1080.0

= 1080

Q 5 [7]



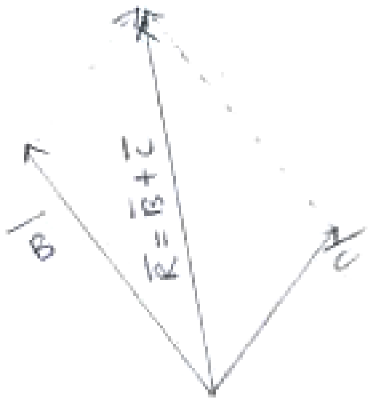
Find $\vec{B}+\vec{C}$ by

(i) triangle law (draw figure)

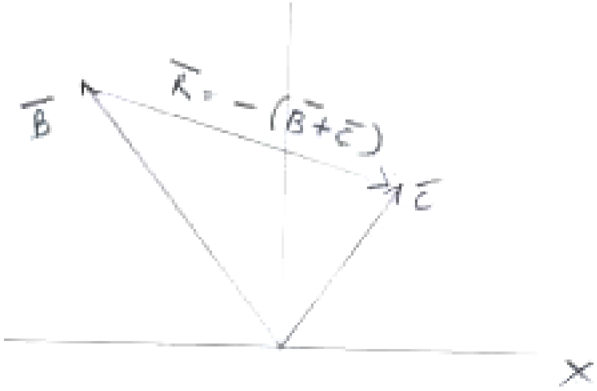
(ii) parallelogram law (draw figure)

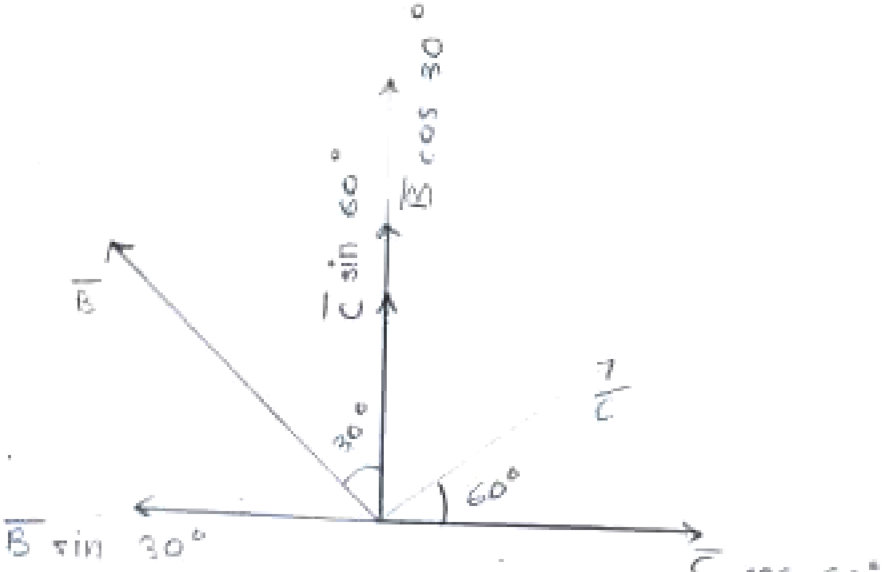
(iii) resolving $\vec{B}$ and $\vec{C}$ into rectangular components.

ii)



i)





KPoP8

	0	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
cosec	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞
cot	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

APPENDIX 2B2 (pp 277-281)

Lagrangian Points L4 and L5

Name: ______________________________

Note:

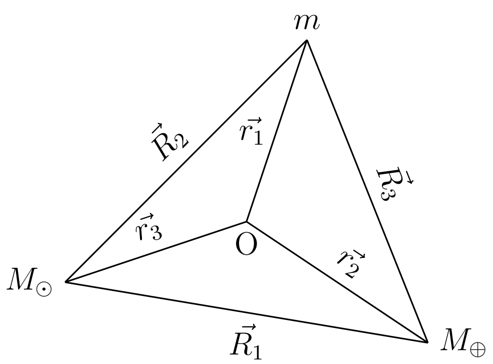
- This derivation is taken from: http://www.phy6.org/stargaze/Slagrng3.htm
- Answer the questions in the spaces provided after each question.
- Your answers will be used in a statistical study about understanding of vectors. This is a science education research project and your answers in this problem sheet do NOT count towards your final score in the OCSC selection. The study is completely impersonal and identity of each student will be kept confidential.
- However, if you still do not want to participate in this study, you may indicate the same by putting a cross mark at the right top corner of this page. It is recommended that you should still solve this problem sheet for your own benefit.

Question:
Let us consider the Earth (mass = $M_\oplus$) and Sun (mass = $M_\odot$) revolving around a common centre of mass O. They complete one revolution in time T. It is possible to keep a test mass m at some specific points in this plane such that it remains in equillibrium with the two bodies. We have already seen three such points (L1, L2 and L3) along the Sun-Earth line. Let us try to find locations of more such points which are not along the Sun-Earth line.

Let us say the test mass is kept at some point, which is not collinear with Sun-Earth.

1. Draw a figure showing the three masses, centre of mass of the system and join all these points by straight lines. We will take object 1 to be the test mass, object 2 to be the Earth and object 3 to be the Sun. Let the vector from the Sun to Earth be $\vec{R_1}$, vector from the Sun to test mass be $\vec{R_2}$ and the vector from the Earth to test mass be $\vec{R_3}$. Let radius vectors from O to P, Earth and Sun be called $\vec{r_1}$, $\vec{r_2}$ and $\vec{r_3}$ respectively.

Solution:



2. All three are revolving around common centre of mass with the same period T. Write expression for the mean magnitude of their angular velocity (ω).

Solution:

$$\omega = \frac{2\pi}{T}$$

3. Write expressions for magnitude of centrifugal force acting on each of them.

Solution:

$$F_{c1} = m\omega^2 r_1 = \frac{4\pi^2}{T^2} m r_1$$
$$F_{c2} = \frac{4\pi^2}{T^2} M_{\oplus} r_2$$
$$F_{c3} = \frac{4\pi^2}{T^2} M_{\odot} r_3$$

4. Recall the fact that the system is in equillibrium. Let us use following notation for mutual gravitational forces acting on these bodies: force on object 1 (test mass) by object 3 (Sun) will be denoted by $\vec{F_{13}}$. Use this notation to write a vecctor equation for equillibrium of each object under the forces acting on it.

Solution:

$$\vec{F_{13}} + \vec{F_{12}} + \frac{4\pi^2}{T^2} m\vec{r_1} = 0$$
$$\vec{F_{23}} + \vec{F_{21}} + \frac{4\pi^2}{T^2} M_{\oplus}\vec{r_2} = 0$$
$$\vec{F_{31}} + \vec{F_{32}} + \frac{4\pi^2}{T^2} M_{\odot}\vec{r_3} = 0$$

5. Add the three vector equations and simplify using Newton's third law. Interpret the resulting equation.

Solution:

$$\frac{4\pi^2}{T^2}(m\vec{r_1} + M_{\oplus}\vec{r_2} + M_{\odot}\vec{r_3}) = 0$$
$$\therefore (m\vec{r_1} + M_{\oplus}\vec{r_2} + M_{\odot}\vec{r_3}) = 0$$

Other forces will cancel each other. e.g. $\vec{F_{13}} = -\vec{F_{31}}$ and so on.
This result is expected as O is the centre of mass.

6. $\vec{r_1}$ can be expressed as sum of other vectors in at least two different ways. Use laws of vector addition to write these expressions.

Solution:

$$\vec{r_1} = \vec{r_3} + \vec{R_2}$$
$$\vec{r_1} = \vec{r_2} + \vec{R_3}$$

Note that both $\vec{R_2}$ and $\vec{R_3}$ are directed towards P.

7. Use the expressions above to find expression for $\vec{r_1}$ in terms of $\vec{R_2}$ and $\vec{R_3}$.

Solution:

$$\begin{aligned}
\vec{r_1}(m + M_\oplus + M_\odot) &= \vec{r_1}m + \vec{r_1}M_\oplus + \vec{r_1}M_\odot \\
&= \vec{r_1}m + (\vec{r_2} + \vec{R_3})M_\oplus + (\vec{r_3} + \vec{R_2})M_\odot \\
&= (m\vec{r_1} + M_\oplus\vec{r_2} + M_\odot\vec{r_3}) + \vec{R_3}M_\oplus + \vec{R_2}M_\odot \\
&= \vec{R_3}M_\oplus + \vec{R_2}M_\odot \\
\therefore \vec{r_1} &= \frac{\vec{R_3}M_\oplus + \vec{R_2}M_\odot}{(m + M_\oplus + M_\odot)}
\end{aligned}$$

8. Let us define $M = (M_\odot + M_\oplus + m)$. Let us resolve $\vec{r_1}$ in two vector components $\vec{\rho_2}$ and $\vec{\rho_3}$, which are along $\vec{R_2}$ and $\vec{R_3}$. Write expression for $\vec{\rho_2}$ and $\vec{\rho_3}$.

Solution:

$$\vec{\rho_2} = \frac{M_\odot}{M}\vec{R_2}, \qquad \vec{\rho_3} = \frac{M_\oplus}{M}\vec{R_3}$$

9. State if following statements are true or false. Justify your answers.
   (a) $\vec{F_{12}}$ is in the same direction as $\vec{\rho_3}$.

   Solution:
   True. $\vec{R_2}$ and hence $\vec{\rho_2}$ as well as $\vec{F_{13}}$ are directed towards P.

   (b) $\vec{F_{13}}$ is in the same direction as $\vec{\rho_2}$.

   Solution:
   True. Justification same as above.

   (c) Centrifugal force at P is in the same direction as $\vec{r_1}$.

   Solution:
   True. Centrifugal force is radially outward from O.

   (d) The vectors $\vec{\rho_2}$, $\vec{\rho_3}$ and $\vec{r_1}$ form a closed triangle.

   Solution:
   True. $\vec{\rho_2}$ and $\vec{\rho_3}$ are componets of $\vec{r_1}$.

   (e) The three forces given above keep the body 1 in equillibrium.

   Solution:
   True. As given in the problem statement.

   (f) The three forces stated here form a closed triangle.

   Solution:
   True. As the forces keep the body in equillibrium, they will add to zero, i.e. they will form a closed triangle.

10. Find ratio of $\dfrac{F_{12}}{F_{13}}$ in terms of ρ_2 and ρ_3.

Solution:
$$\frac{F_{12}}{F_{13}} = \frac{\rho_3}{\rho_2}$$

11. Substitute expressions for these two gravitational forces and $\vec{\rho_2}$ and $\vec{\rho_3}$. Simplify the expression.

280

Solution:

$$\frac{GM_\oplus m}{R_2^2} \times \frac{R_3^2}{GM_\odot m} = \frac{M_\oplus R_3}{M} \times \frac{M}{M_\odot R_2}$$
$$\Rightarrow R_2^3 = R_3^3$$

12. What is relation between R_2 and R_3?

Solution:

$R_2 = R_3$

13. What is relation between R_1 and R_3? Justify your answer.

Solution:

The entire derivation is symmetric about the masses. Same logic will apply if you do simialr analysis at other two vertices.

$$\therefore R_1 = R_2 = R_3$$

14. What can we say about the triangle formed by Sun, Earth and test mass?

Solution:

This is an equilateral triangle.

15. How will the answer vary if we change mass ratio between mass of Sun and Earth?

Solution:

This answer is independent of mass ratio.

16. Will the answer remain the same for circular as well as elliptic orbits?

Solution:

This answer is independent of type of orbit.

APPENDIX 2D1

App 2D1: The conceptual links, broad conceptual areas and sub-conceptual areas outlined in figure 2

Broad Concept Area (BCA)	Sub Concept Areas (SCA)	CL Description	CL Code
Addition of Vectors	Triangle Law (SCA-1)	The understanding of Triangle Law as a law for adding two vectors	1
		The understanding of the importance of directions in adding vectors geometrically using Triangle Law	2
		The understanding of the magnitude of vectors when adding geometrically using Triangle Law	3
	Parallelogram Law (SCA-2)	The understanding of Parallelogram Law as a law for adding two vectors	4
		The understanding of the importance of directions in adding vectors geometrically using Parallelogram Law	5
		The understanding of the magnitude of vectors when adding geometrically using Parallelogram Law	6
	Connecting Triangle and Parallelogram Law	The understanding of how the triangle law and parallelogram laws are performing the same operation.	7
Resolution into Components	Rectangular Components (SCA-3)	Considering the direction of the rectangular components as being determined by the unit vectors.	8
		The understanding that the magnitude of the rectangular components using trigonometric ratios.	9
		The understanding of how to add the vectors as rectangular components.	10
		The understanding of the rectangular components as the parts of the initial vector.	11
	Addition of vectors (non-rectangular components) (SCA-4)	The understanding that when using the triangle law to add two vectors, the initial two vectors are the components of the resultant vector.	12
		The ability to imagine the effect on the magnitudes of the vectors when the component vectors are manipulated.	13
Applying Resolution and Addition	Applying vector operations to Free Body Diagram (SCA-5)	The ability to identify and create a proper frame of reference to solve a given application problem.	14
		The ability to apply resolution of vectors into rectangular components in the context of the force.	15
		The ability to add the resolved forces, to determine the resultant force.	16

Note: The map in figure 2 evolved or analysis purposes in the study (section 4). We present this along with the Table-1 here, as this provides a framework connecting the sections in this paper. The topics that we are interested in are categorised as 3 broad-concept-areas (BCA), which are further categorised in 5 sub-concept-areas (SCA). These constitute 16 links (numbered in fig-2 and listed in table-S1) between various concepts. Link no. 7 is a single link and hence the corresponding SCA was not listed as a separate sub-concept-area.

APPENDIX 2D2 (283)

Rating Sheet Figure in Study-3

5 Sub Concept Areas (SCAs) | 16 Concept Links (CLs) | Pre Test Questions | 5-Scale Rating for the relevant CL-Question Pairs | Irrelevant CL-Question Pair | Irrelevant CL-Question Pair(marked as * by raters) | Pos-test Questions

PRE TEST

Sub Concept Areas (SCA)	CL Description	CL Code	4.a.i	4.a.ii	4.b	4.c	4.d	5.a	5.b	5.c	5.d	6.a	6.b	7.a	7.b	8.ii	8.iii	Total	CL %	SCA %
Triangle Law (SCA-1)	The conceptual understanding of **Triangle Law as a law for adding two vectors**	1	2	-	-	-	-	1	3	-	2	-	-	-	-	-	-	8	40%	38%
	The conceptual understanding of the importance of **directions** in adding vectors geometrically using **Triangle Law**	2	1	-	-	-	-	1	3	-	3	-	-	-	-	-	-	8	40%	
	The conceptual understanding of **magnitude** of vectors when adding geometrically using **Triangle Law**	3	1	-	-	-	-	1	2	-	3	-	-	-	-	-	-	7	35%	
Parallelogram Law (SCA-2)	The conceptual understanding of **Parallelogram Law as a law for adding two vectors**	4	-	1	-	-	-	-	-	-	-	-	-	-	-	-	-	1	20%	23%
	The conceptual understanding of the importance of **directions** in adding vectors geometrically using **Parallelogram Law**	5	-	1	-	-	-	-	-	-	-	-	-	-	-	-	-	1	20%	
	The conceptual understanding of magnitude of vectors when adding geometrically using Parallelogram Law	6	-	1	-	-	-	-	-	-	2	-	-	-	-	-	-	3	30%	
Connecting Triangle and Parallelogram Law	The conceptual understanding of how the triangle law and parallelogram laws are the performing the same operation.	7	-	-	-	-	-	-	3	1	-	-	-	-	-	-	-	4	40%	
Rectangular Components (SCA-3)	Considering the direction of the rectangular components as being determined by the unit vectors.	8	-	-	4	3	-	-	-	-	-	2	-	-	-	3	4	16	64%	64%
	The conceptual understanding that the magnitude of the rectangular components are determined using trigonometric ratios.	9	-	-	3	4	-	-	-	-	-	1	-	-	-	4	4	16	64%	
	The understanding of how adding the vectors as rectangular components.	10	-	-	-	4	4	-	-	-	-	-	-	-	-	-	-	8	80%	
	The understanding of the rectangular components as the parts of intial vector.	11	-	-	-	-	-	-	3	3	-	-	1	-	-	-	-	7	47%	
Addition of vectors (non rectangular components) (SCA-4)	The understanding that when using the triangle law to add two vectors, the initial two vectors are the components of the resultant vector.	12	-	-	-	-	-	-	4	4	-	-	-	-	-	-	-	8	80%	65%
	The ability to imagine the effect on the magnitudes of the vectors when the component vectors are manipulated.	13	-	-	-	-	-	-	-	-	4	-	1	-	-	-	-	5	50%	
Applying Vector operation to Free Body Diagram (SCA-4)	The ability to identify and create a proper frame of reference to solve a given application problem.	14	-	-	-	-	-	-	-	-	-	1	1	3	2	-	-	7	35%	42%
	The ability to apply resolution of vectors into rectangular components in the context of the force.	15	-	-	-	-	-	-	-	-	-	-	-	3	2	-	-	5	50%	
	The ability to add the resolved forces, to determine the resulatant force.	16	-	-	-	-	-	-	-	-	-	-	-	2	2	-	-	4	40%	

POST TEST

CL Code	1 a	1 b	1 c	1 d	1 e	1 f	1 g	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	3	4.i	4.ii	5.i	5.ii	5.iii	Total	CL %	SCA %
1	2	1	3	3	2	3	-	4	4	4	2	*	*	2	1	-	-	-	4	-	-	35	54%	53%
2	1	1	3	3	2	3	-	4	4	4	2	*	*	4	1	-	-	-	2	-	-	34	52%	
3	1	1	3	3	2	3	-	4	4	4	2	*	*	4	1	-	-	-	2	-	-	34	52%	
4	-	-	-	-	-	-	4	-	-	3	4	4	*	*	1	-	-	-	-	4	-	20	67%	62%
5	-	-	-	-	-	-	4	-	-	3	4	4	*	*	1	-	-	-	-	4	-	20	67%	
6	-	1	-	2	-	-	4	-	-	3	4	4	*	*	1	-	-	-	-	2	-	21	53%	
7	-	-	-	-	1	4	4	-	-	4	-	-	*	4	1	-	-	-	-	-	-	18	60%	
8	-	-	-	-	-	-	-	-	4	4	3	-	3	3	-	-	-	-	-	-	4	21	70%	78%
9	-	-	-	-	-	-	-	-	-	4	4	4	3	3	3	5	-	-	-	-	5	31	78%	
10	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	4	4	80%	
11	-	-	-	-	-	-	-	-	-	4	4	4	*	4	-	-	-	-	-	-	5	21	84%	
12	-	-	-	-	1	4	-	-	-	4	2	4	*	-	-	-	-	-	-	-	-	15	60%	59%
13	-	1	-	1	-	4	-	-	-	4	-	4	*	1	-	5	-	-	-	-	-	20	57%	
14	-	-	-	-	-	-	-	-	-	4	-	4	4	3	-	5	4	1	-	-	-	25	71%	54%
15	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	5	1	-	-	-	6	60%	
16	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	2	1	-	-	-	3	30%	

APPENDIX 2D3 (pp 284-288)

App 2D3: Rubric used by the raters to rate the pre-post test written scripts of the students.

CL Code	CL Description	1 (No indication of ability to handle the link)	2 (Very little familiarity with the skill)	3 (Inconsistent Procedure) Trying to impose the textbook understanding without any modification.	4 (Inconsistent Concept/ Procedure- applying with some changes from a regular textbook usage)	5 (Strong Conceptual understanding)
1	The conceptual understanding of Triangle Law as a law for adding two vectors	No understanding of the addition of vectors with respect to the triangle law of vectors.	Students exhibit little familiarity (even procedural) of the triangle law being linked to the addition of vectors.	Students use direct diagrams from their textbook, without any sense of application to the given context.	Students' responses exhibit correct usage or indications of correct understanding (some indirect indicators like a consequent conclusion) of the use of Triangle Law being an addition of two vectors.	Students' response exhibits correct usage (Drawing proper Diagram of Triangle Law with accurate depictions of magnitudes and directions with labels) or indications of correct understanding (some indirect indicators like a consequent conclusion) of the use of Triangle Law being an addition of two vectors.
2	The conceptual understanding of the importance of directions in adding vectors geometrically using Triangle Law	No understanding of the significance of direction in adding vectors using triangle law is indicated.	Students exhibit limited or no specific emphasis on the directions of the vectors concerned in the triangle law.	Has made the correct direction and diagram of triangle law, but no customization to the given problem.	Shifting any of the vectors. Maintaining proper order of the vectors.The procedure is intact with inconsistent conceptual basis.	Some indicator of the strong conceptual basis (shifting the vector laterally or not giving some conceptual basis) of the significance of the directions of the vectors concerned with the triangle law.
3	The conceptual understanding of the magnitude of vectors when adding geometrically using Triangle	No understanding of how the magnitudes of the vectors are related in the triangle law.	Students show a limited understanding of the magnitude of vectors in relation to the triangle law of addition. (inconsistent attempts to use the	Students use some direct application of textbook formulae pertaining to the sides of the triangle, trying to make sense of the magnitudes of the vectors.	Students exhibit a limited conceptual understanding of how the magnitudes of the vectors are related and use a sound procedural understanding of manipulating the magnitudes	Students exhibit a very strong conceptual understanding of how the magnitudes of the vectors in the triangle law are related. These can also be indicated from their responses from the answers which are consequences of the sound

	Law		procedures from the textbooks)		(sides) of the triangle.	conceptual understanding.
4	The conceptual understanding of Parallelogram Law as a law for adding two vectors	No understanding of the addition of vectors with respect to the parallelogram law of vectors.	Students exhibit little familiarity of the parallelogram law being linked to the addition of vectors.	Students use direct diagrams from their textbook, without any sense of application to the given context.	Students' response exhibits correct usage or indications of correct understanding (some indirect indicators like a consequent conclusion) of the use of Parallelogram Law being an addition of two vectors.	Students response exhibits correct usage (Drawing proper Diagram of Parallelogram Law with accurate depictions of magnitudes and directions with labels) or indications of correct understanding (some indirect indicators like a consequent conclusion) of the use of Parallelogram Law being an addition of two vectors.
5	The conceptual understanding of the importance of directions in adding vectors geometrically using Parallelogram Law	No understanding of the significance of direction in adding vectors indicated.	Students exhibit limited or no specific emphasis on the directions of the vectors concerned in the Parallelogram law.	Has made the correct direction and diagram of parallelogram law, but no customization to the given problem.	Shifting any of the vectors. Maintaining proper order of the vectors.The procedure is intact with an inconsistent conceptual basis of the directions in relation to the parallelogram law.	Some indicator of the strong conceptual basis (shifting the vector laterally or not giving some conceptual basis) of the significance of the directions of the vectors concerned with the parallelogram law.
6	The conceptual understanding of the magnitude of vectors when adding geometrically using Parallelogram Law	No understanding of how the magnitudes of the vectors are related in the parallelogram law.	Students show a limited understanding of the magnitude of vectors in relation to the parallelogram law of addition. (inconsistent attempts to use the procedures from the textbooks)	Students use some direct application of textbook formulae pertaining to the parallelogram law, trying to make sense of the magnitudes of the vectors.	Student modifies the textbook procedures with an inconsistent conceptual understanding of the magnitude of the vectors in relation to the parallelogram law of vector addition.	Students exhibit strong conceptual understanding resulting in suitable modification of the procedures to suit the required situation.

7	The conceptual understanding of how the triangle law and parallelogram laws are performing the same operation.	No understanding of any link between triangle and parallelogram law.	The student indicates some limited understanding of the link between triangle and parallelogram law.	The student uses the textbook definitions and algorithms literally to make a connection between the triangle and parallelogram law.	The student uses the textbook knowledge of triangle and parallelogram law to synthesize a link between triangle and parallelogram law, which however is not strongly based conceptually.	Students indicate a strong and correct conceptual basis to the link between the triangle and parallelogram law.
8	Considering the direction of the rectangular components as being determined by the unit vectors.	No understanding of unit vectors as directional entities	Students exhibit a limited understanding of the unit vectors as directional entities of the rectangular components of a vector.	Students exhibit typical textbook explanations of unit vectors, with no strong indicators of conceptual understanding.	Students exhibit typical textbook explanations of unit vectors, with some strong indicators of conceptual understanding.	Students exhibit typical textbook explanations of unit vectors, with many strong indicators of conceptual understanding.
9	The conceptual understanding that the magnitude of the rectangular components is determined using trigonometric ratios.	No understanding of what the rectangular components are and how are they related to the trigonometry	Very limited familiarity (inconsistent or improper usage of textbook mode) with the magnitude of the rectangular components of the vectors.	Students exhibit a typical textbook mode of resolving vectors as $r\cos\theta$ and $r\sin\theta$ directly and mechanically	Students exhibit some limited notions of how the trigonometry of the right triangle is used to obtain the magnitude of rectangular components.	Students exhibit a strong conceptual understanding of how trigonometry of the right triangle is used to obtain the magnitude of rectangular components
10	The understanding of how to add the vectors as rectangular components.	No understanding of how to add rectangular components	Students indicate a limited procedural understanding of how to add rectangular components (like adding like terms, but miss the unit vectors)	Students exhibit typical textbook mode of adding rectangular components of vectors.	Students exhibit a limited conceptual understanding of how rectangular components of vectors add up to give the resultant vector.	Students exhibit a strong conceptual understanding of how the rectangular components of the vectors add up to give the resultant vector.

11	The understanding of the components as the parts of the initial vector.	No understanding of how the rectangular components relate to the initial vector.	Inconsistent procedural indications of how the rectangular components relate to the initial vector.	Students indicate a textbook kind of understanding of the rectangular components relating to the initial vector	Students exhibit a good procedural understanding of the relation between rectangular components and the initial vector but a limited conceptual understanding.	Students exhibit good conceptual and procedural understanding of the relation between rectangular components and the initial vector.
12	The understanding that when using the triangle law to add two vectors, the initial two vectors are the components of the resultant vector.	No ability to link addition of vectors in triangle law with the components notion of a vector.	Students exhibit a very limited understanding of the possibility of the two vectors added by triangle law being the components of the resultant vector.	Students exhibit no understanding of the two added vectors being components of the resultant vector other than having certain mixed claims of applying triangle law to rectangular components or similar ones.	Students exhibit a good procedural understanding of the added vectors being components of the resultant vector but lack the ability to extend this and draw other consequences.	Students exhibit a strong conceptual understanding of the two added vectors being the components of the resultant vector(by showing the possibility of other components etc).
13	The ability to imagine the effect on the magnitudes of the vectors when the component vectors are manipulated.	No indication of understanding pertaining to manipulation of magnitudes.	Students exhibit a limited understanding of the magnitude	No ability indicated to dynamically simulate the vectors	Students exhibit limited conceptual understanding to manipulate the vectors.	Students exhibit strong conceptual understanding to manipulate the vectors.
14	The ability to identify and create a proper frame of reference to solve a given application problem.	No indication of how to handle the vectors in the rotated-frame of reference.	Students proceed without considering any rotated-frame of reference.	Mechanically try to find out the direction of the new vector.	Students exhibit a limited attempt to make conceptual connections from other topics.	Students exhibit a good attempt to make conceptual connections from other topics.

15	The ability to apply resolution of vectors into rectangular components in the context of the force.	No indication of how to handle Force as vectors.	Students proceed with limited ability to handle force as vectors and try resolving into rectangular components.	Students exhibit a direct application of resolution from the textbook but do not show any indications of customization to the given case.	Students exhibit improper attempts to customize the conventional procedure to the given case, indicating a limited conceptual understanding of applying vector resolution in this context of Force.	Students exhibit indications of proper transfer of the notion of resolution of vectors in this context of force.
16	The ability to add the resolved forces, to determine the resultant force.	No indication of how to handle Force as vectors.	Students proceed with limited ability to handle force as vectors and try resolving into rectangular components.	Students exhibit direct application of resolution from the textbook but do not show any indications of customization to the given case.	Students exhibit improper attempts to customize the conventional procedure to the given case, indicating a limited conceptual understanding of applying vector resolution in this context of Force.	Students exhibit indications of proper transfer of the notion of resolution of vectors in this context of force.

APPENDIX 2D4 (2 Pages)

App.2D4 SUMMARY TABLE - Table linking the three limitations of the paper-based medium and the way the new-media design attempts to address those using tasks and the changes in the behaviour seen in the students.

Limitations	Missing SCAs/CLs in the Textbooks (from the textbook analysis in section-2,3)	Hypothesis	Indicative evidence in students & teachers (Section-3)	Design principles for TFV1 (section-4)	Design of Tasks in Intervention	Changes in Student Behaviour (Section 5)
Limited Geometric Manipulation	• Statements in Textbooks ○ Adding vectors geometrically can be tedious. A neater and easier technique involves algebra but requires that the vectors be placed on a rectangular coordinate system (Halliday, Resnick, & Walker, 2013) ○ Although the graphical (geometrical) method of adding vectors helps us in visualising the vectors and the resultant vector, it is sometimes tedious and has limited accuracy. It is much easier to add vectors by combining their respective components (NCERT Grade 11 Textbook Physics-1, 2013) • No Practice problems related to geometric laws, usage of ijk components while solving problems	Dominant Reliance on the Algebraic modes of reasoning and understanding vectors	• High tendency to use rectangular components. • Preference for algebraic approaches while explaining or solving (even when reasoning using geometric methods is best) • Superficial and meaningless usage of algebra • Algebra of scalars as a habit • Gesturing the algebraic manipulations and structures (like a square root in the expression for the resultant vector) • Written scripts filled with formulae and algebraic manipulations • Definitions as statements with verbalized algebra	• Dynamic And Tangible Representations ○ Make the geometrical entities tangible, dynamic and manipulable and linked to algebraic entities. ○ Unpacking the underlying processes of addition and resolution dynamically. • Interactive ○ Active agency in making the manipulations instead of passive visualisations ○ Make the interactions for manipulation consistent with the conceptual links. ○ Circle as an interaction entity	• All the tasks required students to manipulate geometric entities on the screen. Especially Tasks 4, 6, 8, 9 and 10 are completely open-ended and would provide exploratory contexts to manipulate the geometric entities. (See Table-S4) • Students arrive at methods of solving the problems on their own. The vector model is progressively applied and refined during interactions.	• Usage of more geometric entities like diagrams in responding to the questions. • Gestural evidence hinting at the change in the modes of reasoning from more of a mechanical recollection of static formulae to a dynamic imagination of the vectors geometrically (visuo-spatially) along with transformations performed (Karnam et al., 2018).

Serial Ordering	(Karnam & Sule, 2018) • Topics spread across the textbooks. Topics not connected correctly and • sufficiently.	Improper Linkages between the topics	• Order of vectors ignored in Triangle Law of addition • Resolution and addition understood as non-related operations • Establishing equivalence of addition methods (triangle and parallelogram and using ijk components) • Difficulty in integrating the knowledge of trigonometric ratios and geometry of circles, and triangle, while resolving vectors • Difficulty in identifying the ease that rectangular components bring to the algebra of vectors • Difficulty with understanding products	• Real-time integration of the changes in geometry and algebra (denotations and operations) • Order of vectors in Triangle Law of addition • Addition as the inverse of resolution (and the notion of non-rectangular components) • Dynamics in Resolution involving a circle and right triangle and trigonometric ratios • Addition using rectangular components	• A range of tasks – manipulation-based, exploratory, closed-ended (Tasks 1, 3, 5, 7) and open-ended (not single solution Tasks 8, 9, 10) – covering all the concepts related to the conversion of the vector from geometric to algebraic denotation, resolution, and addition using triangle and rectangular components.	• Indicators of change in conceptual understanding especially for those related to components and improvement in triangle law of vector addition and non-rectangular components. • Evidence from the link strength analysis
Opaque Problem solving	(Karnam & Sule, 2018) • Reasons for methods of problem-solving not presented. Heuristics not presented. No explanations in example problems.	Mechanical and algorithmic approaches to problem-solving	• Superficial and Procedural description and attempts to map to standard algorithms of problem-solving	• Present the related topics in an integrated manner enabling the creation of a coherent model for students. (E.g. the unit circle to integrate trigonometry, geometry and vector resolution and addition)	• Students arrive at methods of solving the problems on their own. The vector model is progressively applied and refined during interactions. • All the open-ended tasks (such as 4,6,8,9, and 10) were designed to push students to look for models, beyond the blind application of formulae. This, in turn, could help in applying vectors during problem-solving in mechanics.	• Usage of diagrams and figures and gestures indicate usage of model-based reasoning rather than mechanical formula recital while solving problems, making them less opaque.

APPENDIX 2E1 (291)

Sample Touchy Feely Vector (TFV) Launcher for NCERT Grade 11 Physics Textbook

Name	QR Code	Description	Link to Textbook[1]
TFV Help		Help file for TFV	
Exploratory Tool		Creation of vector, Manipulation, Addition (Triangle law, Parallelogram law), Resolution and addition using rectangular components	Complete Chapter 4
Task 1		Equality of Vectors	Section 4.2.2: Equality of Vectors Page 66
Task 2		Triangle law of Vector Addition	Section 4.4: Addition of Vectors Page 67
Task 3		Parallelogram law of Vector Addition	Section 4.4: Addition of Vectors Page 67
Task 4		Resolution of Vectors	Section 4.5: Resolution of Vectors Page 69
Task 5		Vector addition using rectangular components	Sec 4.6: Vector Addition: Analytical Method Page 71
3D Vectors		Vector manipulation in 3D	Exploratory

[1] All the page numbers are from the above NCERT Grade 11 Physics-1 Textbook

APPENDIX 2F1 (pp 292-299)

Name: **Grade & Div:**
School/ College: **Teacher:**

Did you use the tool in the classroom? Yes/ No

Note: Across this sheet, vectors are denoted as $\overline{P}$ *and their magnitudes are denoted using the small letter p.*

I volunteer to participate in a research project conducted by K DurgaPrasad from Homi Bhabha Center for Science Education, TIFR-Mumbai. I understand that the project is designed to gather my understanding of certain topics in Mathematics and Physics.

1. My participation in this project is voluntary and I was noted about this research by my school.
2. I understand that most participants will find the session interesting and thought-provoking. If, however, I feel uncomfortable in any way during the session, I have the right to decline to answer any question or to end the session.
3. Participation involves being part of the activity facilitated by K DurgaPrasad from Homi Bhabha Center for Science Education, TIFR – Mumbai. I understand that -
   1. The session will last approximately 70-80 minutes.
   2. The session includes a small test of about 70 minutes.
   3. Notes will be written during the interaction.
4. All the data collected will solely be used for the proposed research purpose.
5. I understand this participation does not involve any aspect of formal academic assessment done by the school or any other certifying body affecting my grading.
6. I understand that the researcher will not identify me by name in any reports using information obtained from this intervention, and that my confidentiality as a participant in this study will remain secure. Subsequent uses of records and data will be subject to standard data use policies which protect the anonymity of individuals and institutions.
7. I have read the above points and understand the explanation provided to me. I have had all my questions answered to my satisfaction, and I voluntarily agree to participate in this study.

________________________ ______________________

My Signature *Date*
My Age: ______________
*My Guardian's Name:*______________________
*Contact Number:*__________________________

Q-1. Underline the vectors in the following physical quantities.

mass, volume, velocity, force, time, density, current, torque, speed, energy, work, acceleration

Look at the set of vectors given below and answer the questions following the figure. Q-2 and Q-3 are based on this figure.

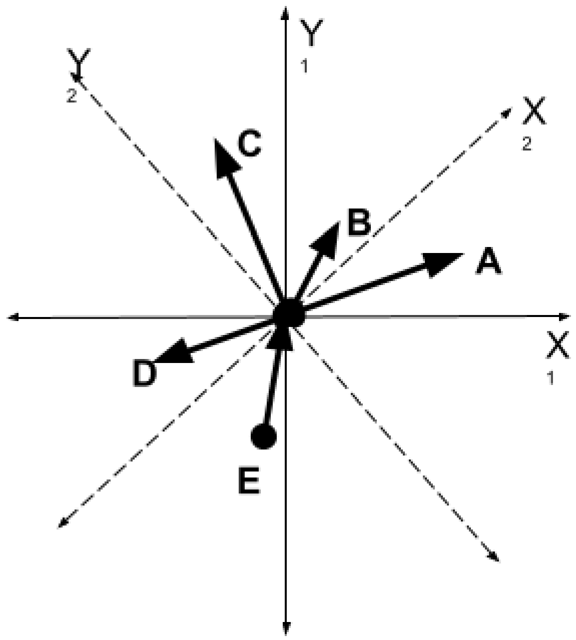


Figure-1 Description:There are two frames of reference X_1Y_1 and X_2Y_2. They are of the same scale. $\hat{i}_1, \hat{j}_1$ and $\hat{i}_2, \hat{j}_2$ are the unit vectors along X_1Y_1 and X_2Y_2 respectively. The magnitudes and the directions of the vectors are denoted using suitable subscripts for all the vectors with respect to both the frames of reference. For example, $\overline{A}$ makes an angle of θ_{A1} with positive X_1 and and θ_{A2} with positive X_2. Magnitude of $\overline{A}$ is r_{A1} and r_{A2} respectively. Similarly, $\overline{B}$makes an angle of θ_{B1} with positive X_1 and and θ_{B2} with positive X_2. And magnitude of B is r_{B1} and r_{B2} respectively. All the vectors have a similar magnitudes and directions.

Q-2. **Draw diagrams of vector addition as required in the table below.** *[Note: Please make rough indicative diagrams (capturing the magnitudes and directions of the vectors as close as possible). They need not be very neatly done using pencils and scale.]*

	Triangle Law of Addition	**Parallelogram Law of Addition**
$\overline{A} + \overline{B}$		
$\overline{A}$ - $\overline{B}$		

$\overline{A} + \overline{E}$		
$\overline{E} + \overline{D}$		

Q-3. Write the vectors $\overline{A}$ in the form of rectangular components (like $r\cos(\theta)\hat{i} + r\sin(\theta)\hat{j}$) along X_1Y_1 and X_2Y_2.

Q-4. State True or False and reasons for you to think that way.

1. The magnitude of the vector changes from one frame of reference to other. i.e., $r_{A1} \neq r_{A2}$

2. $r_{A1} \cos(\theta_{A1}) \hat{i}_1 + r_{A1} \sin(\theta_{A1}) \hat{j}_1 = r_{A2} \cos(\theta_{A2}) \hat{i}_2 + r_{A2} \sin(\theta_{A2}) \hat{j}_2$

3. The y components of B change from one frame of reference to other.

4. The x components of B are same in both the frames of references.

Q-5. Show using a diagram the following additions.

1. $(2\hat{i} + 3\hat{j}) + (3\hat{i} + 4\hat{j})$

2. $(2\hat{i} + 3\hat{j}) + (-3\hat{i} + 4\hat{j})$

Choose the correct answer (multiple answers allowed) and state the reason. (Q-6, Q-7 and Q-8)

Q-6. $\overline{P}$ and $\overline{Q}$ are two vectors at an angle of 45° with each other at their co-initial points are tails, and $|\overline{P} + \overline{Q}|$ (magnitude of their resultant $(\overline{P} + \overline{Q})$) is 5 units. If magnitudes of $\overline{P}$ and $\overline{Q}$ are fixed and the angle between them is increased to 120°, which of the following could be the value of $|\overline{P} + \overline{Q}|$?

a) 3 units b) 5 units c) 8 units d) 9 units e) 2 units

Reason:

Q-7. If $\overline{P}$ and $\overline{Q}$ are two **collinear** vectors, and their resultant is $\overline{R}$.

(**magnitudes** are represented by small letters p,q and r respectively)

1. If p = q , r = 0

   a) Never b) Sometimes Possible c) Always true

   Reason:

2. If p = q , r = 2p

   a) Never b) Sometimes Possible c) Always true

   Reason:

3. r = p + q

   a) Never b) Sometimes Possible c) Always true

   Reason:

4. r = $\sqrt{p2 + q2}$
a) Never b) Sometimes Possible c) Always true
Reason:

Q-8. If $\overline{P}$ and $\overline{Q}$ are two **coplanar** vectors, and their resultant is $\overline{R}$.
(**magnitudes** are represented by small letters p,q and r respectively)

1. If p = q , r = 0
a) Never b) Sometimes Possible c) Always true
Reason:

2. If p = q , r = 2p
a) Never b) Sometimes Possible c) Always true
Reason:

3. r = p + q
a) Never b) Sometimes Possible c) Always true
Reason:

4. r = $\sqrt{p2 + q2}$
a) Never b) Sometimes Possible c) Always true
Reason:

Q-9. $\overline{P}$ and $\overline{Q}$ are two coplanar vectors and their resultant is $\overline{R}$ (i.e $\overline{R} = \overline{P} + \overline{Q}$).

1. If $\overline{P} \perp \overline{Q}$, then $\overline{P}$ and $\overline{Q}$ are called ___________ of $\overline{R}$.

a) rectangular components b) components c) resultant

Reason:

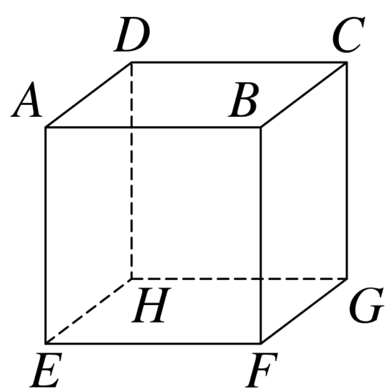
2. If $\overline{P} \perp \overline{Q}$, then p = _______ q = _______. ($\overline{R}$ makes an angle θ with $\overline{P}$)

Fill the blanks with a suitable option from the below.

a) r cos (θ) b) r sin (θ) c) q cos (θ) d) q sin (θ)

Reason:

Q-10. In the cube below, all the edges of the cubes are taken as vectors. The directions of them is given by the order of letters in their names ($\overline{AB}$represents a vector from corner A to corner B). State whether the following statements are true or false along with the reasons.



1. $\overline{AB} = \overline{HG}$: True/False

Reason:

2. $\overline{DH} = -\,\overline{BF}$: True/False

Reason:

3. $\overline{AB} + \overline{AD} = \overline{AC}$: True/False

Reason:

4. $\overline{AB} + \overline{AD} + \overline{AE} = \overline{AG}$: True/False
Reason:

5. $|\overline{AB}| + |\overline{AD}| + |\overline{AE}| = |\overline{AG}|$: True/False
Reason:

6. $\overline{AB}, \overline{AD}, and\ \overline{AE}$ form one set of the rectangular components of $\overline{AG}$: True/False
Reason:

Q-11. How did you find the questions in this test?

1. Difficulty level

a) very easy Q No: ___________________

b) difficult Q No: ___________________

c) tricky Q No: ___________________

d) could not understand at all Q No: ___________________

2. I Could not answer some questions because

a) I forgot formulae Q No: ___________________

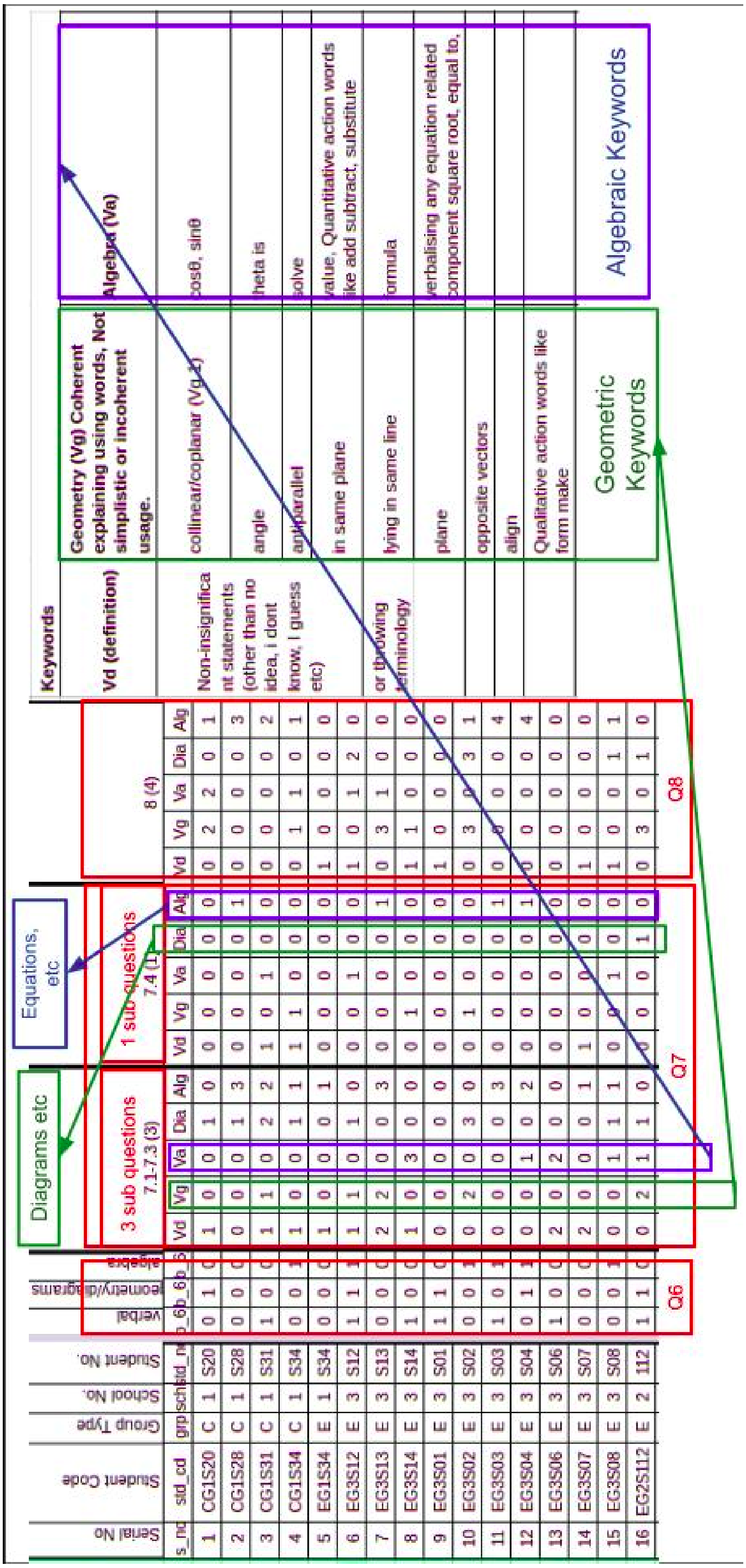
b) I could not understand at all Q No: ___________________

c) I could not imagine Q No: ___________________

d) There was less time. If given time I could have solved. Q No: _________________

APPENDIX 2F2 (p 300)

Sample Rating Scheme of Study-4



Serial No	Student Code	Group Type	School No.	Student No.	Q6 verbal	Q6 geometry/diagrams	Q6 algebra	Q7 3 sub questions 7.1-7.3 (3)					Q7 1 sub questions 7.4 (1)					Q8 8 (4)				
s_no	std_cd	grp	schid	std_no	6b_5	6b_6	6b_6	Vd	Vg	Va	Dia	Alg	Vd	Vg	Va	Dia	Alg	Vd	Vg	Va	Dia	Alg
1	CG1S20	C	1	S20	0	1	0	1	0	0	1	0	0	0	0	0	0	0	2	2	0	1
2	CG1S28	C	1	S28	0	0	0	0	0	0	1	3	0	0	0	0	1	0	0	0	0	3
3	CG1S31	C	1	S31	1	0	0	1	1	0	2	2	1	0	1	0	0	0	0	0	0	2
4	CG1S34	C	1	S34	0	0	1	1	0	0	1	1	1	1	0	0	0	0	1	1	0	1
5	EG1S34	E	1	S34	0	0	0	1	0	0	0	1	0	0	0	0	0	1	0	0	0	0
6	EG3S12	E	3	S12	1	1	1	1	1	0	1	0	0	0	1	0	0	1	0	1	2	0
7	EG3S13	E	3	S13	0	0	0	2	2	0	0	3	0	0	0	0	1	0	3	1	0	0
8	EG3S14	E	3	S14	1	0	0	1	0	3	0	0	0	1	0	0	0	1	1	0	0	0
9	EG3S01	E	3	S01	1	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
10	EG3S02	E	3	S02	0	0	1	0	2	0	3	0	0	1	0	0	0	0	3	0	3	1
11	EG3S03	E	3	S03	1	0	1	0	0	0	0	3	0	0	0	0	1	0	0	0	0	4
12	EG3S04	E	3	S04	0	1	1	0	0	1	0	2	0	0	0	0	1	0	0	0	0	4
13	EG3S06	E	3	S06	1	0	0	2	0	2	0	0	0	0	0	0	0	0	0	0	0	0
14	EG3S07	E	3	S07	0	0	0	2	0	0	0	1	1	0	0	0	0	1	0	0	0	0
15	EG3S08	E	3	S08	0	0	1	0	0	1	1	1	0	0	1	0	0	1	0	0	1	1
16	EG2S112	E	2	112	1	1	0	0	2	1	1	0	0	0	0	1	0	0	3	0	1	0

Keywords

Vd (definition)	Geometry (Vg) Coherent explaining using words, Not simplistic or incoherent usage.	Algebra (Va)
Non-insignificant statements (other than no idea, i dont know, I guess etc)	collinear/coplanar (Vg 1)	cosθ, sinθ
	angle	theta is
	antiparallel	solve
	in same plane	value, Quantitative action words like add subtract, substitute
or throwing terminology	lying in same line	formula
	plane	verbalising any equation related component square root, equal to,
	opposite vectors	
	align	
	Qualitative action words like form make	