MATHEMATICAL EXPLORATIONS AND 'TALK' AT THE MARGINS

A Thesis

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DECLARATION

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions.

The work was done under the guidance of Professor K. Subramaniam and Professor R. Ramanujam, at the Tata Institute of Fundamental Research, Mumbai.

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In my capacity as supervisor of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.

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Date: 15th March, 2024

[R. Ramanujam]

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Abstract

The central concern of this study is ways of mitigating the marginalising effects of mathematics especially for those students who are already marginalised due to their socio-economic and educational backgrounds and “recentering the margins”. Literature highlights the marginalising effects of ‘school mathematics tradition’ with its focus on one right answer, and the stylised language of mathematics with a prevalence of symbols. Moving away from these I sought to design and implement mathematical explorations that enable a rich mathematical experience even in marginalised or low resource contexts. I started with flexibility and accessibility as key design principles guiding task design and identified task features that enable flexibility and accessibility. Following a first-person-classroom-based approach to research, I facilitated and observed students in a low resource context as they engaged with mathematical explorations. I observed students engaging in practices that literature identifies as elements of mathematical thinking. I noted the prevalence of oral communication in informal language and the near absence of symbolisation and formalisation as distinctive features that mark their engagement with such tasks. Moving away from the deficit perspectives that fail to acknowledge the mathematical in such conversations, I sought to define more accommodating acceptability criteria for what constitutes mathematical discourse. Additionally, I look at what it implies for the teacher to enable flexibility without compromising on core disciplinary constraints and suggest teacher support in the form of guidemaps for explorations.

1 Introduction

1.1 Explaining a pattern

In the course of the many “mathematical inquiry sessions” that I used to do with students, I was once presenting a series of examples to students of Grade 7 and 8 that would let them see the following pattern: the sum of k consecutive numbers is divisible by k if k is odd and not divisible by k if k is even. After a few examples of sum of two consecutive numbers where nothing striking was observed, I wrote a number of examples like

$5 + 6 + 7 = 18$

$9 + 10 + 11 = 30$

$17 + 18 + 19 = 54$

and also recorded a few similar student suggested sums on the board. On the basis of these, the group of students concluded that, if we add three “*linewaala* numbers”, that is, numbers that come in a line, the answer will be in the table of three. I quizzed them further to understand what they meant by *linewaala numbers* - whether 2, 4, 6,... or 5, 10, 15, … are *linewaala numbers* as well, and if all of them meant the same thing when they used the term. Further I also wanted to find out if they thought that this would be true of any three consecutive numbers or only of those written on the board. They thought that this would always be true. Saying that I was not convinced of this, I pushed them for a proof. In the examples on the board I could see the numbers and could calculate their sum and verify that it is in the table of three. But how could they make a statement about some three numbers, which may be very large? How would they check if the sum was indeed in the table of three? A student explained that if they give one from the largest number to the smallest number, all the three will become equal and the sum becomes three times the middle number. So it is in the table of three.

1.2 Motivation

The kernel of this dissertation started with instances like the one described above. The activity described was among the first that marked my shift away from regular school teaching, to cover the syllabus and to help students pass exams. I have taught in schools for 5 years. The different schools I worked in catered to students from varied socio-economic backgrounds and I have taught a range of classes from Class 6 - Class 12. The activities as the one described above were aimed at providing opportunities for students to explore and find things out for themselves and to engage in practices of mathematics such as conjecturing, proving, generalising, abstracting, and defining. Such exploratory activities presented a milieu different from my regular classes and triggered a number of thoughts and questions in me. One striking feature of the above description is the rich mathematics that students come up with and the way it is expressed.

In this instance, students refer to consecutive numbers as *linewaala numbers*, meaning numbers that come in a line. In a school context, I would have instinctively corrected them, given the term *consecutive numbers*, and moved on. Their reference to multiples of three as “numbers that come in the table of three” would have evoked a similar response. But then, how important is it to know these terms? What if I let them work with their own terms? What is it that we (the students and I as the teacher) are losing out on if they work with such terms and what is it that we are gaining? Is there anything that I should be wary of when students do mathematics using such language?

The redistribution that they did, by “giving one from the largest number to the smallest”, making them all equal to the middle number was an insightful move, and it establishes conclusively that the sum of three consecutive numbers is divisible by 3. But I as a teacher expected something like:

> Let the three numbers be n, $n + 1$ and $n + 2$
>
> The sum of these numbers = $n + n + 1 + n + 2 = 3n + 3 = 3\ (n + 1)$
>
> A number of the form $3\ (n + 1)$ is divisible by 3.

I suggested this approach to the group of students I was interacting with in the instance described above. Rather than manipulate the variables and bring about an expression that is clearly a multiple of 3, they went back to their initial approach of “giving 1 from $n + 2$ to n to make all three of them as $n + 1$”. Why did I want them to go through the symbol manipulation, when they were unwilling to engage with it?

As noted earlier my goal for the session went beyond observing and justifying that the sum of three consecutive numbers is divisible by 3. I hoped that they would ask what if we add 4 consecutive numbers? Or 5? Or more? Is there a pattern to be seen in the sums? Their term “*linewaala numbers*" suggested another possibility - what if we interpret *linewaala numbers* to include possibilities like 2, 4, 6… or 5, 10, 15…? What patterns emerge when we add a certain number of terms from this sequence of numbers? What if we follow other rules to form the number sequence? While the approach that the students took to show divisibility by 3 allows them to answer some of these questions, it soon becomes inadequate.

I could see that their “redistribution move”, when modified and applied pairwise as shown in Figure 1.1, a sum of an odd number (say k) of consecutive numbers can be shown to be k times the middle number.

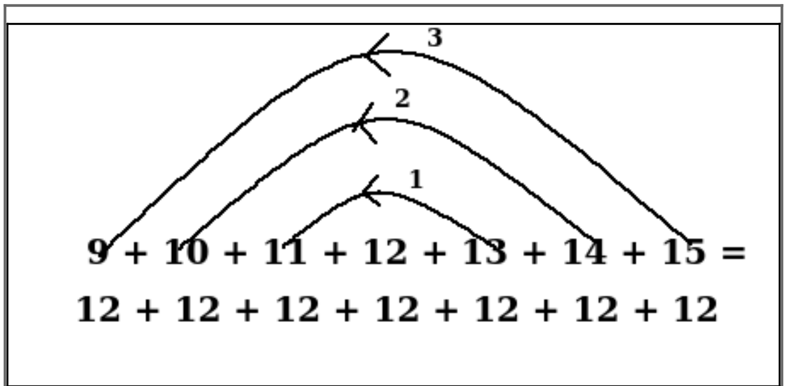


Figure 1.1: Redistribution to find sum of odd number of consecutive numbers

But with an even number of consecutive numbers it does not work so easily. For a sum of an even number of consecutive numbers like 3 + 4 + 5 + 6 and staying within the domain of whole numbers, there is no way that one can “give a whole number *x* from one number to another and make them equal”. It is possible to consider specific sums of 4 or 6 or 8 even numbers and verify that it is not divisible by 4, 6 or 8 respectively. But to establish, when *p* is *any* even number, that the sum of *p* consecutive numbers is not divisible by *p*, is not straightforward by this approach. A complete argument is difficult without algebra.

The generalisation happens as a matter of course with formalisation and is an important part of doing mathematics. Very often a generalisation needs to be expressed as a closed-form expression to enable working with it further. When the goal for the class involves more than solving individual problems, and includes students becoming aware of a general approach to a class of problems or building further on solutions, formalisation becomes critical. Also, perhaps, being steeped in textbook mathematics, I tended to privilege the formal. Students on the other hand did not have a readily available symbolisation scheme for this problem and they had to design one themselves. Rather than coming up with a formalisation of the problem and working with symbols, they drew on reasoning and inference to convince me of their point. While appreciating and accepting the limited generalisability of their approach I wanted to go beyond, considering both the immediate needs of the problem and the role of formalisation in higher mathematics education. One of the key things this thesis is interested in is the role of formal language in the teaching-learning of mathematics and the ways and means students adopt to communicate mathematics when they do not have sufficient access to the formal language.

While formalisation definitely has its merits, as a teacher I am also familiar with the difficulties that students face while reasoning with symbols in this manner. In response to such situations, I have seen students mechanically follow rules like “plus becomes a minus on the other side of the equal sign” or “3 in the numerator on the left hand side goes to the denominator on the right" or disengage completely. The transition from arithmetic to algebra and the symbol manipulation that it involves has been recognised to be one of the problem areas in school mathematics. Even as students have difficulty in making this transition, the above instances highlight that they still generalise, conjecture, and have ways of convincing others, all drawing on informal means. Another concern that this thesis strives to address is how does one strike a balance between being accepting of students' mathematics while keeping formalisation in sight?

In contrast to a 45-minute period where students barely do more than frame and solve 3-4 equations, problem situations like the one outlined here lead to a wealth of patterns. There was experimentation, observing and stating patterns, arguing and justifying that the patterns would "always be so" - all valued practices in mathematics. A key difference that I noted between my usual classes and these is the hum of activity and engagement seen in these classes. Many more students have something to contribute perhaps because these activities allow the freedom to respond in their own ways, rather than those defined by the textbook. If I had to choose between sense-making and engagement in informal means and mechanical rule-following and eventual disengagement, what would I choose? Are there occasions when I need to insist on the formal?

The increased engagement that exploratory activities generated even from students who did not think of themselves as "good at maths" and the rich mathematics that ensued, made me wonder what was it about these activities that led to this difference. What role could they play in reducing anxiety about mathematics performance and nurturing students' confidence that they could do maths as well? What features should these activities have to this end? I looked at mathematics education literature around such exploratory activities and found a number of starting points for such tasks. I also found teacher reports of how the activity progressed in class in the publications of the Association of Teachers of Mathematics (ATM) and a few other scholars who had discussed such tasks. These accounts of classroom teaching were also aligned to the vision for mathematics education articulated in an important policy document of the Indian National Curriculum Framework (2005): the Position paper on Teaching Mathematics. The Position paper proposes a shift in focus "from mathematical content to mathematical learning environments, where a whole range of processes take precedence: formal problem solving, use of heuristics, estimation and approximation, optimisation, use of patterns, visualisation, representation, reasoning and proof, making connections, mathematical communication.…Such learning environments invite participation, engage children, and offer a sense of success" (NCERT, 2006, pp. v–vi). I wanted to investigate what it entails to bring alive this vision and delve deeper into questions like what kind of activities could launch an exploration, what it means to explore, what challenges and benefits do these open up for students and what is the role of the teacher in enabling an exploration.

1.3 Shifting focus to the margins

With these broad goals and questions in mind, I started out designing and implementing tasks in three schools. Given the concern for equity in opportunities for education, shared across the research group consisting of my advisors and me, one of the three schools selected was a school catering to students from socio-economically disadvantaged backgrounds. My prior experience doing a program evaluation of an intervention being implemented in the government schools in India sensitised me to the inequalities that exist between such schools and the schools that cater to the middle-classes not just in terms of opportunities, but also in terms of pedagogy. Based on my belief that everyone can do mathematics, the possibility of producing counterstories that demonstrate the feasibility of explorations in marginalised contexts motivated me.

Due to the difficulty in accessing and scheduling activities in two of the schools which largely catered to middle-class students, most of my teaching sessions happened in the school catering to students at the margins. I also started trying out some of the explorations in yet another school that was similar to this school in catering to students from disadvantaged backgrounds. This sharpened the focus of the study to students studying in schools at the margins. The triumphs and struggles of these students opened up more challenges, dilemmas, and fresh questions.

While the flashes of insight as seen in the instances discussed above were very much there, the way they were expressed was further removed from what I was used to. (Many examples are described in the thesis.) The differences arose from multiple reasons - their limited proficiency with English, the language of instruction, the non-familiarity with technical terms in their own first language Tamil, the differences between the language used in school contexts and in casual conversations at home, and the difference between my language and their language stemming from differences in such social markers as class and caste. On the one hand, it took much more effort and getting used to on my part to listen, hear, and understand the mathematics that unfolded in these classes, and on the other, I was impressed by how these students still managed to communicate their mathematics despite all the differences noted above. Contrary to the prevailing beliefs that engaging with explorations demands a certain mathematical maturity and exposure and that this may not be expected of students at the margins, I observed these students enthusiastically participating and finding things out for themselves.

Not being able to make sense of the "alien" textbook language, these students struggled to make sense of problems and would tell me "Tell us what this means and we will solve it." When the problem was presented to them in an engaging and accessible fashion, they could figure things out for themselves, and doing so gave them a sense of joy and ownership. Recording their findings on their board and labelling them with their names, to be referred to later as “Muthu’s Theorem”, “Kanika’s solution” etc., was a usual practice. Explorations presented themselves as a potential means to enable otherwise marginalised students to engage with mathematics and more importantly offer them an opportunity to bring forth the mathematics that they know. My study of literature pointed to very few studies in this direction.

The study goal became more focused on investigating the potential of mathematical explorations to support mathematical thinking at the margins. The scarceness of existing contexts where I could observe and study this necessitated that I create such a context as well, and praxis became an important part of the study design. As I engaged with students, I became more sensitive to the ways they used language and other means of communication like gestures and diagrams to get across their ideas and language became a concern of the study. What is the role of formal language in doing mathematics? What are ways that one can use informal language to do and communicate mathematics? What makes communication mathematical? This thesis is an attempt to shed some light on some of these questions. In the next section, I give a brief overview of the thesis.

1.4 Organisation of the thesis

This thesis is organised into 7 chapters.

In Chapter 2, I discuss mathematics education literature in relation to the study goals. Noting how mathematics serves to further marginalise those who are already marginalised due to such factors as their class, caste, language, etc., I ask if explorations offer at least some ways of mitigating the marginalising effects specific to mathematics. I also bring up the schooling and school mathematics contexts in India, highlighting the broad pedagogic orientation seen and the frequently observed tendency to have lower expectations of students from marginalised backgrounds. I find the academic motivation for the study in these deficit views and the absence of sufficient counter-narratives in the Indian context.

In an attempt to discuss the literature around the theoretical constructs relevant to the study, I look at constructs such as Mathematical Discourse, Mathematical Thinking, Mathematical Investigations/Explorations and reflect on my stand vis-a-vis these discussions. The formal language of mathematics being a major factor that restricts access to mathematics and therefore a concern area of this study, I discuss the literature around the role of language in teaching-learning mathematics in depth, including the different perspectives on language and the associated dilemmas and tensions, the role of informal language both in school mathematics and the work of research mathematicians. I look at means suggested by scholars to overcome the difficulties posed by the formal language and building on students’ language. This raises questions on how far one can be accepting of students’ language without compromising on disciplinary constraints and their own future prospects and I have tried to address this in

Chapter 5 of the thesis.

In Chapter 3, I formulate my research questions in the light of the literature, and spell out my methodological stance, details of data collected, and the analytical lens. I adopt the stance of a researcher-teacher and approach my research questions from a first-person perspective. The audio recordings and the teacher diary maintained during the two-year teaching stint constitute the major chunk of my data. In addition, discussions within the research team, comprising of a mathematician and an educational researcher besides me, constituted an in-situ analysis in addition to serving as data for later reflective cycles. The analysis consisted of repeated cycles of listening to audio recordings and going over the teacher diary, preparation of further notes, and discussions where multiple perspectives and interpretations were considered until there was consensus among the three members of the research team. These reflective cycles and in-situ analyses contributed to an understanding of the demands that explorations place on a teacher. In this chapter, I also describe the study context and give an overview of the explorations that I facilitated for this study.

I discuss the findings from the study in Chapters 4, 5, and 6. In Chapter 4, I discuss task features that support mathematical thinking and make tasks accessible, especially in a marginalised context. I analysed frameworks that characterise openness of tasks from literature and based on my observations during facilitating explorations at the margins, identified the relevant dimensions of openness in such contexts. Flexibility and accessibility emerged as key design principles and I sought to operationalise these through specific features that could be incorporated in tasks.

In Chapter 5, I describe what mathematical thinking in the context of explorations looks like in a marginalised context. I draw attention to the nature of thinking seen, the processes that students engage in, and view these in the light of characterisations of mathematical thinking drawn from the literature. I also describe the language and other resources students use to communicate their mathematical thinking and note some distinguishing features. Further, I discuss how these support or hinder communication. In this chapter, I also propose an acceptability criterion for mathematical discourse that is more accommodating than those spelled out in the literature. I argue that such a criterion enables a more encompassing view of mathematical discourse leading to acceptability of what might otherwise be considered “inadequate”.

In Chapter 6, I discuss what it entails for the teacher to facilitate an exploration, and the demands that it places on her, especially in a marginalised context. I identify additional challenges that explorations bring - in terms of the content knowledge required, in terms of implementing a pedagogy that is responsive to students' contributions and in listening and understanding students’ mathematics across distances in mathematical background, language, and perspectives on what counts as mathematics and mathematical language. I offer some teacher support in the form of guidemaps for explorations and identify desirable features of such guidemaps and also suggest some ways in which the teacher could disrupt deficit discourses.

In the concluding Chapter 7, I discuss the limitations of the study, some implications of this study, and pointers to future work.

2 Literature review

A striking observation from my initial experiences of facilitating explorations in marginalised contexts was that the very students who eagerly put forward ideas in these sessions struggled to make sense of and solve problems in their textbooks. Perhaps impeded by the textbook language, or the language of mathematics, or textbook mathematics itself, their exam scores did not seem to reflect the mathematics they were capable of doing. The exam performance would limit the further opportunities available to them for higher education and jobs. Thus mathematics served to further marginalise them. In this chapter I examine literature to understand how mathematics marginalises, paying special attention to language, and also look at ways scholars have suggested to mitigate the marginalising effect of mathematics. Although much of the relevant literature was consulted at the beginning of the study, reading and reflecting on the literature continued throughout the study. This helped sharpen my analytical lenses and guided what I focussed on. Among the many ways in which mathematics marginalises, the narrow conceptualisation of school mathematics and the formalised language of mathematics are two key factors that stood out in this literature and were prominent in this study as well.

In Section 2.1, I look at the marginalising effects of mathematics and organise these into three broad dimensions - performative, disciplinary, and language. In Section 2.2, I look at some related ideas, namely deficit discourses, deficit noticing, and framing. Considering the role of mathematical language in alienating learners, I devote Section 2.3 to examining the role of language in teaching and learning mathematics. In this section, I discuss the different perspectives on the role of language in learning mathematics, on mathematical discourse, and the tensions and dilemmas involved in teaching-learning mathematics in marginalised and language-diverse contexts. I also discuss literature on the role of informal language in learning mathematics. In Section 2.4, I look at some ways scholars have suggested to address the marginalising effects of mathematics. In the two subsequent sections, I examine some of these suggestions further. I discuss explorations or open tasks as alternate ways of engaging with mathematics, that enable a broader conceptualisation of what it means to *do* mathematics by focussing on thinking mathematically.

2.1 Mathematics and marginalisation

In this section, I look at the marginalising effects of mathematics. The tendency in the prevailing culture to accord undue importance to mathematics scores and the resultant performance pressure, the focus of school mathematics on repeating procedures to arrive at expected answers, and the symbolic language of mathematics stand out as major alienating factors. I organise the literature on the marginalising effects of mathematics around these factors.

2.1.1 Marginalisation

The concept of marginalisation permeates current educational research literature, but it eludes a unitary definition. Marginalisation takes many forms, not all of which are readily apparent to the observer or even the individual concerned. It occurs at different levels (individual, groups, communities) and may be situated within time and place or internalised to become part of the lived experience of the individual (Messiou, 2006; Mowat, 2015). Marginalisation manifests as disenfranchisement stemming from multiple factors such as: poverty, locale, race, ethnicity, culture, gender, language, disability or ill-health, religion, and other personal circumstances (Chen & Horn, 2022). In the Indian context, the additional factor of caste operates to order social space, marginalising some sections of the population and privileging others. Mowat (2015) draws attention to the feelings that encompass the state of being marginalised. The concerned individual or group has a sense of not-belonging; she may be inhibited in accessing the range of opportunities open to others, and in feeling a valued-member of the community by making valuable contributions within that community.

While education is supposed to be a route out of marginalisation, schools themselves can act as agents of marginalisation. A curriculum that does not take into account individual student strengths and needs, rigid systems and structures, and standards-driven programmes which create winners and losers, all tend to marginalise some learners. Petrou et al. (2009) distinguish between groups which have been formally identified as marginalised according to government policy (relevant order or schedule in the constitution) and those who are marginalised because they fail to conform to the cultural norms and expectations which prevail within schools. Esmonde and Langer-Osuna (2013) take a similar stand, arguing that marginalisation is not assigned or assumed based on identity markers, but reproduced through interactions. Thus marginalisation is not merely a preexisting condition of some students' existences due to some socio-cultural structures, but is produced and perpetuated as individuals and structures interact. Given the interconnectedness of structures and individual lives in reproducing, sustaining and contesting marginalisation, Chen and Horn (2022) suggest the construct of critical bifocality that attends to the interrelationships across structure and agency as an analytic lens to view marginalisation.

2.1.2 Mathematics as marginalising

It is widely acknowledged that mathematics itself can marginalise students (Ewing, 2002; Gates & Noyes, 2020; Warren & Miller, 2016). Several studies indicate large differences between mathematical performance of the dominant groups and the marginalised groups (Akmal & Pritchett, 2021; Goswami, 2022; Graven, 2014; Singh, 2013; Subramanian, 2017; Taylor, 2006), with different countries and educational systems having different criteria for defining the dominant/marginalised groups. These include race and ethnicity in the US, immigrants and other-language learners in several European countries like Spain and Cyprus, rurality in China and many Latin American countries, social-class lens in the UK, and caste-class lens in India (Xenofontos & Alkan, 2022). Mathematics has been variously referred to as "critical filter" as it functions as a gatekeeper to higher education and economically rewarding jobs (Sells, 1978, 1981); "social filter" as access to mathematics is mediated by class- and culture-based language use (Zevenbergen, 2002), and as a "fractional distillation device" that is class reproductive (Ernest, 2020). School mathematics, as part of the wider education system, acts to confirm and/or create the marginalised status of those in society (Jorgensen et al., 2014; Noyes, 2007; Skovsmose, 2019).

Skovsmose (2011) refers to the simultaneous empowering and disempowering capacity of mathematics. On the one hand, given the spectacular applications of mathematics in technology and everyday routines, mathematics education can empower people by providing them with qualifications to participate in a variety of practices and to obtain a good position in the labour market. On the other hand, failure in mathematics, the debilitating anxiety and shame that this induces can be personally disempowering.

Focussing on marginalisation in mathematics classrooms, Chen and Horn (2022) question the assumption that students who are marginalised in society will necessarily be marginalised in mathematics classrooms. While there is truth in this assumption, it also carries the implication that the route to tackle marginalisation in the mathematics classroom is to tackle societal structures that are marginalising. The authors suggest that narratives about who is good (or not) at mathematics are constructed based on the cultures, ideologies, and practices of mathematics and mathematics education rather than filtering into the classroom from broader societal narratives. Elaborating on the lens of critical bifocality, the authors suggest that, in addition to factoring in how individual students, teachers and policy makers participate in the co-creation and reproduction of the societal- structural patterns and their agency in countering them, one also needs to understand the specifics of marginalisation that happens in the maths classroom, the structures specific to the discipline of mathematics that enable such marginalisation and ways of countering them. This study is aligned to the latter goals.

2.1.3 Dimensions of mathematical margins

I now look at some dimensions of marginalisation that are specific to mathematics and how these interact with and are exacerbated by other marginalising factors especially the socio-economic status (SES). I propose that three such dimensions of marginalisation may be identified, that aid in organising research on marginalisation due to mathematics. These are rooted in public perception of mathematics, in the disciplinary canons themselves, including the specialised language of the discipline and how they translate to school mathematics:

1) The performative dimension - stemming from the importance accorded to mathematics in the society, leading to undue importance being accorded to mathematics performance and stigmatisation of non-performance,

2) The disciplinary dimension - stemming from what is generally accepted as ways of doing mathematics, especially in the school context,

3) The language dimension - stemming from what are considered accepted ways of talking mathematics.

While there are considerable overlaps between the last two dimensions I wish to keep the distinction because *talk* (as against writing) happened to be the primary means to *do* and *communicate* mathematics in the contexts where I worked, and hence the language dimension is of particular interest to me.

2.1.3.1 The performative dimension

The performative dimension of marginalisation manifests in mathematics functioning as the gatekeeper to higher education and opportunities thereof, its anxiety-inducing nature that results in negative attitudes to the subject, alienation of learners and their dropping out of maths and sometimes even school, disparities in achievement levels between dominant and marginalised groups, leading to a vicious cycle that maintains the status quo.

Mathematics serves as a “critical filter” (Ernest, 2020; Schoenfeld, 2002; Sells, 1978, 1981) with certification in mathematics being a prerequisite for entry to courses in higher education and professions and jobs. Questions on mathematics find a place in the admission tests for courses and jobs, even when the knowledge of mathematics demanded by the course/profession is not aligned to that asked for in the screening test, as mathematics score is taken as a proxy for “intelligence”(D’Souza, 2021). In India, at post post-secondary level, among the many options available to choose from, the combinations including mathematics are coveted by students as these are seen as a gateway to economically rewarding occupations and upward socio-economic mobility. Students are assigned these courses based in part on their mathematics scores at secondary level. Mathematics scores thus present a serious barrier to job opportunities for students. Noyes (2009a) calls the separating line between those who have the mandated grade C for a pass in the GCSE mathematics to those who do not have, a “magic threshold” to future educational and employment opportunities.The high-stakes nature and the implicit judgement ascribed to maths performance makes it anxiety-inducing for many students. Being good at mathematics implies speed and being competitive with it, and failure to be so leads to feelings of inadequacy and shame in many (Buerk, 1982). In addition, it also instils a sense of low self-esteem in non-achievers. Seen as an obstacle to educational and career advancement, mathematics is feared and avoided by a section of students when there is an option to do so. Ewing (2002) suggests that issues of pacing, speed of content delivery, the linear discipline of mathematics, and the classroom ethos as reasons for students opting out of school mathematics and ultimately school in the Australian context.

Disparities in performance and participation (termed "achievement gap") of different subgroups of the population and underachievement of marginalised groups have been much discussed in literature (Akmal & Pritchett, 2021; Bharadwaj et al., 2012; Borooah, 2012; Darling-Hammond, 1995; Flores, 2007; Noyes, 2009b; Secada, 1992). Studies have shown that one's social and cultural backgrounds deeply influence mathematics performance (Brown et al., 2011) and that such findings have remained relatively consistent over the last 3-4 decades and have been replicated across diverse countries (Zevenbergen, 2002). Low expectations and negative perceptions of students based on class/ethnicity/language contribute to lower achievement (Anyon, 1980; Horn, 2007; Namrata, 2011). Research also points out how these perceptions lead to a differential treatment that further restricts opportunities. For example, perceptions that some groups are ill-equipped to receive anything other than didactic instruction leads to their being subjected to a "directive, controlling and debilitating pedagogy" and "substandard instruction that does not adequately prepare them to function in society" (Solomon, 2008; Strutchens, 2000, p. 7). A study of literature on the achievement gap underlines the entangled nature of these factors, rather than any factor being the primary cause of low achievement.

The role of mathematics as a "social filter" has also been discussed by scholars ((Ewing, 2002; Jorgensen, 2018; Noyes, 2007, 2009b; Skovsmose, 2023). Noyes (2009b) points to the socially differentiated patterns of participation in advanced level mathematics courses in the UK, which one can expect to be replicated in STEM professions. Through two case-studies, Jorgenson et al. (2014) highlight "the subtle and coercive ways in which the practices of the field of mathematics education allow greater or lesser access to the hegemonic knowledge known as school mathematics depending on the cultural backgrounds and dispositions of learners." (p 221). They point out that a vicious cycle is developed when low SES students, who are often also classified as "underachieving", find themselves with a similar cohort. This results in slower progression and continued underachievement in assessments, thus widening the gap between these students and the higher SES groups. Noyes (2007) points to how in spite of an all-ability grouping and common curriculum intended to overcome social differences, the mathematics classroom reinforces the social differences between members of different social groups. This illustrates the socially reproductive tendencies of mathematics classrooms and how it acts to confirm and or create the marginalised status of those in the society.

2.1.3.2 Disciplinary dimension - School mathematics paradigm and textbook culture

Skovsmose and Penteado (2015) identify four characteristics of what might be called mainstream mathematics education - as an enacted curriculum defined by the textbook, practice problems that have one correct answer and all and only the necessary information to solve them, focus on error elimination, and evaluation of performance through end of the year examinations. The focus of the “exercise paradigm” or “school maths tradition” on following a set procedure and repetitive practice to gain mastery discourages alternate ways of thinking and multiple approaches. The approach to teaching mathematics described above finds parallels in the larger teaching culture in the Indian context (Sarangapani, 2020; Subramanian et al., 2015).

Solomon (2008) posits that “traditional mathematics teaching and curricula have the effect of denying many learners access to high-status mathematics knowledge. In particular, it denies them access to meaning-making in mathematics, perpetuating narrow epistemologies, marginalised identities, and a corresponding lack of ownership” (p137). Adiredja and Louie (2020) suggest that views of mathematics as objective lead to mathematical activity in schools being framed as rote practice, involving memorization of established procedures and repetitive computation. This framing does not allow much space for students to demonstrate their mathematical competence as competence itself is narrowly defined. With a narrow definition of competence, the set of people who are considered mathematically competent also narrows down leading to marginalisation. In addition, it also leads to ignoring such aspects as sense-making, experimentation, communication, and creativity that are important to the practice of mathematics and leads to students distancing themselves from mathematics.

School mathematics is perceived to be different from the mathematics encountered in everyday activities and that practised by different cultures and the latter is considered to be outside the purview of “real” mathematics. Conceptions of mathematics as universal, and objective leads to the expectation that all students across the world would learn the same set of mathematical skills and facts using the same curriculum. Hunter (2022) argues that privileging the White middle-class ways of knowing and being in the mathematics classroom leads to devaluing the “funds of knowledge” of some communities and denies the learners from these communities the opportunity to draw on what they know from their lived experience, leading to alienation. She also argues that positioning mathematics as value-free and culture-free leads to narrow views of mathematics as contained only within school settings.

Being able to successfully engage with day-to-day activities that draw on applications of mathematics in ways that are different from the “one right way” of school mathematics or bringing in the approaches learned from everyday activities to solve a problem of school mathematics is not generally considered acceptable. Borba and Skovsmose (1997) drive this point through a typical proportional reasoning question. “What is the price of the food needed to follow a given recipe (for four persons) when nine persons are expected for the party? All and only the necessary information to solve the question being given the student is expected to produce the unique correct answer. However a response along the lines of “I know a slightly different recipe, and if we use some extra carrots, we do not need so much of this, I think it might even taste better…” is appropriate to handle the real-life situation and mathematical as well, but is not acceptable to handle the make-believe real life problem of school mathematics. Cooper (2002) identifies such pseudo-realistic problems as a factor that relatively disadvantages students from lower socio-economic sections. These students are more likely to read such questions too literally and they generate sensible but mathematically unacceptable solutions (Solomon, 2008).

We thus see that narrow definitions of what counts as mathematics also tend to marginalise learners with diverse backgrounds. Conceptions of mathematics as being composed of sequential building blocks to be mastered in a specific order excludes some sections of students by creating hierarchies of “fast learners” and “slow learners” denying opportunities to the so-called “slow learners”. We now look at how the language of mathematics marginalises.

2.1.3.3 Language Dimension

The role of language in teaching and learning mathematics and linguistic structures that are specific to the language of mathematics has been the focus of research since the 1980s (O’Halloran, 1998, 2015; Pimm, 1987; Schleppegrell, 2007). The symbolic and stylised language of mathematics and the inherent formalism have been identified as an entry-barrier to mathematics. A related issue I would like to flag, though not limited to the subject of mathematics, is the difference between Language of Learning and Teaching (LoLT) and the home language of the student. In the current times, multilingual classrooms have become more the norm rather than the exception. This may be attributed to multiple reasons - political, economic, social, etc. A student may have a LoLT different from his/her home language for multiple reasons. The language may be imposed for reasons of national pride, or political and administrative reasons. English has the status of an international language and the language of scientific communication and tends to be the preferred language, especially in former British colonies. The political nature of language and the societal tendencies to prefer some languages over others adds an additional layer to the complexities and leads to marginalisation irrespective of mathematics.

Mathematical language is marked by a liberal use of symbols and equations and an impersonal tone, befitting the universal truths that it is supposed to convey. Scheppegrell (2007) identifies multi-semiotic formations, dense noun phrases that participate in relational processes, the precise meanings of conjunctions, and implicit logical relationships that link elements in mathematics discourse as features that add to the difficulty of mathematical language. The specialised vocabulary with a number of polysemous terms which overlap with everyday usage, the conciseness obtained through the use of symbols and discursive rules that are often not made explicit (Sfard, 2007) makes mathematics inaccessible to those who are not conversant with this language.

In low-resource contexts, few students attain autonomy in access to mathematical language. They use everyday language intermingled with mathematical terms to get their ideas across. Many concepts of mathematics require more than everyday language for a clear disambiguation, for example, the difference between concepts of *difference* and *proportional comparison*. Discussing the challenges involved in adapting everyday language to express mathematical ideas, McGregor (2002) points to the difficulties that even elementary concepts such as expressions of number comparisons generate. Expressing a difference of 4 between two numbers, say 1 and 5, as “there are four numbers in between” or “three digits missing” or “four numbers higher” do not lend themselves to a connection with the operation “add 4” or the idea of “4 more than”. These informal expressions are more likely to be encountered in talk, especially when the ideas being discussed are “in the making” and not yet consolidated. In contexts where fluency in the LoLT or familiarity with academic language is limited, students are more likely to draw on informal versions of their first language or move across languages to communicate ideas. Given that spoken language is perhaps the only recourse to sense-making in such contexts, it becomes an important part of the classroom discourse, integral to making mathematics accessible. However, this language does not figure in the textbook, nor does the teacher have any guidelines as to what is acceptable, or how to manage the transition towards textbook-like discourse.

2.1.3.4 Class - mediated language

Zevenbergen (2000) points out that class- and culture-based language use also mediates access to mathematics via displays of linguistic and cultural capital, which mark out an ascribed status of a competent learner. Zevenbergen (2002) differentiates between the problem arising from lack of familiarity with the LoLT and the lack of familiarity with the language conventions, forms, and styles within the formal school context. She points out that students who are marginalised by their social class, may be native speakers of the LoLT, but the English (or any other language depending on the part of the world in focus) used to teach mathematics and the form of English used in mathematics itself is a very particularised form, that is very different from the English used by students from working class backgrounds. The language used and valued in a formal school context is that of the middle classes. Middle-class discourse patterns are marked by “elaborated codes” that involve syntactical complexity, lexical diversity whereas the working-class discourses are marked by “restricted codes” and simpler syntactic structures (Bernstein, 1990).

The notion of elaborated code involves “embellished language” used in middle-class families. For example, when parents ask their children to locate items, they are likely to use rich positional language - “the red jumper on the top left-hand shelf” - whereas the restricted code of the working class is more likely to be devoid of such contextual clues. The interaction in working-class homes is restrictive in content and prose, and so children encounter a very different experience when they enter school. Depending on the alignment of home and school codes, some students are placed more favourably than others (Zevenbergen, 2002).

> Where students are able to speak, use and understand the language of the school, they are more likely to unpack the messages and content being conveyed by the teacher than students who are less familiar with the language and hence unable to ‘crack the code’ of school English. (p42)

Similarly, some practices are seen as more legitimate than others and students who are able to display or assimilate those practices are positioned more favourably. For example, students who display street talk and skills in street selling may be positioned as marginal within the field of education. The discourse style valued in academic settings encourages elaborated speech, using questioning, hypothesising, and argumentation. Students who experience this style at home are at an advantage and others are at a disadvantage. Therefore, underachievement by some groups of students could be seen as a mismatch between the language of the student and the language of the school rather than as a deficit or lack of ability in the student and ways sought to find ways of addressing this language difference.

I highlight the complexity of the language dimension, marked by multiple dichotomies - that between the first/home language and the LoLT; the class-mediated school language and street/home language; the academic language marked by specific vocabulary and registers and everyday language; and the formal and informal language used in mathematics. All of these dichotomies present challenges to marginalised students. These are central to our study, given that one of our goals is to investigate means of addressing the language dimension of the margins. I revisit the literature on the role of language in mathematics education in Section 2.3 of this chapter focusing on these issues in greater detail.

2.2 Deficit Discourses, framing and noticing

Closely related to the construct of margins is that of deficit discourses. If marginalisation is the exclusion of those who do not conform to the “norms” of the dominant group, deficit discourses imply a value judgement on these differences. Deficit discourses are marked by the tendency to view these differences as dysfunctionalities, as deficits, or shortcomings, that need to be "normalised" by appropriate interventions. Such discourses also tend to take a "blame the victim" orientation, holding the individual/community responsible for their shortcomings (Davis & Museus, 2019b; Valencia, 1997). Peck (2021) defines deficit perspective in educational contexts as "a propensity to locate the source of academic problems in deficiencies within students, their families, their communities, or membership in social categories such as race and gender". Deficit discourses are "systems of meaning that circulate across society, exercising a pernicious influence even on teachers who consciously wish to counter them" (Adiredja & Louie, 2020). Such discourses are prevalent in mathematics education at every level, including instruction from preschool to university as well as research and scholarship, and cause harm to students (Adiredja & Louie, 2020; Davis & Museus, 2019a; Peck, 2021). Deficit discourses do not stand on their own but are reinforced by other narratives, some of them apparently neutral and others which are clearly problematic. In the context of mathematics education, these include narratives about the universality, objectivity, and sequential nature of mathematics and those about the ability, motivation, and needs of students from marginalised groups. They are located not only in the minds of biased individuals but also in systems, institutions, and society at large and are socially, culturally, and historically constructed. These narratives permeate society and are culturally so dominant and naturalised that they are interpreted as common sense. Adiredja and Louie (2020) suggest that deficit discourses at the societal level percolate to the local communities of practice and further down to individual teachers and researchers and hence argue that one needs to examine the systemic nature of such discourses to understand why they persist.

Adiredja and Louie (2020) suggest that prevailing perceptions of mathematics as universal, objective and sequential contribute to deficit discourses, despite appearing neutral. This view that mathematics is universal expects all students to learn the same set of mathematical facts and skills, using the same mathematics curriculum independent of contexts. There is no room to take into account student choices, perspectives, and interests. Also, particular language, symbols, algorithms, and conventions are considered standard, and other ways of thinking and knowing as inferior. The view that mathematics is objective leads to seemingly objective standards of mathematical thinking marked by an over-privileging of formal knowledge such as standard definitions and procedures and a formal mathematical language to encode such knowledge. These standards lead to de-valuing of students' informal mathematical knowledge and language. The perception that mathematics is sequential, composed of building blocks that must be mastered in order from basic to advanced also leads to deficit discourses by producing hierarchical categories of students - those who are ahead, on track, or behind, and those who are strong or weak. Such classifications lead to the "weak" students and the ones "left behind" being denied opportunities to engage in rich mathematics under the pretext that they are "not ready" for it. Thus

Adiredja and Louie point out multiple ways in which seemingly neutral perceptions about mathematics lead to deficit discourses.

Gorski (2011) discusses the layers of socialisation that condition educators (and education researchers), like everybody else, to buy into certain myths and stereotypes. Thus, it is not a purposefully regressive teacher acting in purposefully oppressive ways that says “It is great if these children pass” but one who has been socialised by the deficit hegemony to buy into the myth that children from disadvantaged backgrounds are not educable. Once the blame is pinned on the family circumstances of the student, there is nothing that the teacher (or system) can do to help the student and thus, the teacher’s commitment and willingness to extend herself to provide the support that the student needs to succeed in the exams is diminished.

Peck (2021) identifies some ways in which deficit perspective harms students; it limits access to educational opportunities, results in lowered expectations for students, limits the role the instructor can play in students’ education and more importantly, blinds one to the harm being caused by preventing critical introspection and perpetuates oppression and privilege. It also results in “ability-based” streaming, arguments about the educability of certain groups of students that are based on so-called genetic or cultural deficits, and the defence of the deep unquestioned assumptions of the society (Valencia, 2010). Deficit discourse may lead to deficit noticing - where teachers attend to errors and shortcomings of marginalised students and ascribe these to deficiencies in students, their families or their cultures, ignore their strengths and disregard schooling practices and social structures that limit students’ opportunities to learn and thrive.

Teacher noticing refers to the capacity to attend, interpret and respond to classroom events. Attending involves recognising notable aspects of students’ work, interpreting involves assigning meaning to the work that has been attended to and adopting a specific point of view about students’ understanding, and responding involves proposing teaching strategies based on the observed thinking (Jacobs et al., 2010). What teachers notice depends largely on what they value, and is “tied to their orientations, including beliefs, and resources, including knowledge” (Schoenfeld, 2011). Noticing is influenced by social, cultural and political processes and in turn influences these processes (Scheiner, 2023). The Attending-Interpreting-Responding (AIR) framework fails to acknowledge the influence of culture and power on what a teacher notices. Acknowledging the influence of how teachers frame their object of attention on what they notice, Russ and Luna (2013) suggest that changes in noticing are linked to this framing. Louie et al. (2021) theorise noticing from a sociopolitical perspective and draw attention to social, cultural and political aspects of noticing by including the element of “framing” to the AIR framework of noticing.

Frames provide interpretive contexts that support participants in a given situation to understand what kind of task they are engaged in, what sort of behaviour they are expected to engage in and what kinds of knowledge are valuable. Frames generally refer to the “hidden” layer that shapes what a teacher notices and how she responds. Framing is the interactive process of co-constructing a particular frame and coordinating activities around it (Louie et al., 2021). Framing affects perception by making some aspects of the situation seem more relevant than they would otherwise. Thus what we see and how we interpret a situation is constrained by how we frame the situation. Framings are shaped by cultural attitudes towards nature of mathematics and its teaching and learning, beliefs about the capabilities of different groups of students, the need to address deficiencies in their thinking, and can be seen as socially and culturally defined. Framings orient participants towards what to pay attention to.

Deficit based framing views student thinking as shortcomings or failures, and reinforces social and educational inequalities. This hinders the development of a positive self- perception of mathematical ability among students. Louie et al. (2021) identify three culturally dominant frames which appear neutral but contribute to exclusion 1) Framing mathematics learning as absorption of a universal, objective, fixed body of knowledge; 2) framing students primarily as receivers of mathematics; and 3) framing interactions between students as relatively inconsequential for learning and secondary to individual behaviour and achievement.

Elaborating on how these frames lead to deficit noticing, Louie et al. (2021) suggest that from a frame of mathematics learning as absorption of universal and fixed body of knowledge, teachers may choose to evaluate how well students’ thinking meets the standards, affirm correct answers and remediate errors. Divergent approaches may neither be noticed nor valued. Framing students as receivers of mathematical knowledge erases students' personal and cultural resources and leads to teachers not attending to students beyond their mathematical performances and ranking, labelling and grouping students based on their performance. The frame of school learning as an individual accomplishment leads to teachers discouraging student talk and tolerating it as long as it is “on task”.

Having noted the marginalising effects of mathematics and the ensuing deficit discourses, scholars have suggested various measures to counter deficit discourses, like using students’ language as resource, anti-deficit noticing and framing, and having a more encompassing conceptualisation of what it means to do mathematics. Language is acknowledged to be a stumbling block for access to mathematics in the marginalised contexts (Robertson & Graven, 2020; Sibanda, 2017; Subramanian & Visawanathan, 2023), and this was corroborated by my initial experiences as well. Therefore in the following section, I take a closer look at literature around mathematics and language.

2.3 Language in mathematics learning

The role of language in mathematics education has been conceptualised in different ways. Some scholars consider mathematics itself as language while others consider mathematics as a mode of thinking that is removed from the ambiguities of human languages (Barwell, 2008a). However it is widely recognised that mathematics involves a distinctive form of language use (Halliday, 1978; Pimm, 1987), and that this specialised language makes it difficult to access for learners, more so for the marginalised (Kaplan & Kaplan, 2014; Schleppegrell, 2007; Zevenbergen, 2000). In this section, I look at some key ideas and issues pertaining to the role of language in teaching and learning of mathematics.

2.3.1 Perspectives on role of language in mathematics learning

Moschkovich (2002) identifies three theoretical stances reflected in research into the relationship between language and mathematics learning: learning mathematical language entails acquisition of specialised vocabulary, learning mathematical language involves construction of multiple meanings, and learning mathematics implies participating in mathematical discourse practices.

2.3.1.1 Learning mathematical language entails acquisition of specialised vocabulary

From the perspective that learning mathematics entails acquisition of specialised vocabulary, the technical nature of mathematical terms and vocabulary and the difficulties posed by these are identified as hurdles to learning mathematics (Pimm, 1987; Riccomini et al., 2015; Schleppegrell, 2007). The metaphorical extension of everyday words to mathematical use (for example, 'product as outcome' in the everyday sense becoming 'product as outcome of multiplication' in mathematics) resulting in an overlap between everyday and mathematical language, and vocabulary that is specific to mathematics (Least Common Multiple, tetrahedron) are some problem points. Also, structures of language underlying the vocabulary used in mathematics have been the focus of attention of scholars. The following features have been identified as presenting challenges:

- multiple semiotic systems (symbols and equations; graphs, diagrams and other visual representations; language),
- dense noun phrases (volume of rectangular prism with sides a,b,c),
- a grammatical patterning where a series of *processes* being presented through nouns or noun phrases as if they were *things (* The expression $a^2 + (a + 2)^2 = 340$, or the sum of squares of two consecutive even numbers is 340 can be unpacked into a series of processes - squaring an even number a, squaring the consecutive even number ($a + 2$) and adding these together and equating

this thing to 340.)

- *being* and *having* verbs (X *has* parallel sides versus X *is* a parallelogram) ,
- conjunctions with technical meanings (*o*r usually is interpreted as *exclusive or* in everyday usage, but as *inclusive or* unless specified otherwise, in mathematical use),
- implicit logical relationships (O'Halloran, 1998, 2015; Schleppegrell, 2007).

The view that learning mathematics entails acquisition of vocabulary implicitly presents a simplified view of language as a *lexicon*, prioritising knowing terms and their meanings over how and when to use a particular term. Focusing on students' failure to use a particular technical term, hides how a student constructs meaning for mathematical terms drawing on such resources as gestures, objects and everyday experiences.

2.3.1.2 Learning mathematical language entails using the mathematics register

The second perspective that learning mathematics involves constructing multiple meanings emphasises word meanings, understanding multiple meanings and using language in situations. This perspective draws on the notion of the mathematics register. Halliday (1978) defined register as follows.

> A register is a set of meanings that is appropriate to a particular function of language, together with the words and structures which express these meanings. We can refer to the "mathematics register," in the sense of the meanings that belong to the language of mathematics (the mathematical use of natural language, that is: not mathematics itself), and that a language must express if it is being used for mathematical purposes. (p. 195)

The development of the mathematics register includes more than adding new words. Pimm (1987) suggests that speaking mathematically does not just involve the use of technical terms, but also modes of reasoning and arguing that are characteristic of the discipline. Within this perspective the main language related difficulty encountered by students is the difference between everyday and mathematical registers and ensuing obstacles to communication. For example, the word *prime* which carries different meanings in *prime* number, *prime* time and *prime* rib. Learning mathematics involves in part a shift from everyday to more precise meanings reflecting more conceptual knowledge. This can be understood as movement towards the mathematical register.

Positing the everyday register and the mathematical register as binaries and presenting learning as movement from one to the other may lead to an emphasis on the obstacles in moving from one to another and this can easily turn into a deficiency perspective. Such a perspective also obscures the benefits that can be derived from interweaving both the registers (Forman, 1996). Everyday meanings and metaphors can also function as resources for understanding mathematical concepts. Rather than emphasise the limitations of the everyday register, it is important to understand the different purposes these registers serve.

2.3.1.3 Learning mathematics implies participating in mathematical discourse practices

From the third perspective, learning mathematics is viewed as a discursive activity and as participating in a community of practice (Lave & Wenger, 1991; Lemke, 1990), which includes developing socio-mathematical norms (Yackel & Cobb, 1996) and using multiple material, linguistic and social resources. This perspective assumes that learning is inherently social and cultural and participants bring multiple views to a situation, that representations have multiple meanings for participants and these are negotiated through conversations. This perspective emphasises the situated and socio-cultural nature of language and mathematics learning. Situated and socio-cultural theories of learning have focussed greater attention of mathematics education researchers on the social environment in which learning takes place and the role of language and communication that happens within that environment (Lerman, 2000). Policy documents that highlight the role of communicating mathematically like, The Position paper on Teaching Mathematics, (NCERT, 2006); Principles and Standards of School Mathematics, National Council of Teachers of Mathematics, (NCTM, 2000); US Common Core State Standards Initiative, (CCSS, 2010); UK Department for Education (DfE, 2013), and developments in classroom practice also bring to the fore the need for language-rich activities in the classroom and consequently the need for research attention on the relation between language and mathematics. The increasing multilingual nature of these learning environments and the questions around which and whose language will be privileged have come to prominence following the “social” and “socio-political” turns in mathematics education (Lerman, 2000; Valero, 2004). Also the widening of conceptualisation of language of mathematics from considerations of words and symbols to participating in the practices of mathematics have led many scholars to analyse these practices through the lens of Discourses (Gee, 1996). Moschkovich argues that a situated-sociocultural perspective expands what counts as competence in communicating mathematically and provides an alternative to the deficiency models of students who are not adequately conversant with the academic language in the LoLT by being open to the use of the variety of resources that students use to communicate mathematically and helping teachers to build on these resources.

Defining mathematical language as consisting predominantly of specialised vocabulary or use of a register ((Halliday, 1978), tends to draw attention to incorrect language use by students and may lead to deficit views. Given the aim of this study to come up with broader conceptualisations of mathematical language that are more accommodating of students' languages, I take the stand that learning mathematics implies participating in mathematical discourses. In the following section, I look at characterisations of mathematical discourse in literature.

2.3.2 What constitutes Mathematical Discourse

In this section, I discuss three perspectives on what constitutes mathematical discourse - Commognitive theory (Sfard, 2008), Academic Literacy Framework (Moschkovich, 2015a) and a Dialogic Perspective on discourse ((Bakhtin, 1981; Barwell, 2016).

2.3.2.1 Commognitive theory

Commognitive theory (Sfard, 2008) envisages mathematics as a historically established discourse, and learning mathematics means becoming a participant in this special form of communication. According to Sfard, a discourse counts as mathematical if it deals with mathematical objects such as quantities and shapes. Sfard differentiates between colloquial and literate mathematical discourses and suggests that literate or scholarly mathematical discourses should be the object of school learning. The distinctive features that she marks for mathematical discourses include 1) uses of words that count as mathematical 2) the use of uniquely mathematical visual mediators in the form of symbolic artefacts that have been created specifically for the purpose of communicating about quantities 3) special discursive routines with which the participants implement well-defined tasks and 4) endorsed narratives such as definitions, postulates and theorems. Literate mathematical discourses are marked by the precision and rigour of their routines.

1) Word use: The objects of mathematical discourse are discursively created and are not perceptually accessible. Word use in a mathematical discourse is predominantly structural and impersonal, whereas in a colloquial discourse it is mainly personal and operational. An operational utterance presents the operations as somebody's action. Since actions must have a performer, operational utterances are largely personalised. In a structural presentation, there is no need for a performing subject. The utterance "7 and 8 make 15" is an immutable fact about the numbers which is independent of a person performing the addition, whereas the utterance "I put the 8 down below the 7 and added" is more likely to be about the numerals 7 and 8 and not the corresponding numbers. The second utterance is personal, operational and not objectified.

2) Visual mediators: Colloquial discourses, including colloquial mathematical discourses, are usually mediated by images of concrete objects, whether actually seen or imagined and these are referred to by nouns and pronouns and exist independently of the discourse. Literate discourses on the other hand are

mediated by symbolic artefacts invented for the sake of mathematical communication. Concrete or iconic mediators facilitate production of narratives, but mathematicians regard symbolic realisations as necessary for the endorsement of narratives.

3) Routines: Routines are a set of meta-rules that specify when and how repetitive discursive action is employed. Mathematical routines aim to produce narratives about mathematical objects, whereas practical routines produce changes in discourse-independent objects. Sfard identifies three different kinds of routines 1) exploration, whose implementation leads to an endorsable narrative or substantiates a narrative. 2) deeds, which are routines that involve practical action resulting in a physical change in objects or the environment and 3) rituals, which are sequences of discursive actions that aim to create and sustain a bond with other people. Sfard suggests that deeds and rituals are stages in the development of exploratory routines.

4) In colloquial mathematical discourses narratives are endorsed on the basis of empirical evidence, that is we endorse $2 + 2 = 4$ because whenever we put two pairs of objects and count, the counting ends with the word four. In scholarly mathematical discourse on the other hand, a narrative becomes endorsable if it can be derived according to generally accepted rules from other endorsed narratives.

2.3.2.2 Academic Literacy for Mathematics (ALM) framework

Widening the notion of "mathematical discourse" from the "literate mathematical discourse" characterised by Sfard, Moschkovich (2015a) suggests that academic mathematical discourse is not principally about formal or technical vocabulary, nor should it be confused with the "formal" or "textbook". She takes a more complex view of mathematical proficiency as participation in discipline based practices that involve conceptual understanding and mathematical discourse. She suggests that separating language from mathematical thinking and practices can have negative consequences for the marginalised groups, making them seem deficient, since they may not be able to express their mathematical ideas through language, but may still be engaged in mathematical thinking and participate in mathematical practices that are less language intensive. Drawing on a sociolinguistic perspective and expanding the meaning of "literacy" beyond use of words and language to include broader literacy practices, Moschkovich adds two extra dimensions to academic literacy: a) that it includes the vernacular even when engaging in academic literacy practices b) it draws on a full communicative repertoire that includes multiple modalities. She defines Academic Literacy in Mathematics (ALM), which includes three components namely mathematical proficiency, mathematical practices and mathematical discourse.

Drawing on Kilpatrick et al. (2001), Moschkovich defines mathematical proficiency as comprising five intertwined strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition. These five strands provide a cognitive account of mathematical activity, focussed on knowledge, metacognition and beliefs. Mathematical practices include the “taken-as-shared ways of reasoning, arguing and symbolising established while discussing particular mathematical ideas” (Cobb et al., 2001, p. 126). These practices can be considered to be using language and other symbol systems to think, talk and participate in practices that are the objective of school learning. They include problem solving, sense-making, reasoning, modelling, looking for patterns, structure or regularity, etc. The focus on practices shifts from a purely cognitive account of mathematical activity to ones that assume socio-cultural dimensions as well. This has implications for connecting practices to discourse, as discourse is central to participation in practices. Moschkovich defines mathematical discourse as “communicative competence necessary and sufficient for competent participation in mathematical practices” (p 47). Mathematical discourse is more than language and involves other symbolic systems, artefacts. Meanings develop through participation in mathematical practices. Some general characteristics of academic mathematical discourse that Moschkovich (2015a) marks are: particular modes of argument, precision, brevity, logical coherence, abstraction, generalisation, searching for certainty, etc. She also suggests that everyday and academic registers should not be construed as opposites and that literacy in mathematics goes beyond competence with words. “What makes a discussion mathematical is not the use of formal mathematical words, but mathematical concepts, which can sometimes be expressed using informal words and phrases, and mathematical practices, such as justifying a claim, which are not at the word level” (Moschkovich, 2015a, p. 56).

Moschkovich also highlights the need for tasks that will provide opportunities for students to participate in the full spectrum of academic literacy as defined here and organising classroom instruction so that students actively use mathematical concepts and show their conceptual understanding through explaining and justifying. Moschkovich (2000) suggests that use of everyday language and discourse practices should not be seen only as obstacles to learning mathematics but as resources to be used to communicate mathematically. Several scholars have drawn attention to the interweaving of the everyday and mathematical discourses is seen in the classroom discussions (Barwell, 2016; Forman et al., 1997; Moschkovich, 2003). There are many authentic mathematical discourse practices and such practices may vary across different communities (for example elementary and secondary teachers, or research mathematicians and statisticians), across time, cultural contexts and depending on the intended purpose (Richards, 1991). Hence whether or not student talk sounds mathematical depends on how we understand and view the distinctions between these different genres. There is a need to clarify the differences between mathematical ways of talking and formal ways of talking mathematically (Moschkovich, 2003).

2.3.2.3 Dialogic perspective on discourse

Aligned to the view that *mathematical* does not necessarily mean *formal*, Barwell (2016) critiques the tendency to view progress as movement from the first language to the LoLT, from informal talk to academic talk, and from everyday to mathematics register. Adopting a Bakhtinian dialogic perspective (1981), he suggests that the notions of formal and informal are not absolutes and emerge in relation to each other and to the other aspects of the context in which they are embedded. Barwell argues that a rigid distinction between them is not necessarily productive and that the tension between the formal and informal languages is but an instance of the continual tension between a unitary language and heteroglossia (Barwell, 2005, 2016).

The notion of heteroglossia captures the nature of classroom interaction, especially in a context of language diversity. The classroom interactions include a number of social languages of each student’s background (of class, caste, gender, race and so on), social languages of school (the language of mathematics, of curriculum, of textbooks, etc) and the languages of the teacher. Heteroglossia (Bakhtin, 1981) manifests as the stratification of language into linguistic dialects based on formal linguistic markers; into languages of social groups; “professional” and “generic” languages; languages of generations, etc. Bakhtin calls stratification and heteroglossia the centrifugal forces of language that exist alongside the centripetal forces of verbal-ideological centralisation and unification. Just as self-expression would be impossible without diversity, language would be meaningless without a degree of uniformity. Along with the multiple languages present in a class, there is also the unifying force of the LoLT and the need to communicate a fixed version of mathematics in a recognisable way (Barwell, 2012). The tension between centripetal force of unitary language and the centrifugal force of heteroglossia is present in and shapes each utterance. This tension is inherent in language and the tension between the unified standardised forms of mathematical expression and more diverse idiosyncratic expressions of mathematical meaning is but an instance of this tension. Other manifestations of this tension, especially in a situation of language diversity, include that between the language of instruction in school and the languages used by students outside of school and the tension arising from the differential status accorded to languages by the society (Barwell et al., 2016). These tensions present themselves as “dilemmas” or situations with competing priorities, each with its own advantages and disadvantages, and therefore suggest opposing courses of action, each of which may involve some compromises (Adler, 2002b). In the following section, I discuss these dilemmas and further delve into work relevant to the formal-informal continuum in mathematics.

Barwell (2016) suggests that rather than expect students to follow a linear path from informal to formal mathematical discourse, working with the teacher to expand the repertoire of possible ways to make meaning in mathematics should be considered progress. This flexibility, which allows for the use of varying degrees of mathematical formality and makes rich interactions possible, allows for ambiguity in the mathematics classroom. Barwell (2005) suggests that "ambiguity can be seen as a resource for doing mathematics and for learning the language of mathematics" (p 118).

Even as I accept the need to accept students' informal language, the question arises whether this amounts to acceptance of incoherence and inconsistency, and if yes, how much of it can be tolerated without entirely diverging away from the core epistemic values of mathematics. Further, students need to build on their mathematical talk over time even if they do not rely on formalism that is supplied by the textbook. With these considerations in mind, one of the questions addressed in this thesis is that of acceptability criteria for mathematical discourse. (see Section 5.5)

The progressive broadening of what counts as mathematical discourse across these perspectives presents researchers, teachers and learners and policy makers with "tensions" (Barwell et al., 2016) or situations in which competing influences suggest different or even opposing courses of action. I now take a closer look at the discussions in literature on the different tensions and dilemmas that arise in a language-diverse and marginalised context, especially when one works with students' languages.

2.3.3 Dilemmas and tensions in teaching and learning mathematics in marginalised and language diverse contexts

Barwell et al. (2016) point to three dominant tensions in language-diverse and marginalised contexts.

Tension between the language of instruction in school and the languages used by students outside of school: In a globalised world with increasing mobility of population for political, economic and social reasons, language diversity is the rule rather than exception. In a country like India where a multiplicity of languages are spoken and some of them are recognised as official and the sanctioned medium for public instruction, a conflict between learners' home language and the LoLT at school is inevitable. With English gaining currency as an international language and the language of scientific and technological communications, it is privileged as the medium of instruction and is in tension with the home languages of large sections of non-English speaking learners. Many studies from across the world have drawn attention to this tension (Bose & Choudhury, 2010; Clarkson, 2007; Farrugia, 2009; Halai, 2009; Setati & Planas, 2012) Code-switching and translanguaging have been advocated as possible workarounds (Poo & Venkat, 2021; Setati et al., 2002).

Code-switching involves alternating between two languages, substituting a word or phrase in one language with word or phrase in another. Translanguaging involves purposeful alternation of languages in spoken and written forms. Code-switching is considered to be a ‘responsive practice’ used to respond constructively in the moment to students’ responses, whereas translanguaging is viewed as a planned teaching strategy. This involves intentional attention to working with multiple representations across language and mathematics based on the mathematical topic being discussed. In the context of mathematics, translanguaging involves use of multiple modes to make meaning and a “systematic use of language and registers that go beyond simple substitution of one representation with another” (Poo & Venkat, 2021, p. 45).

Tension arising from differential status accorded to languages by the society: In former colonies and places like India and South Africa, English holds a privileged place. It is seen as a symbol of power and a gateway to success and achievement in life. Consequently, teachers and learners prefer English as the LoLT, driven by concerns of access to social goods and positioned by the social and economic power of English. This comes at the cost of restricted epistemological access to mathematics (Setati, 2008). Also the preference for high-status languages over the languages that students use at home thwarts the use of mitigative measures like code-switching.

Tension between informal language and mathematical language: Use of informal language aids sense-making whereas competence to communicate with the larger community of mathematics calls for formal language. This opens up a decision point for teachers - whether they should pay explicit attention to mathematical language or leave it implicit and transparent so as not to disrupt the mathematical discussion. This tension between the formal and informal has been observed in several studies in different parts of the world (Adler, 2002a; Barwell, 2016; Farrugia, 2013; Khisty, 1995; McGinn & Booth, 2018; Moschkovich, 2008; Nygård Larsson & Jakobsson, 2020). There is no neat resolution to the tension between informal and mathematical language. Insisting on the formalised mathematical language will exclude and disenfranchise many learners who find the formal language of mathematics forbidding. On the other hand, not providing them with the opportunity to learn more formal ways of communicating mathematics, will also in the long run disenfranchise them even if they have a good understanding of mathematics. The teacher’s response to the dilemma may in turn have an influence on students’ participation in mathematics, leading to a vicious circle. Adler (2002b) also points to this tension when she discusses the dilemmas of mediation and transparency.

Adler (2002c) discusses how these tensions present themselves in the classroom, as situations where the teachers perceive a choice of actions, each with its own costs. She discusses three dilemmas: code switching; mediation; and transparency, all of which have to do with both language use in classrooms and the fact that it is mathematics that is being taught.

- *Code-switching: Developing English versus developing meaning.* The underlying concerns here are whether or not and when to switch languages in class, how to grant or assist in gaining access to the particular resources of particular languages, issues that arise when teachers and pupils value and use more than one language in class, switching between the language of instruction and pupils' main/spoken languages.

- *Mediation: Developing mathematical communicative competence (subject-specific-discourse) versus negotiating or developing meaning*. In contexts of shifting pedagogy, where teachers place more emphasis on learners' meaning making and exploring mathematical ideas, some learners need help to express their thinking in English and in mathematical ways. Recognition that specific language help was needed and offering help with ways of speaking mathematically opens up the dilemma of balancing listening to learners' exploratory talk and assisting the negotiation and development of meaning without blocking their meaning by prematurely working on how these are expressed.

- *Transparency: Implicit vs explicit practices, whether or not to be explicit about mathematical language.* Learner-centred curriculum initiatives require the teacher to play a subdued role, letting the situation evolve. Such pedagogies rely on the communicative competence of students which if not sufficiently developed will require mediation and explicit teaching by the teacher. Teachers may need to work explicitly on mathematical language, in the interest of clarity and access to mathematical discourse.The dilemma here is when to focus on mathematical language (making it visible) vs. when to background language and focus on mathematical meaning making (render language invisible).

These dilemmas are not specific to but are exacerbated in multilingual or marginalised contexts.

Tension between the spoken and written modes of communication: In addition to the three tensions that Barwell has pointed out, another tension that I have encountered in the course of this study is the dilemma in choosing between written and spoken modes of communication. The current assessment practices insist on some form of writing. Alternatives to the traditional examination like portfolio or project based assessment require some amount of writing too, perhaps without the supporting cue from a question. Also writing is an important part of doing mathematics. Research has looked at the challenges involved in investigatory writing that is a part of the GCSE classwork and identified such challenges faced by students (Morgan, 1998). Students who were part of this study were reluctant to write except on impermanent surfaces like the classroom floor, desks or the blackboard and erased it as soon as their purpose was served. This created a tension between insisting on writing, perhaps at the cost of student engagement and going with the spoken word, perhaps a compromise from an assessment perspective.

Writing itself could be used as means to support thought processes and to communicate these with oneself or others or as a record to be scrutinised and evaluated by others. Students may have been sensitive to the tension between these different purposes for which writing could be used and chose to stick with the former, frequently writing on erasable surfaces. Writing that is not meant to be evaluated tends to be loosely structured, subject to multiple edits, to be erased once the purpose is served. In the context of this study, well-structured writing was not the students' preferred means to present their thoughts and ideas and alternatives like oral expression supported by minimal writing or diagrams were resorted to. Talk is flexible and allows for more linguistic repertoires and does not have the rigid boundaries of writing and may ease communication. This is not to ignore the importance of writing in education - so a fine line needs to be drawn - questions like when and how much of spoken mode is acceptable, what kind of speech is acceptable, etc., need to be investigated.

Since our key concern in this study is the tension between the formal and informal, I delve deeper into the literature around the roles of and relation between and the formal and informal mathematical languages.

2.3.4 Informal language in teaching-learning and doing mathematics

A number of studies have examined the relationship between mathematical language and everyday language or informal language or colloquial language. Radford and Barwell (2016) in a review of language-oriented papers presented at PME conferences 2005- 2014 identify informal and everyday language as one of the frequently identified theoretical orientations. These papers examine the influence, support and interference of informal language on students' mathematical conceptualisations. Scholars have also documented the linguistic functions that students and teachers resort to while expressing mathematical ideas in informal language. In addition to these I also look at available research on how mathematicians use informal languages in their discussions in this section.

2.3.4.1 The influence of the informal language on mathematical understanding

Kim et al. (2012) studied how the colloquial terms in use influence the discourse and understanding of a mathematical concept. In English the word “infinity” is used in colloquial language and denotes a formal mathematical concept as well. In Korean the mathematical word for infinity is not a formalised version of the colloquial word. Consequently, the authors observed that Korean speaking students' discourse on infinity was more structured and closer to the formal mathematical discourse whereas that of English speaking students was predominantly processual and informal. Cornu (1991) points out the different meanings the word “limit” can have to different individuals at different times. Most often it is considered as an ‘impassable limit’ but it can also mean an impassable limit which is reachable; a point which one approaches, without reaching it; a maximum or a minimum; the end; the finish etc. Students’ use of the mathematical term “limit” is conditioned by these everyday meanings. In a study involving negation of a statement with the universal quantifier “all” that called for recognition of diagrams which implied “Not all A is B” Bardelle (2013) concludes that the interpretation of verbal statements in a mathematical setting may happen based on everyday context and some sentences involving logical connectives evoke meanings that contradict the mathematical interpretation.

2.3.4.2 Informal language as source of interference

One possible source of interference identified by scholars has to do with the overlap that mathematical terminology has with everyday words. Since mathematics itself is not a language, it is taught in a natural language like English or Tamil. Setati (2001) points out that in a multilingual context like South Africa, communicating mathematically means managing the interaction between ordinary English and mathematical English, formal and informal mathematical language, procedural and conceptual discourses and learners’ main language and the LoLT. She describes mathematical English as the English mathematics register. She identifies one of the difficulties in learning to use mathematical English stems from the fact that it is used in speech and writing blended with ordinary English and the distinction between the two languages is often blurred. There are words and phrases which occur in both with different meanings. For example, logical constructions such as “and”, “or” , “if…then”, “some” , “many” appear to belong to ordinary English, but their use in mathematics may have a different connotation. “Or” in natural language is generally used in the exclusive sense – the expression “rain or sunshine” implies that there is rain or sunshine but not both. The use of “or” in mathematics on the other hand does not preclude both occurring together. It needs to be specified as “XOR” or the exclusive or to bring in that sense. Pimm (1987) uses another example to highlight how the word “any” from everyday language has been repurposed for mathematical use. Consider the following questions:

Is there any even number which is prime?

Is any even number prime?

The response to question a) is a clear yes, 2 is an even number and it is prime too. Question b) on the other hand can be interpreted in two different ways - Is any (i.e., one specific) even number prime ? and Is any (i.e., every) even number prime? In mathematics “any”is generally used in the sense of “every” and it is in conflict with the ordinary English usage.

Pimm (1987) identifies three groups of terms in mathematics vocabulary: 1) terms that have the same meaning in everyday and mathematical contexts 2) terms whose meanings changes from one context to the other and 3) terms which are seen only in mathematical context. Terms in category (1) may not be problematic for students and those in (3) may need to be defined as they are not part of the students vocabulary. However those terms in category 2 pose problems for students, more so because of the co-occurence of mathematics and everyday language in the classroom and in textbook definitions as well. There may also be terms used in mathematics and in other disciplines in similar but non-identical ways.

2.3.4.3 Informal language as source of support

Students and teachers frequently use everyday language to understand mathematical concepts. Scholars have investigated the affordances enabled by formal and informal languages and the nature and purpose for which teachers and students draw on different modes. Barwell (2012) observes that in classrooms where multiple languages are used, the formal mathematical terms are presented in the official language or LoLT. Observing classrooms in Pakistan, Barwell finds that words like ‘algebra’, ‘divide’ or ‘axis’ were used in English even in a discussion in Urdu or Burushaski. This phenomenon has been attested to by other scholars as well (Bose & Choudhury, 2010; Setati, 2005). Moreover Barwell observed that English used in mathematics lessons in Pakistan quoted the textbook, either through reading it aloud or through repetition of the textbook content. Students' informal discussion of mathematical ideas were in the regional language. Setati (2005) goes further to state that in primary school mathematics in South Africa mathematics in English tended to be more procedural in nature, while discussions of students' thinking or mathematical ideas were more likely to be in their home languages. We see here the tension between the systemic need for students to develop the “accepted” ways of communicating mathematics and the teacher’s desire to see her students discuss rich mathematics. In the process she makes a trade-off in allowing a certain degree of flexibility to express themselves informally while also gently pushing toward more formal discourses (Barwell, 2012). These examples indicate that the flexibility afforded by the informal and home languages compared to the formal and school languages is an enabling factor to mathematical thinking. While the unified language of mathematics enables communication of ideas, this unified language can be marginalising.

2.3.4.4 Linguistic means used to speak mathematically in informal language

I now look at some ways in which students and teachers use informal language to communicate mathematical ideas. Rowland (2000) draws attention to the deictic use of “it” to refer to and point to mathematical concepts and generalisations which have not or for various reasons cannot be named in the discourse. Deictic terms are linguistic units referring to objects in the universe of discourse by virtue of the situation where the dialogue is carried out. The context of the dialogue determines their referent. For example, in the course of the Magic triangle exploration, when the student says, “If I put a 3 here it will not come”, the *it* refers to a particular value of the side-sum (see Section 5.5.4). Rowland (2000) also points out the use of the second person pronoun “you” as an effective indicator of generalities in mathematical discourse.

Radford (2000) identifies three ways that students use to express generality in natural language: talking about the general through the particular, the deictic function and the generative action function of language. A deictic expression is a word or phrase (such as *this*, *that*, *there*, etc.) that specifies or points to the location the speaker is referring to. In the student utterance *“OK. Alright, look. You . . . one has to add (pointing to a figure on the paper) . . . you always add 1 to the bottom, right? Then you always add 1 to the top”* from Radford (2000), students use the deictic words "top” and "bottom,” to refer to key parts of a perceptual term in order to imagine non-perceptual objects and their mathematical properties. The “generative action function” refers to the linguistic mechanisms expressing an action which is repeatedly undertaken in thought. In the utterance above, the adverb “always” provides the generative action function and generality is implied through the potential for reiteration. Thus, here “always” plays a role similar to the universal quantifier “for all” in more formal language.

2.3.4.5 The informal in mathematician’s work/talk

The role of the informal language in the process of mathematical discovery is well acknowledged in literature (Byers, 2010; Hadamard, 1945; Sriraman, 2004; Thom, 1973). To quote Thom,

> “In practice a mathematician's thought is never a formalised one. One accedes to absolute rigour only by eliminating meaning; absolute rigour is only possible in, and by, such destitution of meaning. But if one must choose between rigour and meaning, I shall unhesitatingly choose the later” (p 203)

During the process of discovery and in discussions mathematicians resort to ill-defined terms, pictures and half- formed ideas. The informal discourse allows greater room for false starts, loose statements and working in a semi-confused state, which is necessary when the solution is unclear or when it may not exist at all. Hadamard (1945) highlights the role of intuition in the process of discovery that happens after long and unconscious prior work. The expression of the result in writing and formal language comes later. Tweney (2012) observed that expert mathematicians tend to use mathematical expressions sparingly and meaningfully, and to use mathematics as a representation rather than as a possible path to an algorithmic solution. Mueller-Hill (2013) suggests that mathematicians may have internalised the rules and principles of formal proving and work in agreement with these rules without explicitly and consciously employing them. Based on interviews with research mathematicians she suggests formalisability as an epistemic feature of discursive proving actions and interprets it as a meta-discursive rule (Sfard, 2008) guiding mathematical discourse.

Barwell (2008b) examines how mathematicians talk about mathematics. He analyses excerpts from a radio broadcast and draws attention to some of the discursive resources that mathematicians draw on in their thinking, and suggests that the discourse used is a *hybrid discourse* incorporating the mathematical and everyday discourses. Among other features like a narrative form and the agentic nature of talk involving doers of mathematics, Barwell also draws attention to the inclusion of everyday discourse in the form of expressions, analogies and references to popular cultures. Barwell points to the interweaving of the everyday and mathematical vocabulary through multiple examples, one of them being the use of terms bagel, torus and hyper-bagel. *Bagel* is an everyday term and *torus* a mathematical one and *hyper-bagel* a conjugation of *hyper* from academic mathematical discourse and *bagel* from everyday discourse. The usage of the term *hyper-bagel* subsumes the everyday term and makes it mathematical. Barwell (2008b) suggests that this kind of hybridity is widespread in the world of professional mathematics. For a mathematician, talking about a bagel can be as mathematical as talking about a torus. Barwell suggests that "any word can be mathematical if used in a mathematical way. Hence, mathematical language is *not* a lexicon, but a way of using language."

Drawing implications from this research for classroom practice, he further suggests that it is valuable to develop a better understanding of the discursive practices of mathematicians and that it may be worthwhile to introduce some of these practices into the mathematics classrooms and to expand the range of genres of spoken mathematics available to a student. Highlighting the complex relationship between everyday and mathematical discourse, Barwell suggests that the everyday does not disappear in mathematics, but used in new and more mathematical ways. He also suggests that the use of everyday in mathematics is not necessarily an indication of an underdeveloped understanding.

2.4 Recentring Margins-

In the preceding sections, we saw that mathematics marginalises along multiple dimensions.

On the disciplinary dimension, we saw that the school mathematics tradition with its right-answer focus does not give sufficient space for students to demonstrate their mathematical competence. On the language dimension, we saw that narrow conceptualisations of mathematical language as predominantly specialised vocabulary or register leads to deficit views and aligned ourselves with the socio-cultural perspectives of learning that view learning mathematics as participation in discourse practices. I also discussed different characterisations of mathematical discourse, the multiple discourses and languages present in the mathematics classroom, the tensions between them and the need to accommodate students' languages, including informal ways of communication to move away from deficit perspectives.

What I termed performative dimension stems from larger societal beliefs and needs to be addressed through systemic measures. Our concern in this project is on ways of addressing the deficit discourses along the disciplinary and language dimensions of the margins as they manifest in the classroom. I do acknowledge that these cannot be isolated, but choose to limit the scope of this study to the pedagogical shifts the teacher can make to address the problem. I now look at some specific steps that are discussed in literature in this direction.

2.4.1 Landscapes of investigation

In Section 2.1.3, we noted that narrow conceptualisations of mathematics as requiring a single right answer, pre-determined by an authority figure such as the teacher or textbook and to be found by following the given procedure are marginalising. Addressing the marginalisation calls for a significant rethinking of how mathematics is taught in school – how teachers and students interact, how students are assessed and how content is introduced.

Skovsmose (2001) proposes landscapes of investigation as a learning environment different from the school maths tradition and the exercise paradigm. The creation of landscapes of investigation is an attempt to organise educational processes in such a way that they allow students and teachers to get involved in exploratory processes guided by dialogic interactions (Godoy Penteado & Skovsmose, 2022). Landscapes of investigations do not specify sequences of problems to be solved, or exercises to be answered, On the other hand, they invite students to engage with inquiry processes – to ask questions, to formulate hypotheses, to try out arguments and to listen to other arguments and ideas. In addition, they facilitate collaboration and shared engagement. Investigations invite students to frame questions of interest to follow, or reformulate a question to make visible a solution approach, or come up with a simplified version of the problem. These are highly relevant practices in mathematics. Also any group of students can engage with exploration of landscapes of investigation (Skovsmose, 2022). The conversation around an investigation is open-ended and does not follow any chartered paths. Several scholars have referred to investigations by different names and discussed their use teaching-learning mathematics (Banwell et al., 1972; Becker & Shimada, 1997; Polya, 1945; Yeo, 2017). I discuss related literature in Section 2.5.

2.4.2 Language as resource perspective

Several scholars have suggested that the use of more than one language in multilingual classrooms is a productive move and a means to challenge the deficit perspectives that portray multilingual learners as less capable of learning mathematics. (Adler, 2002c; Moschkovich, 2000, 2021; Planas, 2018; Planas & Civil, 2013; Planas & Setati-Phakeng, 2014; Setati et al., 2008). This view counters the view that students' home languages are inferior to dominant languages and less suitable for doing mathematics evident in the tensions discussed in Section 2.3.3, and based on the notion of "language as resource".

Drawing on Ruiz (1984), Planas and Setati-Phakeng (2014) suggest the "language as resource perspective as an ideal to work towards for flexible use of student languages. They see language as resource as an "organising principle for classroom practices with the aim of achieving learning opportunities through integration of foci on mathematics and language" and as "the combined strategies, norms, and processes that seek to bring about a balanced integration of these two foci" (Planas & Setati-Phakeng, 2014, p. 887). Research oriented to this perspective puts opportunities arising from the flexible use of language practices at the forefront rather than difficulties and obstacles that arise due to multilingualism in a classroom. While these recommendations were made mainly in the context of multilingual classrooms and the tensions between the learners first language and the LoLT, Adler (2002c) applies the resource perspective to address the tension between the students' informal languages and the formal language of mathematics as well.

Extending Lave and Wenger's (1991) idea that access to a practice requires its resources to be transparent, Adler (1999) proposes the idea of language as a transparent resource in gaining access to mathematical practice. Transparency involves the dual aspects of visibility and invisibility – a resource should be visible so that it can be noticed and used, and invisible so that attention is focussed on the subject matter and not on the tool. The ways of using language in a mathematical discussion should enable learning and therefore be invisible. Simultaneously, learners need to understand the significance of mathematical talk and hence the specificity of mathematical discourse needs to be made visible (Setati et al., 2008).

Further, Moschkovich (2000) talks of the need to consider as a resource the multiple means that students use to communicate mathematically – be it gestures, concrete objects, invented terminology, metaphors or flexible movement across languages. She cautions against taking a deficit perspective that views these as stand-ins for the formal language or obstacles that students face in using formal language, advocating that they be seen as pointers to ways of supporting students to communicate mathematics better.

Scholars have also looked at ways of supporting teachers to design learning material that enhance language for mathematics learning and in being receptive and responsive to student languages. Crespo et al. (2021) suggest that helping teachers to see language diversity as a resource and paying explicit attention to discourse moves that highlight students' language diversity as an asset helps them take up more inclusive and strengths-based approaches. Several scholars have suggested such discursive moves ((Erath et al., 2021; Martinez, 2018; Moschkovich, 2015b; Planas, 2014). These include such principles as engaging students in rich discourse practices, establishing various mathematics language routines , connecting multimodal representations and including students' multilingual resources (Erath et al., 2021).

2.4.3 Anti-deficit noticing and framing

Anti-deficit noticing is noticing that focuses on students' resources and strengths. Louie et al. (2021) conceptualise anti-deficit noticing as

> "noticing that deliberately challenges deficit discourses, intentionally attending to and elevating the humanity, intelligence, and mathematical abilities of marginalised people, not in speeches or statements but in routine instructional interactions. Anti-deficit noticing thus goes beyond a blanket commitment to seeing the assets that all students bring to learning" (p 100)

They suggest that anti-deficit noticing is rooted in framings such as : 1) Students are full human beings with many resources 2) Mathematics learning is a creative exploration of ideas 3) Interactions and interpersonal relationships are integral to learning.

Scheiner (2023) suggests the alternative of strength-based framing which considers students thinking as a resource instead of a deficit. This approach highlights the positive contributions students make to the classroom, without acknowledging the difficulties they may have when learning mathematics. Scheiner reports on a teacher education module designed to encourage prospective teachers to examine how they frame and what they notice about students' mathematical thinking. The module aimed to bring about a shift from a deficit-based to a strength-based approach when noticing students' thinking. Framings like "students' mathematical thinking is a capability to be fostered", "is valuable in its own right and to be cultivated", and a resource to build upon are some strength-based frames that Scheiner identifies. Louie et al. (2021) suggest that more research is needed to create and sustain systems that enable anti-deficit noticing and strength based framing.

Continually expanding what counts as mathematical competence through a more multidimensional framing of mathematical activity to include such practices as sense-making, connection seeking, experimentation, collaboration and argumentation, seeking out and highlighting the resources and strengths of marginalised communities, telling “counterstories” - which challenge the dominant narratives about the inferiority of marginalised groups and the normative superiority of the dominant groups, and an ongoing effort at anti-deficit reframing to counter the pernicious influence of deficit discourses are some steps that Louie et al. (2021) suggest to disrupt deficit discourses. Tasks which have been referred to as Landscapes of Investigation, Explorations or Open-ended tasks in the literature have been suggested as a means to develop a more expanded view of mathematical competence. One of the goals of this study is to investigate the potential of such tasks to engage students in mathematical practices in marginalised contexts. In the following section I discuss literature around such tasks focussing on how they have been characterised by scholars.

2.5 Explorations or Open Tasks

There is a general consensus among educators on the importance of students engaging with tasks other than those that are intended to give them practice in procedural skills taught earlier. Such problems have been referred to by multiple names, but share the objective of fostering in students a belief that they made mathematics their own through exploration. In the 1980s, bodies such as the Association of Teachers of Mathematics (ATM) and the National Council for Teachers of Mathematics (NCTM) promoted the view that problem solving should be the focus of school mathematics and that a problem perceived as a situation to explore is a more valuable mathematical task than one involving a reproduction of a ready-packaged method applied to recognisable problem set (Orton & Frobisher, 1996).

Open tasks that differ from those commonly encountered in textbooks have been referred to by different terms in literature: Open problems (Pehkonen, 1997a), open-ended problems (Boaler, 1998), mathematical investigations (Ernest, 1984; Jaworski, 1994; Mason, 1978), ill-structured problems (Shuk-kwan, 1997). Yeo (2007) notes that different people use the same term to mean different constructs or use different terms to refer to the same construct. Some scholars use *open* and *open-ended interchangeably*, while others distinguish between them. Some scholars use the synonymous term “exploratory problems” to avoid confusion with the unsolved problems of mathematics. The term has been used to refer to pure maths based investigative tasks and to authentic-real life tasks. The latter have also been referred to as real-life landscapes of investigation (Skovsmose, 2001) and as environmental problems (Orton & Frobisher, 1996).

The concept “open problem” has been the subject of many scholarly discussions and there have been many attempts at clarifying both the words, “open” and “problem”, that constitute this term.

2.5.1 What is a "problem"?

Schoenfeld (1985) defines a "problem" as a situation that does not have a ready answer, it is more than merely doing exercises which can be completed using known procedures. Echoing this Pehkonen defines a problem as a " situation where individuals are compelled to connect known information in (for them) new ways, in order to accomplish a task. If they can immediately recognise the actions needed to do the task, then it will be a standard (routine) task " (Pehkonen, 1997b, p. 75). Whether a situation is a problem as per these definitions depends on the individual trying to solve it and not some features inherent to the situation. Schoenfeld clarifies,

> The difficulty in defining the term problem is that problem solving is relative. The same tasks that call for significant efforts from some students may well be routine exercises for others and answering them may just be a matter of recall for a given mathematician. Thus being a "problem" is not a property inherent in a mathematical task. Rather it is a particular relationship between the individual and the task that makes the task a problem for that person. The word problem is used here in this relative sense, as a task that is difficult for the individual who is trying to solve it. Moreover that difficulty should be an intellectual impasse rather than a computational one. (Schoenfeld, 1985, p. 74)

There is also the alternate view that a mathematical problem is characterised by its nature and purpose. Scholars have suggested classifications of tasks based on these aspects. Orton and Frobisher (1996) distinguish "problems" and "investigations" based on the presence of a specific and recognisable goal. They call tasks that do not have a prescribed goal and readily known or recallable mathematics procedure to make immediate progress as "investigations". Shuk-kwan (1997) distinguishes between well-structured and ill-structured problems based on the "givens" (objects and operators) and the "goal state" of a problem. A well-structured problem is one where both the "givens" and the "goal-state" are well-defined. Ill-structured problems open up opportunities for problem posing, with the problem poser having to define the givens, the goal state or both. Yeo (2007) points to distinctions along multiple dimensions - problems v/s exercises, problems v/s investigative tasks, Investigations v/s guided discovery learning, academic, semi-real or real-life tasks. Skovsmose (2001) offers six learning milieus based on whether the tasks refer to pure mathematics, semi-reality or real-life or and whether they fall into an exercise paradigm or investigatory paradigm. Pehkonen (1997b) defines a category of tasks called "problem field" as a sequence of problems that are connected to each other. A problem field is generated through changing the conditions given in the task and given to students gradually, continuing based on the approaches they take and the solutions they come up with. Two key characteristics of problem fields that Pehkonen marks are a) generativity of further problems b) problems across a range of difficulty levels, with a problem field being suitable across classes and abilities.

Yeo (2017) distinguishes between the following four types of tasks based on their purpose:

a) *Standard textbook type task* or routine procedural task, exemplified by the task

> Solve the quadratic equation: $x^2 + 2x - 3 = 0$ (Task 1, Quadratic) (Yeo, 2017, p. 177)

The main purpose of such a task is routine practice of procedural skills for students who have already learnt the quadratic formula. This may be a problem to students who have not encountered the procedure to solve a quadratic equation, or having encountered the procedure, are yet to acquire sufficient fluency to apply the procedure appropriately. But with enough practice, this task can become a routine exercise for all students.

b) *Problem-solving task,* exemplified by the task

> At a workshop, each of the 100 participants shakes hand once with each of the other participants. Find the total number of handshakes. (Task 2, Handshakes)(Yeo, 2017, p. 177)

The purpose of such tasks is to make use of some problem-solving heuristics, such as looking for patterns to solve it. Unlike the first problem which only requires the application of a procedure to solve it, this task requires some creative effort and higher-level-thinking to solve. Hence such problems are classified as problem solving tasks, though they may not actually be problems for students who have already encountered such tasks. Another notable feature is that it has a clear goal - namely to find the total number of handshakes. This is the key difference that Yeo marks between problem-solving tasks and investigative tasks

c) Investigative Task, exemplified by

> Powers of 3 are 3^1, 3^2, 3^3, 3^4, …Investigate. (Task 3, Powers of 3) (Yeo, 2017, p. 178)

The purpose of such tasks is for students to investigate and discover the underlying patterns or mathematical structures. The distinguishing feature is that they do not specify a goal in their task statements and therefore few mathematics educators would classify investigations as problems (Orton & Frobisher, 1996).

d) Real-Life Task exemplified by

> Design a playground for the school. (Task 4, Playground) (Yeo, 2017, p. 178)

The purpose of this kind of task is to learn and apply mathematics in real-life situations. Such tasks create opportunities for students to learn and utilise mathematics such as measurement, geometry, costing and spatial visualisation. Engaging with such tasks implies that in addition to thinking through the problem, students will need to find out the relevant information such as dimensions of a swing or slide, cost of building these etc.

Yeo (2007) proposes that different types of tasks have different pedagogical uses and a clarification of the features and uses of these tasks will enable teachers to choose tasks appropriate for their students and their needs. It also helps researchers to define more clearly the tasks they are investigating and delimit their research accordingly. However some educators do not distinguish between these tasks. Pehkonen (1997a) suggests that the concept “open problem” be used as an “umbrella” class for problems which contains all such classes as: investigations, problem-posing, real-life situations, projects, problem fields or problem sequences, problems without question, and problem variations or “what-if” method.

2.5.2 What makes a problem open?

Pehkonen (1997a) explains the concept of what he calls “open problem” in these words.

> “We will begin with its opposite and say that a problem is closed, if its starting situation and goal situation are closed, i.e. exactly explained. If the starting situation and/or the goal situation are open, i.e. they are not closed, we have an open problem. (Pehkonen, 1997a, p. 8)

Cifarelli and Cai (2005) define open-ended problem situations as those where some aspect of the task is unspecified and requires that the solver re-formulate the problem statement in order to develop solution activity. This is similar to Shuk-kwan’s (1997) notion of an ill-structured problem and Orton and Frobisher’s (1996) definition as a problem that leaves the goal unspecified, as an open decision yet to be made. Becker and Shimada (1997) define open-ended problems as “problems that are formulated to have multiple correct answers” (p 1). Absence of complete specification and multiplicity of possible answers that this gives rise to are the key features that stand out in these definitions. Yeo (2017) suggests that these are not the only two factors that make a task open. He proposes a framework to characterise the openness of mathematical tasks. This framework has five dimensions namely answer, goal, method, complexity, and extension.

Open Answer: Tasks which have only one (or a fixed number of) correct answer are termed closed in its answer. Tasks 1 and 2 above have only one correct answer, whereas tasks 3 and 4 admit of multiple answers. Task 3, the Powers of 3 task, is open in answer because any pattern in powers of 3, be it in their units digits, or some of digits or their frequency in a given interval, all would qualify as correct answers. Also it would not be possible for anyone to claim that they have found out all the correct answers and in this sense the answer is “indeterminate” according to Yeo (2017). Similarly, the Playground task also admits of multiple answers in that there are more than one way of designing a playground. In this case, in addition to the answer being indeterminate, the correctness of the answer is subjective as well. Yeo terms this as an “ill-defined answer”. Thus Yeo defines a task to have a closed answer if the answer is determinate, that is it is possible to determine all the correct answers. Otherwise the task has an open answer, which could be either well-defined and objective or ill-defined and subjective.

Open Goal: Tasks which do not clearly specify a goal in the task statement are said to be open in terms of goal. Tasks 1, 2 and 4 above have well-specified goals. Investigative tasks like Task 3 typically do not have a goal in their task statement. The goal of an investigative task is supposed to be a “general goal” which is to investigate. Students can choose any specific goal to investigate - for example finding whether there is a pattern in the last digits in the powers of 3. If the goal is open, one would expect the answer to be open and indeterminate as well; however it is possible to have a task that has a closed goal but is open with respect to the answer. The playground task discussed above is an example.

Yeo (2008) points to the difficulties that a task with ill-defined goals pose to students who have not encountered such tasks before and raises the question whether the ill-defined goal of an investigative task can be clarified sufficiently to help students understand the task requirement and yet keep the goal open.

Open Method: Yeo describes a task as having an open method if there are multiple methods of solution involving problem solving heuristics rather than mere application of known procedures and closed if otherwise. Since the method is in the “middle” of the goal and answer, such tasks have also been referred to as “open-middle” tasks. While a task may be open with respect to the method in-principle, most students using one method or teachers teaching only one method may make it effectively closed. This can be addressed by reframing the task in such a way that openness with respect to the method becomes a task-inherent feature and does not depend on students and teachers. For example, the Handshakes task discussed above could be reframed as,

> At a workshop, each of the 100 participants shakes hands once with each of the other participants. Find the total number of handshakes using as many methods as possible. Discuss which methods are ‘better’ and in what ways they are ‘better’ (Yeo, 2017, p. 183)

Task Complexity: Yeo terms a task open along the complexity dimension if it is too complex for the students it is addressed to, and there is not enough scaffolding in the task statement to allow them to get started. A task that is simple enough for students is closed in this dimension.

Extension: A task is open along this dimension if it can be extended , that is extending the task would lead to discovery of more underlying structures. A task that cannot be extended or leads to unrelated tasks if extended is closed in this dimension. Extensibility of a task can also be subject dependent as in the case of the Handshakes task, or task-inherent as in the case of Powers of 3 task. The Handshakes task is in-principle extendable, in that the number 100 in the problem can be varied and the problem generalised in this dimension. However the way the task is framed students may not think of generalising it, nor would teachers expect it. With the Powers of 3 task on the other hand, it is more likely that students investigate if a pattern they found is true of powers of other numbers as well, or they are expected to do so. Therefore Yeo terms the extensibility of this task task-inherent.

In this study, I use the term *Mathematical Explorations* to refer to an open-ended and loosely-defined mathematical problem situation, that involves students asking their own questions, choosing the ones that interest them, following different paths to find answers and asking further questions. We preferred games and puzzles which invite student engagement over real-life problems that often have a hidden 'curriculum agenda' where the shadow of assessment and non-performance dominate student perception. I drew on Yeo's framework in the design of modules and my attempt was to design tasks that were open along the dimensions of method, goal and answer but not along the complexity dimension. I reinterpreted task extensibility as generative of more questions, albeit not directly related as Yeo suggests (see Section 4.2). I discuss in detail the task features in Chapter 4.

2.6 Mathematical Thinking

Explorations shift the focus from the one right answer to the ways of thinking and reasoning adopted by students and practices such as sense-making, experimentation, argumentation, etc. Focussing on mathematical thinking allows for a broader conceptualisation of what it means to do mathematics. Mathematical thinking gives attention to process rather than content though both are important for learning mathematics and are represented in school mathematics curricula (Goos & Kaya, 2020). There are many different definitions and interpretations of the term mathematical thinking. In this section I review literature on mathematical practices and mathematical thinking.

2.6.1 What is mathematical thinking?

Schoenfeld (1992) describes mathematical thinking as a point of view that values the process of mathematisation and abstraction, and competence with and using the "tools of the trade" to understand mathematical structure. Schoenfeld (1985) offers an explanatory framework for examining "what people know, and what people do, as they work on problems with substantial mathematical content" (p 11), that includes the four elements - the resources of mathematical knowledge and skills that students bring to the task; the heuristic strategies that the students use; the control that the student exerts in guiding the problem solving process in productive directions; and the beliefs that students hold about mathematics.

Kilpatrick et al. (2001) introduced the notion of “mathematical proficiency” consisting of five intertwined strands as a comprehensive framework for what is necessary for anyone to learn mathematics successfully, namely

- conceptual understanding—comprehension of mathematical concepts, operations, and relations
- procedural fluency—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- strategic competence—ability to formulate, represent, and solve mathematical problems
- adaptive reasoning—capacity for logical thought, reflection, explanation, and justification
- productive disposition—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy (Kilpatrick et al., 2001, p. 116)

Mason et al. (1982) identify four fundamental processes, in two pairs - specialising and generalising, conjecturing and convincing, and show how thinking mathematically proceeds by alternating between them. Mason et al. (2010) add to these core elements of mathematical thinking four additional process pairs: imagining and expressing; stressing and ignoring; extending and restricting and classifying and characterising. They claim that mathematical thinking is about learning to use these processes or “natural powers”, which every child has, in mathematical ways and in the exploration of mathematical problems. In addition they also suggest themes of doing and undoing; invariance in the midst of change; and freedom and constraint, as markers of mathematical thinking.

Many educators and mathematicians agree that the mathematical practices and thinking to be encouraged in learners of mathematics should mirror the practices of professional mathematicians (Bass, 2005; Moschkovich, 2015b; Ramanujam, 2010; Schoenfeld, 1983).

Bass suggests that

> “the school curriculum provide opportunities for learners to have some authentic experience of *doing* mathematics, opportunities to experience and develop the practices, dispositions, sensibilities, habits of mind characteristic of the generation of new mathematical knowledge and understanding – questioning, exploring, representing, conjecturing, consulting the literature, making connections, seeking proofs, proving, making aesthetic judgments, etc.” (Bass, 2011, p. 3)

Ramanujam (2010) identifies processes such as selecting between or devising new representations, looking for invariances, observing extreme cases and typical ones to come up with conjectures, looking actively for counterexamples, estimating quantities, approximating terms, simplifying or generalising problems to make them easier to address, building on answers to generate new questions for exploration, etc., as vital to mathematicians' practice but missing in school mathematics. Other scholars have drawn attention to such practices as empirical exploration, logical deduction, seeking relationships, verification, reification, formalisation, locating isomorphisms, comparing arguments for accuracy, insight and efficiency, abstraction, symbolisation, modelling, etc. (Bell, 1976; Cuoco et al., 1996; Watson, 2008).

Drawing on Krutetski’s study, Watson and Barton (2011) mark these tendencies as distinctive of acting mathematically: grasp formal structure; think logically in spatial, numerical and symbolic relationships; generalise rapidly and broadly; curtail mental processes; be flexible with mental processes; appreciate clarity and rationality; switch from direct to reverse trains of thought; and memorise mathematical objects. They also highlight the quality of “sustained niggling” or persistently working on a problem, trying out multiple approaches.

2.6.2 Burton’s framework for mathematical thinking

Building on Mason et al’s, (1982) process-pairs, Burton (1984) proposes a framework for mathematical thinking that includes operations and dynamics in addition to processes. Drawing a clear distinction between mathematical thinking and the body of knowledge (content and techniques) described as mathematics, Burton (1984) suggests that teaching mathematics content like algebra or geometry or trigonometry compulsorily and over years does not necessarily provide the conditions through which students develop their mathematical thinking. She argues that mathematical thinking is not thinking about the subject matter of mathematics, but “a style of thinking that is a function of particular operations, processes and dynamics, recognisably mathematical.” (p 35). Starting from the “axiom” that “Thinking is the means used by humans to improve their understanding of and exert some control over their environment” (Burton, 1984, p. 36), she claims that mathematical thinking uses particular means, arising from or pertaining to the study of mathematics, and describes these in terms of operations, processes and dynamics of mathematical thinking.

a) Operations of mathematical thinking: Burton terms any event that can provide a stimulus to begin thinking as an element. Thinking requires that the elements be acted on in some way. In mathematical thinking the methods or operations used to act on elements are recognisably mathematical. Enumeration, repetition, iteration, study of relationships, ordering, making correspondences, substituting, transforming, adding, subtracting are some mathematical operations that have wider relevance as well. When a child encounters a collection of objects, and asks “how many?” the child is considering the mathematical operation of enumeration.

b) Processes of mathematical thinking: Burton identifies four processes, namely specialising, generalising, conjecturing and convincing, as central to mathematical activity.

- Specialising means considering a simpler case or a special case in order to observe what one is doing when examining a particular instance or case in order to recognise relationships that might generalise to all other cases. The purpose of specialising is to become aware of structural relationships that are generalisable. Specialising can be done randomly to get a feel of the questions; systematically to prepare the ground for generalising or artfully to test the generalisation (Mason et al., 2010).

- Conjecturing about a relationship that connects a number of specialised examples observed is the first step of an inductive approach to learning. Conjecturing is the outcome of exploration of patterns, their expression and substantiation.

- Generalising implies moving from the consideration of a given set of objects to a larger set, containing the given one. Mason et al. (2010) describe it as the “process of seeing through the particular, by not dwelling in the particularities but rather stressing relationships” (p 232). Recognition of pattern or regularity provokes a statement of generalisation. Burton refers to such statements as building blocks used by learners to create order and meaning out of an overwhelming quantity of data.

- Convincing is the means by which a generalisation is validated. This involves the thinker convincing themselves of the truth of the generalisation and then a friend and a sceptic. Proofs are attempts at constructing convincing arguments.

c) Dynamics of mathematical thinking : The dynamics of mathematical thinking consists of repeated cyclic movements through the stages of manipulating an object or idea, getting a sense of pattern and articulating that pattern symbolically, each cycle building on the awareness and understandings achieved from previous cycles. Burton describes this process as follows:

> The process is initiated by encountering an element with enough surprise or curiosity to impel exploration of it by manipulating. The element may be a physical object, a diagram, an idea, or a symbol, but it must be encountered at a level that is concrete, confidence inspiring and amenable to interpretation. A perceived gap between what is expected from the manipulation and what actually happens provokes tension that provides a force to keep the process going until some sense of pattern or connectedness releases the tension

> into achievement, wonder, pleasure, or further surprise or curiosity that drives the process on. Although the sense of what is happening remains vague, further manipulating is required until the sense can be expressed in articulation (Burton, 1984, p. 40).

An achieved articulation crystallises the sense of pattern achieved through manipulations and becomes available for further manipulation and more complex thinking. The complexity may be due to increasing generality, refinement or abstraction.

Burton illustrates the various elements of her framework through the annotated response of a person to a problem drawn from Mason et al. (1982), which I summarise here. The problem is the following:

> At a warehouse, I was informed that I would obtain a 20% discount on my purchase but would have to pay 15% sales tax. Which would be better for me to have calculated first, discount or tax?

The person's initial response was to guess that it would be better to calculate the discount first, as that would lead to the tax being calculated on a smaller amount. He specialises and calculates the amount he would have to pay both ways for an item priced £100 and finds that he gets the same amount. The fact that the order of calculation did not make a difference came as a surprise leading to the question if it would be the same for another amount, say £65. The second instance of specialising confirmed earlier observation. This led to the conjecture that it would be so for any price. Having convinced himself that the order is indeed immaterial with a price of £X, the next step was to subject this confirmed conjecture to further manipulation. The person wondered if the order would still be immaterial if the rates of discount and tax were different and further for any rates.

This example displays all the four processes and cycles of manipulation, getting sense and articulation. the person specialised by manipulating particular numbers (100, 65) to get a sense of what was going on and to generate an articulation of generalisation. He then convinced himself of the truth of the generalisation and subjected this to further manipulation.

Burton points out that a "model" or formalised answer would have suppressed all evidence of mathematical thinking and removed examples of negotiation of meaning through specialising and the recognition of constraining factors. She goes on to say,

> "The mathematics is presented as a closed manipulation of techniques, whereas the mathematical thinking demonstrates open inquiry. An over-conscientious concentration on content of mathematics would therefore be expected to obstruct the development of the kind of awareness on which mathematical thinking is based. (Burton, 1984, p. 44)"

Differentiating developing mathematical thinking from generating problem solving approaches in the classroom, she suggests that making the processes overt and concentrating on them so that they become the focus of the learner's attention is key to the former.

The characterisations of mathematical thinking seen in Section 2.6.1 highlight certain dispositions and practices. Burton's framework on the other hand also includes the dynamics of thought processes, in addition to operations and practices. The framework draws attention to the repeated cycles of manipulation, getting a sense of pattern and articulation that students go through as they engage in mathematical thinking. That is a pattern or result that is articulated becomes the subject of further manipulation in the next stage. These repeated cycles lead to increasing levels of abstraction, which captures an important element of doing mathematics. Thus Burton's framework makes apparent the extent to which students progress on the path to abstraction. I therefore draw on Burton's framework to discuss mathematical thinking as it happens in a marginalsed context.

2.7 The Indian context and the academic motivation

In this section I present a brief idea of the educational context in India and how it motivated this study.

It is widely acknowledged that the schooling system in India contributes to the reproduction of societal inequalities. Despite the Indian schooling system being unified and uniform in principle, with limited variation in curriculum or syllabus, schools vary a great deal. Some key dimensions of variations are the management structure (whether the school is financed and run by the government, privately managed with some degree of government intervention, or entirely financed by school fees and/or corporate grants), medium of instruction, and school costs. There is an unstated 'hierarchy' among schools based on these parameters, with the privately managed, high fee charging English medium schools occupying the top of the hierarchy (Majumdar & Mooij, 2012).

Majumdar and Mooij (2012) also identify factors such as cost of education and admission policy, and school choice and voice exercised by the more well-to-do and educated parents in favour of private schools, as reproducing segregation and segmentation. The selective admission process and the high fee charged by the schools at the top of the hierarchy favour the wealthier and high-status sections of the society. In addition to having the choice of schools at the top of the hierarchy, these parents also use their voice to monitor school quality and complain if there is a deficit. Unlike this section of the society, the poor and the uneducated parents are usually neither able to insist on quality from mal-performing schools nor able to choose a "better" school. Thus, the multiple segregating factors identified lead to class, caste-wise sorting and streaming of children into different categories of schools with the schools themselves becoming socio-economically stratified. The government schools happen to be the only option for the socio-economically disadvantaged sections.

Sarangapani (2018) studied the “quality” in the variety of school types and found a “complex picture of pedagogy” emerging. Based on factors like teacher’s aims for students, their expectations regarding potential parental support in their children’s education, methods the teachers employed to teach and to ensure learning and the disciplinary culture of schools, this study arrived at an understanding of pedagogy across school types. Sarangapani notes that at the lower end of the social spectrum, “pedagogy was mostly massified with focus on disciplining students and forming citizenship” while at the upper end, the focus was “all round development of students and becoming an individual with autonomy” (Sarangapani, 2020). The textbook discourse was the dominant pedagogic discourse and the teachers seemed to function on the implicit belief that their role was to mediate between the textbook and the student (Vijaysimha, 2013).

The dominant role played by the textbook in the Indian classroom has led to the term “textbook culture” (Kumar, 1988; Sarangapani, 2020; Vijaysimha, 2013). The variations of textbook culture captured in research are marked by themes of teacher centred classrooms, strong pacing arising from the teachers’ goal of completing the portions of the prescribed syllabus, emphasis on repetition, drill and word-for-word recall. Sarangapani (2020) notes that it is widespread in the government schools in India accounting for as much as 55 to 75% of class time. She also notes the layered and stratified nature of variations wherein rote learning and drilling is evident in the classrooms for children of the poor, while “answer in your own words” is found in classrooms with students from higher socio-economic groups. It has also been noted that in schools catering to lower socio-economic groups traditional instructional methods dominate, whereas ‘student-centric’ and active learning may be encountered in classrooms where facilities are better (Sankar & Linden, 2014; Sarangapani, 2018).

Vijayasimha (2013) also marks this difference emerging in an ethnographic study of pedagogic recontextualisation in different school types.

> “Control over students was highest in government schools, present to a lesser extent in the private school and least visible in the international school, indicating that poor children were perceived as requiring more disciplining. Poor children were also constructed as passive and probably unwilling/undeserving recipients of worthwhile knowledge, whereas affluent children were seen as capable of discursively constructing this knowledge” (Vijaysimha, 2013, p. 67).

In addition to being subject to a directive and controlling pedagogy, students from government schools are also subject to negative attitudes and deficit perceptions.

> “Research on teachers working in Indian government schools frequently report that they have stereotyped and ‘deficit’ perceptions of children from marginalised communities. They perceive of themselves as working in ‘deficit situations’ and cite ‘inability of children to engage with schooling on account of low IQ or ability’, ‘lacking in parental support and interest in schooling’, ‘lacking concentration’, and ‘being disruptive’” (Sarangapani, 2020)

These may also be attributed to cultural stereotypes and prejudices carried by teachers into the classroom. Vijayasimha (2013) goes on to suggest that the pedagogic discourse experienced by students from government schools in conjunction with systemic problems results in creating an educational disadvantage for these students.

There are very few research studies on mathematical attainment of children in schools catering to marginalised groups. There are two major periodic large-scale assessments – the National Achievement Survey (NAS) conducted by the National Council of Educational Research and Training (NCERT) and Annual Status of Education Report (ASER) conducted by an independent organisation PRATHAM. According to the National Achievement Survey 2021, 90% of Class 8 students in a wide-ranging sample drawn in Tamil Nadu are at ”basic” or “below basic” level in mathematics. However, such reports have been critiqued by scholars (Johnson & Parrado, 2021) and also reinforce the deficit perspectives leading to lowering of expectations from marginalised students.

There are a few studies that look at mathematical thinking in marginalised settings in the Indian context. These studies describe the sense-making and mathematical thinking engaged by students during curriculum focused classroom learning. Subramanian and Verma (2009) discuss how students relate to the fractional symbol in a meaningful way and employ a variety of methods to compare fractions drawing on multiple interpretations of fractions. Menon (2015) highlights the variety of strategies used by 8+ year olds in solving addition and subtraction problems and argues that this is indicative of an understanding of what is being done, rather than a mechanical execution of a single taught procedure. Rahaman (2022) draws attention to the process of construction of area-concept happening in a classroom with students proposing varied and contending solutions to a given problem and engaging in collective argumentation. She also notes the intense student engagement that marked this process. These studies which report on interventions aimed at creating spaces where students could actively engage with mathematics and offer evidence of curriculum based interventions fostering mathematical thinking and understanding in marginalised contexts.

Apart from mathematical thinking engendered by appropriately designed curricular interventions, Subramanian et al. (2015) draw attention to the oral and informal mathematical computations, including solving simple algebraic equations that students in marginalised contexts can perform, and talk of the need to recognise these competencies. They suggest a rethinking of the upper primary curriculum so that it is relevant to students who have the necessary reasoning skills but have not been trained in written mathematics. However I did not find any study that discusses similar competencies of secondary school students in the Indian context. The central aim of this study is to explore ways of creating opportunities for students to engage with the process of discovering mathematics on their own and recognising the oral and informal competencies that secondary school students bring to this process.

Aiming to design alternative approaches, this study builds on mathematical explorations as a means to move away from the exercise paradigm. The absence of a privileged discourse in an exploratory context foregrounds student talk as a means of doing mathematics, especially in contexts where students do not have sufficient background in or access to formal mathematical language. The study intends to investigate the potential of mathematical explorations to support mathematical thinking and mathematical talk especially in marginalised contexts.

I have not come across studies of students' involvement with exploratory processes on a sustained basis in the Indian context. Also, given the complex dimensions that produce marginalisation in the Indian mathematics classroom, there are no studies that address the possibilities of overcoming deficit discourses by privileging students' language. Thus, this study aims to examine the potential of mathematical explorations and talk in recentering the margins.

2.8 Summary

In this chapter I identified and examined literature around the different dimensions along which mathematics marginalises and associated deficit discourses. Among these I identified narrow conceptualisation of school mathematics and its right-answer focus (the disciplinary dimension) and the formalised and class-mediated language (the language dimension) as points of concern and undertook a more detailed analysis of literature around the role of language in mathematics education and alternate focal points centered on mathematical thinking and ways of framing mathematical activity to overcome deficit discourses.

We noted that scholars suggest:

a) Organising educational processes in such a way that they allow students and teachers to get involved in exploratory processes guided by dialogic interactions (Skovsmose, 2022). Open tasks, landscapes of Investigations or mathematical explorations are means to this end.

b) Multidimensional framing of mathematical activity to include mathematical thinking and practices such as sense-making, connection seeking, experimentation, argumentation, etc., that highlight the resources and strengths of those at the margins (Louie et al., 2021)

c) An orientation that views students' language and the multiple means that they use to communicate, including oral language, as resources instead of shortcomings to be addressed (Adler, 2002c; Moschkovich, 2018; Planas & Civil, 2013)

d) Developing a better understanding of the discursive practices of mathematicians and introducing some of these practices into the classroom, thereby expanding the genres of spoken mathematics available to students (Barwell, 2013).

The chapter also discussed the different frameworks relevant to the points of interest of this study, namely explorations or open tasks, mathematical thinking, and mathematical discourse. Among these, I draw on Yeo's framework of open tasks (discussed in Section 2.5.2) as the analytical background for task analysis in Chapter 4.In Chapter 5, I draw on Burton / Mason's framework of mathematical thinking (discussed in Section 2.6.2) to analyse student moves towards abstraction and Sfard's Commognitive framework and Moschkovich's ALM framework (discussed in Section 2.3.2) to analyse student talk. I draw on Mason and Davis's work on noticing and listening as background to reflect on the challenges faced by a teacher in noticing students' mathematics.

3 Methodology

3.1 Research Questions

Having set myself the goal of moving away from deficit discourses including the perception of school mathematics as “truths” to be memorised and procedures to be replicated, innateness of mathematical ability, and consequently the impossibility of some /many students being good at it, I asked myself if students, especially those at the margins, could be given a different mathematical experience. If yes, what would this involve and how would students respond to this experience? By margins, I refer to the ‘mathematical margins’ along the dimensions as described in Section 2.1.3, comprising of students whose mathematics “achievement” is not what is expected at their grade level, who have not had any prior mathematics experience other than the ‘school mathematics tradition’ and/or those who are not conversant with the formal language of mathematics. As discussed in Chapter 2, there are large overlaps between the socio-economic margins and the mathematical margins, given the influence of a students’ socio-cultural background, especially language, on their mathematics performance. The broad aim of this study was to address the dimensions of marginalisation stemming from narrow conceptions of mathematics and the use of formal language in the classroom experience of mathematics among the students at the margins. Drawing on literature that suggested “landscapes of investigations” as a learning environment that differs from the school maths tradition and the Open University/ATM ideas of explorations, I intended to study the potential of explorations to enable mathematical thinking, especially at the margins. I also noted the potential of informal and home languages of students to support meaningful discussion of mathematical ideas as different from the procedural mathematics that a more formal language seems to encourage (as discussed in Section 2.3.4.3) and wanted to investigate this further. I also hoped to understand the challenges involved in designing and facilitating an exploration from the perspective of a teacher and what it takes to be attentive to students’ mathematics which may be expressed in ways unfamiliar to me.

The goal of studying the potential of mathematical explorations to support mathematical thinking in marginalised contexts, called for observation of institutionally-marginalised students engaging with mathematical explorations. I needed to observe students as they engage with mathematical explorations and understand the kind of thinking they engage in when not constrained by the textbook language and curricular goals, their ways of communicating their thinking, the mathematical and linguistic hurdles to engagement and communication, the multiple resources including linguistic resources that they draw on to overcome these hurdles. The position of the teacher is central to this enterprise and I wanted to represent the interpretation of events and to ground evolving questions from this perspective. I was also sensitive to the constraining factors like the demands on the teacher, prior knowledge requirements both for students and for the teacher and other pragmatic constraints in making explorations a part of the schooling experience of a student. I wanted to examine what it takes to engage with these explorations both for the students and the teacher.

Our research team that brought together the first hand experience of a teacher, the perspectives of a mathematician and an educator opened up special possibilities for insight and understanding and generated further questions. While we anticipated that the flexibility offered by explorations could be an enabler for mathematical thinking, we also felt the need for constraints so that flexibility does not contravene considerations fundamental to the discipline. We felt the need to redefine boundaries for what counts as mathematical thinking and mathematical discourse such that they balance disciplinary considerations and the need for flexibility. This raised questions like: What constitutes mathematical thinking in such contexts? What constitutes acceptable mathematical discourse in the context of explorations? These questions call for a conceptual answer informed by the practice of mathematics. The mathematician in the research team brought in this perspective.

The broad concerns of this study were: a) design features of mathematical experiences that deviate from the school mathematics tradition and support mathematical thinking, especially at the margins, b) documenting student response to such experiences including the resources they draw on and the challenges they face, c) disciplinary considerations in allowing flexibility, and d) implications for the teacher.

The research questions that my study addresses are:

RQ 1. What task-features support mathematical thinking at the margins?

RQ 2. What does engagement with mathematical explorations entail at the margins?

a) What is the nature of mathematical thinking seen in these contexts?

b) How do students communicate their mathematical thinking?

c) How does language support or hinder mathematical communication?

d) What counts as mathematical discourse in such contexts?

RQ 3. What does it entail for the teacher to facilitate mathematical explorations at the margins balancing the need for flexibility and the need to adhere to disciplinary considerations?

In the course of my study, the school authorities requested me to take some sessions that would help students understand curricular content better. This created an opportunity for me to teach curricular content without being tied by the need to "complete the syllabus" and therefore to extend some learnings from the facilitation of explorations to a curricular context. This created the opportunity for the study to address a further question:

RQ 4. What *could* mathematical engagement look like in a curricular context?

RQ1 entailed that the study involved a design experiment, where exploratory tasks in mathematics suitable for students at the margins would be developed. RQ2 entailed observations in the classrooms as students engaged with exploratory tasks. Since exploratory activities in mathematics are rarely taken up in classrooms situated at the margins, the study also needed to implement such activities. Hence I had to play the role of both the teacher and the researcher. RQ3 and RQ4 necessitated that I reflect carefully on my own experience as a teacher, supported both by records of what occurred in the classroom as well as my interactions with the research team in designing and implementing the exploratory activities.

Methodologically, this study interweaves the empirical approach of a practitioner rooted in a reflective-practice and conceptual-analytic approach of a scholar. It calls for a research design that aligns with what Kelly and Lesh (2000) call "teaching and learning experiments". This type of research design "distinguishes itself by its conscious breaking down of the researcher-teacher divide. The role of the researcher is recast, sometimes as a teacher, always as a co-learner. Similarly, the roles of students and teachers often are recast as collaborators in the search for critical issues, promising perspectives, relevant data, or useful interpretations. In all cases, the characterization of research is transformed beyond that of a remote viewing of classroom life in which the researcher acts by judging the classroom life against a prefabricated ideal" (Kelly & Lesh, 2000, p. 118). Studies that follow this design focus on development that occurs within conceptually rich environments designed to optimise the chances that the intended developments will occur in observable forms. They involve a component of design research along with classroom observation.

3.2 Classroom-based research from researcher-teacher first person perspective

I adopt the methodological stance of classroom-based-research (Kelly & Lesh, 2000) from what Ball (2000) calls "researcher-teacher first person perspective", with elements of design research. This involves blending the construction and analysis of practice. Within this paradigm, the site for research moves out of the researcher's laboratory setting and into real classrooms and the researchers are not simply "disinterested observers" but are significantly involved in projects aimed at improving instruction. The roles of teachers and researchers become blurred with researchers functioning as teachers or co-opting teachers as researchers. Ball (2000) emphasises ways that researchers may benefit from the viewpoint of “insiders" - by adopting the role of teachers.

The focus of studies based in this stance is on “what is possible" rather than in “what is typical” in ordinary classrooms. Rather than test whether a given design or model is true or false, or better or worse than another, the focus is more on the “extent to which a model or design is sufficiently meaningful and detailed to be useful and powerful for specific “customers" to achieve specific goals” (Kelly & Lesh, 2000, p. 222). The goal is to develop descriptions of existing situations, or conjectures about possible situations, and the research results are “existence proofs” or designs of alternate learning environments.

Ball (2000) discusses three paradigmatic examples of classroom-based research where the researchers use their own teaching as a centrepiece of their inquiry, with three different focal areas and three different designs. These three examples are Lampert (1986) who studied student learning, Heaton (1994) who studied teacher learning and Simon (1995) who studied teacher-educator learning, all in the context of their own teaching. Calling these as special cases of the genre of qualitative case studies, Ball points out the common underlying purposes of such research as illuminating a border case, probing a theoretical issue and developing an argument or framework. In all three cases, the researchers played an important role in constructing the fundamental features of the context of study, the issue studied being at once situated in everyday challenges of practice and in a larger scholarly discourse as well.

> “Instead of merely studying what they find, they begin with an issue and design a context in which to pursue it. The issue with which they begin is at once theoretical and practical, rooted in everyday challenges of practice but also situated in a larger scholarly discourse, and they create a way to examine and develop that issue further. What they ultimately focus on may emerge out of the situation and its unfolding, but they have an important hand in constructing fundamental features of the arena of study” (Ball, 2000, p. 236).

Lampert (1986) examined ways of intertwining a computational focus and a focus on mathematical structures and principles in an elementary mathematics classroom. The approach to teaching multiplication in which she was interested was rare in practice. So she created an instance of such teaching and sought to examine how it works and to describe it so that pursuing it becomes viable in any classroom. In investigating these questions Lampert used her own teaching as the site for research. She wanted to shape teaching in particular ways and to adjust her design to enable a particular kind of learning. Having another teacher do the teaching would mean adding an additional layer of teacher learning that would interfere with the aims of the project. Being the teacher as well as the researcher afforded Lampert a space in which to work that is not easily available otherwise. In a broader sense Lampert asked how disciplinary dispositions, like a focus on mathematical structures, can be acquired and used her teaching as evidence of what is possible. Lampert's creation of an instance of teaching multiplication differently is also generative of questions on what such creation involves, broadening the relevance of her research.

An experienced and skilled teacher Ruth Heaton, examined in a disciplined way her own struggles to change her teaching from a traditional approach to one grounded in reform ideas about good teaching. In her dissertation research, *"Creating and Studying Practice of Teaching Elementary Mathematics for Understanding"* (Heaton, 1994) draws on her own teaching to answer the questions "What would it take to teach elementary mathematics in ways envisioned by the current reforms in mathematics education? What struggles would be experienced by teachers as they transform their teaching?" In documenting the struggles that she faced during her year of teaching, she sought to make claims regarding challenges that *any* teacher might face attempting to engage in reform teaching. Thus they go beyond one teacher's struggles to transform her own practice and is therefore of broader interest. Heaton documented what she found difficult and why, in terms of content and pedagogy, and what helped her learn.

Problematising the common view that the role of a teacher within constructivist theories of learning as one of noninterference in students' learning, Simon focussed on the role of deliberative and design aspects of practice involved in constructivist teaching such as selecting tasks and choosing representations. He sought to address the question "what is the nature of design work when constructivist theory underlies pedagogy?" through his own teaching of prospective elementary teachers within an experimental teacher preparation program. Analysing his own actions as a teacher in this context, he developed a provisional theoretical model that captures the design processes that underlie teaching rooted in constructivist learning theories.

The concerns of all three studies necessitate studying teaching-learning from a first person perspective. Their concerns are rooted in practice, but are relevant from the larger scholarly discourse as well. The goals of this study overlap with the goals of these three studies to varying extents. With Lampert I share the goal of designing a phenomenon of interest and describing the possibilities that follow from the new design. In describing what her students did in the context of her approach, Lampert offers insights into what learning of other students might look like with a similar approach. By describing the mathematical thinking observed as students from marginalised contexts engage with mathematical explorations, I hoped to offer insights on how other students from similar contexts might respond to such tasks. By documenting what students could do and the kind of thinking they engaged in, and how they expressed their thinking, I hoped to present "counterstories" which challenge the deficit perspectives about the kind of tasks and modes of instruction that are considered appropriate for students at the margins. As in

Lampert’s situation, this generated further questions on what is entailed in designing and implementing the novel learning environment. With Heaton I share the concern for challenges in implementing a different pedagogy called for by the flexible and accommodating learning environment I imagined and the nature of support needed to mitigate these challenges. With Simon, I share the concern for documenting what teacher moves supported or hindered the progress of an exploration.

3.3 Insider Research as my methodological choice

Studying teaching and learning from the inside may not be a fruitful approach for all research goals. For a given research goal, one therefore needs to ask if the first-person perspective has an advantage to offer and if yes how may the perspective be used. Ball (2000) identifies three crucial questions in considering the appropriateness of first-person research for a particular research agenda,

> First, does the phenomenon in which the researcher is interested exist? Does the researcher have a conjecture or image of a kind of teaching, an approach to curriculum, or a type of classroom that is not out there to be studied? And, if it is this need to create the phenomenon that underlies the impulse to engage in first-person research, does the researcher think he or she is particularly well equipped to be designer, developer, and enactor of the practice or would an experienced practitioner be a more reliable partner in this construction? Second, is what the researcher wants to know uniquely accessible from the inside or would an outsider be able to access this issue as well, or perhaps better? Third, is the question at hand one in which other scholars have an interest, or should have an interest, and if so, will probing the inside of a particular design offer perspectives crucial to a larger discourse? (p240).

Though the idea of explorations itself is not uncommon, instances of marginalised students engaging with explorations for a long enough period for me to undertake an in-depth study of the phenomenon is rare if not absent in the Indian context. Despite having a wide network of connections with schools and teachers through my institution and parent research group (mathematics education at the Homi Bhabha Centre for Science Education) we were not aware of such instances. Further, the curricular imperatives and the consequent time-constraints for the regular teacher; and the prevalent deficit perspectives around what these students can or cannot do, make it improbable to find a classroom where the students are engaging with mathematical explorations in anything more than a one-off instance. I therefore had to create the phenomenon I wanted to study. I needed to design modules or tasks that offer flexible ways of engagement with mathematics and a corresponding pedagogy. I conjectured that flexibility - be it in the kind of language used; preferred means of expression (talk over writing); methods and procedures followed; all are important in enabling explorations in such contexts. At the same time, the flexibility admitted needs to be within disciplinary boundaries. So I needed to adopt a pedagogy that is accepting of students' ways of communication and ways of doing mathematics while keeping the disciplinary considerations firmly in view. I imagined tasks and pedagogy that balances flexibility and disciplinary constraints.

Given the improbability of finding a teacher willing to invest the time and effort the project demands, and the added layer of teacher preparation, I decided to capitalise on my experience as a teacher who has taught in marginalised contexts, and facilitated explorations albeit in more affluent and resource rich contexts. Our research collaboration which includes a teacher-researcher with prior teaching experience, a mathematician who also brings in a rich experience of interacting and facilitating explorations with students across levels, and an educational researcher, is ideally positioned to support my study, to design exploratory contexts and undertake a “first-person inquiry’ into teaching to understand the pedagogical elements involved in enabling such a context.

The schools that were part of this study not being “exceptional” in any way, I argue that studying how the students engage with explorations in these schools gives us a window to “what might be" in schools with a similar context. Also the process of designing an alternate learning milieu allows us to ask what is involved in designing this milieu? What kind of tasks support flexibility and accessibility? What challenges is a teacher likely to face in designing and implementing such modules? These questions are of wider relevance and not limited to the two schools that were part of the study. Moreover the larger goal motivating this study, namely the potential of explorations to address at least some dimensions of the marginalisation stemming from mathematics, I argue, is of importance to the larger community. While the expertise available in the research team is more difficult to replicate across contexts, and limits generalisability of the study, I argue that this very expertise adds value, given the exploratory nature of the study and the goal of unearthing possibilities.

We planned to design a number of explorations, with me as the teacher-researcher facilitating explorations in a school catering to students from socio-economically disadvantaged backgrounds. This we anticipated, would allow us to probe what students are capable of doing when not forced to follow the expected norms. Rather than offer a generalised trajectory along which mathematical explorations could progress or conclusions about having students at the margins engage with explorations, my aim was to produce an image of what students engaging in explorations would look like - a pointer to “possibilities”. Drawing on approaches that worked across the explorations I also hoped to offer some guiding principles and noteworthy points while facilitating explorations.

In the following sections, I look at our study context, the data corpus and the analysis carried out.

3.4 The study context

The schools that were sites for this study were chosen based on their approachability through contacts and their willingness to take part in the study. As mentioned earlier in Chapter 1, two of the schools that were initially chosen did not continue due to the constraints they had in accommodating the teaching sessions for the study in their schedule. The school which forms the primary source for my data is a “Corporation School” run by the city government in Chennai in South India. The school complex consists of a Primary school for Grades 1 to 5 which feeds into a High School for Grades 6 -10. Both schools function from the same compound and are two separate units for administrative purposes. My interactions were with the High school section. This section has a strength of approximately 350 students, with 150 boys and 100 girls across classes 6 -10. There are about 65 - 80 students at each class level, with two divisions for each class. The class sizes vary from 30 - 40. It caters largely to students from socio-economically disadvantaged backgrounds, mostly from the oppressed social classes. The students qualify for fee-exemption and mid-day meals and most of them are first generation school-goers. This implies that there is almost no academic support available at home, should they face difficulties in school-learning.

Tamil is the first language of most students in the class and language of teacher-student and student-student conversations. The school had a few students who opted for Tamil as the medium of instruction, but most parents preferred English (about 10 students in Tamil medium and around 50 in English in Grade 9 in 2018 - 2019). Tamil as a medium of instruction has now been discontinued because of the decreasing number of parents opting for it. In the years that I taught, because of the fewer number of students who opted for Tamil medium, both the sections were frequently combined and taught together. The students who opted for English or Tamil medium have their textbooks in the respective language, but the classroom teaching itself often happens together, with Tamil being the preferred language. The students who were part of this study came from the section where the medium of instruction was English. The only exposure the students have to English is their textbooks and what they learn as part of their English lessons at school. With this, they understand when spoken to in English. Some of them can carry on a rudimentary conversation in English, but none are fluent speakers. Even the high-scorers amongst the group of 13-14 year olds struggle to independently read and comprehend their subject textbooks in English. Literary or written Tamil is starkly different from the conversational versions, which vary regionally and carry markers of caste, class and community. The official medium of instruction being English, these students do not have an opportunity to gain familiarity with the literary Tamil and the disciplinary terms in the language, doubly disadvantaged and denied access to academic language in either their home language or the language of instruction.

My own fluency in Tamil when I started to teach was limited to speaking, and that too a variant that was typical of a particular caste from a different region. My familiarity with the language was limited to informal conversations at home. I could understand literary Tamil to a large extent but not speak, and I had rudimentary reading skills and no writing skills. I have no prior experience in teaching in Tamil, and started out with a reluctance to do so given the variant of the language that I speak and the social markers that it carries. My teaching was in a mix of English and Tamil, with me switching to English whenever I had difficulty expressing myself. Over a period of time I gained comfort in Tamil, became sufficiently familiar with the language to speak the regional conversational variant (or a close enough version!) and my language became dominantly Tamil with technical vocabulary alone being in English. I also picked up reading skills.

In addition to the primary site, I also had a second school, catering to students from similar backgrounds, with the difference that the first is a school run by the local government and the second run by a trust and charges a nominal fee. School 2 has classes from Pre-KG to Class 12, with a total strength of about 650 students. Of these around 300 are girls and 350 boys. There were around 160 students - 60 girls and 100 boys - in the High school section (Classes 6 to 10) and the class size varied between 25 and 35. English was officially the medium of instruction, and classroom teaching happened in a mix of English and Tamil. Going forward, I will refer to the first school as School 1 and the second as School 2.

3.5 The data

The teaching for the study spanned 3 academic years. The classes were in the nature of after school enrichment classes. In the first year, I implemented one exploration in School 1. In the subsequent years I taught twice a week in School 1 and once a week in School 2 for most part of the academic year. There were breaks in classes during school exams, or other activities and when I was travelling. The schools requested that I devote a few sessions to help students understand curricular content better. The total hours that I engaged with students in these schools, including the curricular sessions are as shown in Table 3.1. Other than this I also helped a group of 20 students with the preparation for the school leaving examinations in School 1, (involving 26 hours of engagement) and a small group of 5 students in School 2 for their annual science project (involving 8 hours of engagement), both at the request of the respective schools.

Table 3.1: Total hours of teaching

Year	**School 1**	**School 2**
2017 - 2018	8 hrs	
2018-2019	22 hrs	15 hrs
2019-2020	21 hrs	17 hrs

In both schools, the students were assigned to the study by the school authorities based on interest expressed by the students, time constraints of students coming from involvement in other school activities, and the concerned teacher's perception of "students who would benefit from participating in this study". Marks obtained in the school exams was not a criteria for selecting students as attested to by the teachers.

Table 3.2: Student information

		Academic year	**Approx number**	**Gender Distribution**	**Grade**	**Remarks**
School 1	Cohort 1	2017 - 2018	15	14 girls, 1 boy	Grade 9	
	Cohort 2	2018 - 2019	20	10 - 15 boys, 3 -5 girls	7-8 Grade 8 students & 12 - 13 Grade 9 students	
	Cohort 3	2019 - 2020	20	10 - 15 boys, 3 girls	Grade 9	Includes the Grade 8 students from Cohort 2
School 2	Cohort 1	2018 - 2019	15	Roughly even gender distribution	5-6 of Grade 8 and rest Grade 9	
	Cohort 2	2019 - 2020	10	Roughly even gender distribution	Grade 9	Includes the Grade 8 students from Cohort 1

The primary focus was Grade 9 students, but both schools also assigned a few Grade 8 students to the group in the year 2018 - 2019. These students continued to participate in the study in the following year, as ninth graders, along with a few more students who were freshly assigned. In School 1, I interacted with three cohorts of Grade 9 students. Each cohort had 15-20 students. The cohort of 2017-2018 consisted almost entirely of girls - 14 girls and 1 boy. The later cohorts were predominantly boys with 3-5 girls and around 12-15 boys. In School 2, I interacted with 2 cohorts of ninth graders and the gender distribution was more even. The group in School 1 that I worked with for the Grade 10 exams was the same group that was with me the previous year. So in both the schools I worked with a good proportion of students for two academic years. The student details year wise are as in Table 3.2.

As noted earlier, the school authorities in School 1 requested that I do some classes related to their curriculum as well, in addition to exploratory tasks that I had planned for. Responding to this request, I devoted some hours of teaching to textbook related content like percentages and discounts, mensuration, algebra and set theory. The attempt in these curricular sessions were to engage in more open-ended problem solving. In general I repeated the explorations and tasks done in School 1 at School 2 as well, both exploratory and curriculum related. During the later stages, the teacher at School 1 explicitly requested to do "revision problems" from specific topics, which I complied with. These sessions, though not part of the initial design of the study, contributed to my evolving understanding, and I draw on some salient moments from these classes in my analysis.

I facilitated a total of 11 explorations across these two schools, some of them being repeated with different cohorts and some being done with only one cohort. During this span of 2 years I also facilitated explorations with other groups of students - as part of summer schools and talent nurture camps and as an enrichment activity in a third school where some of these 11 explorations were repeated. I draw on implementation of 6 of these explorations for the analysis for this study. These 6 explorations were chosen based on the number of times they were repeated including outside the project schools, the availability of audio recordings and a detailed teacher diary entry from the project schools. Although the data analysed here is only from School 1 and 2, the repetition of these explorations in other sites led to my increased familiarity with them. Table 3.3 shows a snapshot of these 6 explorations and the number of times they were done in the project schools and elsewhere.[1] A more detailed description of these explorations and their development process is discussed in Section 3.6

1. The other 5 explorations are NIM game, Clapping game, Views of Solids, Arithmagons, and Consecutive numbers.

Table 3.3: Snapshot of explorations

Exploration	Details	Cohorts In Project schools	Elsewhere
Matchstick geometry	Students explore matchstick shapes with focus on the concepts of similarity, congruences, constructable shapes, shapes that can be replicated without measurements, etc.	1	2
Magic triangle	A puzzle that involves writing numbers along the perimeter of polygons such that the sum of numbers along each side remains the same. The starting point is a triangle with numbers at the vertices and at the midpoint of the sides.	2	4
Guess the colour	A game based exploration where a 5 x 5 square grid, is divided into two rectangles (horizontally or vertically) and each coloured differently, and students guess the division by asking an optimum number of questions. The problem is extendable and generalisable.	1	4
Polygons	In this task students figure out the maximum number of right angles possible in a polygon. Task variation is achieved by interpretations of “polygon” and “right angle”.	2	2
Leapfrogs	Another game based exploration, where the primary task is to interchange a set of black and white tokens arranged in a straight line with a gap of one space in between, following specified rules of movement.	1	3
Partitions and cells	Students imagine “partitions” that slot into one another, in a crate, to create “cells” to hold bottles. The exploration involves exploring the number of cells possible with a given number of partitions and optimising the number of cells and partitions.	2	0

The exploratory sessions were audio-recorded by placing two recorders at vantage points. I maintained a teacher diary during the entire teaching span. These and the audio-recordings constitute the major data sources for this study. Most of the sessions were done in the presence of an observer. The sessions in School 1 in the years 2017-2018 and 2018-2019 were observed by the students' mathematics teacher. There were post-class discussions with the teacher on problems faced by specific students/groups and what worked/did not work in the particular class. These discussions were noted in my teaching diary and when necessary course corrections were done based on the input received from the teacher. An observer was recruited for the project in the academic year 2019 - 2020, who wrote observer notes in addition to participating in post-class discussions. However, for the explorations chosen for analysis in both schools, observer notes were not available and hence these notes were not included in the analysis, although they were consulted. At a later stage in the study, noting the rich discussions that happened in the curricular sessions and how they add value to the larger study, I audio recorded these sessions as well though not part of the initial design. Table 3.4 is a consolidated table of the data collected.

The teachers and headmistress in School 1 took a keen interest in the study and were very co-operative. This allowed for smooth scheduling of activities and most of the planned explorations were done here. The scheduling of activities in School 2 was not very smooth and consequently some of the explorations could not be done there. Therefore the data for this study is mostly from teaching sessions in School 1. The data sources from School 1 available for each of the explorations analysed are as in Table 3.5.

In addition I also draw on the implementation two explorations - Magic triangle and Partitions and cells in School 2. Detailed teacher diary entries were made for both these explorations and the Magic triangle exploration was audio-recorded (a total of 2 hrs and 25 minutes of recording).

Table 3.4: Data sources[2]

School	Session Type	Number of sessions	Audio recorded	Teacher diary written	Observer notes available
School 1	Explorations analysed (all those listed in Table 3)	20 sessions	14 sessions (totalling approx 12 ½ hrs)	20 sessions	0 sessions
	Other Explorations[3]	11 sessions	6 sessions (totalling approx 4 ½ hrs)	9 sessions	7 sessions
	Curricular sessions	17 sessions	5 sessions (totalling approx 4 ½ hours)	15 sessions	6 sessions
School 2	Explorations analysed (Magic triangle and Partitions and cells)	5 sessions	3 sessions (totalling approx 2 ½ hours)	5 sessions	0 sessions
	Other Explorations[4]	13 sessions	7 sessions (totalling approx 5 ½ hours)	10 sessions	3 sessions
	Curricular sessions	10 sessions	5 sessions (totalling approx 4 hours)	7 sessions	4 sessions

2. Some sessions could not be audio recorded (or the quality of recording was such that it was not usable) because of unforeseen disruptions. Also the teacher diary was not written for some classes when there were observer notes, and no significant events marked for discussion. Video data was collected for school 2 but not analysed to maintain parity with school 1.

3. The other explorations done are NIM game, Clapping game, Views of solids, Arithmagons.

4. Other explorations done are Views of solids, Clapping game, Consecutive Numbers and Arithmagons and a simpler version of Magic triangle.

Table 3.5: Data sources from School 1

	Exploration	Approx Module duration	Audio recording duration	Teacher diary
1	Matchstick geometry	7 hrs[5]	6 hrs 18 mts	Y
2	Magic triangle	3 hrs	2 hrs 19 mts	Y
3	Guess the colour	3 hrs	2 hrs 58 mts	Y
4	Polygons	3 hrs	51 mts	Y
5	Leapfrogs	2 hrs	N	Y
6	Partitions and cells	2 hrs	N	Y

Students were requested to record their written work in the loose sheets and later in a separate notebook provided so as to collect written work. But as mentioned earlier in Section 2.3.3, there was reluctance to write and very little (less than a total of 10 pages of written work per student for the entire teaching span) written work was collected. I encouraged them to write in the notebooks provided but did not insist they write in any specified form, in the interest of keeping them engaged. I collected whatever they wrote in the material provided and also occasionally captured images of written work on the blackboard and classroom floor. I also tried to insist on writing on two occasions - once in the Matchstick geometry exploration and once in the Views of solids exploration - by giving them worksheets to be filled in. I discuss the worksheet related to the Matchstick geometry exploration, which is one of the explorations I have chosen to analyse, in Section 5.3. The written work collected was also looked at along with audio recordings of corresponding sessions and these enabled better sense-making of the recordings, especially the pointing words.

Another important data that I drew on were the notes from the regular discussions with others in the research team. The design of the explorations, the steps of implementation to be followed in class, the progress of the exploration, any salient moments that stood out for me in the class and my reflections on them were all discussed with the team. I noted the key points from these discussions in my teacher diary and they contributed to the later analysis. In addition, a record of the versions of the explorations tried out

5. This was the first module designed and an experimental one, involving multiple activities and worksheets. Hence the comparatively longer duration.

were maintained and analysed.

3.6 The Explorations

As mentioned in the previous section a total of 11 exploratory modules were done in the project schools. Given our goal of moving away from the rigidity of the textbook and formal mathematical language we looked for tasks with multiple approaches and branching questions. While games and puzzles are engaging and provide easy opportunities to incorporate mathematics, we needed to ensure that the mathematics involved is accessible and relevant to the mathematics that students learn. We had to look beyond solving the puzzle or coming up with a winning strategy for a game to mathematising the puzzle or game as well. While the Magic triangle and Leapfrogs exploration draw on algebra, Clapping game[6] draws on properties of factors and multiples and modular arithmetic. We also strove to design explorations around curricular concepts as well. The explorations on Matchstick geometry, Polygons and Views of solids[7] are examples. We experimented with different curricular areas like algebra, geometry, number theory, etc. This raised questions on the ease with which particular topics lend themselves to designing explorations, whether some content areas lend themselves to more accessible explorations in marginalised contexts and the feasibility of curricular starting points for explorations. We observed that an exploration rooted in curricular context tended to have prior knowledge requirements and privileged the textbook language. Multiple aspects had to be considered during the module development phase.

The explorations were developed and refined iteratively based on discussions within the research team and insights drawn from repeated implementation. The module development process was as follows: Based on an idea or question generated by the research team, picked up from literature, or suggested by the mathematician in the team from the rich repertoire of explorations he has done with students on multiple occasions, I would come up with an initial version of the exploration. This would then be tried out informally with colleagues and friends. This was done with the aim of getting pointers to multiple possibilities, ways in which the exploration could branch off. The tasks which were picked up from literature needed to be solved, adapted to the mathematical maturity of the particular group of students who would be engaging with the exploration, variations thought through and difficulties anticipated. This would then be discussed within the research team and refined if necessary and implemented in class. The evolution of the exploration in the classroom would also be discussed and course corrections done as necessary and changes incorporated for the next implementation. Thus my implementations of the

6. This exploration, which is not analysed here, involved N students sitting in a circle and every nth student clapping. The starting point is to investigate for what values of N and *n* do all students get to clap and state the relation between N and *n* in such cases.
7. An exploration that involves investigating multiple views of solids built from unit cubes, building solids with given views and investigating compatible and incompatible combinations of views.

explorations outside the project schools and the mathematician's implementations with different student groups, though not done with the research agenda in mind, contributed to a richer understanding of the possibilities that could open up while implementing these explorations, and learnings from them informed the evolution of the explorations and the study. The reflection notes written after each implementation, discussions on the learning from these implementations and versions of the explorations served as the basis for identifying task features that support flexibility and accessibility (Chapter 4). I now describe the 6 explorations which constitute the bulk of the data source for the findings reported in Chapters 4, 5 and 6 of this study in some detail.

1) Matchstick geometry: The thought behind this exploration was motivating students to explore by making the familiar unfamiliar. The exploration involved students playing with matchstick shapes and exploring how these shapes are similar to and different from the shapes of Euclidean geometry they are familiar with. The key discussion points planned were: the idea of congruent shapes and similar shapes; and use of mathematical terminology in describing shapes; replicable/constructable shapes – shapes that can be replicated without resorting to measurement. A series of tasks were planned that would bring to focus these ideas as points of discussion. The main tasks included

i) Freeplay by making matchsticks shape of their own choice

ii) Replicating the given shapes. The shapes included some shapes with only right angles and could be replicated based on side-lengths alone and others which required measures of specific angles to be replicated. The intention was to call on students to justify that the shape that they made was indeed identical to the one provided, thereby opening up a discussion on “when are two matchstick shapes identical?”.

iii) Describing a given shape – This was intended as a task where students work in pairs with one of them describing a given shape to the other, who would have to make the shape being described with matchsticks. This task was intended to create opportunities for students to use mathematical vocabulary like horizontal/vertical, parallel/perpendicular and use unambiguous referents.

iv) A further task on replicating shapes intended to identify and discuss non-constructability of shapes which involve non-integer lengths. For example, argue for the non-constructability of the shape in Figure 3.1 with integral multiples of matchstick lengths.

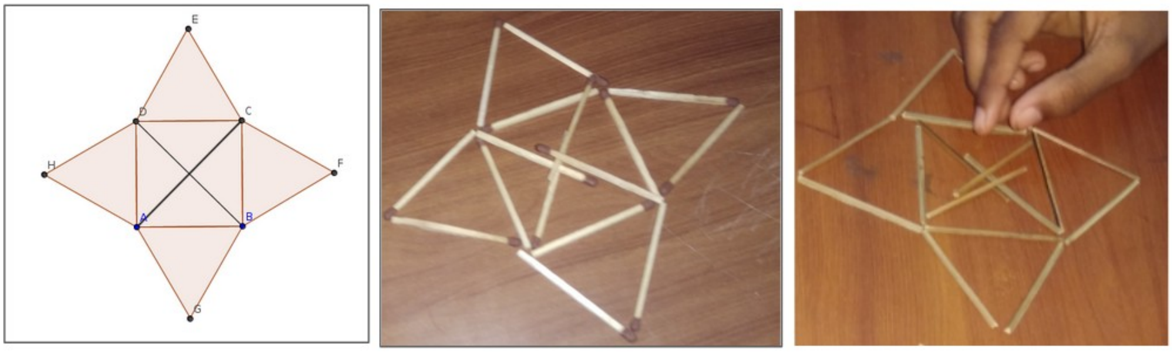


Figure 3.1: Matchstick geometry:Replicating shapes

2) Magic triangle: This is a puzzle based task – finding ways of filling up the circles along the perimeter of an equilateral triangle as in Figure 3.2, with numbers in such a way that the side-sums are equal.

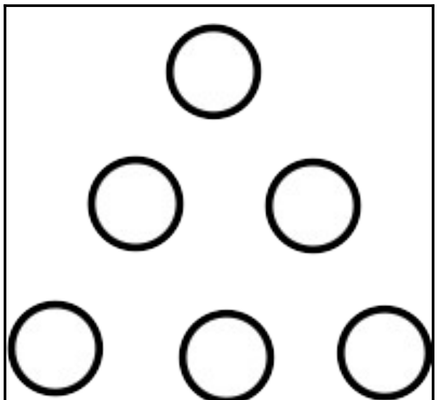


Figure 3.2: Magic triangle

This puzzle admits of 4 distinct solutions – when consecutive numbers 1 to 6 are used, the side-sums 9, 10, 11 and 12 are obtained. There are multiple approaches to the solution – namely trial and error, exhaustive listing of possible ways of filling, using parity rules to eliminate some possibilities and solving algebraically.

Assuming the 6 numbers to be say *a, b, c, d, e* and *f*, and noting that adding the side-sums is equivalent to adding the six numbers being used with those at the corners being added twice one can obtain a relation between the side-sums and corner-sums as 3S = C + 21. This implies that the corner-sum has to be a multiple of 3, and knowing the minimum and maximum possible values, one can find out all possible values of the corner-sum and hence the side-sums. Now solving the problem reduces to finding out ways of obtaining the permissible side-sums. This vastly reduces the number of possibilities and allows for a quicker solution. The variations of the problem include using a different set of consecutive integers, or using sets of numbers in arithmetic progression, or using any set of six numbers. These variations give rise to further questions like – Will there always be solutions? What is the condition for existence of solutions? Will existence of one solution guarantee four different solutions? If not, under what conditions will this guarantee 4 distinct solutions? and so on.

The problem can be extended to other polygons and with more nodes than 3 along each side and is referred to in literature as “Perimeter magic polygons” (Trotter, 1972, 1974). The algebraic solution outlined above may be applied to solve the generalised perimeter polygon puzzles (and puzzles involving open curves) and the relation between the side-sums and corner-sums may be modified according to the sets of numbers being used and the number of sides of the polygon. Thus an algebraic solution proves to be a method that can be used to solve a class of problems rather than the specific problem posed.

3) Guess the colour: Given a 5 x 5 square grid, which was divided into two rectangles (by a horizontal division or a vertical division) and each coloured differently, say blue and green as in Figure 3.3, the task was to guess the division by asking questions. The task admits of multiple questions types – for example questions which ask for a particular fact (colour of a particular cell, or number of cells of a particular colour, or the direction of division etc) or ones which are answerable by a yes or no. For example the following questions give sufficient information to find the dominant colour in the grid:

- What is the dominant colour in the grid?
- How many grid cells are coloured blue?
- Is blue the dominant colour?
- What is the colour of the cell at the centre?
- Is the cell at the centre green?

The goal was to come up with an algorithm to minimise the number of questions. This would involve figuring out the key information required to “crack” a grid (Position and direction of partition, dominant colour and its relative positioning) and coming up with an optimal set of questions from which this information can be inferred. The task was further generalised to larger square grids.

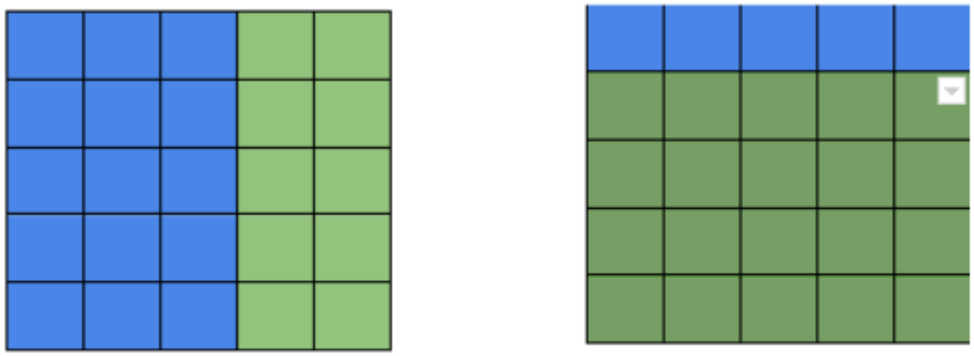


Figure 3.3: Guess the colour

For an *n x n* grid, the position of the partition can be figured out by using binary search and consequently, the optimal number of questions required will be bounded above by log *n*. The remaining pieces of information could be asked for by one question each. Thus in a worst case scenario, one can guess the partition with utmost 3 + log *n* questions.

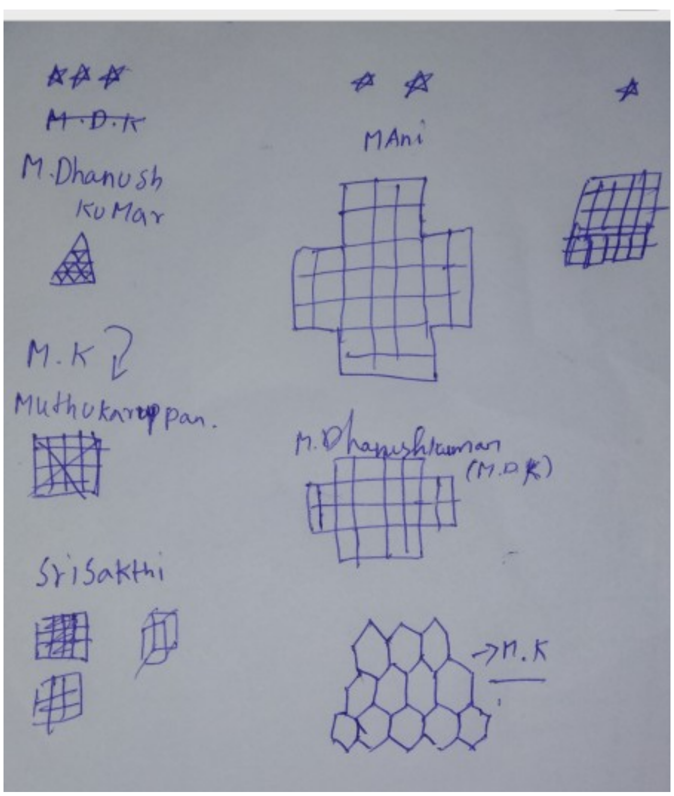


Figure 3.4: Guess the colour: Student given variations

Other extensions/variations to the tasks include more than one division and grids of varying shapes, allowing for other question types and variations in the kind of partitions allowed (for example removing the restriction that the partitioning line has to be edge to edge) or rules of the game. Some variations which students came up with along with their guess of the difficulty involved in solving the problem, 3 star rating being the most difficult, are shown in Figure 3.4. Some of these were discussed, but none solved.

4) Polygons: This task starts with the question “How many right angles can a polygon have?” (Fielker, 1981). The question leaves the term polygon and right angles open to interpretation – it could mean convex or concave polygons. Right angles could be internal angles or external. So the first step in this exploration is to clarify the question and spell out the goal. In the convex case it can be shown that for n >= 5, the max number of right angles possible is 3, and in the case of concave polygon it is the integer just less than 1/3(2*n* + 4), where *n* is the number of sides. There are multiple ways of arriving at and proving the answer – using the exterior angle property, or using angle sum property and algebraically arriving at an upper bound for the number of right angles, or by generalising the pattern obtained by counting the number of right angles in polygons with different number of sides (using induction), etc. This tasks lends itself to experimentation, and we have noted students coming up with many different shapes in an attempt to maximise the number of right angles (Figure 3.5).The task lends itself to variations through possible reinterpretations of the key terms involved – for example polygons may include looping polygons or crossed polygons etc.

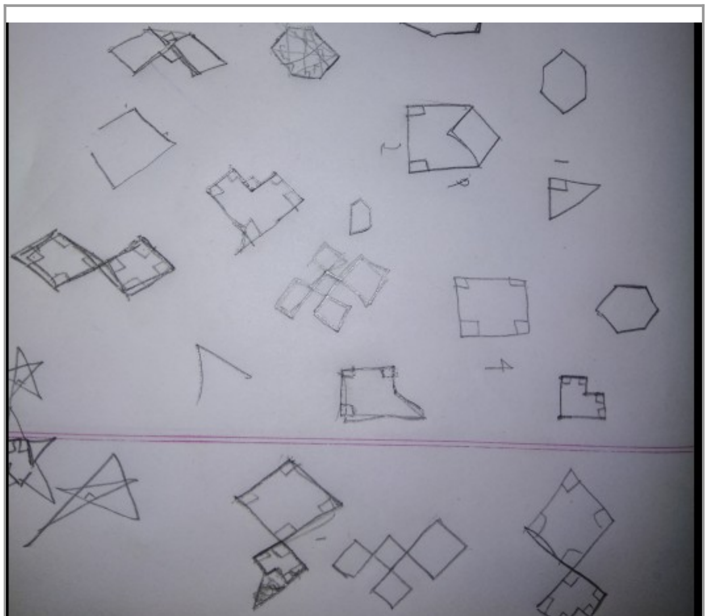


Figure 3.5: Polygons: Student work

5) Leapfrogs: Drawn from Mason et al. (2010) the game of Leapfrogs is a single player game whose goal is to interchange the black and white tokens arranged linearly, with one blank space in between as in Figure 3.6, in an optimal number of moves .

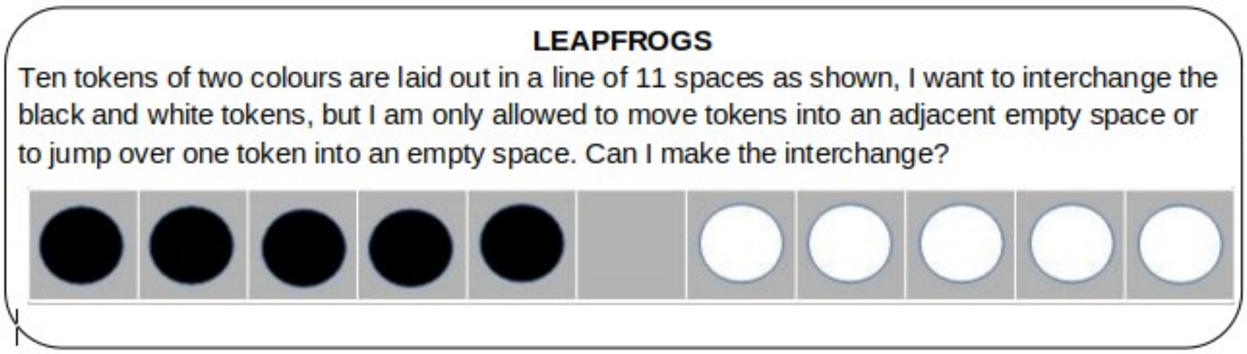


Figure 3.6: Leapfrogs exploration

Task parameters like the number of tokens on each side of the blank space, the position of blank space, rules of movement, the initial and final configurations all lend themselves to change resulting in variations of the task. The task affords multiple formalisations and solution methods.

In Figure 3.6, If one were to label the positions left to right as 1 to 11, with the blank space being at the 6th position, in the target position, the black tokens should occupy places 7 to 11 and the white tokens 1 to 5. The move is effected by the tokens of each colour moving in line, en masse, as a train – that is the black token at position 5 moves to position 11, the one at position 4 moves to position 10, ... and the one at position 1 moves to position 6 and there is a similar shift for the white tokens. That is each token needs to shift position by 6 places, and there are 10 tokens – which means there needs to be 60 shifts of positions. Every black token has to jump over every white token to reach its designated place and no two tokens of the same colour need to jump over one another (This will disturb the train, lead to backward moves, and disturb optimality). There are 10 tokens in all and each has to jump over 5 tokens, making it 50 jumps. But in this each jump is counted twice and so there needs to be 25 jumps, which contribute 50 shifts. The remaining 10 shifts need to be effected by slides and so there needs to be 10 slides and 25 jumps totalling 35 moves in all. The particular sequence of moves is obtained by choosing that move which avoids jumping over a token of the same colour from each configuration. There is a unique move at each step that guarantees this. The argument can be generalised for n tokens on either side. Interchanging n tokens requires $2n\ (n + 1)$ shifts of which n^2 are jumps, resulting in $2n^2$ shifts, and the remaining $2n$ slides. So the total number of moves required is $n^2 + 2n$ or $(n + 1)^2 - 1$. It is also possible to solve the problem inductively by observing and extending patterns in the slides and jumps required to make a transition as the students are seen to be doing in Figure 3.7.

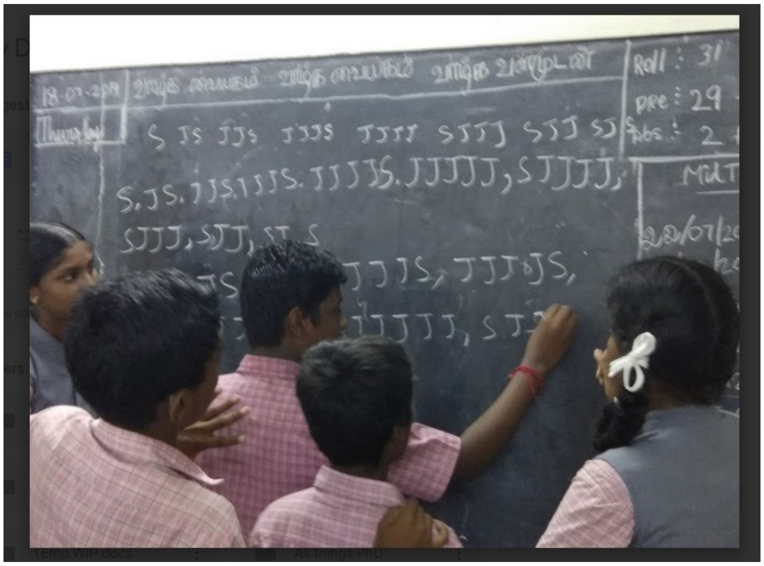


Figure 3.7: Leapfrogs: Student work

6) ***Partitions and cells*****:** In this problem students investigate the number of cells that can be formed by placing a number of slotted partitions in a crate as in Figure 3.8.

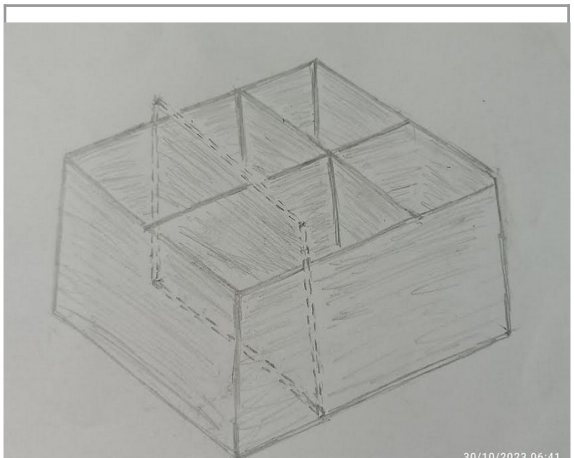


Figure 3.8: Partitions and cells

By trying out multiple alignments, it can be seen that for a given number of partitions placing them so as to form a square grid or a grid as close to a square as possible gives the maximum number of cells. Students are expected to find a relation between the number of cells and partitions and use it to find one given the other. Given *n* partitions where *n* is even, the maximum number of cells obtainable is given by

$\left(\frac{n}{2}+1\right)^2$. The problem can be abstracted to one involving lines and squares (Hardy et al., 2007) and to counting squares of larger dimensions as well instead of just the unit squares in the contextualised version involving crates.

3.7 Data Reduction and Analysis

Among the 11 explorations designed, implemented and observed, I focus on 6 of them in the analysis. An opportunistic sampling based on 1) the total number of times the exploration was implemented in the project schools and elsewhere 2) availability of audio recording and the details captured in the teacher diary, was done to reduce these 11 explorations to 6. From these 6 explorations, a further selection of explorations to be analysed for particular chapters was done based on the availability of data relevant to the theme of the chapter. Table 3.6 shows the particular explorations analysed for each of the chapters where I discuss the findings from the study and the rationale for the choice. The repetitions outside the project schools gave room to try out variations in task formulation and implementation and also increased my familiarity with the exploration. Therefore these explorations offer a better basis for our inferences on task features. For instances that describe student responses to the explorations, the criteria for choice of explorations to be analysed was the availability of audio recordings and detailed entries in the teacher diary. I choose to anchor the analysis in Chapter 6 (on demands that an exploration makes on the teacher) on an exploration whose implementation was observed by the mathematician in the research team. This brought to light the gaps in my implementation and the opportunities I missed out enabling a reflection on the challenges I faced and the demands I had to navigate as a facilitator.

The selection of instances to be discussed from both the exploratory contexts and the curricular sessions, were especially guided by the interactions that were salient in my memory as a teacher. The choice of instances was also guided by my insider perspective and intuition as a teacher on the significant events in the class. Thus the teacher’s subjectivity was used as an instrument in guiding interpretation and analysis.

Cochran-Smith and Lytle (2006) suggest that in forms of research where the roles of the teacher and researcher are blurred, what counts as data and what counts as analysis are often different from those of more traditional modes. Reflection-on-practice ((Mason, 2001; Schon, 1983) and stimulated recall were the key analytical means adopted. Salient moments (Helliwell, 2017) that stood out because of the mathematical thinking or student agency displayed were revisited in the audio recordings and transcripts prepared. These two aspects were central to our study given the research goal of investigating a broader framing of what it means to do mathematics in the course of teaching and learning in marginalised contexts.

Table 3.6: Explorations analysed - chapterwise details

Chapter	Theme of Chapter	Explorations analysed	Rationale for choice
4	Task features	Guess the colour, Partitions and cells, Polygons and Magic triangle[8]	Availability of data relevant to module revisions and repetitions of the exploration in other sites
5	Students' mathematical engagement, language use	Magic triangle, Matchstick geometry	Availability of audio recordings and detailed teacher diary
6	Demands on the teacher	Leapfrogs	Module implementation observed by the mathematician leading to "missed opportunities" being brought to light and discussed.

The first step in the analysis happened in-situ in the weekly discussions with the mathematician in the research team who was aware of all the explorations carried out and contributed to their design and the post-class discussions with the observing teacher. The evolution of the exploration as it was being implemented in class was discussed with focus on moments that stood out for me and their implications for the next class and the study goals as well. Notes of these discussions were maintained in the teacher diary. The regular diary entries spurred reflection-on-practice and also contributed to the in-situ-analysis. Parallely, I listened to audio-recordings multiple times, prepared annotated notes and transcribed selected episodes where students' mathematical thinking was expressed. This enabled stimulated recall and recapturing detailed accounts of what happened in class which suggested further questions and points of discussion. The time-lag between the class itself and the revisit also enabled reinterpretation of some instances, sensitised me to missed-opportunities and suggested possible alternate approaches for future enactment. The formalisability and putative formalisation of students' arguments and explanations also guided interpretation.

The initial review of data, transcription and discussion was followed by further discussion of the transcripts with the research team. Alternate interpretations and perspectives were offered in this post-

8. These explorations: Leapfrogs, Clapping game and Views of solids have been used to point to features use of physical material, curricular dependencies and game based module design.

facto analysis and instances where the team (of 3 members) did not concur on the interpretations were dropped. This process led to the final selection of instances/ episodes to be focussed on in detail. Thus the subjectivity arising from my insider-perspective in data reduction was constrained and validated through post-facto discussions with the research team.

From the systematic observation and documentation of students and their sense making, the analysis was collectively constructed and emerged from conjoined understandings of the researcher-teacher and others committed to the study, taking into account multiple layers of context, multiple meaning perspectives and wide-range of experiences in and outside of their immediate contexts of practice (Cochran-Smith & Donnell, 2006).

3.7.1 Ensuring validity

Cohen et al. (2007) suggest that validity of qualitative data may be addressed through honesty, depth, richness and scope of data achieved, the participants approached, the extent of triangulation and the disinterestedness and objectivity of the researcher. However, regarding forms of research where subjectivity of the researcher is a central aspect the authors note:

> "We, as researchers, are part of the world that we are researching, and we cannot be completely objective about that, hence other people's perspectives are equally as valid as our own, and the task of research is to uncover these. Validity, then, attaches to accounts, not to data or methods; it is the meaning that subjects give to data and inferences drawn from the data that are important. 'Fidelity' requires the researcher to be as honest as possible to the self-reporting of the researched"(Cohen et al., 2007, p. 134).

On a similar note, Mason (2001) suggests that validity rests on whether others can recognise what is being described or suggested through resonance with their own experience, and whether they find that their own sensitivities to notice what is being described are enhanced so that their future practice is informed.

Fidelity was maintained in self reporting of classroom instances. These were corroborated with other data sources like the audio-recordings, written work of students and observer notes when available to guard against misrepresentation. The interpretations and meanings assigned to the instances were arrived at through multiple reflective cycles and discussed with observers, fellow researchers and the research team, with due consideration given to alternate interpretations. Through these discussions, I also noted that these experiences resonated with others thus adding to the validity of the study. Also, the extended period of engagement and the different purposes for teaching allowed for the interpretations to be confirmed or rejected.

By consciously engaging in introspection and reflexivity throughout the study, I engaged with my own positionality, critically examining my own biases and assumptions. Thus I strengthened the validity of our research by ensuring that my interpretations were grounded in self-awareness and an understanding of my role in shaping the study.

3.8 Ethical considerations

Discussing the special ethical concerns when the teacher and researcher roles are conflated and there is a close relationship between the researcher and participants, Creswell (2012) acknowledges the sensitivity that this requires. He points to the importance of the data collection to be non-coercive and the need for students to be able to opt out of the study if they so desire without being penalised. In such situations, Creswell goes on to advocate "covenantal ethics" established on the basis of caring relationships among the researcher and the researched. "This commitment entails open and transparent participation, respect for people's knowledge, democratic and nonhierarchical practices, and positive and sustainable social change among the action research community." (p 588). He also talks of the need to involve participants in as many phases of the research process as possible.

Cotton (2008) draws attention to the gap between the "researcher and the researched" in mathematics education. In this regard, he points out the tendency to relegate the voices of the intended beneficiaries of research (teachers and students) to "play the role of clipped commentators, allowed in only so long as they offer sound bites that sit neatly in the researcher's preferred story". Moving away from this mould, I imagined an open space that allows all participants in the study to think about alternate paradigms. Guided by the larger goal of recentering the margins, I hoped to create a participatory environment where the students and teachers were part of the research rather than as subjects who were researched on.

Given this, I tried to involve the teacher, who taught the students regularly, in the study, as much as his/her time permitted. In School 1, during the sessions in 2017 - 2018 and 2018 - 2019, the teacher was present in most of the classes. I had post-class discussions with him and noted his inputs. The time constraints of the teacher who taught them in 2019 - 2020 limited her role to coordinating classes and occasional participation. She also discussed with the students occasionally on what transpired in these classes, and passed on feedback from these discussions to me. The curricular topics that I taught intermittently were based on teacher requests/suggestions. Thus the concerned teachers kept themselves aware of what was happening in these sessions either by themselves observing or by talking to students about the sessions and reciprocally involved me in the regular teaching in topics where they felt I could contribute.

Processes like seeking informed consent, permissions for data use and anonymity of participants while presenting data were adhered to. Video data was not collected respecting the concern at School 1 about the identities of students being compromised.

The project aim, the nature of data being collected and the ways it would be used were shared with the students and consent sought in addition to consent from school authorities. The students actively sought clarifications on these with questions like “the future implications for them” - whether their “performance” in these sessions would anyway influence their exam scores, or would be “reported to the headmistress” etc. A few students (all girls) who were assigned to the study by school authorities were not comfortable being recorded and opted out of the study. This was the reason for the skewed gender distribution of participating students in School 1. There was also a boy who was drawn to these sessions by the initial warm up sessions that I did for the whole class, but did not want to be recorded. For the most part of day 1 of the study, when I started audio recording classes, he would neither leave the room, nor speak anything. He insisted on staying and communicating through gestures because mathematics interested him. Towards the end of the session, he took the recorder in hand saying “It is my turn to talk now” and spoke straight into the recorder. He continued to be an active participant for the first few sessions and then dropped out in favour of music classes whose timing clashed with these sessions. Thus students’ voices were listened to and agency respected and encouraged in all matters - from participation in the study to the nature of participation. As mentioned in Section 2.3.3, their reluctance to write in the notebooks provided was respected and writing on other surfaces in the classroom accepted. This choice resulted in very little written work being collected.

The students’ help was enlisted in the data collection process as well, by assigning them to handle the recorders. There was competition amongst them (at least in the initial days when the novelty of the experience had not worn off) to be “the recorders for the day”, and I had to assign students in turn. They volunteered to capture board work, to write observer notes, in enforcing class-norms - in general to help in any way they could. This reciprocity enabled the established a relationality between the researcher and researched and the study results need to be interpreted in this light.

3.9 Researcher positionality

This study draws on a first person perspective of research and it is of critical importance to spell out where I and the researchers who mentored me in this study stand in relation to the subject of study and how our experiences and viewpoints influence the study. Acknowledging the unique perspective, experiences and biases that we bring to the study enriches the depth of qualitative research.

As an experienced teacher with a deep interest in mathematics and some prior background in educational research I come with a special combination of experience. As a teacher, I have interacted with students from varied backgrounds and have engaged in a range of teaching from “teaching to the test” in a low resource school to facilitating explorations in nurture camps for the “gifted”. In the course of teaching in a low resource school, I have seen first hand students’ struggle with mathematics as a “must pass” subject and have had an otherwise meritorious student take her own life because of failure in the 12th grade mathematics examination. So the problem addressed in the study has a deeply personal connect to me.

As a researcher, I have taken part in the program evaluation of a large scale intervention in government schools. The school visits, observations and discussions with co-researchers done for this gave me a good understanding of low resource government schools and an appreciation of the inequality between these schools and the schools catering largely to the middle-class, not just in terms of infrastructure but also in terms of pedagogy.

During the course of this study I had the opportunity to be trained in the maths circle pedagogy by Prof. Robert Kaplan. His credo that “mathematics is for all” and that learning mathematics is like learning to speak in “our lost native tongue” and the interactions and mentoring received from him as also from Prof. Ramanujam, a member of the research team influenced me and shaped my teaching for the study.

The mathematician who was a part of the research team brings the rich experience of facilitating explorations with students across levels (middle school students to doctoral researchers) and backgrounds in addition to his research and teaching experience. As a person deeply concerned about equitable opportunities for education and the role of language of instruction in enabling this, the language focus of this study could perhaps be traced to my interactions with him. The education researcher in the team, brings a deep understanding of how students learn, ways of uncovering student thinking and on the demands on the teacher to engender learning. He brought in the domain expertise in planning and implementing this study.

At the core of my teaching is a deep concern for the well-being of my students and the need to foster in them a deep understanding of mathematics. The anxiety-inducing nature of mathematics often puts these two goals in conflict and in such situations I find myself prioritising care for the student over that for mathematics. Perhaps the freedom that I allow in the class, stems from this core belief and is perhaps the point of resonance for my argument for flexibility.

3.10 A note on the pedagogy

The pedagogy followed was a mix of whole class teaching and small group work. Students would work on the proposed tasks in small groups, with whoever they chose to work with. They were free to move around in the class, look at others' work and join other groups as well. The seating arrangement in the class was flexible, with students moving around and sometimes preferring to sit on the floor. I and the observer when present would move around to get a sense of who was doing what and sometimes provide appropriate hints/supports as required. One practice that was followed was that whenever an individual student or a group came up with a finding, it would be shared with the rest of the class. If I came across some student work which I felt could be the starting point of a discussion, I called upon the student to share their work as well. Usually, the student or a student from the group which came up with the result would explain to the whole class. The teacher or other students would ask clarificatory questions. This practice allowed for interweaving of group work and whole class teaching and ensured that no group or student was left behind without making any headway. I also encouraged them to write/record their findings on the blackboard with their name. This practice was eagerly taken up and there was a sense of pride and ownership about their work. Any further reference to this work happened with the student name - Mani’s solution or Neha’s method, etc. One ground rule which was framed as a group was that only one person would speak at a time (when talking to the whole class) and the others had to listen when someone was presenting. This was perhaps the only “rule” for the class, which the rule breakers would be reminded of - either by me or sometimes students.

4 Task features that support mathematical thinking at the margins

In Chapter 2, we discussed the different ways in which mathematics contributes to marginalisation. As described in Chapter 3, the overall goal of the study was to address especially the disciplinary and language dimensions of marginalisation in the classroom. Consistent with this goal, I aimed to create a teaching learning environment that is recognizably different from the familiar routines of the mathematics classroom. Marginalising factors belonging to the disciplinary dimension include the primacy of the textbook and exercises therein, often having one right answer. When there is one right answer to be found and the student fails to come up with it, it is not favourably viewed, be it in the classroom or in the assessment. Repeated failure to come up with the expected answer through the intended means leads to a deficit view of the student. Another major stumbling block for students to engage in mathematical thinking is the formalism which is expected in assessments and is dominant in textbooks. I asked if I could open more pathways for students to make their mathematical thinking visible in ways that do not rely on formal language and provide opportunities for “undirected mathematical play” (Barton, 2008, p. 9). Therefore a guiding principle in designing the explorations was flexibility.

Besides flexibility, accessibility of the exploration to students at the margins was critically important to our enterprise. While explorations already bring in a certain amount of flexibility in comparison to textbook based tasks, a more pertinent concern in marginal contexts was to ensure that students could engage meaningfully with the tasks. We aimed to design tasks such that every student could make some progress within the exploration and feel a sense of achievement at having figured out something for themselves. This meant that we needed to design explorations which a student could get started on irrespective of whether they have the “grade appropriate content knowledge”. The exploration needed to be such that it did not hinge on knowing a particular theorem, result or approach in such a way that no progress could be made unless this result is known. We had to side-step such dependencies, if they were unavoidable, by anticipating the consequent difficulties and planning workarounds. In general, we needed to create “problem situations” which allow for entry at multiple levels.

Guided by the two principles of flexibility and accessibility, we aimed to design explorations that create opportunities to engage in mathematical thinking. In the following section, I recall the module development process followed (discussed in Section 3.6) and spell out the analytical lens adopted. In Sections 4.2 and 4.3, I bring out the task features that enable flexibility and accessibility through a discussion of some of our explorations and their evolution both in terms of the design and pedagogy. In addition to listing out and describing the desirable features for explorations, I also describe instances from the classroom that illustrate how these features facilitated student engagement with the exploration.

4.1 Module design process and analytical lens

Flexibility being central, we envisaged the end product of our design process as consisting mainly of a context or starting point, some initial tasks and a map of the possibilities and guidelines to navigate rather than a well-defined sequence of tasks or trajectory that a teacher could follow when implementing a particular exploration (see Section 6.2.4). We found games and puzzles to be good starting points and also considered alternatives like an activity or a generative question/problem related to the curriculum. As described in Section 3.6, we thought through the possibilities these opened up and spelt out mathematical questions that could be posed in the context and ways these could evolve and branch out. I got a broad-based understanding of the students from conversations with their teacher and my own observations from the few classes that I engaged them before formally starting the study. Thus mathematical background of the students and the context in which the exploration was being implemented was also factored in. Thus we had a "conjectural plan" for the exploration, which was then discussed within the research team and reshaped as necessary. This version was then implemented in class. While the exploration was being implemented in class, the progress of the exploration was discussed within the research team, course corrections made and hitherto unanticipated possibilities mapped out. The exploration continued to evolve based on iterations outside the project schools as well.

An important initial decision to make was regarding the length of the modules. We decided on planning for three hours of engagement with a module in class, spread over 3 weeks. This was a decision we came to considering the need to challenge the students, give them sufficient time to engage and move on as the interest level of the group wanes. This was more a guideline rather than a hard and fast rule. The possibility of some students continuing with the exploration beyond the three hours was open and there have been instances when we have stopped short of three hours because of waning interest.

Deciding on *flexibility* and *accessibility* as key design principles to aim for in explorations that support mathematical thinking at the margins, we needed to elaborate and operationalise these principles with respect to the module design. As discussed in the preceding paragraphs, the module development followed an iterative process, with later implementations being informed by the progress of and discussions about the earlier implementations. The discussions during the module development phase and the post-implementation discussions constitute an in-situ analysis which sensitised us to the underlying considerations that guided the choice and design of tasks and their implementation. A post-facto analysis of the different versions of the explorations, the notes of the research team discussions on module development when available, and reflection on the changes incorporated in successive implementations and the rationale for these changes further clarified and helped identify the task features that we focussed on. In the process we also arrived at our operationalisation of the guiding principles of flexibility and accessibility. In this chapter, I examine the following questions:

i) What task features afford flexibility in tasks?

ii) What task features make them more accessible for students at the margins?

It must be noted that I am not suggesting that the explorations must *necessarily* have all these features. These features are presented more as desirable features and an exploration that incorporates more of these features being preferable to one which incorporates fewer features.

The instances are drawn from the design and implementation of 6 explorations described in Table 3.3, chosen based on the number of times they have been repeated (in project schools and elsewhere) and availability of audio recordings and teacher diary entries. Further, as noted in Section 3.7, the choice of explorations to be discussed in this chapter was based on an opportunistic sampling from among the explorations implemented, based on availability of data pertaining to revisions of the module, that is, versions of the modules, records of design related discussions within the research team and comparative notes on implementation of different versions. “Guess the colour” is an exploration where discussions happened on the module design, and the design was consequently changed. The versions and rationale for change have been recorded. Two different versions of “Partitions and cells” were implemented in the two project schools and detailed entries were made in my diary marking the differences between these implementations. There is a discussion of the design elements of the “Polygons” exploration in the literature (Fielker, 1981) describing the potential paths opened by a framing that leaves some details of the problem open to interpretation. The choice of these three explorations were based on the above considerations. The choice of instances from the classroom implementation of these explorations, for the purpose of illustrating how the identified features facilitated student engagement with the exploration, was guided by their illustrative power to show student engagement consequent to the described module feature.

4.2 Flexibility in tasks

Textbook problems are generally well specified and have one correct answer. Typically the problem statement contains all the information required to solve the problem and no more. The main purpose of such tasks is to apply concepts and practise procedures taught earlier. Yeo (2017) terms such tasks “closed mathematical tasks”. As discussed in Section 2.5, there is a growing discussion around and support for other types of tasks, referred to variously as open problems, open-ended problems, mathematical investigation and ill-structured tasks in mathematics education literature. I build on the literature around open tasks and identify features that add to task flexibility. I build on Yeo's (2017) framework (discussed in Section 2.5.2) to characterise the openness of mathematical tasks and reinterpret the framework elements for our context and purpose.

Yeo identifies five dimensions along which a task could be open – goal, answer, method, complexity and extension, and differentiates task-inherent openness and subject-dependent openness for the dimensions of complexity, methods, and extension. Given our concern for accessibility, our goal was to design tasks that give sufficient guidance to students on how to solve the task, rather than ones which students may not know how to start. Therefore task openness along the complexity dimension was judged to be not relevant to our study context.

On the dimension of extensibility, some task formulations have evident possibilities for generalisations, making extensibility task-inherent, whereas other formulations may require the teacher or students engaging with the task to deliberately vary the task parameters to extend or generalise the problem, making extensibility subject dependent. Similarly some task formulations may explicitly suggest multiple methods, whereas others may leave it implicit leading to one preferred method being adopted. Given that the implementation of the explorations in this study were teacher mediated, the distinction between task-inherent openness and subject-dependent openness with regard to the dimensions of extensibility and method was of limited relevance: only to the extent of sensitising us to the need for the teacher to be aware of the multiple possibilities. Also, since we privileged talk as means of doing and communicating mathematics and students engaged with the explorations through discussions, necessarily bringing in multiple voices and methods, these nuanced distinctions that Yeo makes were judged as not pertinent to this study.

Another difference I wish to mark is that Yeo specifies that the extension of a task be another task related to the initial task. I do not differentiate whether the subsequent tasks are related to the initial one. Going beyond generalisability, I reinterpret openness along the extensibility dimension as "generativity of problems". I look for tasks that give rise to further questions, irrespective of whether they are "related" to the initial task as Yeo indicates. Thus I consider explorations on integer geometry, rectilinear polygons or describability of matchstick shapes, all as offshoots from the exploration of matchstick shapes though they may be "unrelated" in Yeo's terms.

Yeo terms a task statement that does not specify a goal as open along the goal dimension. Such a statement affords flexibility in that it allows students to choose their own goals. The following task, which Yeo terms an "investigative task", is used to exemplify the many dimensions along which a task could be open.

The powers of 3 are 3^1, 3^2, 3^3, 3^4, … Investigate. (Yeo, 2017, p. 178)

The purpose of such a task is for students to investigate and discover the underlying patterns or mathematical structures. With such a task formulation, students could choose to investigate if there is a pattern in the last digit of successive powers, or the sum of their digits, or in the number of digits in successive powers or any other interesting pattern that may occur to them. They may also investigate if similar patterns exist for powers of other numbers. However, Yeo (2008) reports that even "high-ability" students in Singapore did not know how and what to investigate and were unable to pose their own problems to investigate when the task statement did not provide any sample problems to investigate. Task statements that give some pointers as to what to investigate, with sufficient room for interpretation, without being overly specific may enable flexibility without compromising on accessibility. Thus one of the challenges in designing and implementing the exploration is to achieve a balance between specificity and openness, indicating that sometimes there is a tension between the goals of accessibility and flexibility. While flexibility is important as noted above, openness must be balanced with specificity in such a way that satisfactorily addresses this tension. While it may not be possible to specify general features, detailed discussion of selected cases of explorations will, I hope, illuminate task features that have a bearing on this issue.

We now look at task features that afford flexibility in tasks.

4.2.1 Openness in task formulations: room for interpretation and choice of goals

Based on the evolution of three of our explorations a) Guess the colour b) Partitions and cells and c) Polygons and how they panned out in class, I examine how to balance the specificity/flexibility in the task statement, ways of allowing space for students to interpret and reformulate the tasks and the extent of information/direction to be specified in the task statement in order to support such reformulation. I also describe instances to illustrate how such a formulation facilitated student engagement.

Guess the colour: In the initial conceptualisation of the exploration, the main task was framed as follows:

> Given a 5 x 5 grid of squares, divided by a single horizontal or vertical line into two rectangles of two different colours, say blue (B) and green (G), the goal of the puzzle is to find out how the grid is divided (i.e., the colouring pattern of the entire grid).
>
> 1. The colours of how many grid squares would you ask about to solve the puzzle?
> 2. What is the minimum possible number of grid squares, whose colour must be revealed through questions so that the puzzle is solved?"

Implicit in this task statement is our expectation that students would figure out the colouring pattern by asking questions about the colour of individual cells. Based on this, I planned scaffolding tasks, involving inferring the colouring of as many cells as possible, given the colour of some cells as can be seen in Figure 4.1. I drew attention to relevant "clues" through questions such as whether it makes a difference if the revealed cells are the same colour, different colour, or how they are positioned etc. I also pointed to the patterns that would help them make inferences. For example, in Figure 4.1 (a) from the information given it can be inferred that the entire rectangle with the pair of diagonally opposite corners marked by the Bs will also have to be coloured blue. From Figure 4.1 (b) it can be inferred that the square has been vertically split into a 5 x 2 rectangle coloured blue and a 5 x 3 rectangle coloured green.

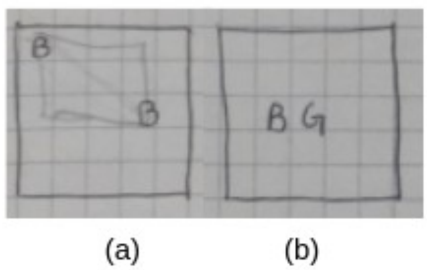


Figure 4.1: Guess the colour: initial version

There were questions that drew attention to relevant facts such as the colour of the centre square being the dominant colour and diagonally opposite corners being differently coloured. Other questions asked students to think of positions of "revealed squares" that would allow them to infer the colours of more squares and the kind of inferences that could be made if the "revealed squares" were of the same or different colours. The expectation was that by engaging with such questions, students would figure out the combination of cells to ask for that would reveal the maximum information, thereby solving the problem.

Two points that emerged during the discussion of this version of the task within the research team were:

1. The formulation allows students to ask for only one kind of question - namely what colour is a particular grid-cell. Students would also need to precisely specify the cell whose colour they wanted to know - for example the cell in the third row, fourth column etc.

2. From inferring additional information based on what is given (as in Figure 4.1) to figuring out the combination of grid-cells that will reveal the entire colouring is perhaps a leap, which might

> make the task inaccessible at least to some students. More importantly it privileges one solution method and the sub tasks “funnel” towards this privileged method. In that sense it is goal-directed and prescriptive in a manner similar to a textbook problem.

Based on the discussions, the initial formulation was changed to

> I have a 5 x 5 grid. I have divided into two rectangles and coloured each with a different colour - One rectangle is red and the other is blue. You have to guess how exactly I have coloured the grid. You can ask me questions.

The kind of questions that could be asked by the students to obtain information about the colouring in the grid were left open. We anticipated that at least some students might have difficulty in identifying questions to ask. We planned to get around this difficulty through a demo game where the teacher asks questions and guesses the colouring the students had in mind. It was decided to observe what kinds of questions students ask and how they refine the questions if they had to do it in fewer turns. We now look at some of the questions that were asked in School 1 and how they were subsequently refined. A sample of questions that came up in the first few attempts of the game were:

- How many blue coloured squares have you coloured ? (I answered 15)
- Straight line or cross line?
- Would *it* be across or down? {with hands gesticulating. The intention was to ask if the blue strip was going across the grid or down the grid.}
- Will blue be upside-down or side-side?
- Blue up-to-down or side-to-side?
- *Blue sideaa varuma straightaa varuma?* (Will the blue be on a side or will it be straight?)
- Is the 15 upside or downside?

Terms like horizontal/vertical, row/column, did not emerge at all and were clearly not a part of the students’ active vocabulary. In articulating the questions, the students gestured and drew figures to clarify their point. For example, when I asked for clarification of the question: straight *line or cross line?*, the student came up to the board, drew Figure 4.2 and asked which of these divisions I had in mind.

Figure 4.2: Guess the colour:
"Straight lineaa cross lineaa?"

This was addressed in subsequent runs of the game by their coming up with their own terminology (standing/sleeping lines for vertical/horizontal lines) and by my giving them the necessary vocabulary (row/column, horizontal/vertical), which they took up. The questions were refined not just in terms of language, but they had to be sharpened to get unambiguous information as well. Among the questions that came up in later runs were

- Does the first row have blue?
- Is blue there in the first row on the left side? (Meaning if there is blue in the first column)

A “yes” to the question “does the first row have blue?” could come from a horizontal division with blue at the top of the grid, or a vertical division with the first row having both blue and red cells. However, a “no” clearly indicates a horizontal division. Unlike an answer of “no”, a “yes” to either of these questions does not give unambiguous information about whether the division was horizontal or vertical. Students initially took a “yes” as indicative of the entire first row/column being blue, leading them to an incorrect guess of the colouring. This was eventually sorted out by asking if it was “fully red/blue” or pairs of questions,

- Is there red in the first column?
- Is there red in the last column?

OR

- Is there blue in the first column?

- Is there red in the first column?

Yet another opportunity for refinement that came up was in terms of avoiding redundant questions. In one of the runs of the game, the series of questions and answers went as follows

- How many Blue? 20
- Is blue in the first column? Yes
- Is red in the first column? Yes
- How many blue in the first column? 4

From the first three questions, it is possible to infer that it is a horizontal division, with four blue rows and one red row. The relative positioning of the rows is what remains to be figured out, which can be done by asking for the colour of the first or last row. The fourth question is redundant in that the answer can be figured out from the first three. I drew attention to this and said that going forward I would not answer questions, if the answer could be inferred from questions already answered. This step brings to the fore the need to infer additional information from what is known. This was the motivation behind the scaffolding tasks in the initial version and is key to guessing the colouring with an optimal number of questions.

In later runs of the exploration elsewhere (in talent nurture camps and summer schools), more complex and language intensive questions like, questions like “Is the length of the blue rectangle along the top/bottom/left or right edge of the square?” or “is the number of cells in either of the rectangles a multiple of 3?” came up.

Based on the variety of questions that came up and the refinement that happened both in terms of sharpening the question to extract relevant information and more precise references to objects, we concluded that the revised and more “open-beginninged” formulation that did not restrict the kind of questions admissible was more in line with our goals than the initial formulations of the tasks. The wide range of admissible questions created more opportunities for students to talk and articulate their own ideas than when there was a template to be followed with only one kind of question to be asked. The refining of the questions itself provided opportunities for mathematical thinking and to develop ways of expressing ideas, refining these ways and expanding vocabulary.

After a few runs of the game, we chose to restrict to questions that can be answered with a yes/no. We felt that the “how many” questions was giving the game away too soon (For example, students asked questions like how many blues in the first row, followed by how many blues in the first column) and that some restriction would keep the challenge level high. Narrowing down the scope of questions also enables better planning for further progress of the exploration and offers an obvious path to generalisation than when any kind of question is admissible. Thus, for this exploration, an approach that did not overly specify the kind of admissible questions, giving students room to work with their own interpretations of what could be asked, and narrowing down questions as the game progressed depending on the context gave flexibility to the task, without compromising on accessibility and being generative of further questions. This has been corroborated by our experience with other tasks as well.

Partitions and cells: The motivation for this task was the task titled "Lines and Squares" in Hardy et al. (2007) (see Figure 6.1), which requires students to find out the minimum number of lines, vertical and/or horizontal, needed to make a certain number of squares in a rectangle and asks students to "investigate further". The intended goal is to let students explore possibilities, notice patterns in how the number of lines and squares vary, articulate and generalise these patterns. We found this task suitable for our setting with some adaptation. For a start, we restricted to counting only unit squares as opposed to including squares of larger dimensions as in the original, and contextualised the task (see Section 3.6). The intention was to make the starting point simpler. We added the context of crates that hold soft drink bottles (assuming of course that the partitions could slot into one another!) as in Figure 3.8. We started with a small number of partitions inside a crate or cardboard box and counted the number of cells formed, so that students could get started. One of the possible results we hoped students would see was that the maximum number of cells are formed in a square array. The task presentation in the two schools where I implemented the task varied in the extent to which I foregrounded this goal. In School 1 the presentation was geared towards this goal, and was limited to square arrays. In School 2 on the other hand, I went with a more open formulation, where students were asked to explore the possible number of cells they could form inside a crate with a given number of partitions. The exploration evolved to finding out and listing the number of possibilities for this case and eventually to the optimality question. I now share some excerpts from the teacher diary describing the task formulations and the differences in how these were received by the students.

In School 1, the task was presented in a form that had a unique answer, but lent itself to extensions and generalisations. In Yeo's (2017) terms, the task was not open in terms of admissible answers and goal specification, but was open in terms of extensibility and methods that could be adopted. The problem was posed in terms of the number of "cells" that could be created with a given number of "partitions", such that the "cells" form a square grid or fit into a square crate. Quoting from the teacher's diary entry written after the exploration was done in School 1, that describes the task presentation.

> *The problem I gave today was that I have a cardboard box with a square shaped base and I wanted to make it into the kind of box in which Pepsi/coke bottles come, by placing partitions. I showed them if I have 2 partitions I get 4 cells, If I had 4 - I had 9 cells and asked them how many cells would I have if I had 16 partitions. The idea was to move to rectangular boxes, the multiple possibilities therein for a given number of partitions and get them to find out the min/max cells/bottles possible with a given number of partitions and find 'formulae' for one in terms of the other. Didn't get that far.* (Diary entry dated 4th Feb, 2019)

Given the restriction of the square box, they soon realised that there must be an equal number of horizontal and vertical partitions and based on this arrived at an algorithm to find out the number of cells given the number of partitions, and vice versa. They did not want to pursue my suggestion of coming up with a closed form expression for these and were not enthused enough to explore the case of rectangular arrays of cells. I did not pursue the task any further in School 1.

In School 2, I tried a different presentation, which may have contributed to the higher engagement levels and further progress on the exploration. The excerpts below are from the teacher diary from School 2 and describe the difference in formulation and engagement levels.

> *I did the same Pepsi/carton Problem as on Monday - but this evolved very differently today. From my part, I kind of started in a more open way and I found the class leading me on to the rectangular array first - For a moment I debated within if I should restrict to square grid - but let the class flow as it will. And I think it was a good move - I saw almost all of them engaged - number of happy faces and comments at the end and persisting after the bell - obviously had a rub off on me as well.*
>
> *I started with the same Pepsi Crate - but said with 2 partitions, we could either make 4 cells or 3 and drew. Asked them what if we have 3 partitions - they came up with multiple ways - The usual 4 cells (linear arrangement) and 6 cells (2 x 3 arrangement) Then they started drawing 3 inclined lines, and 3 lines which didn't go from edge to edge of the carton. I ruled out the oblique line case saying that we will take it up later and drew the rest on the board. Ninan objected to the one where the partition was not going edge to edge and said that cells in that was not uniform and so he wouldn't allow it. I crossed out and said we will stick to partitions where all cells are identical.* (From diary entry dated 6th Feb, 2019)

Since the initial formulation did not specify a square grid, students had the opportunity to try out other options and as a group,the students and I narrowed down the problem to be investigated. In the course of this session, students investigated the various possibilities for a given number of partitions, had a way of exhaustively listing them and came to the conclusion that the maximum number of cells is obtained when there was a square alignment and when that is not possible, when the length and breath differed by as less as possible. They also noticed some patterns and structure in the possibilities for a given number of partitions and how the number of cells increase both when the given number of partitions are realigned and when the number of partitions changes. In a subsequent session they went on to find an expression for the maximum number of cells when the number of partitions was *n* even producing the closed form expression $(n/2 + 1)^2$. In addition, there were indications that they were engaged and enthused and interested in pursuing similar problems. Another excerpt from the diary

> *Now RK came and asked me why I was making them do all this - Is it to get them interested in Maths, or to make them think or for fun. I asked if all this was happening and I got a resounding yes from 3 girls (and beaming faces too) They said when I leave them with those questions - they are driven to find the answer - "we want to find out how it works" . They also asked me where I get these problems from, and if they can try and solve these on their own as well, if I am not there.* (From diary entry dated 6th Feb, 2019)

The task presentation in School 1 was more as a problem to be solved, rather than an invitation to explore. The presentation focused on finding the number of cells for a given number of partitions and generalising for any number of partitions. The presentation in School 2 was a more tentative presentation, where the goal evolved in the course of the session with contributions from the students as well. Robert Kaplan captures this difference with these words "no eliciting of answers according to pre-ordained schema, but the free flow of invention and zaniness, with goals of your own kept in mind (these may change as the conversation takes unexpected turns)." Perhaps this led to higher levels of engagement in School 2. Also students observed other patterns[9] over and above the relation between the number of partitions and cells in the case of a square array. These indicate that open formulations offer more scope for mathematical thinking, especially at the margins where a formulation that specifies a hard and fast goal could be daunting.

Another instance where I have observed a task formulation that allows multiple interpretations enabling mathematical thinking is the Polygons exploration, which is inspired by Fielker (1981).

***Polygons*:** The task here was formulated as "What is the maximum number of right angles possible in a polygon?" The question leaves the words "polygon" and "right angles" open to interpretation. Feilker points to certain questions as recurring in his experience of multiple groups of teachers engaging with the task, such as "must the polygons be convex?" or "can the right angle be outside the polygon?" and the self imposed restrictions of "having all sides horizontal or vertical " or "at least as many as possible".

9. For example: With 5 partitions, it is possible to get 6, 10, or 12 cells .
With 7 partitions, it is possible to get 8, 14, 18, 20 cells.
With 9 partitions, it is possible to get 10, 18, 24, 28, 30 cells.
These numbers of possible cells form a series, with the differences between two consecutive terms in each series being the decreasing sequence of even numbers - 8,6,4,2 in the case of number of cells with 9 partitions.
With an even number of partitions, the possible cells come out as series that differ by a sequence of odd numbers. For example with 10 partitions one could have 11, 20, 27, 32, 35, 36 cells, with the difference between consecutive numbers being 9,7, 5, 3,1.

These questions also came up when I tried the exploration with students in School 1. In addition, I also encountered questions such as: Are we talking of a specific polygon, say a triangle, square or hexagon? Is it a polygon in which all angles are right angles? Does it need to be a regular polygon? Clarifying the problem was the first step in engaging with this problem and intentionally so. If I had spelt out these details in the task statement, it might have been a problem which the students had not encountered before, but with a specific goal that they could pursue. In Feilker's words

> But, it seems a pity firstly to stifle all this originality which interprets the problem in other ways, but more importantly it seems better that the solver should make decisions about which interpretation is worth pursuing. In this way it is more likely to become the solver's problem, rather than someone else's with implications of a solution already lurking which the solver is trying to divine. (p.11)

Aligned to this view, I see asking for and clarifying the required details and reframing the question with the desired level of specificity itself as a mathematical practice that is worth engaging in.

The loose formulation that allowed for alternate interpretations led to students working with definitions of polygons that differed from conventional textbook definitions. In School 1 where *polygon* was understood intuitively as "many-sided figure" some students did not exclude shapes with intersecting sides or looped shapes from polygons and explored the possibility of these having more right angles. The "lack of grade appropriate knowledge" which might have been viewed as a handicap opened up an opportunity to explore a different track. Not having access to analytical methods to address the problem, was compensated for by experimentation, and students drew various shapes with the goal of maximising the number of right angles - some of them with sides crossing over as seen in the Figure 4.3. While no results were obtained on this, other reinterpretations led to results and their justification.

Fielker (1981) notes the differences that stem from the two statements - the "sides of the polygon are at right angles" and "the polygon's angle is a right angle". While the former interpretation leads to an external-right angle being counted as a right angle as well, the latter does not. In the process of drawing polygons so as to explore ways of maximising the number of right angles, rectilinear polygons (polygons all whose angles are either 90^o or 270^o that is all whose sides meet at right angles) invariably appear and some students raise the question whether they could include right-external angles of the polygon as well. This was the case in School 1 as well, and led to observations such as "in an even sided polygon, the maximum number of right angles is equal to the number of sides" and in turn to questions about polygons with an odd number of sides. These alternate interpretations proved to be opportunities to observe and justify patterns and ask further questions. While a result like the above obtained by counting the exterior-right angles as well, may not be part of any standard textbook or of much importance mathematically, I see value in them as opportunities for students to figure out something by themselves and to justify their finding.

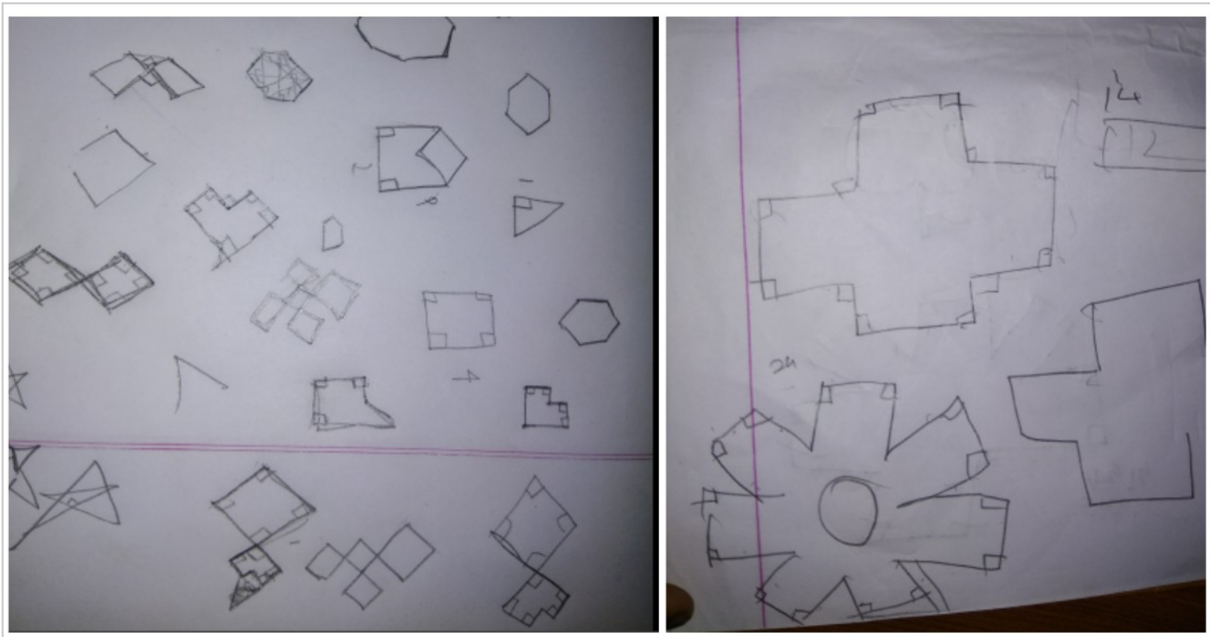


Figure 4.3: Polygons: Student work

All three examples described above illustrate how a task formulation that does not over-specify what needs to be done, allowing for alternate interpretations and choices of goals to pursue, provides opportunities to engage in mathematical thinking and sustaining student engagement. I therefore suggest that a task formulation which points to potential directions of inquiry without spelling out a specific goal or preferred approach, affords flexibility. Also, not spelling out all information necessary to solve the problem opens up room for multiple interpretations, approaches, and therefore room for more engagement and mathematical thinking.

Another task feature that I identify as supporting mathematical thinking at the margins is the affordances to function at multiple levels of formalisation.

4.2.2 Affordances to function at multiple levels of formalisation

As discussed in chapter 2 it is well-recognised that the symbolic representations and the formalism of mathematics are entry barriers to the discipline. Affordances to work at different levels of formalisation is a feature that we looked for to make the task more flexible and allow students to make some progress, even when functioning at informal or semi-formal levels. For example, game based problem formulations allow for solutions within the context of the game while demanding mathematical thinking. Students working through the game may come up with a winning/optimisation strategy as required, intuitively, with hardly any formalisation visible. I now illustrate the levels of formalisation afforded by some of our explorations. I choose the Magic triangle exploration to highlight the possibilities and the Guess the colour exploration to show the levels of formalisation seen as the exploration was implemented.

Magic triangle: In this puzzle, the starting point is the goal of arranging distinct numbers 1-6 along the sides of a triangle, such that numbers along each side sum to the same total. We expected students to try out different possibilities and see that there are four distinct solutions to the problem. We also expected them to systematically eliminate possibilities and enagage with the processes of conjecturing, proving, and symbolising. That there are only four solutions to the puzzle needs justification which can happen at multiple levels of formalisation. One might do a brute-force listing out of all possible arrangements of the numbers and observe that there are only four solutions, or use parity arguments to rule out some possibilities right away. For example an alignment that has the odd numbers 1, 3 and 5 at the positions as in Figure 4.4 is not possible, as this will give an even sum to two sides and an odd sum to one side.

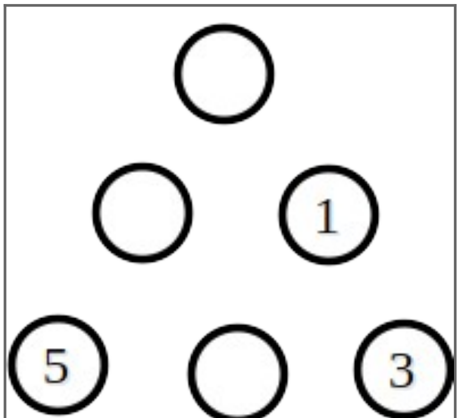


Figure 4.4: Magic triangle: An impossible arrangement

The odd numbers need to be positioned in one of the ways as shown in Figure 4.5 for all side sums to have the same parity.

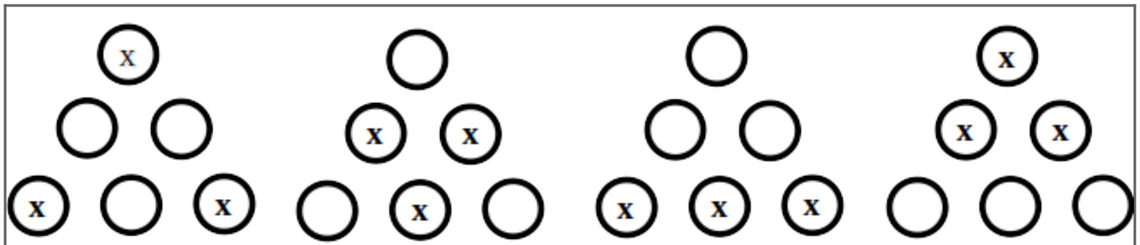


Figure 4.5: Magic triangle: Possible positioning of odd numbers

This argument vastly reduces the number of possibilities compared to a brute force listing and with some systematic work can lead to the four arrangements that give equal side sums. Neither of these approaches require formalisation.

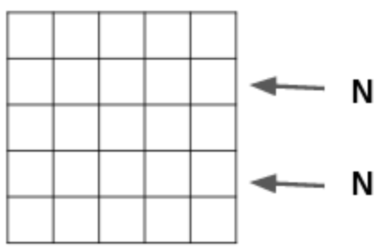
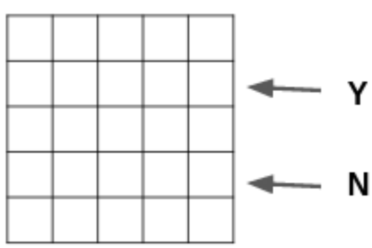
Another approach is by finding upper and lower bounds for the side-sums. The numbers at the vertices are two of the three numbers that contribute to a side-sum. So one can argue that the minimum side-sum is obtained when the numbers 1, 2 and 3 are at the vertices and the maximum side-sum when the numbers 4,5 and 6 are at the vertices. Reasoning further one can see that the minimum side-sum is 9 and the maximum side-sum possible is 12. This drastically reduces the possible arrangements to be considered and leads to a justification that there can only be 4 solutions. Though more “sophisticated” than the earlier arguments this can still be carried through without resorting to symbolisation (see Section 5.5.4). The algebraic approach described in Section 3.6 on the other hand is more general and solves a larger class of problems and can be extended to solve the variations of the problem as well.

Even if the formalised approach is not within reach, the student can still solve the problem, obtain the four distinct solutions, and justify that there are exactly four. The multiple approaches that lead to the solution, ranging from brute force listing that requires no formalisation, to semi-formal reasoning to a symbolised formalised approach ensures that formalisation is not an entry barrier to the task. I will discuss students’ responses to this task in more detail in Sections 5.1.1 and 5.2.1.

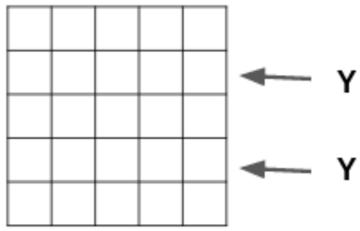
Through the above example, I illustrated the potential of one of the explorations that we chose to enable functioning at multiple levels of formalisation. I now look at Guess the colour exploration to illustrate how the potential was actualised in the classroom. The game based formulation of this exploration allows students to engage in mathematical thinking within the game context drawing on informal reasoning. In Section 4.2.1, we saw the informal language that students used in framing the questions in Guess the colour exploration and the adoption of formal terminology over a period of time. I now look at some of the students' strategies, how they expressed these strategies and the approach that they took to generalising the problem. In this I draw attention to the different levels of formalism that was visible in the class. The strategies were described referring to a figure and hence there was frequent use of deictics. The generalisation was in the form of an execution of an algorithm which was not explicitly articulated. The number of questions required was not formulated as a closed form expression.

Guess the colour*:* One of the claims that came up was that the colouring can be guessed in 4 questions. Abhi[10] of School 1 who came up with this strategy said that he would ask “is there blue in this row/column?” for each of the 2nd and 4th columns and 2nd and 4th rows. He later said that there were holes in the strategy and changed it to asking for the middle row/column, followed by the first and last row/columns. The various cases that could come up, on following Abhi’s initial strategy are as shown in Table 4.1.

Table 4.1: Guess the colour: Unpacking a student strategy

Case	**Possibilities for “Is there blue in the 2nd and 4th rows? “**	**Implications and Further questions**
1		• This implies that the 2nd, 3rd and 4th rows are red and either the 1st or 5th row is blue. (assuming that the entire square is not coloured with a single colour). • Further question needed to find out which of the 1st or 5th rows are blue to get the division.
2		• The case where the 4th row has blue and the 2nd does not is identical up to a rotation by the symmetry of the situation. • For the case on the left, the given information implies that the first two rows are blue and the 4th and 5th rows are red. • Further question needed to clarify the colour of the middle row to get the division.

10. All student names used are psuedonyms.

Case	Possibilities for “Is there blue in the 2nd and 4th rows? “	Implications and Further questions
3		● This could either mean a vertical division or a horizontal division with either of the 1st or the 5th row being red and the remaining rows blue. ● Further questions to clarify if it is a horizontal or vertical division and if vertical questions to fix the position of the division.

On probing further it was seen that Abhi had thought through these cases and said that he was asking for the colours of the 2nd and 4th column to address the possibility that the division is vertical. But he did not clearly disambiguate the possibilities for this case. Considering the case that one of these columns has blue and the other does not he says “ *appidinnu paatha, mukkavashi chance vandhu ithukulla, ithu randukkula blue irukka chance irrukku*” (“In this case there is a three-fourths chance that there is blue within/ between these two”). Unlike in the first two cases, here he only offers a possibility and does not make a clear statement. His later statement that “there is a hole in the strategy”, and that it didn’t work out, may perhaps be because he was not clear of how the vertical division case could be tackled. Perhaps he noted that the additional questions that he had to ask over and above the four that he initially said he would need were about the central row/column and the extreme row and columns. This seemed likely as his revised strategy started with asking about these.

Another strategy came from a student, Saju, who said that he could guess the colour in 5 questions. He would ask “Is there red?” in the 3rd row, followed by the same question for either the 2nd or 4th rows. Depending on the answer the third question would be of the form “Is this your division?” where he would propose a horizontal split where either the first two or first three rows are of the same colour. He took a “no” to this question as an indication that the split was vertical and proceeded to ask the same question for the 3rd column and either the 2nd or 4th column.

Evidently, both these are well thought out strategies. The students did consider a number of hypothetical situations and branching in the form “If they answer yes, then…, if they answer no, then…” , but they missed out a few possibilities as well. On being probed further with a “what if…?” question, they realised the need to consider these possibilities as well. All this was done orally. They did not write the strategy down listing out all the possibilities or show their decision processes diagramatically. The articulation was accompanied by pointing to the diagram at hand with a frequent use of deictics. The students frequently focussed only on one of either horizontal or vertical division and were unclear about how to incorporate both possibilities in their chain of questions. For example, a question “is this row blue” would come up even before ascertaining that a single colour could be ascribed to the row. All this points to the informal expression of mathematical ideas. For the students, it was a game they were playing, trying to guess the division with fewer questions than others and strategies were arrived at and refined in that spirit. Perhaps they did not feel the need to represent their decision tree through a flow-chart or consider the benefit of doing so when they could play the game very well without any of these. Moreover, neither strategy discussed above is “complete” in the sense of laying out all possibilities, nor are they optimal. But the exploration affords the possibility to come up with such “partial solutions”, and further discuss and refine them unlike typical textbook exercises. Thus I suggest that an exploration that affords students the opportunity to engage with mathematics at multiple levels and, not limited to arriving at the predetermined answer, supports mathematical thinking.

Moving beyond the game to generalising the problem of guessing the division for larger grids they came up with an algorithm to find out the number of questions required. This was evident in action - given a grid, they would first ask two questions to clarify a) if I had drawn a “standing line or a sleeping line” (i.e vertical and horizontal lines) and b) if I had more of red or blue. This was followed by a set of systematic questions intended to fix the dimensions of the rectangle- for a 20 x 20 grid, they would ask, have you done a 19 by 1 division, have you done a 18 by 2 division, have you done a 17 by 3 division and so on till they reach the halfway mark. This was followed by a question on whether the dominant colour was at the top/left of the grid depending on whether the division was horizontal or vertical. To respond to how they would go about guessing in a 8 x 8 grid, they felt the need to draw a grid of the given size, point to appropriate grid lines as they asked, have you done a 7 x 1 division, have you done a 6 x 2 division etc. With a few attempts they got to asking the questions without the grid, but not to verbalising the strategy. (Describing what they would do in words such as “We will ask for such and such information and then identify the particular division through these questions” etc. as opposed to actually asking those questions and executing the algorithm). Going further, one could have gone on to get them to verbalise the strategy, give a count of questions that would be needed for a particular grid size, articulate the procedure that would give such a count or give a closed form expression for such a count. This group of students were not ready or rather chose not to take this path and stopped with the execution of the algorithm. Even this gave them a sense of achievement at having figured something out for themselves. The exploration allows the flexibility to function at multiple levels of formalisation, and students could engage at whatever level they are comfortable with, moving to more mathematically sophisticated ways as they gain comfort.

4.2.3 Incorporating multiple trajectories

Choosing tasks that can branch out along multiple paths possibly to multiple content domains of mathematics is another way of bringing in flexibility. One way of incorporating multiple trajectories is through variations on task parameters (Brown & Walter, 2005). In a game based exploration like Leapfrogs, one could vary the initial configuration, rules for operations allowed or the desired end configuration. Other tasks may give room for such questions as whether a property that is true for a restricted class of objects may hold for a larger class (e.g. whether a property that is true for a regular polygon is still true of any polygon? Convex polygons?) and if it does not, how may one modify (weaken) the statement so that it holds for a larger class or can one characterise the largest class for which it holds and so on. This gives students choice to pursue a trajectory that they find interesting and accessible.

I now list out the different parameters that can be varied in the Magic triangle exploration and some possible trajectories that evolve from this.

Numbers used: We start with consecutive numbers 1-6. We could vary this and use a different set of consecutive numbers. We could also relax the condition that the numbers must be consecutive. The defining property of consecutive numbers is that they differ by 1. We could use a set of numbers that differ by a constant number other than 1. That is, we could use a set of 6 numbers in arithmetic progression. We could further relax the condition of constant difference and use any 6 arbitrarily chosen numbers. With numbers 1-6, we see that there are 4 distinct solutions. The question comes up whether there would be 4 distinct solutions whatever be the set of numbers we use. If not, under what condition would there be 4 distinct solutions, Can there be more or fewer number of solutions than 4, can we predict the number of solutions for a given set of numbers, are there sets of numbers for which a solution does not exist at all, under what conditions does a solution exist, all these could be potential points of investigation. One might even admit negative numbers and rational numbers.

The triangle shape: We could vary the shape and consider numbers along the sides of other regular polygons like square, pentagon etc. We could also have open curves like in Figure 4.6 and insist that the sum of numbers along all arms are equal.

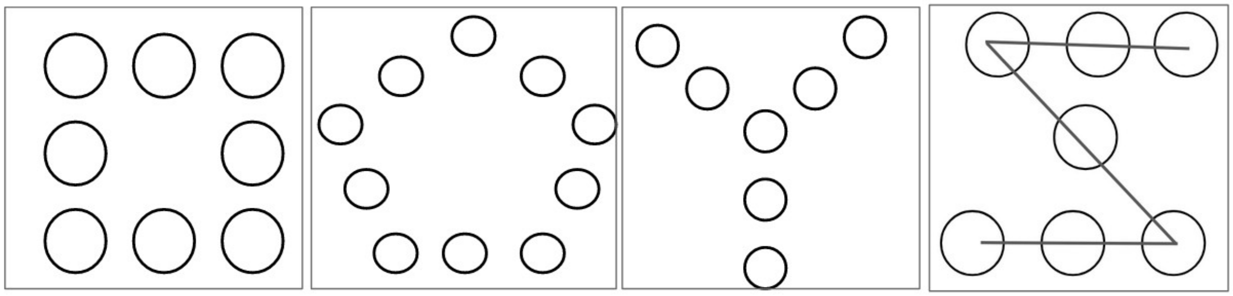


Figure 4.6: Magic triangle: Variations

We could have star polygons or even three-dimensional shapes as in Figure 4.7[11]. In 3-D shapes, corresponding to side-sums, we have two different notions namely edge-sums and face-sums. We could choose to equalise either of them or both.

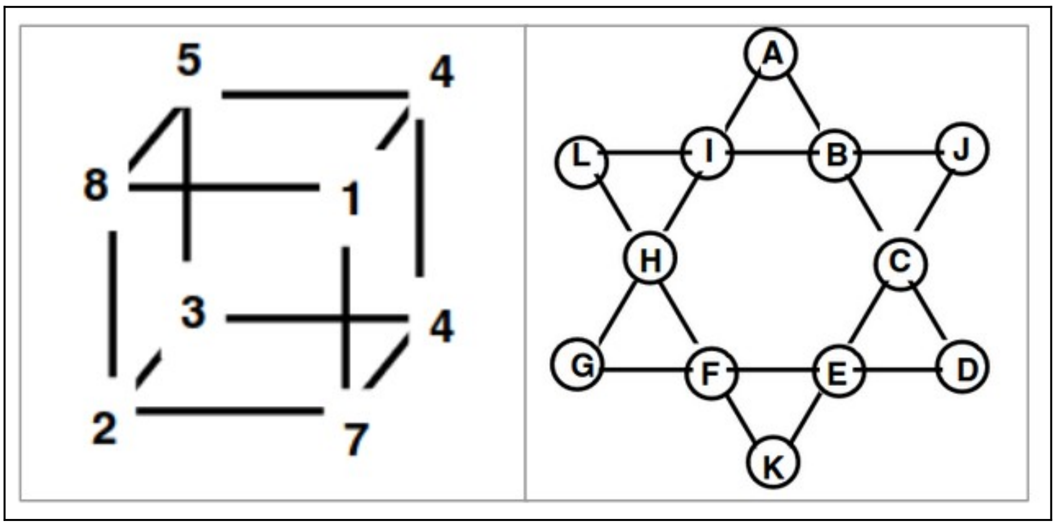


Figure 4.7: Magic triangle: More variations (Reproduced with permission)

Number of numbers per side: We start with 3 circles per side. This could be 4 or 5 or more.

The condition of equality: Instead of insisting that the side-sums be equal, we could specify conditions like the side-sums be all even, or all odd, or consecutive-numbers or even that all side-sums be distinct. It is possible that solutions cannot exist under some of these conditions - and that could be a point of investigation, what choice of numbers can afford what relations between side-sums .

11. Thanks to Mr. Joseph Eitel for permission to reproduce image from https://amagicclassroom.com/uploads/3/4/5/2/34528828/permeter_magic_polyygons_introduction.pdf

Other possibilities: It can be observed that some transformation of solutions leaves the solutions invariant whereas others yield new solutions. As shown in Figure 4.8, flipping the triangle about the median line (corresponding to a reflection) or moving the numbers around by two positions, (corresponding to rotation by 120^o) gives the same solution in that the side-sums and corner-sums remain the same.

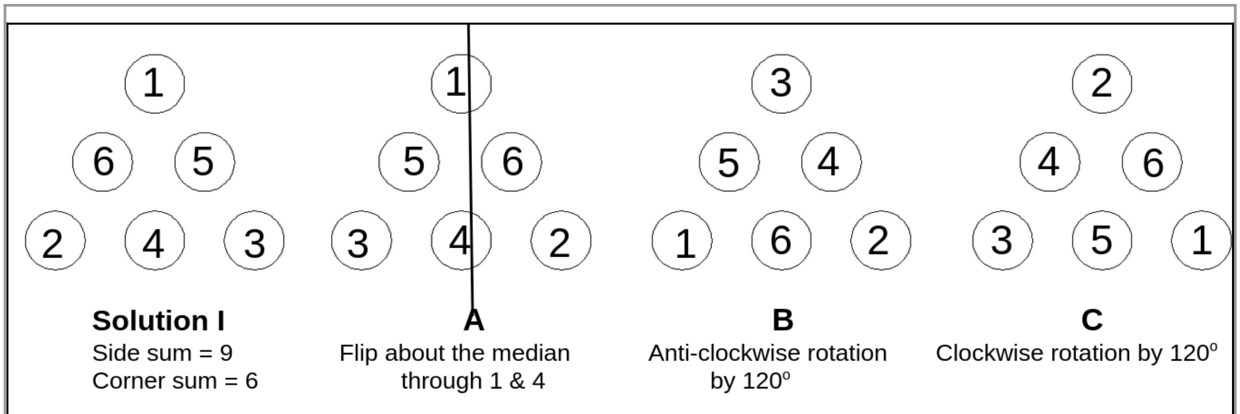


Figure 4.8: Magic triangle: Solution preserving transformations

Moving the numbers around by one position, or swapping the numbers along the median line gives a different solution (Figure 4.9). Applying the same transformation on the new solution, we get back to the initial solution. This is to be expected as rotation by 60^o twice, corresponds to rotation by 120^o degrees which is a symmetry of the equilateral triangle and swapping the numbers along the median a second time can be considered the inverse of the initial transformation.

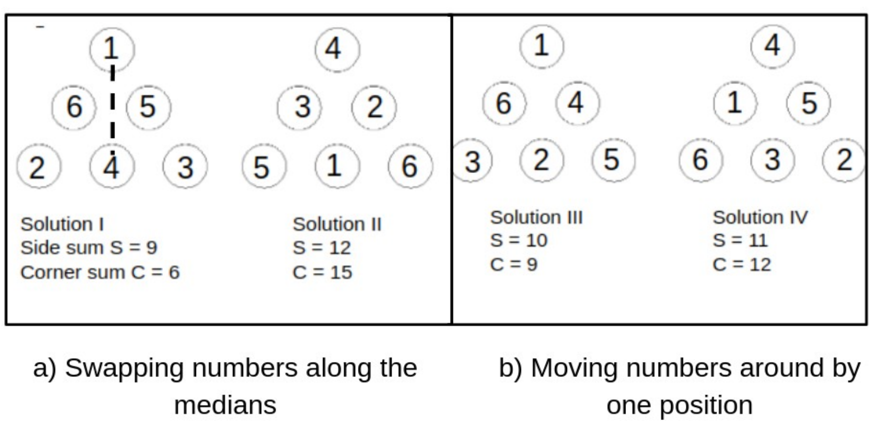


Figure 4.9: Magic triangle: Transformations that give a different solution

Observing these transformations could lead to questions like how many permutations exist for a given solution? What transformations leave the solution invariant and what do not? What transformations yield new solutions and what do not? Do the arrangements that are derived by transforming one solution share any common properties?

Some of these variations may come up naturally from students and others may be suggested by the teacher through a “what if..” or “what if not…?” question. Thus the goal of the exploration need not be having students find the four distinct solutions and have them prove that there are only four solutions. Students may explore transformations of solutions, even before finding all the solutions. In fact, we have had students using the transformations to generate more solutions having found one solution, especially when working with larger numbers (see Section 5.1.1). Similarly if they are not comfortable working with justification and proofs they could move on to other shapes and solve them or observe patterns in solutions working with different sets of numbers. The different trajectories available makes it more likely that even when one particular path does not enthuse a student or is not accessible, there is another that can engage them.

Variations may come up that may not be accessible to students or the teacher. Some might even lead to problems yet unsolved by the community of mathematicians. But these need not hold back students thinking through these variations. One practice that I generally followed is to have students think of possible variations irrespective of whether they can engage with the variations. Our observation has been that students engage well with this task and come up with well thought through variations. The teacher could choose the ones that are accessible and engaging for students to pursue.

While doing the Guess the colour exploration in school 1, questions of whether they could start with a triangular grid instead of a square grid came up very early on in the exploration. Later when it was set up as a game between two groups within the class, the spirit of the game prompted students to try out other variations - students asked if they could divide into three rectangles or if they could divide the 5 x 5 grid into two by drawing a smaller square within as opposed to a straight line that was proposed by the task, so as to challenge the opponents. I also explicitly asked students to think of variations for the game, especially to make it “more difficult for opponents to guess” with a few questions. Figure 4.10 below shows some of the variations that students came up with when asked to do so.

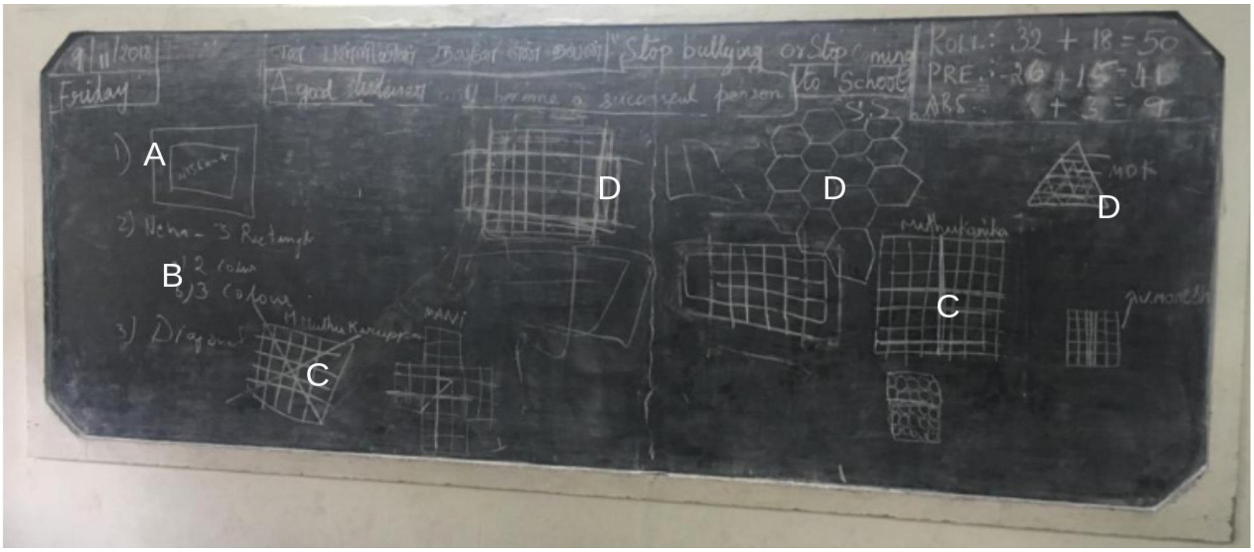


Figure 4.10: Guess the colour: Board work on variations

Dividing the grid by drawing an inner rectangle or square that is mentioned above is the first variation recorded on the board in the part marked A. The second was to divide the grid into 3 rectangles. Within this an additional layer of variation was suggested by using either 2 or 3 different colours to colour these (part B). There were also suggestions to divide the square grid into 4 squares and a diagonal division (parts marked C). Some of the variations used grids of other shapes - a triangular grid, a hexagonal grid, a shape that is made of two-rectangular grids with their central regions overlapping etc (parts marked D). Some of these were presented without sufficient thought, like what would the division of the hexagonal grid be like or in the diagonal division what exactly was to be found out. But some details were worked out for other variations like the square inside a square or a three-way-division with 2 colours. None of these were taken up either in class or later, in the sense of working through all possibilities, but I believe that the very act of thinking through what could be varied and how these could be varied calls for mathematical thinking. We chose to deliberately allot time and space for this in our explorations. Also the Figure 4.10 shows that students used the board to record their variations and wrote their name against it. They often attached their names to their findings and subsequently referred to these by name. So I have had “Nishant’s Theorem”, “Mani’s game” etc. These are indicative of a sense of ownership and joy in their discoveries/creations. I made it a point to encourage this practice as mentioned in Section 3.10.

In addition to flexibility in tasks, there needs to be flexibility in pedagogy as well, in terms of what is considered acceptable, what is considered worthy of building on, preferred approaches to problem solving and communication, etc. For example, in the Magic triangle exploration, if the teacher’s goal is to have students prove that there are four and only four solutions, investigations into the symmetry and transformations of solutions may be limited to the extent that they contribute to identifying distinct solutions. Similarly the incomplete and not very precisely articulated strategies in the Guess the colour exploration may be labelled incorrect and a deficit view taken of the students. The teacher needs to shift the goal post from "finding the solution" to accommodating these lateral explorations. In addition to tasks being designed to accommodate shifting goal posts, the teacher needs to play along as well, so that the design comes alive in the implementation. Being sensitive to the mathematical potential of investigating the symmetry of solutions or being accepting of partial-strategies, seeing the mathematics in these and working with students to refine them can be very challenging for a teacher. I discuss these challenges and supporting measures in Chapter 6.

We see through these examples that, flexibility incorporated in explorations by making available multiple trajectories, multiple goals to pursue and opportunities to work at informal or semi-formal levels is a move away from right-answerism and an attitude of intolerance to student error. The discussion also illustrates how flexibility and specificity could be balanced to allow for student engagement. Going further, I now look at other task features that make tasks more accessible.

4.3 Accessibility of tasks

The general perception is that explorations are meant to challenge the mathematically inclined and that explorations cannot be sustained at the margins where even the "grade appropriate content knowledge" may be lacking. I now ask how we can counter these deficit views and design "low threshold" tasks that are accessible to students at the margins, without sacrificing the potential to elicit mathematical thinking.

4.3.1 Limiting prerequisites

A key consideration while designing tasks has been to minimise dependence on specialised prerequisite knowledge or algorithms to get started on the task. I have observed that at times what may be considered "grade-appropriate content knowledge" or theorems that students are supposed to have learnt as part of their curriculum can prove to be stumbling blocks to progress when an exploration crucially depends on them. Students may not be able to recall or apply these results in an unfamiliar context, or may not have understood them sufficiently to retain and use them beyond the requirements of the end of year exams. When such a dependence is unavoidable, we sought to incorporate alternate paths based on what the students know and the nature of the result in question and how it is expected to be used in the exploration. These may vary from the facilitator "giving" the result as something to be taken for granted and used, to having students arrive at it through a process of guided discovery, or having them "look up" the result from other sources or drawing on the knowledge of the few in the class who may be aware of the result, etc. Most of our explorations discussed in Section 3.6 do not require any prior content knowledge to get started.

The Magic triangle exploration is a puzzle that assumes hardly any curricular knowledge. Finding the four solutions could be done by trial and error. The effect of transformation of solutions, those that preserve side-sums and those that generate new solutions could be identified intuitively. Students could also engage with extensions to other polygons, and prove the existence of 4 and only 4 solutions using simple reasoning. There is much that can be done without any specialised prior knowledge. However, a facility with algebra suggests a solution strategy that applies across polygons and leads to an elegant solution.

The explorations Guess the colour, Leapfrogs, and Clapping game are structured as games, enabling students to engage as they would in a game. Even with this they could try and refine strategies, think of optimisation and attempt generalisation. As in the case of Magic triangle, algebraic reasoning supports and greatly eases the generalisation process. Making progress in the Leapfrogs exploration hinges on coming up with and working with an appropriate representation. The same holds for Clapping game as well – a presentation of the observations from multiple trials of the game as a table makes it easier to see the inherent patterns. Even though the prerequisite content knowledge is limited, the explorations enable engagement with a number of mathematical practices and were found sufficiently challenging by students.

Clapping game draws on the curricular concepts of factors, multiples, common factors, least common multiples, etc. However it is possible to engage with the exploration with an intuitive understanding of these concepts. These are concepts which students would have encountered by Class 9, but may need brushing up. Not being able to recall the term factor, I have had students work with the Clapping game exploration referring to the factors of a number *n* as "those numbers in whose table *n* appears". I consider this acceptable, but not desirable and try to "slip-in" the term at an opportune moment. Without "abstracting" the concept of factor/divisor, referring to and working with the concepts of "Greatest Common Divisor (GCD)" or "co-prime", which is necessary to solve the initial problem posed in the exploration, may be cumbersome. Having an understanding of "common divisor" may make it easier to understand and work with the concept of GCD.

The exploration on Views of solids hinges on visualisation, being able to "see" and draw the different views of a solid and working with them. Students in School 1 had not met these as part of their curriculum. But working hands-on with unit cubes and blocks, they could easily get the required familiarity to work with this exploration.

Thus none of the explorations discussed above crucially depend on any curricular concept. Even when there is a need to draw on such concepts it is possible to brush up or build the required knowledge easily in a short time. Such skills as reasoning, visualisation, representation, which can be expected of all students, are more central for these explorations. Algebraic reasoning is a crucial factor, which determines progress in the exploration, but not having facility in it does not function as a barrier to engagement. The students did not have any readily available solution methods for any of these tasks and could arrive at some methods with some thinking and reasoning. Thus the tasks were not "watered down" and did succeed in challenging the students.

The Polygons exploration was an exception in that it crucially depended on knowledge of Angle Sum Property of polygons. Students cannot engage with the task deeply unless they are familiar with this result. With one group of students in School 1, realising that this was not something which they could draw on readily, I had students arrive at the property through a series of guided questions and then had them use it to obtain the maximum number of right angles in the polygon. This was laboured progress and many students found it difficult to engage and did not want to continue with the exploration. In School 2, the students were not familiar enough with the notion of polygons and the exploration had to be dropped altogether. Thus the task may need to be adapted, or dropped altogether depending upon the level of preparedness of students with whom it is done.

4.3.2 Using physical material

Some tasks can be made more accessible by starting with hands-on experience using physical material. While the use of hands-on material is a recommended practice at primary levels, use of such material tapers off in middle and secondary school. I suggest that use of hands-on-material has its benefits even at secondary school level, especially as starting points for explorations at the mathematical margins. The question itself may be framed in terms of the activity and may evolve to be framed in more abstract/general terms. A formulation in terms of the physical material allows for a solution in terms of the material, which does not rely on formalism. This makes the task more accessible at the margins. It can serve as a first step in the transition to a more formal framing and solution. The transition from a formulation in terms of physical material to more abstract formulations also creates affordances to work with multiple representations.

For example, moving tokens around in the case of Leapfrogs helps students "see" the problem. The tokens in Leapfrogs offer a way to try out multiple moves and retrace them if required without any penalty. After playing a few times with actual tokens students go on to devising means of representing the positions of tokens and the moves through various means and gain comfort in working with representations, but the experience of playing with the tokens helps them arrive at these representations. The act of manipulating matchsticks to make shapes also aids the exploration. In School 1, in the initial stages the questions framed were all with respect to the matchstick shapes - for example, having observed that they cannot fit a diagonal in a unit square, asking whether it is possible in a larger square is the first step to generalisation, taken in the context of matchstick shapes. The physical matchsticks available helped students to actually try out the construction. However, in this case it led to the misleading conclusion that it is possible in larger squares. In the case of views of solids, the difficulty of visualising the various views of a three dimensional solid was eased considerably by working with models. Thus while concrete material, visual representations all have been observed to enable access to tasks one needs to be wary that statements made of physical objects may not be necessarily true of abstract mathematical objects.

On a similar note, I have observed that posing the task in a game context also has some benefits. This makes the task more engaging for the student. For example, the Guess the colour exploration could have been framed as a question of the form - what information would you need to guess the colouring and how many questions you would need to ask to get this information. But framing it as a game and having groups or individuals playing the game against one-another creates a spirit of friendly competition that helps in maintaining engagement levels.

4.3.3 Multiple entry points, answers or approaches

Another means of making tasks accessible is by allowing for multiple entry points, answers or approaches. I discussed multiple trajectories from the perspective of task extensibility and available methods in Section 4.2.3. I now suggest that a sufficient number of these, or ways along which the task could progress, should be at a low threshold.

In the example of the Guess the colour exploration discussed above, the formulation which does not specify the kind of admissible questions, allows for multiple approaches to the problem. Also there is no one set of questions that need to be asked to elicit the required information. There are multiple ways of doing this. Students are likely to find out one way or another to solve the problem, making it more accessible.

Similarly, that there are four ways of filling in the numbers in Magic triangle ensures that a student is able to find at least one of them and feel a sense of accomplishment. Many of the trajectories outlined in Section 4.2.3 are easily accessible and can be the entry points to the task. There are multiple ways of proving that there are only 4 solutions, from informal counting strategies to algebraic proofs.

4.4 Summary

To summarise, I suggested that flexibility and accessibility as key design principles in designing tasks that support students who are marginalised by/in mathematics to engage in mathematical thinking. Flexibility in tasks makes it possible for teachers to suitably tune the activity and tasks so that students sense an invitation to engage. Also students feel encouraged to bring in their *own* ways of thinking. Thus flexibility enables access. However flexibility must be accompanied by some degree of specificity as well, so that students are not at a loss as to which direction to take or what problem to address. In the various explorations in this chapter, this specificity is brought in through appropriate interventions and suggestions by the teacher.

The analysis offered in this chapter contributes to an understanding of how flexibility is enabled through various task features. An evident aspect of flexibility is “openness” of the tasks. But openness can be along different dimensions. Openness along the dimensions of goals and extensions that could be pursued emerge as important from this study. Openness of goals implies a task that does not specify a unique goal to be reached but offers a choice of goals that could be pursued. The dimension of extensibility implies that the provided starting points are generative of further questions and the task could potentially branch out to multiple trajectories. When students are unfamiliar with formal modes of expression and have a limited formal repertoire, affordances to function at multiple levels of formalisation becomes important. This is a more subtle aspect of the task that may not be evident or emerge through the teacher's own explorations. It is in the interactions with students and the ways they engage with the tasks that the different levels of formalisation at which students may engage with the task comes to light. I did not anticipate the ways in which students articulated their strategies in the Guess the colour exploration discussed in Section 4.2.2. They emerged in the course of the implementation of the task.

Besides flexibility, there are other task features that contribute to making the task accessible. These are especially important in working with students who are marginalised even through their experience of mathematics. An important way of making tasks accessible is by minimising dependence on prior content knowledge. This is achieved by choosing and framing tasks that allow for multiple solution approaches, at least some of which do not hinge on specialised content knowledge. Working with physical material like matchsticks, tokens, etc., used in some of the explorations gives students a “feel” for the task and enables a solution to be found at the concrete level. This may then be represented visually and eventually formalised. The move from concrete to abstract has its benefits at the secondary level too. Having a sufficient number of low threshold entry points is another feature that enables access.

While there is a need to make tasks at a low threshold, this should not be at the expense of making available opportunities for mathematical thinking and making it challenging to students, especially the ones who are eager to move on and take more challenges. Designing tasks that are accessible and approachable does not mean reducing the intellectual challenge. Challenging extensions and variations need to be built on a low-threshold starting point. The aim is to strike a balance between making tasks accessible and engaging for all learners, and at the same time providing access to rich mathematics. Incorporating multiple dimensions of generalisation and multiple trajectories leading on to multiple domains are some means of keeping the challenge level high without compromising on access.

5 Engagement with mathematical explorations at the margins

In the previous chapter I posited accessibility to mathematics in less structured ways than required by a textbook as a requisite for supporting mathematical thinking at the margins and identified design features that have the potential to make tasks flexible and accessible in these contexts. I now look at what engagement with mathematical explorations that incorporate these features entails at the margins. As noted in Chapter 2, while there is ample literature around students at the centre engaging in mathematical explorations, the same is not true of the margins. The general perception of explorations is that they challenge and hone the creative skills of the mathematically inclined, more so because they call for students coming up with their own approaches, procedures and discoveries rather than replicating previously taught procedures. For this reason, from a deficit view, their usefulness for students at the margins may not be acknowledged, and there is scant literature on such students engaging with explorations in the Indian context. The potential of explorations to enable mathematical thinking in such contexts has not been sufficiently studied. One of the aims of this study was to explore how students at the margins engage with mathematical explorations.

As discussed in Section 3.7, I draw on the implementation of two explorations - Magic triangle in Schools 1 and 2 and Matchstick geometry in School 1 - to describe the nature of mathematical thinking seen at the margins. The choice of these explorations is based on the availability of multiple data sources that allow for a thorough study of student engagement. Audio recordings of all the sessions and detailed notes in the teacher diary were available for these explorations. As noted in Section 3.5, Matchstick geometry was one of the explorations where I insisted on written work through a worksheet, making available a sample of students' writing as well. In the Magic triangle exploration, I observed students engaging with multiple variations and taking different trajectories and coming up with rich insights. In the Matchstick geometry exploration, I observed them working with and rediscovering familiar concepts in a slightly unfamiliar context. The deep engagement that I saw in these explorations also added to the illustrative power of the chosen instances. Burton (1984) identifies, as have other mathematicians and educators, the study of relationships and transformations as being central to mathematics. Therefore I have chosen student talk and writing around the theme transformations in the selected explorations to point to some features that I noticed in the way they communicated their mathematical thinking. The choice of explorations to be discussed in this chapter was guided by the above factors.

The data sources for this chapter are the teacher diary, the audio recordings of these sessions and the written work collected in these sessions. The audio recordings were listened to multiple times and annotated notes prepared. Moments that stood out saliently for me as the teacher, marked by student agency and/or mathematical thinking, were further discussed with the research team and selectively transcribed. Thus, the instances discussed below were selected for their power to illustrate aspects of the students' engagement and to show its feasibility, while I as the teacher revisited the data and reflected on my experience.

In this chapter I describe the nature of mathematical thinking that I observed as students at the margins engaged with explorations and the means that they adopted to communicate their thinking. I saw that while using means other than the formalised language of mathematics to express mathematical thinking was liberating, these also hampered progress in explorations in some ways. I document such instances. With a view to better understanding how one could work with and build on informal mathematics in marginalised educational contexts, I looked to the practice of research mathematicians. Drawing on this, I suggest an acceptability criteria for informal discourses in an educational setting, especially in the context of explorations.

I address the following questions:

1. What does engagement with mathematical explorations entail at the margins?
    a. What is the nature of mathematical thinking seen in these contexts?
    b. How do students communicate their mathematical thinking?
    c. How does language support or hinder mathematical communication?
    d. What counts as mathematical discourse in such contexts?

I begin the chapter with extended descriptions of student engagement with two of our explorations, Matchstick geometry and Magic triangle, with transcripts of student discussions where relevant. In Section 5.2, I use these descriptions to draw attention to the elements of mathematical thinking as identified by Burton (1984). In Section 5.3, I draw attention to the notable features of the ways of communication students adopted. In Section 5.4, I discuss how these support or hinder engagement with and progress in the exploration. In the light of these and drawing on the practice of research mathematicians, in Section 5.5, I identify central features of mathematical discourse in contexts of discovery and suggest an acceptability criteria for mathematical discourses that aligns with these features.

5.1 Two Explorations

In this section I give descriptions of student engagement with the two selected explorations - Magic triangle and Matchstick geometry. For each exploration, I give an overview of how the exploration evolved and how far it progressed and zoom in on the relevant parts to give a more detailed description. I largely draw on implementations in School 1 and a few instances from School 2 for the Magic triangle exploration.

5.1.1 Magic triangle

As discussed in Section 3.6 this is a puzzle that involves arranging distinct numbers 1-6 along the sides of a triangle as in Figure 5.1, such that numbers along each side sum to the same total.

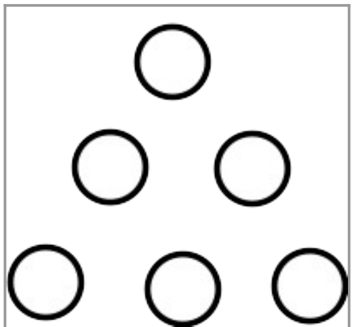


Figure 5.1: Magic triangle

I briefly describe how the implementation of the exploration progressed in School 1. This narrative also illustrates how the variations and extensions of the task described in Section 4.2.3 were realised in class. Within the first six minutes of posing the problem, students came up with three solutions with side-sums 9, 10 and 11 and some permutations of these solutions. They noticed that some solutions were similar in that the same combination of numbers appeared on the sides, and some students also noticed that the side-sum was the same in these cases (Transcript excerpts presented in Section 5.3). It was decided to count those solutions that have the same side-sum as the same solutions.

A student Krithi raised the question if they could get all numbers as side-sums. I picked up the question and also asked if they could come up with more distinct solutions. Some students argued that side sums of 8 or less are not possible. Two approaches that were seen are: a) They made a sum of 8 on a side with three numbers, say 5, 2 and 1 and argued that the remaining three numbers cannot be placed in the remaining circles so as to make a side sum of 8. b) The side on which 6 is present will have two other distinct numbers from 1 to 5 and so cannot have a sum of 8 . Some students also tried to argue that a side-sum of 12 is not possible as well, using an argument similar to a) above, but others found a solution with side-sum 12. There were also students trying to get 13 as side-sum and others trying to argue its impossibility (Details and analysis presented in Section 5.5.4). Based on their not finding any more solutions, some students claimed that there are only three distinct solutions. I asked for a justification. In the meanwhile, in an attempt to get a side-sum of 13, some students changed the numbers and used 7 instead of 1. This opened up the possibility of changing numbers and students chose different sets of numbers and went on finding solutions. Using numbers 2 - 7, they found solutions with side-sums 13, 14 and 15.

By now there were multiple groups working on multiple things - Some students trying to find more solutions with numbers 1-6, others working further on solutions already found and yet others varying the numbers to other sets of consecutive numbers and solving the puzzle with these numbers. A student, V2. [12], came up with what he called a “theorem” - namely that if one exchanges the vertex number with the middle number of the opposite side, one gets another solution (transformation of solutions as discussed in Section 4.2.3). This was termed “V2’s Theorem” and was used by him and by other students subsequently to find more solutions. They also experimented with other transformations on the solutions to see if they result in new solutions. In the process V2 came up with what he called “V2’s second Theorem”, which was that moving the numbers around by one position yields another solution. Working with the original set of numbers from 1-6, V2 used this exchange or transformation on the arrangement that had a side-sum of 9 to get another solution with side-sum 12, thus finding the 4th solution. He also found that applying the same transformation on the solution with side-sum 12 gives back the solution with side-sum 9. Using these transformations, experimenting with other sets of numbers and not finding more than four solutions strengthened their conviction that there would be four and only four solutions for the initial problem.

Noting that the side-sum of his initial arrangement was 9 which is currently established to be the minimum possible, and that he obtained a “maximum” 12, V2 claimed that there can be only four solutions, one each with side-sums 9, 10, 11 and 12. He later came up with a more convincing argument that this is indeed the case. (I analyse this proof in later sections - Sections 5.2.1, and 5.5.4)

Intending to get them to prove formally that there would be four and only four solutions with any set of consecutive numbers as they had been observing through multiple examples, I introduced algebraic notation n, $n + 1 \dots n + 5$ for the six consecutive numbers that they were using. This move did not work as intended - the students were not very comfortable with algebraic manipulation. Also I realised that it would be a better option to prove that there are only 4 solutions for a specific set of consecutive numbers, rather than for any set and presented and explained the proof outlined in Section 3.6. Students had difficulty in following the proof. Also they were not enthusiastic to go on any further with the exploration and hence it was decided to move on to another exploration.

The implementation of the exploration in School 2 was similar in that students came up with the transformation of shifting the numbers by one position to get another solution early on and actively

12. Pseudonym chosen by the student himself is being used for this student.

looked for other such transformations and patterns. A number of statements related to parity were made - some of them applicable only to the specific solution being considered and others more general. Some examples of such observations are: if odd numbers are at the corners, the side-sum would be 10, if there even numbers are at the corners the side-sum would be 11. Finding solutions with different sets of numbers kept students occupied for a good amount of time in School 1, while this was done for a comparatively shorter time in School 2. But patterns were observed and side-sums with other sets of numbers predicted without actually finding the solutions - For example, one student predicted that the sum of the maximum and minimum side-sums possible using numbers 3-8 would be 33 and that with 4-9 would be 39, with an increase of 6 for every increase of 1 in the starting number (A detailed analysis of this can be found in Section 5.2.1). Here also, I explained the proof outlined in Section 3.6 to them. A few students used this to find possible side-sums and solutions for a square arrangement of numbers 1-8 and arrangement of numbers 1-7 in the form a Z, but many of them found it hard to follow and the engagement levels waned.

Across the many iterations of this exploration with other groups of students across different contexts, we have noted some common "stages" - finding different solutions, proving there are only four, working with different sets of numbers, looking for patterns in solutions and transformations, extending and generalising to other figures. However the order and extent to which the different groups engage with each stage differs. For example, between School 1 and School 2, School 1 spent more time working on solving the puzzle with different sets of numbers and came to the conclusion that there had to be four solutions based on these attempts whereas pattern finding was a prominent activity in School 2. Elsewhere, where students had sufficient facility with algebra and found the algebraic proof that there are only 4 solutions accessible, more of the extensions and variations were explored. The exploration has the potential to engage students at different levels of mathematical proficiency and is what is called a Low Threshold, High Ceiling (LTHC) task.

5.1.2 Matchstick geometry

The first task in this exploration was to replicate some matchstick shapes. Students were given three kinds of sticks to use - matchsticks, toothpicks and a third variety of thin sticks cut into identical pieces. They were expected to replicate the shapes using one of these. This activity was used to anchor the question "when are two matchstick shapes the same?" Students considered conditions such as number of sticks used to make the shape, the number of sticks per side, the "size" and shape itself, the length and breath of the shape and angles. There were also instances where the replicated shapes did not retain the proportions of the original shape. Responding to the need to further discuss these points, a worksheet was designed bringing these to the fore (Figure 5.7 on page 143). This had two sets of shapes, and students were asked to spell out what they consider “same shapes” and compare each shape with the first shape in the corresponding set and write whether they considered these shapes the same as the first shape and why or why not. The points of discussion that I hoped would come up as students worked through this task were whether students would consider shapes transformed through the following ways same or different:

a) Scaled shapes, with scaling being done by:

i) using a different number of unit-lengths per side.

ii) keeping the number of unit-lengths identical, but using “units” of a different length.

b) Transformed shapes - with shapes being rotated and reflected along different axes.

Different students had different definitions for what they would call “same shapes” - some considered the total number of sticks, some considered the number of sticks per side and others considered the number of sticks per side and their lengths as well. This led to their formulating competing criteria for when they would consider two matchstick shapes the same and raised questions about why one should be preferable to another (I examine some student work and some turns of conversation from this discussion in Section 5.3).

The second task was to describe a shape such that a friend who has not seen the shape could replicate it without seeing it. This called for precise description and use of mathematical vocabulary such as parallel, perpendicular, horizontal, vertical, adjacent, midpoint etc. The third task was again on replicating shapes, but this time the shapes were drawn (as opposed to the first task where matchstick models were presented) and they included shapes which could be made with matchsticks and those that could not be made as can be seen in Figure 5.2. The number of matchsticks per side not being specified opened up the possibility for experimentation with students trying to make the shapes with different numbers of matchsticks per side. This led to discussions on the need to keep the proportions between the different edges/sides of the shapes constant. These discussions also fed into the worksheets mentioned above. Questions on the diagonals of squares and other shapes like rectangles and rhombuses were discussed at length – whether these could be made with matchsticks without gaps or overlaps and if yes for what dimensions. I examine the discussion and student work around the question of constructibility of a diagonal of a square in Section 5.2.2.

The final task of the exploration was about identifying shapes that could be described to another without resorting to any measurements. The last two tasks opened up possibilities to investigate “constructable” and “describable” shapes when the allowed “steps of construction” was restricted to laying matchsticks end to end, with no gaps or overlaps, without resorting to measurements. This is a further restricted version of the ruler and compass constructions in Euclidean geometry, where the use of compass is restricted too. The shapes that can be constructed are contingent on the restrictions in place.

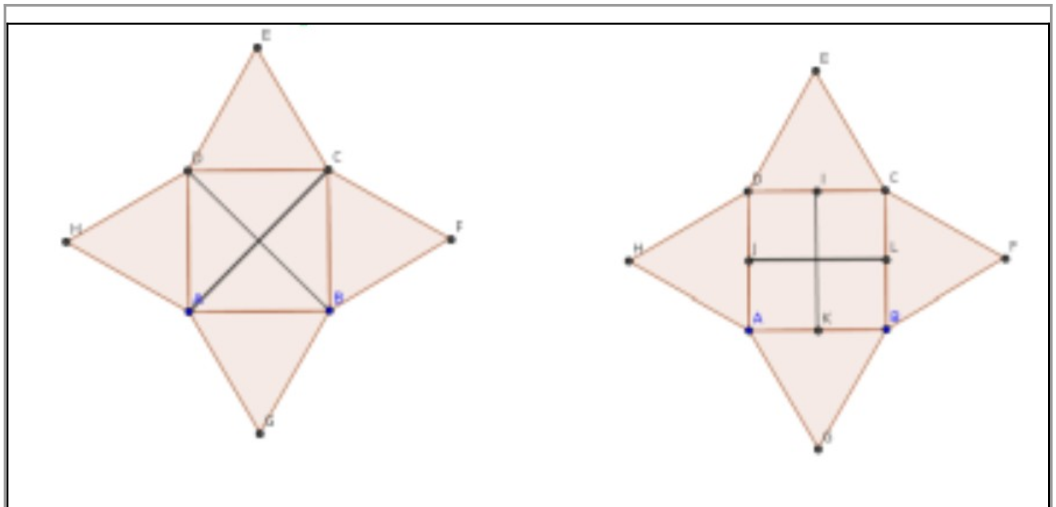


Figure 5.2: Matchstick geometry: Some shapes for replication

5.2 Nature of mathematical thinking seen at the margins

As discussed in Chapter 2, (see Section 2.6.2), Burton (1984) terms an idea, an observation, a happening or any event that can provide a stimulus to begin thinking, as an element on which mathematical thinking operates. Operations constitute the first of the three components of mathematical thinking and include enumeration; iteration; study of relationships such as ordering, correspondence, equivalence, inverse, converse, etc.; transformation by combination, substitution. Burton identifies the second component of mathematical thinking as consisting of four key processes - specialising and generalising, conjecturing and convincing. The third component in Burton’s framework is dynamics of mathematical thinking which consists of cyclic movements through the stages of manipulating an object, getting a sense of pattern and articulating the pattern and making the articulated pattern the subject of further manipulation. I now discuss these components of students’ mathematical thinking as they engaged in the explorations described above.

5.2.1 Students’ mathematical thinking: Magic triangle

I anchor my analysis of students’ mathematical thinking in this exploration around the process of making claims and conjectures. I distinguish a conjecture from a claim or a conjecture-in-action in terms of the attention paid to examining its validity or truth. A claim would become a conjecture if in the course of students' exploration, they either succeeded in refuting or proving it or at least attempted to do so, even if to a limited extent. If a claim is merely stated or is a conjecture implicit in the students' actions, it does not become a conjecture. I noted many instances of claims and conjectures in the Magic triangle exploration. These are as follows:

S - 1 The Magic triangle puzzle has only three solutions for a given set of consecutive numbers.

S - 2 Larger the numbers used in the triangle the fewer the number of solutions.

S -3 The side-sums corresponding to different solutions for a given set of numbers are consecutive numbers.

S - 4 The sum of the maximum and minimum side-sums obtainable using numbers 3-8 is 33.

S - 5 The Magic triangle puzzle has four and only four solutions.

S - 6 Nine is the minimum side-sum possible with numbers 1-6, and 12 is the maximum side-sum possible

Of these S - 1 and S - 2 were unsubstantiated claims, S - 3 was an unarticulated conjecture implicit in action, S - 4 was a conjecture that was made building on a few other conjectures but not proved and S - 5 and S - 6 were eventually proved as can be seen from the following discussion.

In School 1, students found solutions to the puzzle with side-sums 9, 10 and 11 within the first 5 minutes of posing the problem, but the fourth solution with side-sum 12 was found after a gap of about 25 minutes. The difficulty experienced in coming up with the 4th solution may have been the basis for statement S - 1 above. Students tried to solve the puzzle with different sets of consecutive numbers and in both the schools, they found it harder to equalise the side-sums when using larger numbers like 10-15. Based on this experienced difficulty, they thought it plausible that there would be fewer solutions when larger numbers are used. Similar to S - 1, S - 2 is also based on the experienced difficulty in solving the puzzle. Neither of these statements were pursued further and remained unsubstantiated claims. These were in effect refuted when S - 5 was proved later, but an explicit connection was not made.

S - 3 was not articulated but seen in action and can be termed a conjecture-in-action. Students noted that the side-sums in the four solutions found for the initial version of the puzzle with numbers 1-6 were consecutive numbers 9, 10, 11 and 12. Perhaps they expected this to be true of other sets of numbers. I noted that having obtained a few side-sums, they were trying to bring about contiguous numbers as side-sums. For example, with numbers 2-7, having found solutions with side-sums 13, 14 and 15, 12 and 16 were the side-sums they tried for. It can be considered a pattern noticed but not articulated. There were other patterns that were noticed, articulated and built-on as well.

S - 4 above is one such. Velan, a student of School 2, conjectured that the sum of maximum and minimum side-sums obtainable with numbers 3-8 is 33. This conjecture itself is the result of cycles of manipulating solutions, getting a sense of pattern, articulating the pattern and further manipulating these patterns - what Burton calls the dynamics of mathematical thinking. The starting point for this conjecture was manipulating the obtained solutions to the puzzle. Moving the numbers around and observing the resulting arrangements, students noticed two kinds of transformations - i) those that result in the same solution (or the side-sums being preserved) and ii) those that lead to a different solution. (see Section 4.2.3 for a more detailed discussion of transformations of solutions). To recall, the transformations that lead to other solutions are- a) interchanging the number at the vertex with the one at the midpoint of the opposite side (Figure 5.3 (a)) b) moving the numbers around cyclically by one position (Figure 5.3 (b)). Both these transformations have the effect of moving the numbers at the three vertices or corners of the triangle to points on the midpoints of the side and vice-versa.

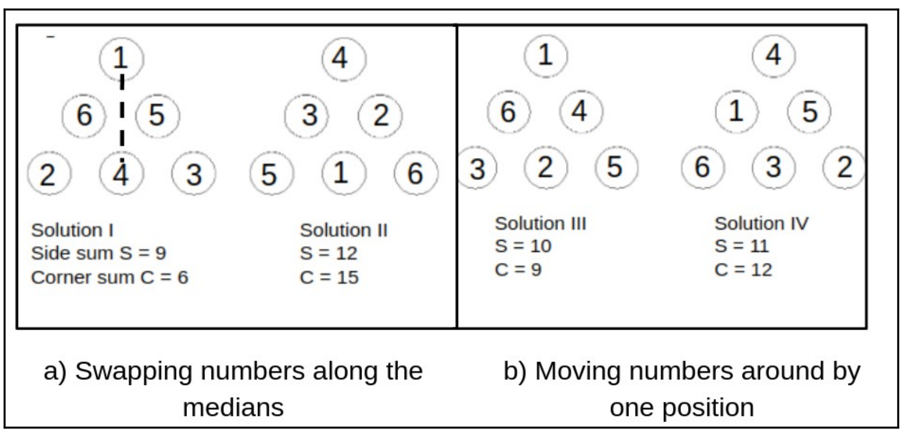


Figure 5.3: Magic triangle: Transformations that give a different solution

Applying either of these transformations on a solution with side-sum 9 gives the solution with side-sum 12. Transforming the solution with side-sum 12 in a similar fashion gives back the solution with side-sum 9. Thus the solutions with side-sums 9 and 12 are “paired” in that transforming one results in the other. These two happen to be the maximum and minimum side-sums possible with numbers 1-6. Solutions with side-sums with 10 and 11 are similarly paired. (Figure 5.3 (b)). The sum of these paired side-sums is a constant, 21 (9 + 12 = 10 + 11 = 21).

Velan noticed this “pairing” of solutions, thereby establishing a correspondence (a mathematical operation in Burton’s framework) between solutions that can be obtained from one another through well-defined transformations. He also noted that the solutions with maximum and minimum side-sums are paired in this fashion. These pairs of solutions then became the object of his attention and he identified a property that they share, namely that the sum of their side-sums is a constant. Having got a sense of this pattern, he verified the pattern with another set of numbers, 2-7. He saw that the pattern holds in this case as well, with the side-sums of paired solutions summing to 27. Manipulating and building on the pattern that "sum of side-sums of paired solutions remains a constant for a given set of numbers" he also conjectured a rule as to how this sum varies with the set of numbers being considered. Thus S-4 above is a complex conjecture that is based on the following observations and intermediate conjectures.

1) Starting from a solution of the Magic triangle puzzle, interchanging the numbers at the vertex with those at the midpoint of the opposite side, leads to another solution.

2) The solutions of the Magic triangle puzzle with a given set of consecutive numbers are "paired" in a certain way - transforming a solution as in 1) above leads to the other solution of the pair. The side-sums of these solutions form a corresponding pair.

3) The maximum and minimum side-sums obtainable are so paired.

4) The sum of paired side-sums is a constant for a given set of consecutive numbers.

5) The constant sum in iii) above increases by 6 for contiguous sets of 6 consecutive numbers

Velan's conjecture that "the sum of the maximum and minimum side-sums obtainable using numbers 3-8 is 33" follows from 4) above and the observation that the sum of the paired side-sums with numbers 1- 6 is 21. The students did not attempt to prove[13] this, but arriving at it itself involves mathematical operations, processes and cycles of manipulation, getting a sense of pattern, articulating pattern and further manipulating articulated pattern. We now look at S - 5 and S - 6 above, which were proved. The proof that the students came up with is described in the following paragraphs.

V2, of School 1 suggested that the maximum side-sum is obtained by placing the three larger numbers

13. The proof is as follows:
We noted that the transformations in Figure 5.3 have the effect of moving the numbers at the three vertices or corners of the triangle to points on the midpoints of the side and vice-versa.
Therefore an arrangement with C as corner-sum gets transformed into one with (21 - C) as corner-sum. We also saw in Section 3.6 that the side-sum S and corner-sum C are related by the equation
$3S = C + 21$ (A)
If S' and C' are the side-sum and corner-sum after a transformation, $C' = 21 - C$ and we also have
$3S' = 21 - C + 21$ (B)
Adding equations (A) and (B), we see that $S + S' = 21$. That is the sum of "paired side-sums" is 21, which is also the sum of the 6 consecutive numbers being used. One can explain the "increase of 6" that Velan observed in this sum, based on this observation.

from among those given at the vertices of the triangle and placing the remaining numbers in such a way that the side- sums are balanced. In Figure 5.4, the side with 5 and 6 has the maximum partial side-sum and so the smallest of the remaining three numbers needs to go there. Similarly the with 5 and 4 has the smallest partial side-sum and so the largest of the remaining numbers goes there. By following this algorithm we find the maximum side-sum to be 12. By a similar argument, placing the three smaller among the given set of six numbers at the vertices, and balancing the side-sums, we get the arrangement with the minimum side-sum, 9 here. V2 came to this argument by considering multiple examples (or by the process *specialising* in Burton's framework) and then *generalising* (also a process) the pattern across cases.

Another student Krithi in an attempt to convince herself of this noted that the numbers at the corners get counted twice when calculating the side-sum and hence if the larger numbers are placed at the vertices one gets a larger side-sum than otherwise. Thus V2 came up with an inductive argument which is complemented by an explanation from Krithi. Having obtained the maximum and minimum possible side-sums (9 and 12) and solutions with the side-sums in between (10 and 11), V2 proved that 9, 10, 11 and 12 are the only possible side-sums and hence there are only four solutions. Here we see an instance of the process that Burton terms "convincing".

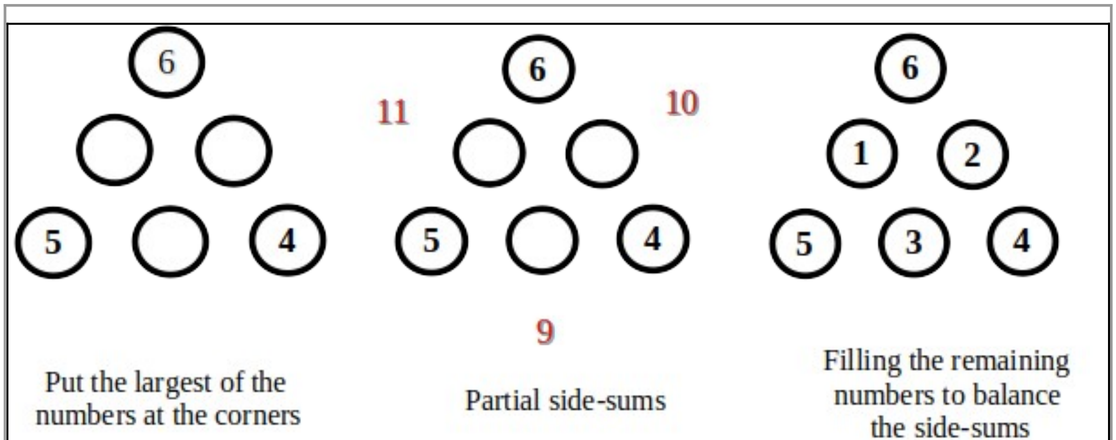


Figure 5.4: Magic triangle: Student strategy to get maximum side-sum

5.2.2 Students' mathematical thinking: Matchstick geometry

I now look at the discussion and student work as they argue that a diagonal cannot be placed inside a unit square without gaps or overlaps, then consider a larger square and then generalise to any square. This is from the Matchstick geometry exploration and was the fourth of the six sessions of this module.

Trying to accommodate two sticks along the diagonal, they said

1. *"Ithu edamme paththamatenguthu"* (There is not enough space at all).

With one stick they said

2. *"ithu romba chinnatha irukke, intha end ukku varamatteguthu"* (this is too small, it doesn't come to this end) which I revoiced as "corner to corner varallaye" (It is not coming from corner to corner).

One student suggested that they make a bigger square (with more than one matchstick per side). I encouraged them to go ahead and explore this. In the meanwhile another student said that it is not possible to fit the diagonal whatever the size of the square, and I asked to be convinced.

In the following turns of conversation, Maariya, Priya, Aashika and Megha are students, J is the teacher, and T is their regular teacher who was observing. Translation into English is given in parentheses. Turns that were judged to be not relevant and have been omitted are indicated by "…".

3. Maariya: *diagonal vanthu eppovme.*.(The diagonal of the square is always…)

4. Priya: *Square-oda diagonal is not equal to the side of length, squarennudayathu* (The diagonal of the square is not equal to the side-length)

5. Maariya: *ithodu lengthum athodu lengthum equal aa ve irrukkathu* (The length of this and this will not at all be equal.)

6. Maariya and Priya: *Diagonal eppovme vanthu,* diagonal is greater than the side of the square (Diagonal is always, diagonal is greater than the side of the square,)

7. J: How much greater?

8. Maariya: *Eh? How much aa?* (What? How much?)

…

9. T: you tell me how much bigger, Maariya and Priya?

10. Maariya:Double

11. J: *Double aa? Appo rendu kuchci vacha varannume?* (Double? then two sticks should fit)

12. Maariya: half double.

13. J: *half double na?* (Half double means?)

14. Maariya: One and a half

…

15. Priya: Miss, Pythagoras Theorem. This square plus this square is equal to this square

16. Aashika: *Hey, yaar sollithantha?* (Hey, Who taught you?)

17. Priya: *nangalle kandupidichom* (We found out ourselves.)

18. J: Ok, So you are telling me that you cannot make this?

19. Priya: *huh huh* (No No)

20. Megha: *Pythagorus theorem thane?* (It is Pythagorus theorem isn't it?)

21. T: *Appadiya?* (Is it so?)

22. Priya: This square plus this square is equal to this length

23. T!: *Enn?* Why?

24. Priya: Because it is a right angle. Yes, *Squarekku right angle sir*. (A square has a right angle)

(Someone claps)

In the meanwhile, the group who was making a larger square fitted a three-unit diagonal to a 2-unit sided square. I drew attention to this and asked the group Maariya, Priya, Aashika and Megha if they wanted to reconsider their stand that a matchstick diagonal cannot be fitted to any square. Priya and others pointed out to Maariya that she was wrong. The diagonal could be longer than the sides and yet it may be possible to fit in a matchstick diagonal. There was confusion - Some of them sensed something was wrong, but did not connect it to Pythagoras theorem that was mentioned earlier and tried to figure out what was wrong. One student felt that the 2-unit square with the 3-unit diagonal may be flawed ("not perfect") in some way and wanted to measure the lengths. I suggested that the lengths are obvious - the matchsticks along any side could be counted. A boy pointed out the possibility of the matchsticks being slightly different in lengths and wanted to check the exactness of lengths through a pencil and paper construction. In the meanwhile a group of girls writing on the floor as shown in Figure 5.5 drew on Pythagoras Theorem again to justify that the square was indeed "flawed" - Pythagoras theorem gives the value of 4 + 4 = 8 for the square of diagonal, whereas the matchstick square on the floor has the value 9.

Figure 5.5: Matchstick geometry: "4 + 4 is not equal to 9"

In order to see if there could be other squares with integer sides and integer diagonal lengths, they looked for perfect squares which when added to themselves would give another perfect square. They tried out specific examples and evaluated the square roots by long division method (see Figure 5.5, Part C). At this point I intervened to suggest using factorisation and writing the square root as a surd in $a\sqrt{2}$ form instead of evaluating it as a decimal. They had been introduced to the surd form as part of their curriculum and could do this. Based on a student's statement that $\sqrt{2}$ is a never-ending decimal, I pointed out that the diagonal is a non-integer multiple of the side and hence not constructible using matchsticks.

The processes of specialising, generalising, conjecturing and convincing are evident in this description as well. Students first consider the case of a unit square (specialisation), manipulate matchsticks and observe that it is not possible to fit in a matchstick diagonal without gaps or overlaps. They articulate the observed pattern for a special case, namely the unit square. The observation that it is not possible is explained by the statement that the diagonal of a square is always greater than its side. Here we see students trying to prove a statement (matchstick diagonal cannot be accommodated) using already known facts (diagonal is longer than the side) - a distinctive feature of mathematical proofs. While a few steps are missing in the proof, it is clearly an attempt to convince. I sought to fill in the missing steps with the prompt - “how much greater?” . The prompt is heard but not pursued; instead the students draw on the Pythagoras theorem to justify their claim. They then build on this observed pattern and conjecture that it is impossible to fit a matchstick diagonal in any square. However, a group of students manage to make a 2-unit square with a 3-unit diagonal with matchsticks. This was contrary to expectations and led to what Burton terms a "surprise fueled attack" (that is, exploration motivated by a surprising finding; see Section 2.6.2 and (Burton, 1984, p. 43)). They again draw on Pythagoras theorem to convince themselves and others that the 3-unit diagonal that seems to fit inside a 2-unit square had a flaw. The 2-unit side a square is another special case that was considered to exemplify the impossibility of matchstick diagonals. They eventually generalise the conjecture to all squares and attempt an inductive proof. They note that twice the squares of 3, 4, 5… are not themselves perfect squares and therefore matchstick diagonals are not possible in squares of these sides. They conclude that the conjecture is true based on the fact that they did not find a counterexample. I intervene to present and explain a deductive proof.

5.2.3 Students' mathematical thinking: What was seen? What was missing?

The nature of thinking seen in the instances discussed above aligns with Burton's description of mathematical thinking and I pointed to evidence of mathematical operations, mathematical processes and cycles of manipulating ideas/objects, getting a sense of and articulating patterns and building further on them as students engaged in explorations. In addition to the two examples of such cycles highlighted in the previous two sections, V2 coming up with his "theorems"(see Section 5.1.1) and these theorems being used by V2 and others to find more solutions is another example of an articulated pattern being built-on and manipulated (Mason, 1989). V2's initial work was the search for a pattern/transformation through the process of manipulating solutions. Once the pattern/transformation is found and articulated, it becomes an entity in its own right, usable by others to find more solutions. We have observed other instances, both in this exploration and in other explorations where students demonstrate engagement with the elements of mathematical thinking as described by Burton.

Also notable is the way V2 terms it a "theorem" - though technically it is a conjecture - perhaps based on the familiar textbook terminology. V2 attaching his own name to the "theorem" and the class accepting and referring to the transformation as "V2's theorem" is indicative of ownership and a sense of pride in their own achievements. Similar feelings are reflected in "we found out ourselves" in turn 17 of the conversation shared above and positive emotions are visible when others acknowledge the effort by clapping. An important fallout of the openness of explorations is that they provide opportunities for students to "make and own" their own mathematics.

While the instances described above show "little mathematicians at work" (Ramanujam, 2010), I also draw attention to two instances where we thought that they went one step too short. Formalisation is one aspect that we saw very little of in these examples. That apart I point to a couple of instances which were puzzling for us - raising the question “having gone thus far, why was the next step a struggle for them?”

a) Maran in School 1 correctly argued that the side on which 6 appears needs to have two more distinct numbers, the minimum possibilities being 1 and 2. So the minimum side sum has to be 9. I encouraged them to use a similar argument to prove that 12 is the maximum possible side-sum. I suggested that they start from the side on which 1 appears. The side that has 1, needs to have two more distinct numbers, the maximum possibilities being 5 and 6, and therefore the maximum side-sum possible has to be 12. However neither Maran nor others in the group who heard Maran’s argument could adapt the same reasoning to explain the impossibility of 13 as a side-sum.

b) While justifying the impossibility of a matchstick diagonal for a unit square, though students in School 1 started with the reasoning that this is because the diagonal has to be longer than the sides, they soon saw that as an application of Pythagoras theorem. Having made the connection, they were still confused by the two-unit square with the three-unit diagonal fitted in. After some time, one student came to the conclusion that Pythagoras Theorem is being violated in this arrangement as well.

Perhaps this may be because the “shift from articulating to manipulating” which Mason calls abstracting has not happened here? In Maran’s case, what he offers is a reasoning to explain an observation. The shift from “a reason to explain an observation” to “a *reasoning* to explain some observations”(abstracting a pattern of reasoning), akin to the shift from “*a sequence with a property* to a *property satisfied by what sequences*” that Mason (1989) points to has perhaps not happened. The same can be said about the use of Pythagoras Theorem by the students working on Matchstick geometry. Their reference to Pythagoras theorem was with respect to the triangle at hand - “this square plus this square is equal to this square.” That they did not refer to the theorem in terms such as “sum of squares on the perpendicular sides of a right triangle is equal to square on the hypotenuse”, independent of the triangle at hand and that applying it to a different example was not an instinctive response, may perhaps be because they have not “abstracted” Pythagoras theorem to be able to manipulate it further.

In this section, we saw examples of students making their own mathematics, or “re-making” previously met and perhaps forgotten mathematics. In the following section, we look at how they communicate their mathematical thinking. To this end I share some transcripts of instances from the classroom already referred to in the previous section and identify features that mark these discourses.

5.3 How do students communicate their mathematical thinking?

The key observation here is that *talk* was the primary means that these students adopted to communicate their thinking. As pointed out in Section 3.5 , in School 1 there was a preference to write on impermanent surfaces like the blackboard, classroom floor or the desk and erase what was written after the task was completed. Though notebooks were provided so as to make available records of student writing for analysis, they were hesitant to write in these, and very little writing was collected. The writing that did happen was done more as an aid to thinking through and working out, rather than for presenting their work to a wider audience. The impression that I got on going through written work was that it was scanty compared to the rich conversations that happened in class and that the writing that they produced was not an indicator of the mathematical thinking they were capable of. As an example of the writing seen, I discuss the worksheet on Matchstick geometry (described in Section 5.1.2) here. Of the 15 students who completed the worksheet, 8 had answers with reasons stated in some detail. The rest either did not state reasons or wrote only 2-3 sentences in all. In this section, drawing on student talk and writing on the theme of transformations, I highlight some features I noticed in the ways students communicated mathematics. The characteristics of mathematical discourse identified by Sfard (2008) and Moschkovich (2015a) discussed in Section 2.3.2 provide points of reference for the analysis. I briefly recall these characteristics.

Sfard (2008) identifies the distinctive observable features of mathematical discourse – namely (a) word-use (b) visual mediators (c) routines and (d) endorsed narratives. Word-use as described by Sfard is not limited to technical vocabulary but implies objectified word use. This impersonal and reified use of words is indicative of awareness of the discursive objects signified by them. Visual mediators are the visible means that support communication. While informal mathematical discourses are mediated by real-life objects either seen or imagined, academic mathematical discourses are mediated by symbolic artefacts like graphs or algebraic notation created specifically for the purpose. Routines are recurrent forms of communication actions and the meta-rules governing them. Academic mathematical discourse is marked by routines that aim at knowing objects, i.e. producing endorsable mathematical facts about them, whereas informal discourses often have routines resulting in practical action. Endorsed narratives are sets of propositions produced using the given language, mediators and routines and are endorsed as potentially useful or true by the given community. The meta-discursive rules that govern the endorsement of narratives or even the response of discursants are specific to discourses. In mathematical discourse only those narratives are endorsed that can be logically deduced from narratives already endorsed. Though concrete or iconic mediators facilitate production of factual narratives through appropriate routines, symbolic realisations are necessary to warrant these narratives' general endorsement in academic mathematical discourse.

Moschkovich (2015a) defines mathematical discourse as "communicative competence necessary and sufficient for competent participation in mathematical practices". Mathematical discourse is embedded in mathematical practices and draws on hybrid resources – oral and written text, multiple modes, representations (gestures, objects, drawings, tables, graphs, symbols, etc.), and registers (school mathematical language, home languages and the everyday register). The ALM framework admits of the vernacular even when engaging in academic literacy practices and draws on a multimodal repertoire. Meanings are situated and develop through participation in mathematical practices. It is also marked by precision, brevity, logical coherence, particular modes of argument and tends to value abstraction, generalisation and search for certainty.

I now analyse how students communicated their mathematical thinking on the theme of transformations, keeping the features identified by Sfard and Moschkovich as a background.

5.3.1 Transforming solutions of the Magic triangle

In the following turns of conversation students are explaining what they mean by “same solution” in the context of the Magic triangle exploration. This conversation happened pointing to this part of the board, where students had recorded their solutions and labelled them with their name.

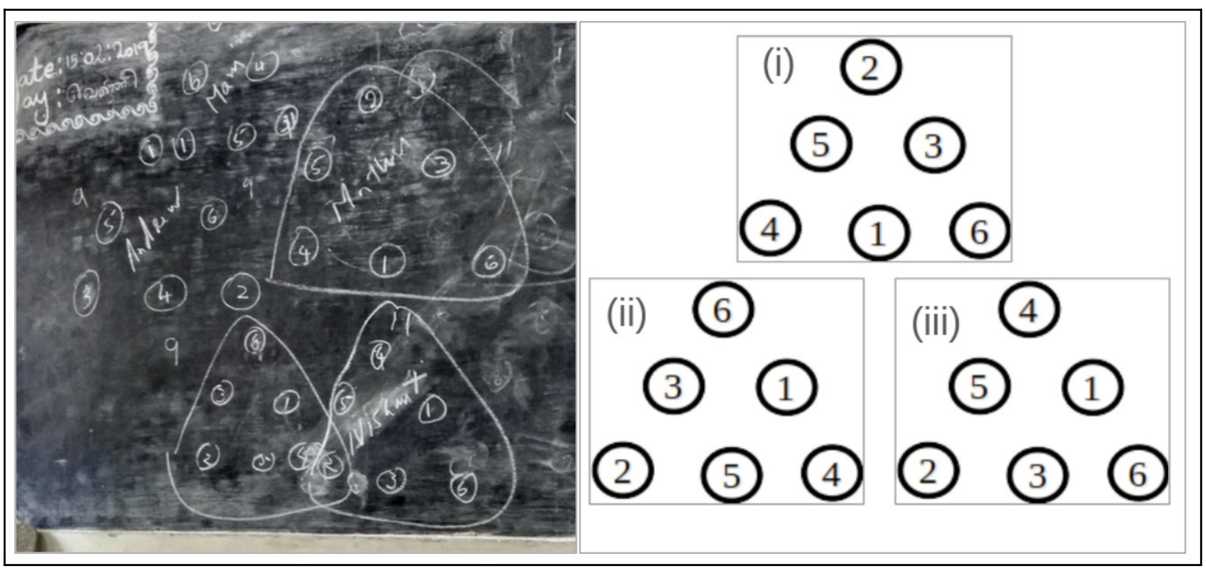


Figure 5.6: Magic triangle: Same solutions

I recreate the solutions circled on the board for clarity in Figure 5.6.

Through this discussion, students point to two properties of solutions which they would call the same. The first is that the side-sums in all these would be the same though the positions of the numbers may change (turn 4 and 22). The second, which may also be considered an elaboration of “*numbers maathi maathi varum*” (numbers will change) in turn 4, is that the three numbers that appear on any given side stay together, but they may appear on any other side of the triangle, in any other order as well. For example the numbers 6, 1, 4 appear along the base of the triangle in part (i) of Figure 5.6 (b) , along one side of the triangle in (ii) and (iii), and are differently positioned, But all three numbers are on the same side of the triangle, whichever side it may be (turn 11). This ensures that the side-sum remains the same. Adding 6. 3 and 2 will give 11, whichever direction (on whichever side) they are written. (turn 22).

Transcription conventions followed: … indicates irrelevant turns omitted. [] indicates indistinct words. Translation with sentences completed for understanding where required are shown in brackets. Smaller font is used for explanatory commentary where required.

1. J: Nitin is saying that these three things (referring to the circled solutions on the blackboard) which I have circled here are the same solutions - *En Nitin?* Explain (Why Nitin? Explain)

2. Nitin: *Ore number irundha ellam ore mathiri dhan irrukkum* (If there is the same number, all will be alike)

3. J: Ore number irundhanna? enga ore number iruntha? (What does it mean, 'if there is the same number'? Where should there be the same number?)

4. Nitin: *Ore..11 aa iruntha .. Ellathukkume 11 aa iruntha, intha munnume onna than varum. Numbers maathi maathi varum* (The same .. if it is 11.. if it is 11 for all, these three will have to be the same. Numbers will change.) The 11 being referred to here is the side-sum in the three solutions marked

5. J: *Puriyalle ennakku* (I don't understand)

6. Sneha: *Miss naan sollaren miss.* (Let me say, miss)

7. Abhi: *place vere vere maathi pottirukkanga*. (They have changed the places and put in different places)

8. Raju: Pointing to appropriate places on the board and referring to the change in position of numbers 2, 5 and 4 in parts (ii) and (iii) in Figure 5.6 (b) above. *inga irukkara intha number [..] ithu ippadi irrukkutha? intha 2 a tukki inga pottirukkanga miss, inga 5 pottirunkkanga, aana inga 4 pottirukkanga [..]* (The number that is here.. This is like this? They have picked up this 2 and put it here miss, they have put 5 here, but 4 here)

9. J: enna, enna, enna? (What, what, what?)

10. Maran: *ithu straightaa potirukkanga miss, ithu tiruppi pottirukkanga* (They have put this straight miss, they have turned this) Reference to the sides with numbers 2, 5, 4, being horizontal in (ii) and "turned" in (iii)

[]

11. Nitin: ithu *6,1,4 inga irrukku, ithu 6,1,4 inga irrukku, ithu 4,5,2 inga irrukku, ithu 4,5,2 inga irrukku miss, 2,5,4 inga irrukku miss* Pointing to the solutions on the board - In this 6,1,4 is here, in this 6,1,4 is here. In this 4,5,2 is here, in this 4,5,2 is here, 2, 5, 4 is here)

12. J: *Ok.. so sum 11 varuthu* (Ok, so the sum is 11)

…

12. J: *Athinala intha muunumme patha vithyasama irunthalum munnum…* (So even though they look different, these three…)

13. Nitin: *onnu than* (... are the same)

14. J: *ore solution*. (same solution)

15. *J: So oru solutionlenthu innuru solution eppadi kondu varathu?* (So how do we bring one solution from another?)

…

16. Maran: *Number mattum maathuvom miss, vere number poda kodathu. Antha numberkku badil* (We will only change the number. We should not put a different number instead of that number)

17. J: *Entha numbera maathuvenga? Eppidi maathuvenga?* (Which number will you change? How will you change?

18. Maran: *Edatha eppidi venunna maathikkalam.* (We can change the position in any way)

19. Nitin: *Ultava podalam, ippadi podalam, eppidei venna maathikkalam.* (We could put it inversely, we could put it like this, we could put it in any way.)

20. Maran: *Indha linela intha number irunthunna, itha inga pottu, Itha tuuki ippadi pottu, eppadi pottalum 11 than miss varum.* (If this number is on this line, if we put this here, and this here, it will be 11 whichever way you put)

21. J:*eppidi pottalum?* (Whichever way you put?)

…

22. Maran: *Ippovanthu, 6 um three um 9 miss, athila 2 add pannina 11 miss, Athe entha directionla*

pottalum antha 11 than miss varum. (Now 6 and 3 is 9, If you add 2 to that it is 11. It will be 11 in whichever direction you put these numbers)

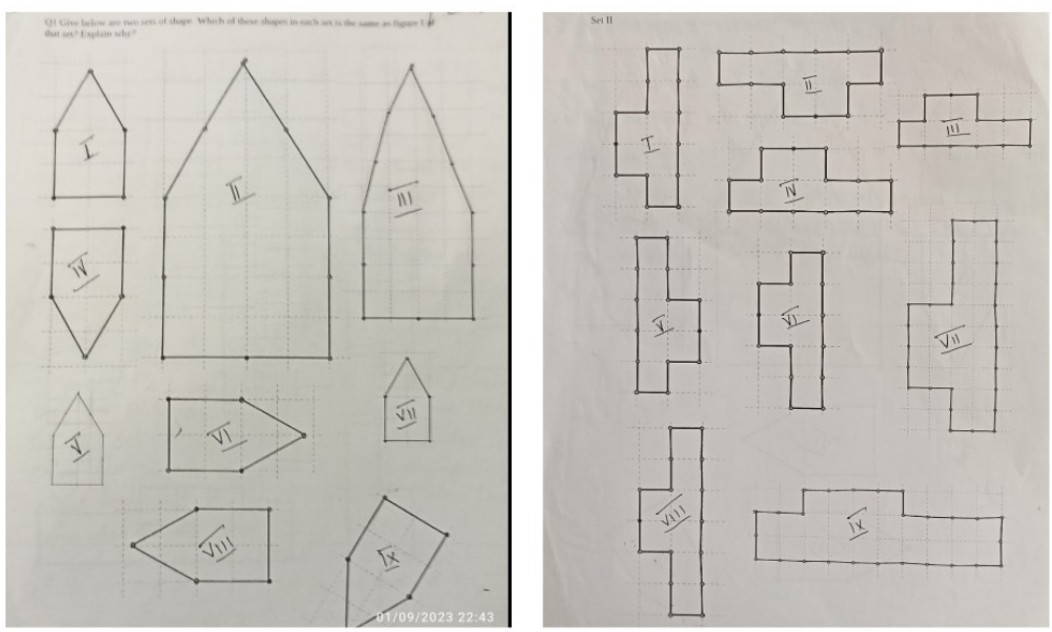
The two characteristics shared by same solutions and referred to in these turns by the students, namely side-sum being the same and triplets of numbers staying together on whichever side they appear, is a complete characterisation of two solutions being the same, but the way it is articulated is based on the image on the board and as it applies to this specific example. The objects of discussion here are some permutations of the numbers 1-6 around a triangle. The talk is visually mediated by a pictorial representation of the situation as opposed to a symbolic or algebraic notation for the permutation. My statement that "I don't understand", intending to push them to clarify their articulation led to more pointing to the board. Students do not talk of the triangle or the arrangement of numbers being rotated or reflected, but as some unknown agent picking up the numbers on one side and putting them on another side. Even when they do use the word *turn (tiruppal),* it is the agent who is turning a line of numbers which they had written straight earlier (turn 10). Word use here is not objectified or impersonal. Rotation here is expressed through the act of moving the numbers around. The talk is about the imagined/practical act of moving the numbers in groups of 3, in such a way as to maintain side-sum, and not about the "fact" that rotation leaves the side-sum invariant, as in Sfard's (2008) characterisation of mathematical discourse.

While students are aware of rotation as the transformation that keeps the sum invariant, not having the abstract language to talk about it, they communicate it in an operational sense, by moving the numbers around. I suggest that the purpose of the routines leading to an action on numbers (imagined entities) is to express a fact or truth that side-sums are preserved in the process. Also they have not yet assigned a name/label to the concept "side-sum" here and refer to it by the specific value, 11 in this case (turn 4), but they do see side-sum as something that remains invariant through these transformations. Another example of non-standard word use seen is the use of "*ulta*" (which means reverse) to refer to a reflected shape.

Many of the utterances are vague and part sentences. For example by "*numbers maathi maathi varum*", Nitin is referring to the change in position of the numbers in the different arrangements that give the same side sum. Abhi tries to clarify with the statement "*place vere vere maathi pottirukkanga*" meaning that the places of numbers have been changed. Even with the clarification, it is difficult to fathom what they mean. Turn 16 seems self contradictory "We will change only the numbers. We should not put other numbers instead of these". The first instance perhaps means "position". So "number" has been used to mean position in both turns 4 and 16. The generous use of deictics makes it impossible to make sense of the conversation without referring to the diagram they are pointing to. My attempt to get them to articulate the transformation verbally, through the question “what would you do to get one solution from another?” also gets a vague answer that one could realign the number in any way. Implicit in the response however is the need to preserve the side-sum. The conversation lacks the characteristics of precision and brevity that Moschkovich (2015a) identifies in mathematical discourse. Even with the kind of word use and expression seen here, evidently the students are making sense of the underlying ideas.

5.3.2 Transforming shapes in Matchstick geometry

We now look at how a group of students of School 1 discuss transformations of geometric shapes in the Matchstick geometry exploration. This was in the context of the worksheet created for the purpose, as described in Section 5.1.2. The worksheet had two sets of figures Set I and Set II as shown in parts (a) and (b) of Figure 5.7. that included rotations, reflections and scaled-up and scaled-down versions of the first shape and 1-2 figures that were different. Students were asked to write when they would consider two shapes the same, and compare each shape with the first one and write whether it is the same as the first one or not as per their definition, with reasons. Students’ concern here was how Shape I of each Set is related to the other shapes in the Set and what transformation takes Shape I to the others in the Set.



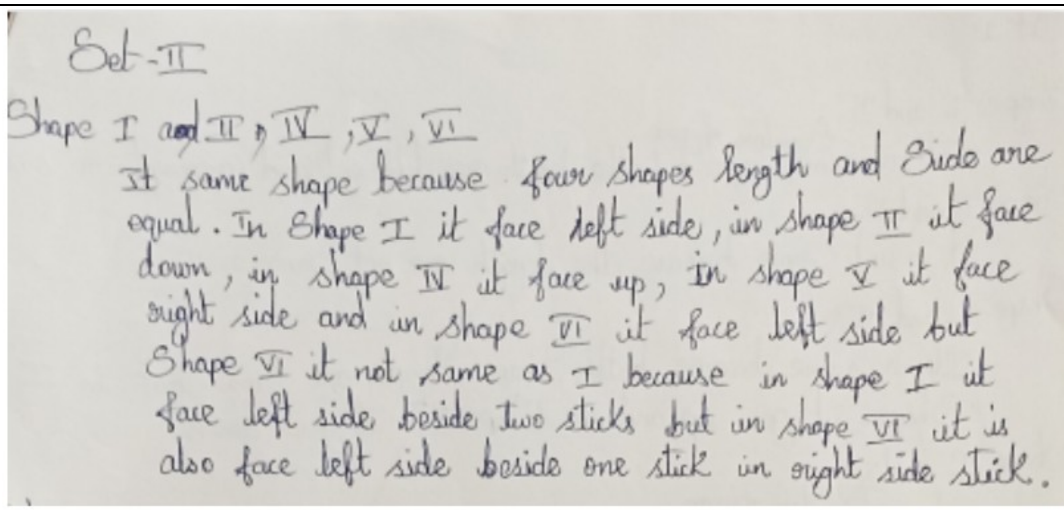
a) Set I b) Set II

Figure 5.7: Matchstick geometry: Worksheets

While they considered rotated or reflected shapes to be the “same shape”, they did not use the language of rotations or reflections to refer to these transformations and justify their stand. They distinguished between the shape and size of a shape and referred to scaled- up and scaled- down figures as being of the “same shape but different size”.

Talking of rotations and reflections: We now look at how some students compared Shape I of Set II (Figure 5.7(b)) with Shapes II, IV, V and VI. Shape I is rotated by 90° in anti-clockwise and clockwise directions to get shaped II and IV respectively, and reflected about the vertical and horizontal edges of the longer rectangle (that is part of the shape) to get Shapes V and VI respectively.

a)


Set-II
Shape I ~~and~~ II, IV, V, VI
It same shape because four shapes length and side are equal. In Shape I it face left side, in shape II it face down, in shape IV it face up, in shape V it face right side and in shape VI it face left side but Shape VI it not same as I because in shape I it face left side beside two sticks but in shape VI it is also face left side beside one stick in right side stick.

b)

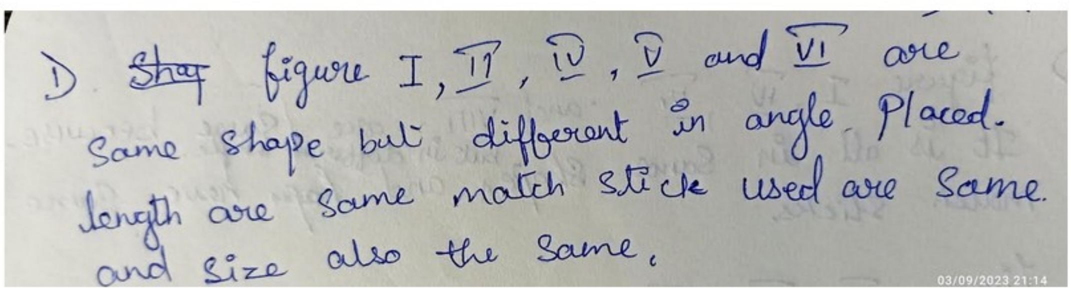

1) ~~Shap~~ figure I, II, IV, V and VI are Same shape but different in angle Placed. length are Same match stick used are Same and Size also the Same.

Figure 5.8: Matchstick geometry: Samples of written work - 1

The instruction to them was to compare “each shape” with Shape I of the corresponding sets and many of them grouped some figures together as can be seen in the write-ups in Figure 5.8. This indicates that they identified a common property across these shapes. In the first write-up in Figure 5.8, the student remarked on the change in orientation of the shape (as compared to Shape I) by referring to the side of the longer rectangle where the smaller one is attached as the “direction it faces” (Shape I to the left, Shape II down etc), rather than view these shapes as a rotation or reflection of shape I. She marks a difference between Shapes I and VI - though the smaller rectangle is placed to the left of the longer one in both, the positioning is different. She has not seen it as a reflection about one of the horizontal edges of the longer rectangle. Another student just marks that the shapes are placed at a different angle (the second write up in Figure 5.8). Other phrases that were used to articulate the same idea are “turned to that side”, “facing different direction”, “opposite to each other”, “upside down to each other”. Thus while the students recognise and mark the change of orientation of the shapes, they do not use the language of rotations or reflections to talk about them. They point to the transformations using everyday language. Also in Figure 5.8 (a) the student uses the pronoun “it” without specifying the referent and is perhaps using the preposition “beside” instead of “below” or “beneath”. Some of these may be because of not being fluent in English.

While the transformations done on Shape I to get the other shapes in Set II may not be immediately obvious and needs some mental manipulation of the shapes, those on Shape I in Set I are obvious. There were a few references to rotation in the student responses to this set: Comparing Shapes I and IX in Set I a student writes *“ Shape I and Shape IX are equal shapes, but it turn little a side, but it looking same.”* Comparing shapes I and VI another student writes *“I and VI are same because they have the same length and angle. Even though it is facing another side we can rotate it”.* And yet another student said *“In Shape I it goes up and shape VI it goes right”.*

Talking of similarity or scaling: We now look at how students communicated the idea of similarity and the transformation of scaling. The following are some samples of student responses where they describe this relation.

“Shape I and VII are semicular (sic. meaning similar) figure because same diagram and different length and small sticks are used.” (Set II)

“Shape I and Shape III are semicular figure but the matchstick size is small.” (Set II)

“One and two are not same. Why because in the figure one they kept at 2 cm distance in the figure two they kept 4 cm. So second figure is bigger than the first figure and the angle were not same” (Set I)

“Figure 1 and 2 are not same because the length is different and matchsticks used differently. If the length and matchsticks are used samely it will be same.” (Set I)

“I and VII are not same because each stick of figure 7’s length comparing to first figure is different” (Set I)

The term "similar" was a "met-before" for them as part of the curriculum, but was perhaps forgotten. I gave them the term for what they referred to as "of the same shape but different sizes". They adopted the terminology and here we see a student reusing the term as "semicular". While students seem to have noticed the difference in side-lengths between similar or scaled-up figures, they have not expressed that the side-lengths need to be proportionately scaled up or down in the above statements. However, I infer that they are aware of this requirement. One student used a series of diagrams (Figure 5.9) to explain what she understands/means by "same shape" where proportions between the sides figured explicitly.

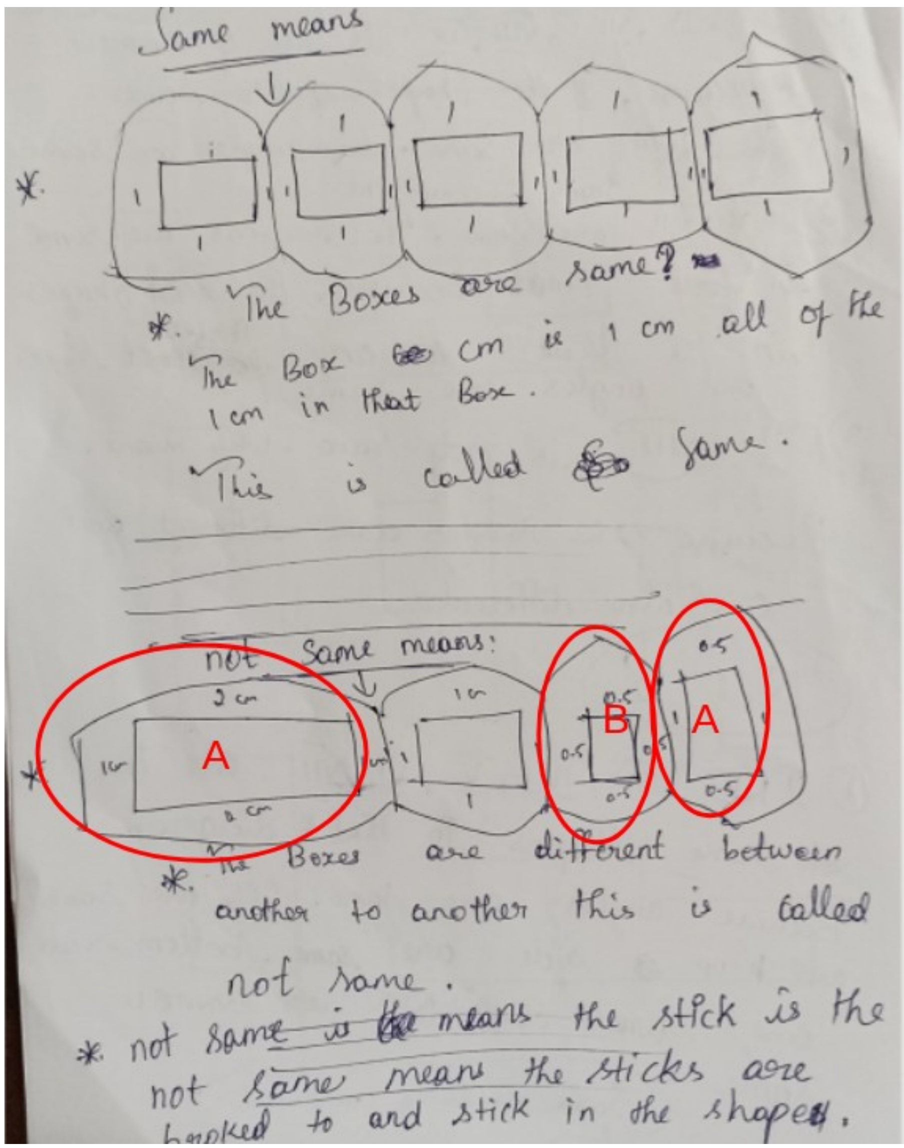


Figure 5.9: Matchstick geometry - Samples of written work - 2

Considering the unit square as reference, the parts marked A tell us that shapes with different proportions are not considered the same by this student, whereas the part marked B indicates that the dimensions also need to be the same in addition to the proportions. Some students noted the number of sticks used to make the figures (5 and 10 respectively for Shapes I and II of Set I, 14 and 28 for Shapes I and VII respectively of Set II). But they did not express scaling in multiplicative terms, perhaps because they did not have the language necessary to express a multiplicative relationship.

Though none of the students explicitly verbalised the need to conserve proportions, all of them noticed that Shape III of Set I has not been scaled up proportionately. Some ways in which they expressed this is as shown in Figure 5.10.

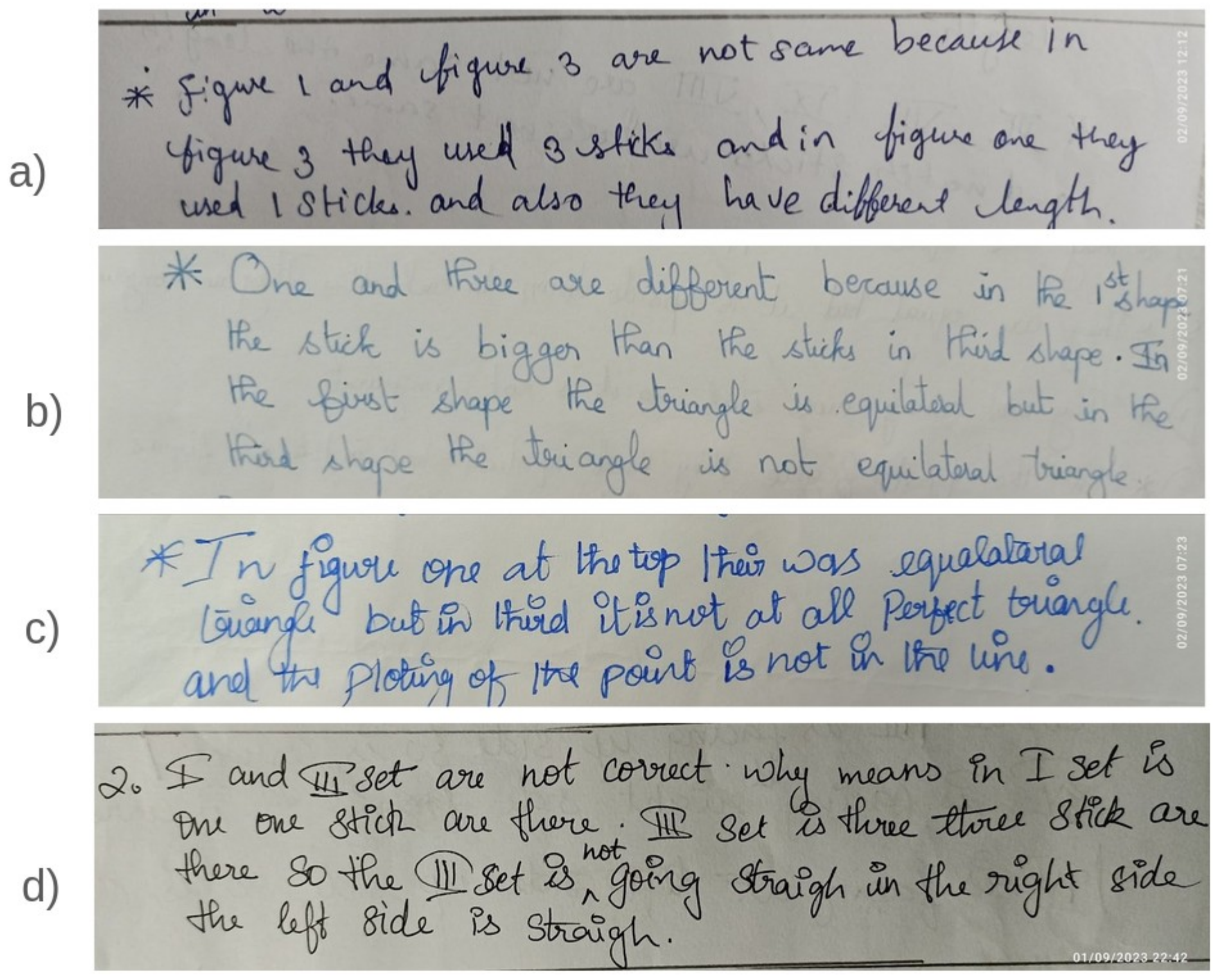


Figure 5.10: Matchstick geometry: Samples of written work - 3

Worth noting in Figure 5.10 (a) is that the first two uses of “they” refers to some external agent who made the shapes, whereas the third “they” stands for the sticks. Figure 5.10 (b) and (c) point out that the triangular part in Shape III was not equilateral unlike in the other shapes (Set I Figure 5.7). It can be inferred that these three students have noticed that the proportions of Shape I are different from that of Shape III. In Figure 5.10 (d), the student suggests that Shape III was "incorrect", because the lines don't look straight. On probing this student and another student on how to make it correct, they suggest either removing two sticks one each from the two sides of the triangular part, or adding a stick to each side of the square part. This adds strength to the inference that these students were sensitive to the differences in the ratios between sides of these shapes. Also they are clearly participating in mathematical practices. We now look at the turns of conversations with these two students Vidya and Rima.

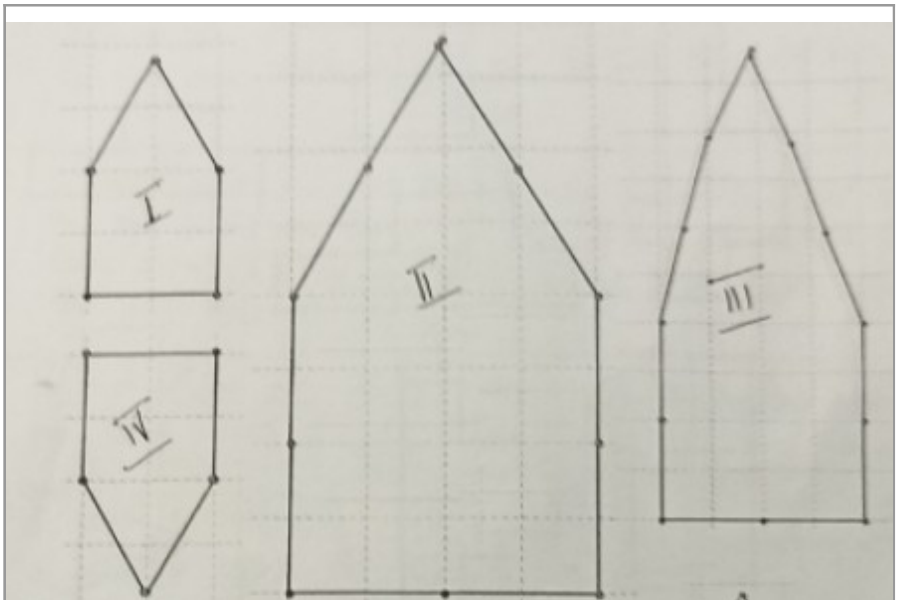


Figure 5.11: Matchstick geometry: The "wrong" shape

1. Vidya: *Intha figure vanthu …konchum… ithuva irukku. Ithu thappu.* (This figure is a little .. This is wrong.) Referring to shape III in Set I (Figure 5.11)

2. J: *Thappunna?* ("wrong" means?)

3. Vidya: It is … 1, 2, 3… So it is not correct. Referring to the three sticks per side in the triangular part of the figure.

4. Rima: *Appadiya sideaa pokuthu. Ippidi. Neenka sonnengale ippadi poy…innu. Athumathiri pokuthu ithu.* (It goes to one side. Like this. Like you said earlier, it goes like this)

5. Vidya: This is not same to this. Meaning that Shape III is not the same as Shape I.

6. J: Overlap with Vidya …same to this. But this, is same to this? Referring to shapes I and II

7: Vidyaand Rima: Similar

8: J: *ithu samum alla, (*This is neither same…) Referring to shape III

9. Vidya and Rima: similar figurum illa. (not a similar figure)

10. J: *similarum alla.* Can you make this similar? What will you do to make this similar? (...Nor similar.)

11: Vidya: *Naan itha eduthittu, nan ithaappadiya join panniduven.* (I will take this out and join this). Indicating that she would remove two of the sticks from the triangular part of the shape III and close the triangle with the rest of the sticks.

12:J: *Sari, naan itha thodakoodathunnenna?* How will you make this similar? (Ok, What if I tell you that you cannot touch this?) Indicating the triangular part

13. Rima: *Itha edukkama itha mattuma?* (Without taking this, only this?)

14: J: *Aa, itha edukkanuma, Itha modify panni, appadi sameaa pannuvenga?* (Yes, Without taking this, how will you modify this to make the figure the same?) Indicating that only the lower part of the shape III in Figure 5.11 is to be modified

15: Rima: *Itha Innum konjam intha ithila eduthinnu varanam*. (This has to be taken a little bit to this side.)

16: J: *Enna? Ethila*? (What? Where?)

17: Vidya: *Innum oru stick randu stick eduthu, inka, chi, inka vachu, Intha edathilla three sticks vachu.* (one more stick, taking two sticks, here, no, keep here, keep three sticks in this place) Vidya is talking off adding a stick each to the vertical sides of the square part of Shape III in Figure 5.11 and making the horizontal side 3 sticks long

18: J: *Intha edathila*? (in this place?)

19: Vidya: *three sticks vachu, congruent aa pannidalam, similar, simlaraa pannidalam* (Keep 3 sticks, and it can be made congruent…similar. It can be made similar)

20: J: Similar. *Intha edathilla 3 stick vacha ithu ennakum?* (Similar, What happens if you place 3 three sticks here?)

21: Vidya *Expand pannum koncham.* (It will expand a little bit)

22: Rima: *So ithu koncham viriyum, ippadi aakum.* (This will open out a bit, It will become like this)

23: J: *Ithum viriyum, ithum viriyum*. So 3 sticks, 3 sticks, 3 sticks. 3 sticks, 3 sticks na, Similar figure. So this is different from all the rest. (This is open out and this will open out).

In this conversation, students offer two ways of making Shape III similar to the rest, namely removing a stick each from the triangular part, or adding a stick each to the square part, both actions in the real world. To my question in turn 20, where I expected a response that expresses why the modified shape is similar to the rest, perhaps in terms of the proportions involved, they respond that the shape "expands" or "opens up" in some parts. Scaling proportionally does not get talked about. Thus these students engage in routines that produce a narrative about an action in the real world and not an endorsable fact about the shapes. The purpose of the action is however to articulate the fact, for which they do not have the language. Another example of an action in the real-world meant to articulate a fact can be seen when a student suggested that they measure the side-lengths of the matchstick triangle in Section 5.2.2, he was talking about the real-world triangle created with matchsticks and not the abstract triangle which Sfard (2008) considers a discursive object that constitutes the subject of mathematical discussions. The narrative that is produced is also about the real-world object.

In summary, from the above analysis of student talk and writing on transformations, it can be seen that students are aware of transformations, but do not verbalise it. These examples underline the need to *listen* to the mathematics implicit in student conversations, though the surface features may not indicate it.

5.3.3 Students' mathematical talk vis-a -vis scholars' characterisations

In this section, I analyse student talk related to transformation of solutions of the Magic triangle puzzle and transformations of geometric shapes, including rotations, reflections and scaling. In the first instance, rotation and reflection were articulated as a realignment of numbers preserving a certain order and the side-sum. In the second instance, it was seen that while students have a grasp of transformations, they used everyday language and terms like "facing" a particular side, "opposite" and "expand" to express their ideas. The awareness of proportionality came through in the way they say they would modify the shape to make it similar to the rest. These can be considered as actions on "imagined objects" if not real objects and are everyday routines in Sfard's terms. In these examples, students came up with deductively *endorsable* (as compared to *endorsed*) narratives, using words that are not objectified, mediators that are diagrammatic, operational reasoning and informal routines on imagined objects. Thus the talk seen in the contexts of this study differs widely from Sfard's (2008) characterisation of mathematical discourse.

Students drew on multiple resources - like oral and written text, diagrams, gestures, everyday and mathematical languages, Tamil and English - to communicate their ideas and the discourse was embedded in mathematical practices as suggested by Moschkovich (2015a). They were seen to be engaging in the mathematical practice of defining - or coming up with a definition of what they mean by "same solutions" or "same shape" and the operation of classifying solutions and shapes according to this definition. Different students characterised "same shape" differently and saw that these lead to different classifications, and raised questions on why one should be preferable to another. Though the group did not go through the process of evaluating these definitions and their consequences, and choosing between them, there was a felt need for a shared definition. However the features such as precision, brevity, deductive reasoning which Moschkovich identifies as markers of mathematical discourse are not very prominent in student talk seen here.

Issues stemming from not being comfortable enough with English are prevalent in the written work. I pointed to some of the issues like use of pronouns (it, they) without specifying referents. Others include tense mismatch, agreement issues and issues of sentence construction and punctuation. I acknowledge these language-level issues and do not analyse them.

5.3.4 Features of mathematical talk

Drawing on the analysis in the preceding sections, I now highlight the distinguishing features of the talk that I encountered in the course of this study.

Code-mixing: The language used by both the teacher and students was a mixture of Tamil and English, sometimes switching between both languages within the same sentence. More often than not, mathematical terms like triangle, diagonal, pentagon, congruent, etc., were retained in English and the rest in Tamil.

Interaction drawing on multiple modes: Students drew on multiple modes to communicate - speech, writing, diagrams, gestures, etc., with talk being the preferred mode. There was very little writing seen other than when insisted upon. The writing that was seen was informal. I noted students resorting to gestures and pointing. For example, turns 4 and 22 in the excerpt shared in Section 5.3.2 were accompanied by gestures and almost all turns involved pointing to the figure at hand. That Shape III is not identically proportioned as the other shapes of the set of shapes can be conveyed through words alone, but not having the language - mathematical or otherwise - to convey notions of proportionality, students resorted to pointing and gesturing and managed to get across their point. Figure 5.9 is an instance of usage of diagrams to make a point. Similarly the turns of conversation shared in Section 5.3.1 were accompanied by frequent pointing to the board and gesturing as well. The word-use in these conversations is not "reified" as in Sfard's (2008) characterisation.

Lack of precision: Precision and brevity are two key features of mathematical discourse that Moschkovich (2015a) marks, but were missing in these conversations. I pointed to some instances of ambiguity and lack of precision in the preceding analysis. For example, a student listed out the following features should be the same for two shapes to be the same: angle, size, what the diagram looks like, and added length, breath, volume and everything. “Size” is a layman’s concept that cannot be captured in mathematics by a single or unique measure. Similarly, “what the diagram looks like” and “everything” is vague as well. Words like “bigger”, “greater” were used when the intention was to compare lengths. These words convey the intended meaning in everyday conversations, but the word “longer” may be more appropriate in a mathematical context. The idea of superposition to define sameness was conspicuous by absence.

Informal expression : The conversation was informal, with some mathematical terms like square, diagonal, etc., and symbolism being absent. They frequently drew on the everyday register to express themselves. For example, my expectation when I asked the question in turn 20 in the conversation analysed in Section 5.3.2 was that they would respond in terms of the shape becoming proportional to the remaining ones in the set. But they talked of the shape “expanding” and “opening out”. In the first 2 turns of the conversation analysed in Section 5.2.2, they conveyed the idea that the length of the diagonal is less than 2 in terms of there not being enough space (*“idam pathale”*) to accommodate 2 sticks, very much tied to the physicality of the context. They also fumbled for words to express themselves and came up with self-created words. The word “semicular” is derived from “similar” in an attempt to internalise and use the teacher suggested terminology.

One aspect of the “insisted-upon-written-work” that was striking is that it resembled talk more than writing. One would expect more formal language and well-formed sentences in writing, something that is distanced from the immediate context. Student responses to the worksheet problems on the other hand were written in conversational style - as they would talk to the teacher. For example, consider phrases “...is not at all perfect triangle…” (Figure 5.10 (c)) or “... is not correct. Why means in I set…”(Figure 5.10 (d)). The construct “why means” might possibly be a word by word translation from the Tamil “*en na*” for “because”. I also found two examples, one where the writing included a couple of phrases written in Tamil and another where a sentence was interspersed with a few transliterated Tamil words. In terms of structure, tone and word-use, the writing was more like talk than writing (written talk!). The spontaneous written work that they did was often on impermanent surfaces (Figure 5.5 on classroom floor) and seemed more to aid thinking than expressing organised thought for others. There was very little text seen and it was not well-organised.

A focus on how the discourse differs from the expected characteristics of mathematical discourse may lead to a deficit perspective that fails to acknowledge the mathematical in such discourses. Though the discourse itself deviates from what is considered mathematical discourse in literature, there are "family resemblances" it shares with mathematical discourses and definitely show elements of mathematical thinking, though expressed in divergent ways. This brings home the need for more flexible acceptability criteria for mathematical discourse that focuses on the resemblances that they bear to mathematical discourse. I take this up in Section 5.5.

5.4 Language limitations that hinder mathematical thinking

This study corroborates the findings of other scholars (Barwell, 2016; Bose & Choudhury, 2010; Moschkovich, 2008; Setati, 2001) that the simultaneous presence of the home language and the school language, formal and informal mathematical language supports students in expressing mathematical ideas meaningfully. The instances described in the previous sections exemplify how students use a mix of mathematical and everyday language, LoLT and home language, spoken and written language and means such as diagrams and gestures to make a mathematical point. However we also have instances where language limitations have been a hindrance to further progress. Our observation has been that while students may be able to solve the problem presented as the starting point for the exploration, extensions and generalisation of the problem are easier with formalisation.

One instance where this comes out strongly is the Magic triangle exploration. Students solved the problem, looked for patterns in solutions, and came up with some transformations that give other solutions or preserve solutions, all in informal terms. When different sets of numbers were used, especially larger numbers, balancing the side-sums became harder for students and brute-force approaches were not very effective. Students need to get a sense of the structure of the problem and formalisation helps in this. This is more so when polygons with more sides, or more numbers per side are considered. Even in the case of square, finding all 8 solutions using brute force can be time consuming if not challenging. The formal approach to the solution explained in Section 3.6 is extendable to all these cases and supports extensions and variations of the initial problem. Based on our implementations of this exploration in summer camps, talent nurture camps and in some teacher workshops, our observation has been that an algebraic approach to the solution is crucial for extensions of the task. In the implementations in School 1 and 2 discussed here, neither group came up with the algebraic solution and both groups had difficulty with algebraic manipulation. Consequently, the student engagement with extensions and generalisations of the problem seen elsewhere, was minimal here. I did some explicit teaching, explaining the algebraic solution to both the groups. After this, students in School 2 tried solving the problem for a square and a Z shape. In School 1, this explicit teaching was done when students' frustration levels were high and perhaps because of this, they did not build further on this.

In the instance where students tried to prove that a matchstick diagonal cannot be accommodated in a unit square (discussed in Section 5.2.2), the student Maariya was taken aback by my question of how much bigger and does not immediately respond to the question. The language necessary to express a multiplicative relationship could have been the stumbling block. We also noted the difficulty in expressing the scaling transformation in multiplicative terms in the excerpts shared in Section 5.3.2. The relation between the side-length and the diagonal of the square was expressed in less precise terms as "not equal" and "greater". Being able to express it multiplicatively would have made the solution to the problem obvious and allowed for generalisation as well. This is an instance where less precise and informal language hindered progress.

On a similar note, Pythagoras theorem was mentioned when the students argued for the impossibility of the diagonal of a unit square, but was not articulated precisely or in symbolic form. Even when the symbolic form was mentioned in the context of rectangles for which diagonals could be fitted, the students did not clarify what the symbols meant. Their statement that the diagonals could be fitted only for those rectangles for which $a^2 + b^2 = c^2$ indicates an incomplete understanding of the equation and the range of possible values the variables a, b, and c could take. Having mentioned Pythagoras theorem in one specific case, the realisation that the same holds for another specific case (that of square of side 2) was not immediately obvious to them. When looking for squares within which a matchstick diagonal could be fitted, their approach was to examine specific cases. No amount of trying out will yield a positive example here, but not being able to find an example does not prove its impossibility and further progress in the problem is blocked. Had they formalised the problem as for a given x, finding y such that $2x^2 = y^2$ the general solution would have been within reach. The lack of formalisation proved to be a stumbling block to solving the problem. I have had similar observations in other explorations as well, and infer that while informal language helps to get started on an exploration, lack of formal language may hinder progress beyond the initial stages.

More importantly, formal mathematical language is also tied to questions of access. While formal mathematical language functions as a deterrent for many students to learn mathematics, it is also empowering by providing the skills valued in a technological society to obtain a good position in the labour market (Skovsmose, 2011). Several scholars who have highlighted the need to view students' language as a resource rather than as a problem have critiqued the tendency to view progress as movement from the informal to formal academic talk, and suggested that a rigid distinction between them is neither necessary nor productive (Barwell, 2016; Moschkovich, 2000; Planas & Setati-Phakeng, 2014).

While informal language helps sense-making and provides epistemological access, formal mathematical language is essential for communication with the larger community of mathematics, for opportunities for higher education and the prospects that it opens up. Thus not providing students opportunities to learn more formal mathematical language will also disenfranchise them in the long run by restricting access to social goods such as higher education and employment (Barwell et al., 2016; Setati et al., 2008).

Insisting on the highly formalised mathematical language makes mathematics inaccessible to many learners. While accepting and encouraging informal mathematical language improves access, not providing them sufficient opportunities to learn more formal mathematical languages will eventually deny these learners access to higher educational and professional opportunities. Thus there is a tension between the formal mathematical language and informal language in teaching-learning of mathematics.

5.5 What counts as mathematical discourse?

To address this tension between the formal and informal in educational contexts, I look to the practice of research mathematics, which on the one hand insists on formal communication as a necessary condition for acceptability of assertions, and allows for many different levels of formalism on the other. Formalisation is important in the practice of research mathematicians in catching contradictions as they arise and building coherence. As discussed in detail in Section 5.5.1 below, mathematicians also take advantage of the freedom afforded by the informal during the process of discovery, keeping formalisation in sight by insisting on *formalsability*. Mueller-Hill (2013) identifies formalisability as one of the epistemic features of proofs in research mathematics. Being mindful of the epistemic features of mathematics while being accepting of student contributions, I propose "*coherent formalisability*" as an acceptability criterion for mathematical discourses (Jayasree et al., 2023). There are two components to the criterion I propose here: that of formalisability, and that of coherence. Informal discourse is formalisable if, given sufficient additional mathematical resources as support, it can be restated in formal terms. However, such a criterion could be trivial at the level of individual statements or assertions; hence the qualifier of coherence, that binds statements together into a whole. I describe these components and present an illustration of what the criteria implies in some detail below.

5.5.1 From "formal" to "formalisable"

Formal communication is considered a necessary condition for the acceptability of mathematical assertions in the practice of research mathematics[14]. However, there are many levels of formalism in mathematical communication: a machine-checked proof in a formalised theory represents one extreme of

14. The observations on mathematical practice made in this section draw on the experience of the research mathematician in the collaboration.

formalisation; a Bourbaki-style rigorous definition – proposition – lemma – theorem exposition is formalised but yet not as definitive as the machine checked proof; research papers in mathematical journals, while striving for rigour, often admit more informal discourse than the Bourbaki style authoritative text; graduate and undergraduate textbooks are even more informal; seminars and classroom lectures are far more informal. Yet, all these would be considered formal in comparison to the language employed by mathematicians during discussions and discovery when pictures, half-formed ideas and ill-defined terms often dominate discourse. Thus, there is no single mathematical language that can lay a hegemonic claim to being “formal”.

There are many possible ways of formalising an informal notion or argument, and invariably there are cycles of progression in mathematical practice, where some particular formalisation is seen to be “wrong” or “inappropriate” or “unhelpful”. This does not mean that a formalism is discarded and another is chosen, but invariably, this is seen as signalling a need to proceed further in informal terms, strengthen some intuition and then attempt a reformulation. For instance, when an informal notion is formalised using a definition, using it in proofs may lead a researcher, after many false steps, to realise that the definition is too stringent and allows a much smaller class of structures to work with than the informal outline had originally assumed. At this stage, the researcher does not discard the definition, but typically harks back to the informal discourse and examines whether the problem is with the definition or the original strategy. This process may well be said to constitute the bulk of time and energy spent in mathematical research.

The process of formalisation and mathematical writing is geared towards making apparent any assumptions that may be inconsistent. While research mathematicians use pictures and highly ambiguous terminology and notation during discussion and discovery, they proceed to use a more formal language in writing definitions, assertions and proofs, mainly to discover possible inconsistencies, or possible gaps in proofs (Hadamard, 1945). A question that naturally arises in such a scenario is this: if formalisation is essential, why do mathematicians privilege informal discourse, going back to it, even when an inconsistency (or other intellectual obstacle) is encountered? The obvious answer is that informal discourse allows much greater room for false starts, loose statements, and working in a semi-confused state, which is necessary when the solution is unclear (and indeed when no solution may exist). A deeper answer is the confidence that mathematicians have, that the informal discourse they employ is *not in itself the source of difficulties they encounter:* that it is *formalisable*, and if there is an obstacle to proof, it is in the way that they have envisioned the structures they work with and the proof strategies they employ. Without such confidence, they would be extremely reluctant to hark back to informal terms, preferring to live with the stringency of the formal in order to ensure the safety of their approach.

Mueller-Hill (2013) identifies *formalisability* as one of the epistemic features of proofs in research mathematics. She goes on to suggest that formalisability of discursive proving actions, be seen as a metadiscursive rule, in the sense of Sfard (2008). With Mueller-Hill, I suggest that formalisability, with appropriate reinterpretation, is of relevance to mathematical discourse at school as well, allowing multiple levels of formal discourse and maintaining coherence. I propose *coherent formalisability* as the delineating feature of mathematical discourses. I see formalisability and coherence as the "family resemblances" (Wittgenstein, 1953) that binds the informal discourses in these contexts to the discourses of research mathematics.

When we consider students' mathematical work in school, we have some similarity with such discourse and a marked contrast: they too do not see solutions in sight, and informal discourse helps them to try out strategies in a flexible manner. However, often the teacher (or the textbook) has one clear and formally expressed formulation at hand. Moreover, neither the teacher nor the students are sure whether difficulties encountered are due to the informal discourse employed or due to other reasons. It is precisely in this context that our suggested criterion of *formalisable* informal discourse makes sense: it mimics the process internalised by mathematicians, providing an external certification of formalisability that sustains this process.

5.5.2 Coherence

The foregoing narrative places consistency at the centre of formalising mathematical discourse, and this is indeed a fundamental requirement in the development of mathematics. However, in the context of students' work, I suggest a weaker requirement, that of coherence. While consistency is the absence of logical contradictions, coherence is meaningfulness, manifested in consistency of construction, representation, etc. Coherent discourse may lead to propositional inconsistencies but yet contain sufficient structure to make it easy to detect inconsistencies when they arise. When mathematicians work, their training is expected to ensure such coherence, and hence the more stringent criterion of consistency is demanded. In students' work, coherence cannot be assumed, but needs to be established.

By coherence, I mean that formalisation of disparate elements hangs together as a *meaningful* whole. The classroom discourse includes many parts pertaining to definitions, visualisations, representations, conjectures, exemplification, providing counterexamples, justification and refutation, etc. When some of these are supplied by students, it is likely that their meanings do not mesh correctly. For instance, students may choose to represent quantities by integers, and then divide them in context, without realising that division by zero may lead to meaningless terms. Even if this is fixed, they may well proceed with integer operations without realising that they should now be working with rational numbers. The criterion of coherence is to principally form a unified whole without internal inconsistencies and logical contradictions, while also allowing for inconsistencies to build meaningfully so that they can be detected easily and corrected, resulting in learning.

5.5.3 Formalisability

Mueller-Hill (2013) defines a formalisable proof as “a proof that can be transformed into a formal derivation in a consistent axiom system”. The transformation itself could happen in wide ranging ways – from an “independent formal derivation in a consistent axiom system” to “being translatable step by step into a formal proof” and other variations in between.

I reinterpret formalisability as the potential of a section of discourse to be mapped to a formal one by supplying missing terminology, definitions and reasoning, in a uniform manner. Aiming for something in between the extremes of an existence of an independent formalisation and a step by step translation to a formalisation, I expect some structural similarity to a formal discourse, though not all elements of the structure may be present explicitly. It is possible that there are multiple ways to augment the discourse and map it onto a formal one. Thus, the conversation is a template, which when mapped in such a way that the structure is preserved, yields formal discourse. Such a mapping is not unique.

The proposed criterion applies to the set of utterances that arise in the course of a mathematical exploration and is theoretical. Hence it should be seen as certification of formalisability by an abstract observer who has access to all the relevant linguistic and mathematical resources required to make such a formalisation. It is not essential that such capability be evident in the teacher in the classroom context, but I expect that recognition of the criterion would make teachers more reflective and prepare them better to handle mathematical discourse in the classroom. I consider coherently formalisable discourse as a stage in the transition to more formal discourses and opens up possibilities to work with student contributions not being bound by the rigidities of formal school mathematics.

5.5.4 What does coherent formalisability look like?

We now look at the different arguments that students came up with to convince themselves and others that side-sums 8 and less are not possible as are side-sums 12 and above.

1) Maran argued as *“Eppidiyum ethachi oru circle la 6 use panniye aakanum. 6 use panninalum inga motham munnu circle irukku. Appo randu onnu vantha than ithu vanthu 8 aakum.”* (6 has to be used in one of the circles in any case. If we use 6, there are three circles on that side altogether. So we can get a sum of 8 only if we have two ones). 7 was also eliminated using a similar argument and numbers 1–6

were eliminated right away, 6 being one of the numbers used, thus establishing 9 as a lower bound.

Maran's argument expressed in informal language with no symbolic mediators can be mapped step by step on to the following formalisation:

> Considering the problem as finding three subsets A, B, C of S = {1, 2, 3, 4, 5, 6} such that each has three distinct elements, their union is S and pairwise intersections are distinct singletons and the sum of the elements in all subsets are equal, proof 2 can be translated as
>
> $6 \in S$,
>
> $6 \in$ at least one of A, B, C, say A
>
> A has two distinct elements other than 6, ie, $A = \{a, b, 6\}$ where $a,b \in S$
>
> If there is an arrangement with side-sum equal to 8, $a + b + 6 = 8$
>
> This is not possible unless $a = b = 1$

Thus, there exists at least one formalisation that is coherent and consistent. Note that the intention here is to point to one possible formalisation. This is tied to a particular set of numbers and to the particular side-sum 8. It needs to be modified appropriately to cover the general case.

2) Krithi's proof - attempt: Trying to come up with a solution whose side sum is 8, Krithi saw that 5 + 2 + 1 is one way of making 8, and on a diagram wrote these along one side of the triangle as shown in Figure 5.12. She then tried out possibilities for the number that could be in the circle marked X, and saw that whatever number she writes there the side sum exceeds 8. Given the condition that the numbers have to be distinct, the only options available to her are 3, 4 and 6. Of these, she immediately ruled out 4 and 6, as the side-sum obviously exceeds 8. She ruled out 3 as well, for there has to be a (non-zero) third number on the side as well, which would make the side-sum exceed 8. Having convinced herself that none of the available numbers fit in in the position marked X, she concluded that it is not possible to get a side-sum 8.

The proof is incomplete in that it does not exhaustively consider the ways for forming a sum of 8 using three distinct numbers from 1 to 6, and their possible arrangements along a side, and rule out all of them. 5 + 1 + 2 is but one way, which is being ruled out by Krithi. If one probes deeper into what the underlying "proof scheme" that Krithi might have adopted, 'look for combinations of numbers that make a desired side-sum' seems likely. Having found one such combination (5, 2, 1) , she looks for another one that includes 5 and notes that such a combination does not exist. What she missed out is exhaustively considering all such combinations. So if we modify her proof scheme as "look for *all possible* combinations of numbers that make a desired side-sum", her proof scheme and argument would have been sufficient to prove that 8 and numbers less than 8 cannot be the side sum. Krithi does use the scheme in a limited way when she exhaustively ruled out all possibilities for the position X. Thus by augmenting her proof scheme to include all possible combinations and arrangements of numbers along a side, her proof attempt is coherently formalisable.

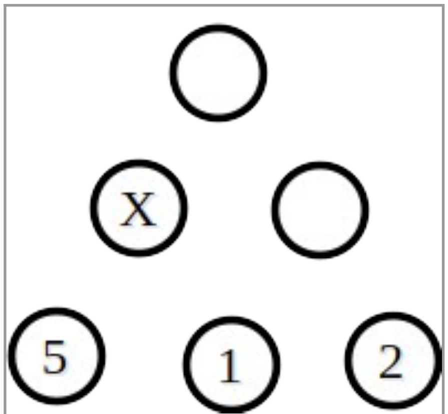


Figure 5.12: Magic triangle: 8 cannot be the side-sum

3) We now look at how V2's argument (see Section 5.2.1, S - 5 and S - 6) that the maximum side-sum is obtained by placing the larger of the three numbers at the vertices of the triangle and the minimum side-sum by placing the smaller of the three numbers at the vertices and placing the remaining numbers in such a way that the side-sums are balanced can be formalised.

Assuming the numbers to be a, $a+1$, ... $a + 5$, making explicit the implicit definition of what V2 called "highest" numbers as $\{a + 3, a + 4, a+ 5\}$, and placing these numbers at the corners and subsequent placing of the remaining numbers to balance the side-sums as articulated by V2, would give configuration A (Figure5.13), with the side-sums $3a + 9$. Similarly, appropriately defining and placing the smaller numbers at the corner would give a side-sum of $3a + 6$ as in configuration B in Figure 5.13.

Taking this further, Krithi argued that V2's process guarantees that each side has 2 of the larger numbers and one of the smaller when the larger numbers are at the vertex, maximising the possible side-sum. The move of explicitly symbolising the 6 consecutive numbers as a, $a + 1$, ... $a + 5$, that might happen as a matter of course in a context where students were more fluent in algebraic manipulation, remained implicit here. With the added definitions for the ill-defined term "highest numbers", V2's intuitive algorithm and Krithi's argument can be formalised. The nature and extent of augmentation required in each proof is different and is an indication of the distance from the formal proof.

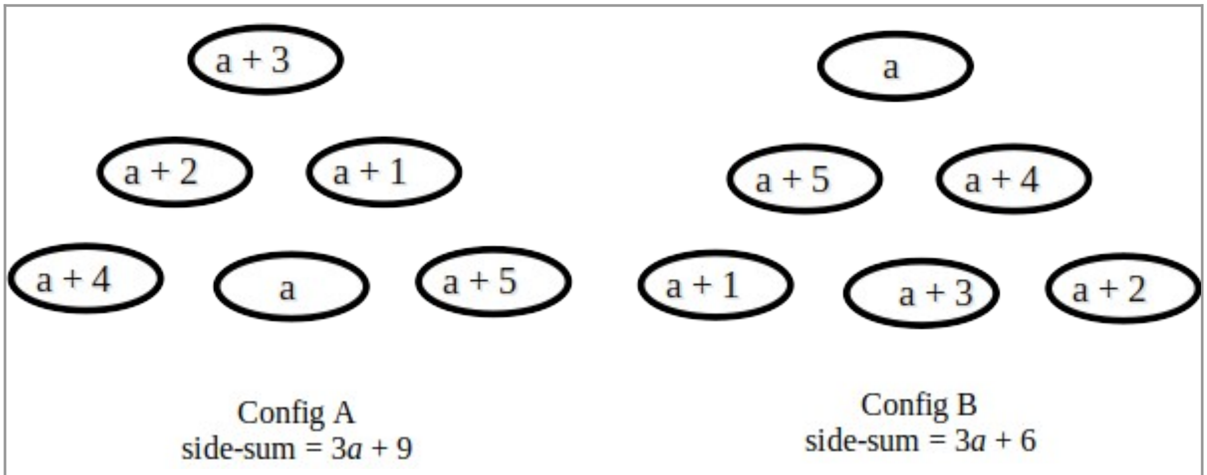


Figure 5.13: Magic triangle: Maximum and minimum side-sums - generalisation

All three proof attempts give a sense of “understanding and conviction” and the proof production seems to be led by a “key idea” (Raman, 2003, p. 323). A key idea, according to Raman, is a hueristic idea which one can map to a formal proof with appropriate sense of rigor. We now look at an example of another proof attempt to understand what a proving action that is not coherently formalisable looks like.

4) Student Dheer argued that 6, 5 and 2 add up to 13. The remaining numbers (1, 3 and 4) sum up to only 8 and hence a sum of 13 is not possible.

The proof seems to draw on the intuition that “large numbers” need to be used to make a sum of 13 and having used up 6 and 5 to form a sum of 13 on one of the sides, there may not be enough large numbers to draw on for the subsequent sides. Dheer may have come to this argument based on his observations of trying out particular combinations of numbers that form a target side-sum, here 13. Also, there is an unstated assumption that the numbers remaining after the target side-sum is obtained on one side, should also have a sum at least equal to the required sum. The proof scheme underlying this proof could be spelt out as “add the remaining numbers and check that their sum is not less than the required side-sum”. However, if this proof scheme is extended to other side-sums and combinations of numbers, it leads to a contradiction. If we choose 6, 5, 1 as the initial triplet that forms a side-sum 12, the remaining three numbers, namely 2, 3, 4 add up to 9 which is less than 12. Yet it is possible to find an arrangement that has side-sums 12. Thus the proof scheme in this case does not yield a valid proof.

In the above examples I examined the coherent formalisability of proving actions in the context of explorations. I further suggest that the criterion is applicable in curricular contexts (see Section 6.5.2) as well as in the use of mathematics in everyday life. In a curricular context, there is an already given formalisation in the textbook that shapes conversation and guides discourse, perhaps limiting freedom in the process. Algorithms encountered “on the street” have an informal basis, formalisable over domains of limited validity. I also suggest that coherence and formalisability apply to a wider range of elements such as definitions, representations, algorithms, etc. For example, the many definitions of “same shapes” that we saw in Section 5.3.2 are all formalisable. But only one where the number and length of units per side, and angles are preserved coheres with the definition of congruence in school geometry. With other definitions it is possible to maintain coherence within the domain of the delineated exploration, but requires the teacher to watch out for developing incoherence as students explore further beyond the delineated boundaries. In the next chapter, I examine what coherent formalisability as acceptability criterion for mathematical discourses entails for the teacher.

5.6 Summary

To summarise, in this chapter I described students’ engagement with mathematical explorations, both in terms of the thinking and reasoning seen and in terms of the ways of communication adopted to communicate these. In the multiple instances described, we saw students engaging with operations of mathematical thinking that Burton (1984) identifies - looking for similarities and differences, classifying, making correspondences, studying relationships, experimenting, recognising and continuing patterns, etc. We also saw them engage with the processes of mathematical thinking like abstracting, defining, conjecturing and convincing, generalising and specialising, etc. They used already found results to find further results as they solved problems, asked questions and found things out for themselves, rather than follow taught methods. The kind of thinking and sense making and the intense engagement seen with the explorations establish the feasibility of mathematical explorations in marginalised contexts and the potential of exploratory tasks to enable mathematical thinking.

The ways in which students communicated their mathematics was different from what one would expect in a school context. Most of the mathematics was done orally and communicated with minimal use of symbols and formalism. I noted a reluctance to write in a presentable fashion. This underlines the need to privilege talk as a means to express and communicate mathematics, especially in contexts where insistence on formal writing may hinder participation. The talk that students engaged in differed from what literature describes as mathematical discourse - especially in the extent of objectification seen in word use, the reliance on real-word objects and non-symbolic mediators to make a point, narratives being endorsed on the basis of examples seen (inductive means as different from deductive means that is the norm in mathematics) or practical action. The nature of talk used to communicate mathematics was marked by use of hybrid languages - Tamil and English, home language and school language, formal and informal mathematical language - and other resources like diagrams, gestures and unstructured writing.

The talk was imprecise and vague at times and included part sentences and phrases. It was highly contextualised by use of self-created and reappropriated words and pointing gestures.

Given the high levels of student engagement seen and the richness of mathematics discussed, my study underlines the need to be accepting of this “unconventional” means of communicating mathematics. At the same time, the study also points to the limiting nature of such communication in that it hinders progress in explorations to an extent and as attested to by the literature eventually leads to limiting access to educational and professional opportunities. This creates a tension between being accepting of students’ mathematics and ways of communicating it and insisting on the disciplinary norms.

I looked to the practice of research mathematicians who balance the need for flexibility during moments of discovery with the need for rigour to enable spotting any developing inconsistencies and gaps. Echoing the criterion of formalisability that they use as guiding principle to achieve this balance, I proposed coherent formalisability as acceptability criterion for mathematical discourse in educational contexts as well. This criterion focuses on the core aspect of formalisability and is accommodating of the many “deviations” seen in students’ ways of communicating mathematics. This enables teachers to take a non-deficit view of students’ languages. I look at what this entails for the teacher in the next chapter.

6 Mathematical explorations and “talk” at the margins - What does it entail for the teacher?

In the last chapter we examined mathematical thinking as seen at the margins and the means students adopt to communicate it. Keeping in view the prevalence of informal means through which students communicate mathematics and the role and importance accorded to the formal in different facets of mathematics - in school mathematics, in the work of the research mathematicians and as a part of prevailing culture - I suggested an acceptability criterion for student mathematical discourses. Loosening the tight grip of the formal in school mathematics, I suggested that “formalisability” and coherence are sufficient criteria for acceptability of mathematical discourses. In Chapter 4, I posited flexibility and accessibility as key guiding principles that support mathematical thinking at the margins and looked at task features that make tasks flexible and accessible in these contexts. Together these chapters argue for flexible boundaries for the mathematics that may be taken up in the classroom and admissible ways of talking mathematics. This has implications for the teacher and presents multiple challenges. I mark some key differences for the teacher in this respect from typical classroom teaching.

A) Unlike in a curricular situation, where there are well-defined boundaries to the mathematics that the teacher is likely to encounter or is expected to deal with, explorations entail that the teacher needs to be prepared to go beyond the prescribed curriculum, sometimes to mathematics she may not have learnt as part of her institutionalised mathematics learning.

B) While contingent situations where the teacher has to handle an unexpected student response is a characteristic of any classroom teaching, I suggest that in an exploratory context the teacher is more likely to encounter contingent situations - stemming from both the mathematics that students come up with and the ways that they talk about it.

C) In a curricular context, there is a privileged mathematical discourse, the discourse of the textbook, that guides the classroom discourse and implicitly functions as a norm to be followed. Having a more relaxed/flexible acceptability criterion in place, and in a culturally diverse classroom, a teacher is likely to encounter ways of communicating mathematics that are alien to hers. Attending to, interpreting and responding to mathematics expressed in differing ways without taking a deficit perspective can be challenging.

Thus I suggest that teachers face additional demands in terms of the content knowledge required; noticing the mathematics in student contributions, and interpreting these even when articulated in unfamiliar ways; and in resisting and countering the influence of pervasive deficit discourses. In this chapter, I elaborate on these additional challenges that facilitating an exploration at the margins entails and suggest some pointers to mitigate these challenges.

This study did not include a deliberately designed methodology aimed at answering questions on the implications for the teacher. However there were aspects of the study that threw light on these questions. The active collaboration with a practising mathematician and an educator that guided my interactions in the class may not be available to a typical teacher setting out to do explorations with marginalised students. Similarly the time and effort I could spend in working through the tasks that I proposed to do with students as an explorer myself, may not be feasible for all teachers. These factors together with the opportunity to return to the “data” and reflect on my experiences made it possible for me to become aware of aspects that supported me in my understanding and others that may have posed hindrances. For e.g., an abiding interest in recreational maths, puzzles and reading popular mathematics helped in finding starting points that could pan out into explorations. Prior teaching experience helped in establishing a relationship with the students. Being from a different class, caste and region from the students I interacted with and not being fluent enough in their dialect posed some challenges. However, the experience of having taught students across the board and the developing familiarity with the students that comes with long term engagement helped mitigate some of these challenges. Drawing these reflections together, in this chapter I attempt to lay out pointers for a teacher hoping to engage students with explorations.

The basis for this chapter is my experience of facilitating explorations in a marginalised context. The struggles that I experienced, the struggles and triumphs of the students that stood out for me in the the course of their engagement, the discussions I had with the mathematician of these and the day-to-day progress or lack of it of the class, the course corrections I did based on these discussions, and the records maintained of these discussions in the teacher diary and email exchanges form the data for the discussions in this chapter. These are also shaped by the mathematician's rich experience of facilitating explorations with students, especially those at the margins and my facilitation of explorations in places other than the project schools, not necessarily at the margins. As discussed in Section 3.7, the teacher diary and the discussion notes maintained were revisited multiple times, discussed within the team considering alternate perspectives and interpretations till a consensus was reached among the team members. In addition, I also draw on relevant literature around teacher support in the reform teaching context; knowledge demands for mathematics teaching; and noticing and listening to children’s mathematics and extend these to the context of explorations, to draw some implications for the teacher facilitating explorations at the margins.

In the following section, I elaborate on the challenges that a teacher is likely to face in facilitating explorations at the margins, based on my experience of doing so. I examine challenges stemming from demands on content knowledge, and those on the need for responsiveness in class especially when the language and mathematics may border on the unfamiliar. In Section 6.2, drawing on the ideas of Hypothetical Learning Trajectories (Simon, 1995) and Local Instruction Theories (Gravemeijer, 2004), I propose guidemaps prepared by practising mathematicians or "seasoned explorers" as teacher support for explorations. In Section 6.3, I draw on Mason (2015) and Davis (1994, 1997) to highlight the importance of responding and listening to students in-the-moment, with the intention of understanding and building on student contributions, and with a willingness to question our own biases that shape our perceptions and actions. In Section 6.4, building on literature on strength-based and anti-deficit framing to disrupt deficit discourses, I suggest reframing perceived "gaps" as "distances" to be traversed and illustrate how coherent formalisability could be an indicator of potential distance. In Section 6.5, I describe what mathematical engagement could look like in a curricular context. I also illustrate the possibility of noticing potential distances not just in exploratory sessions, but in curricular context as well and the possibility of traversing the distance between unfamiliar and familiar mathematics.

6.1 Challenges that explorations bring

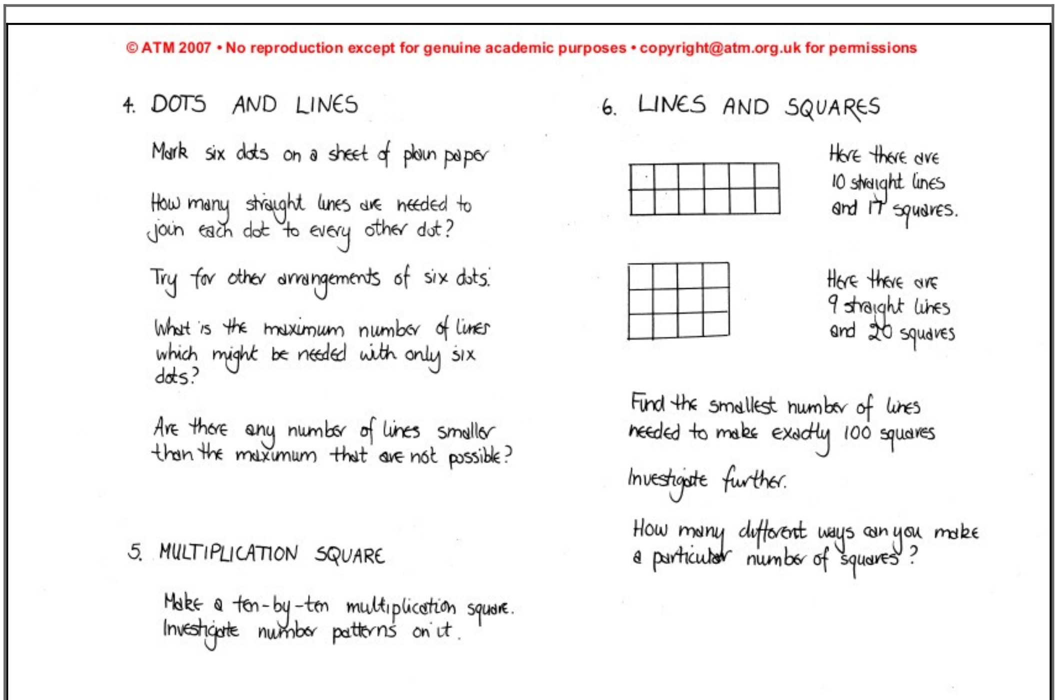
In this section I draw on my experiences of the challenges in designing and implementing mathematical explorations with students and my reflections on how I navigated these challenges. I focus on those challenges that are likely to be faced by a typical teacher seeking to implement such explorations in marginal settings.

6.1.1 Absence of ready-to-use material comparable to the textbook in a curricular context

One of the first challenges that I faced as I started out to do explorations is choosing an appropriate task. In Chapter 4, I discussed several features that make a task appropriate as a starting point for an exploration. Besides being aware of these features, one also needs to have knowledge of sources where material for designing explorations may be found. However, there is no equivalent to a "standard textbook" for explorations or an "exploration-bank" that the teacher could look into or guide herself by in the Indian context.

Other than my primary source of task suggestions from the research mathematician in the research team, the sources that I drew on were a) Task collections brought out by Association of Teachers of Mathematics (ATM) of the like *Points of Departure* (Hardy et al., 2007), *Starting Points* (Banwell et al., 1972), b) Popular mathematics articles and puzzle collections of Robert Kaplan, Ian Stewart, Martin Gardner, etc., c) Newspaper and e-zine columns of Alex Bellos (The Guardian), Dan Finkel (The Hindu), Pradeep Mutalik (Quanta), and d) rarely, books like those of Bearden (2016) and Sally and Sally (2003). Each of these sources require further work of a different nature to adapt the available content to a format usable in class. The popular mathematics articles need to be framed as tasks or activities that students could engage in. The newspaper columns on the other hand come with a built-in task framing that may be usable with some changes and they also give solutions.

The ATM collections offer a rich collection of usable tasks, but are limited to a problem statement and a few suggestions for directions in which the exploration could take off. A typical page from the collection *Points of Departure* is as shown in Figure 6.1.



4. DOTS AND LINES

Mark six dots on a sheet of plain paper

How many straight lines are needed to join each dot to every other dot?

Try for other arrangements of six dots.

What is the maximum number of lines which might be needed with only six dots?

Are there any number of lines smaller than the maximum that are not possible?

5. MULTIPLICATION SQUARE

Make a ten-by-ten multiplication square. Investigate number patterns on it.

6. LINES AND SQUARES

Here there are 10 straight lines and 17 squares.

Here there are 9 straight lines and 20 squares

Find the smallest number of lines needed to make exactly 100 squares

Investigate further.

How many different ways can you make a particular number of squares?

[15]

Figure 6.1: Points of departure - Examples (Reproduced with permission)

As a teacher intending to use these tasks in my class, I had to engage with the task as an explorer myself, solve the task, think through different approaches students might take and the possible branching points, the kind of nudges or hints I could give without giving away the solution and point to further explorable questions. There is no “solution booklet” where I could look up the solution. Some of these tasks needed modification before they could be used in the class. These include making the task easier or more difficult depending on the students who would be working on it, contextualising it if necessary, or spelling out goals more clearly. I have used modified versions of tasks 5 and 6 shown in Figure 6.1. I modified Task 5 by suggesting some examples of number patterns that could be investigated - how the numbers change/grow as they move along horizontal, vertical or diagonal lines, row and column totals in smaller squares drawn on the grid in different places and how they vary and so on. Task 6 in the figure was the

15. Thanks to ATM for permission to reproduce image.

inspiration for the “Partitions and cells” task discussed in Section 4.2.1. Instead of counting squares of all dimensions as intended by the version here, I started off with a simpler version that asked students to count only unit squares. With this starting point, one of my implementations further restricted the task to square grids and their generalisation, whereas another implementation explored rectangular grids as well, and focussed on questions of optimality as seen in Section 4.2.1. (What is the smallest number of lines that could make a given number of squares, or what is the maximum number of squares that could be made with a given number of lines etc.). The ATM collections offer a rich variety of tasks, but they are limited to *points of departure* or *starting points* as the titles indicate. This leaves a lot to be done by the teacher, including solving the task, and adapting it to her class.

Since explorations and “investigatory writing” are part of the regular school assessments in the UK, I found some teacher support material developed in the UK context. The series of 8 booklets by Midland Examining Group, Shell Centre for Mathematics Education (Maddern & Crust, 1989) give topic specific investigatory tasks (from such topics as practical geometry, pure mathematics, statistics and probability, etc.) and one task in each booklet is presented in a ready to use form, including detailed teacher notes including possible task variations, a case study describing a teacher’s reflection on doing the task with students, and samples of student work which demonstrate achievement at different levels. Each booklet also has six alternative tasks, with a student version of the task and brief teacher notes, which include pointers to the role of the teacher, but the task solution is absent. While this is definitely a valuable resource, it is still insufficient given that detailed notes are present only for one task of seven, and the teacher still has to work through the solutions of all tasks. The other sources of tasks mentioned like newspaper or e-zine columns or popular maths articles also require content knowledge beyond the school curriculum, knowledge of practices of mathematics and considerable work on the part of the teacher to adapt it into a task usable in the classroom.

6.1.2 Demands on content knowledge and practices of mathematics

Given the potential to branch out into multiple trajectories, possibly onto different domains of mathematics, and the salience of practices in an exploration, mathematical content knowledge and knowledge of practices become crucial to facilitate an exploration. Explorations place intense demands on what has been theorised in literature as Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK) (Ball et al., 2008; Carrillo-Yañez et al., 2018; Shulman, 1986). The practice-focus also calls on the teacher to demonstrate “mathematical modes of seeking, using and exemplifying understanding” and to “enact mathematics” (Watson & Barton, 2011) as a mathematician.

An exploration admits of multiple trajectories, multiple approaches, and affordances to function at multiple levels of formalisation. Some of the trajectories may lead to “trivial” or “obvious” results while others may require specific content knowledge and working through. Some of these may be easy starting points, especially at the margins while others may lead to yet unsolved problems. Being sensitive to potential trajectories and their difficulty levels calls for deep content knowledge which the teacher may not have access to. In a curricular context where the teacher may be teaching something which she has taught multiple times, and may have herself learnt she may have the required content knowledge. But this may not be so in the case of explorations where she may be on unfamiliar ground. Deciding which of these trajectories would be appropriate for a group of students calls for Knowledge of Content and Students (KCS) in Ball et al.’s (2008) classification. For example, in the Matchstick geometry exploration discussed in Section 5.2.2, the question being investigated was whether certain lengths - like the diagonal of a unit square - are “constructible” with the allowed steps of construction, namely laying unit-lengths end to end. The natural extension, asking what lengths are constructible with the straight-edge and compass constructions, involves non-trivial mathematics. Similarly replicating some shapes like a rhombus brought forth the question of how one would ensure that the angles are congruent without measuring and further onto what angles are constructible in matchstick geometry and still further to the corresponding extension to Euclidean geometry. I, as the facilitator, did not have a satisfactory answer then (or now) and being aware of the difficulties involved, did not explore this track with the students. Similarly, the final task of the exploration, on describability of shapes (see Section 5.1.2) also soon led to mathematics which I did not know. Being unfamiliar terrain for me, integer geometry and properties of rectilinear polygons were two other potential trajectories which I read and explored for myself in preparation for this exploration. This equipped me to anchor additional trajectories. This preparatory work was helpful when the exploration evolved to include the trajectory of rectilinear polygons in an implementation outside the project schools.

The multiplicity of approaches and the levels of formalisation possible also place demands on the teacher’s content knowledge. The teacher may have one or at best a few approaches to solve a problem and the student might come up with one, which is different and draws on a different content domain which is unfamiliar to the teacher. In addition, this may be expressed in informal terms, making it that much harder for the teacher to interpret and respond to the student contribution. For example, in the Leapfrogs exploration, my preferred method of solution was an algebraic approach, but a student came up with a graph theoretic formulation discussed later in this chapter in Section 6.1.3. In the Polygons exploration a student came up with what may be called “canonical constructions” to solve the problem, which was not part of the three or four solution approaches that I had anticipated (see Section 3.6). Both these approaches were not familiar to me, and hence went unnoticed. The mathematician who was observing these classes drew my attention to these approaches, which I missed on my own.

Mathematical practices are central to an exploration and an important goal of explorations is to create opportunities for students to engage in these practices. Different tasks may privilege different practices - while coming up with an appropriate representation may be key to one task, another may hinge on visualisation. Being question driven, coming up with mathematically significant questions, clarifying and reframing a vaguely framed question spelling out assumptions and conditions under which solution is sought are all valued practices in explorations. These may be relatively unfamiliar to the teacher given the focus on answers in the curricular context. The teacher needs to be aware of the expected/anticipated practices and be alert to the practices being engaged in by the students so as to draw attention to these practices and enable students to internalise them. These call for what Carillo-Yanez et al. (2018) call Knowledge of Structure of Mathematics (KSM) and Knowledge of Practices of mathematics.

As discussed in the previous chapter, in the course of a mathematical exploration, students will need to engage in such practices as conjecturing, justifying, refuting (may be through a counterexample), defining, exemplifying, generalising and specialising, drawing analogies or connections, optimising, recognising and backtracking from unproductive approaches and dead ends, etc., as they engage in explorations. Students may come up with a variety of conjectures - evidently false/true conjectures, plausible conjectures, some that may need to be restated for more specific/general domains etc. For example, among the claims/conjectures discussed in Section 5.2.1, S - 1 (that the Magic triangle puzzle has only three solutions) is false, S - 2, (that larger the numbers used in the triangle, fewer the number of solutions) looks plausible and S - 3 (for a given set of numbers in the Magic triangle, the side-sums for different configurations are consecutive numbers) is true only in the specific case when the numbers used in the Magic triangle are consecutive numbers. In the instance described, this was taken for granted because the group had not considered the possibility of filling in the triangle with other numbers. The teacher has to respond depending on the nature of the conjecture - some may require being ready with a counter example, some may need to be tested for extreme examples and the teacher may need to suggest such examples that could potentially add credence to or disprove a conjecture; the obviously true ones may need to be proved and so on. For example, in Section 4.2.2, we saw students conjecturing about the optimal number of questions and a strategy that will allow them to guess the partition in the Guess the colour exploration. I had to evaluate the strategy and produce a counterexample that would point to the situation where the suggested strategy would not work. As noted in the section, students themselves were doing this leading to the statement that “there was a hole in his strategy”. Similarly students may come up with a definition that differs from the one the teacher has encountered priorly and she needs to be able to anticipate the implications of working with an alternate definition. We saw an example of this in Section 5.3.2 where students came up with different definitions for “same shapes”. The teacher may also need to make a reasoned - choice between competing definitions, representations or strategies and needs to be able to spell out her rationale for the benefit of students. That is, the teacher needs to function as a mathematician, mediating between school mathematical experiences and disciplinary experiences (Watson & Barton, 2011), which is a challenge to the teacher.

6.1.3 Need to recognise and respond to mathematics in students' contributions

Students are engaged in the process of discovering things for themselves as they engage in explorations. As we saw in Section 5.3.4 students use multiple languages, draw on multiple modalities like visual, oral, written, gestural, use informal ways which may be vague or imprecise to express themselves. The teacher needs to listen for and understand the mathematics communicated through such means. For example when a student, Sneha, offered a phrase "*moodamudiyathu miss*" (it cannot be closed) by way of explanation that there cannot be two right angles in a triangle, I had to interpret this as: "A figure that has 3 straight sides and two right angles cannot form a closed shape and hence is not a triangle." The accompanying gesture of moving her hands up and down, palms facing each other and parallel to each other helped me see what she meant - that when there are two right angles, the two sides of the triangle will be parallel to each other and would not meet (close) to form a triangle. I was discussing this question in the context of the Polygons exploration and was expecting that they would use the angle sum property of a triangle to prove this, so that I could generalise to the case of other polygons. While Sneha's argument made sense to me, it didn't meet my expectations, nor did I immediately recognise the inherent mathematics. With later reflection and discussions in the research team I realised that what initially appeared to be an intuitive argument was actually an articulation of Euclid's fifth postulate.

When the student's approach to a problem is different from that of the teacher and perhaps one she may not be familiar with, making sense of an incomplete articulation can be challenging. I illustrate this with an example where as the teacher, my own mathematical knowledge was inadequate to interpret and respond to students' mathematics. In the course of the Leapfrogs exploration (see Section 3.6, and Section 6.2.4) with 3 tokens per side, a student argued for the minimality of 15 moves by considering the possibilities available to her at each step, and eliminating the ones that lead to wasteful moves. That is, if there are 3 moves that she could make from a given state of the game, and the first two possibilities effected the required transposition in 16 and 17 moves, say, and the third one in 15 moves, she would choose the third option. She says *"So after trying out the whole thing, If suppose there are three possibilities, and for the first possibility I try out, get 16, second I get 17 and third I get 15, then I try out the 15 one. So like according to me 15 one is the best possibility."*

To me this student's explanation sounded like a trial and error method, where she tried out the available moves at each stage, carried them through and chose the one that gave the minimum number of moves.

Had this been the case, an exhaustive trial of the possibilities would be needed to establish optimality. But the mathematician who was observing this session, could see her explanation leading to the formalisation of the situation as a graph, with each vertex representing possible states and each edge representing possible moves. In this formalisation, the problem transforms to one of finding the shortest path between the initial and final states. The existence of a unique best move at each step is the necessary condition for the existence of the shortest path and he drew the attention of the class to this through clarificatory questions to the student. Recognising the larger mathematical idea in the student's argument, identifying the element that she was perhaps leaving implicit (existence of a unique best move at each stage) and explicating it through questions requires a depth of content knowledge that comes from deep immersion in the discipline.

I also relate a similar episode when the mathematician was interacting with a classroom very similar to the one that was part of this study. Responding to the question "How many right angles can a polygon have?", a student Muthu replied: "*naalu thadava suthuna thiruppi angiyethaan varanum. naduvule ethana thadava venumnalum veliye poyittu varalam*". (If you turn four times you are back where you started, but in between you can go out and come in any number of times). Muthu is grappling with an intuitive picture of convexity and convex hull of points. His reference to exit and re-entry is the consideration of non-convex polygonal shapes. The coherent interpretation of Muthu's utterance is the assertion that the sum of exterior angles of a convex n-gon is 360 degrees, which is 4 right angles, and any traversal must complete the cycle on the fourth, whereas arbitrarily many zig-zags can be inserted in between in the non-convex case. Thus Muthu is making a complex assertion about linear traversals of convex and concave polygons, with clear mathematical thought underlying the intuition, while being entirely informal. In this case, the facilitator being a researcher trained in geometric algorithms was familiar with traversals as legitimate means of constructing induced n-gons, and could therefore perceive Muthu's strategy which was constructive rather than analytical.

Apart from mathematical difficulties in recognising the formalisation implied in an informal expression and providing appropriate support for a more mathematical articulation illustrated above, the informal language that students use could prove to be a challenge as well. In a classroom in a marginalised context, where there is "tension" between the home language and the LoLT, privileging talk over written mathematics and having a flexible acceptability criterion of coherent formalisability for students mathematical talk exacerbates the mathematical challenges discussed above and brings challenges that are specific to such contexts. We now look at the additional challenges posed by informal language use at the margins.

6.1.4 Listening and responding at the margins

The difficulty in listening, understanding and responding to students' mathematics is well acknowledged in literature. "Listening effectively and responding to children's mathematical thinking is surprisingly hard work. Research indicates that years, not months, are required to develop the personal resources needed to teach in ways that incorporate responsive listening" (Empson & Jacobs, 2008). Listening is at the core of using students' mathematics productively, but teachers may be unprepared to hear and see things the way students do. They may not have seen students solving problems in a similar way, nor would they have solved it in this fashion themselves. This poses challenges to listening. "Use of children's mathematics in teaching is a specialised skill and, for most teachers, requires a significant shift in how they conceptualise their role" (Empson & Jacobs, 2008, p. 259). This is more so when the teacher's socio-economic and mathematical background differs from that of students. Non-standard terminology that students use, incomplete or inappropriate articulation, unstated assumptions and intentional hedging and vagueness that students bring in when they are uncertain, all pose challenges in terms of language.

Students may use self-created terminology when they are not aware of the standard term for a concept, reinterpret terms or when not aware of the "standard" definition of a concept redefine them in their own ways. We saw numerous examples for such usage in the previous chapters - For example students use of "standing and sleeping" for "horizontal and vertical", reinterpreting "polygons" to include self intersecting shapes, (Section 4.2.1), use of the word "half-double" for one-and-a-half in Section 5.2.2, use of the word "tiruppal" for rotation, without being specific about what is being rotated, "ulta" or opposite for mirror image in Section 5.3.1, different students using the word "same" to mean different things including similarity and congruence in Section 5.3.2, etc.

Terminology apart, the way the student articulates her insight/finding with missing words and dietetics whose referents are not clear may make it difficult for the teacher to understand what the student is saying. For example, in the Magic triangle exploration, here is how a student Sumi articulated the transformation of moving the numbers around by one position (Section 4.2.3, Figure 4.9 (b)) to give a new solution.

Sumi: *Inthe numbers vanthu lineaa ippidi exchange pannite vantha puthussu puthussa varuthu* {If these numbers are exchanged in a line like this new new < > are coming}

About a minute later she re-words this as

Sumi: *Numberse appadiye rotate pannitte vantha puthu puthu solution varuthu* (If we go on rotating the numbers like that, new new solutions are coming}

The student is using “*lineaa* exchange” to mean shifting position by one space and “rotating the numbers” refers to the cyclic nature of the move. The second articulation captures the transformation better, and clarifies that the noun that the adjective “new new” qualifies is a solution, but is still not a clear and precise articulation. When pushed for further clarification, the student chose to explain what she means through an example rather than articulate it in clearer terms. The class and I had to make sense of what this student was saying based on the example and contextual cues.

Sumi, exploring transformations further, suggested interchanging the numbers in the inner and outer triangles as another transformation that might give a different solution. She articulated this tentatively as “Intha threeyum intha threeyum interchange panninna …” (if we interchange these three and these three…), pointing to her notebook.

Swapping the numbers along the median as shown in Figure 4.9 (a), Section 4.2.3 does indeed amount to interchange numbers in the inner and outer triangles with some rearrangement as can be seen from Figure 6.2.

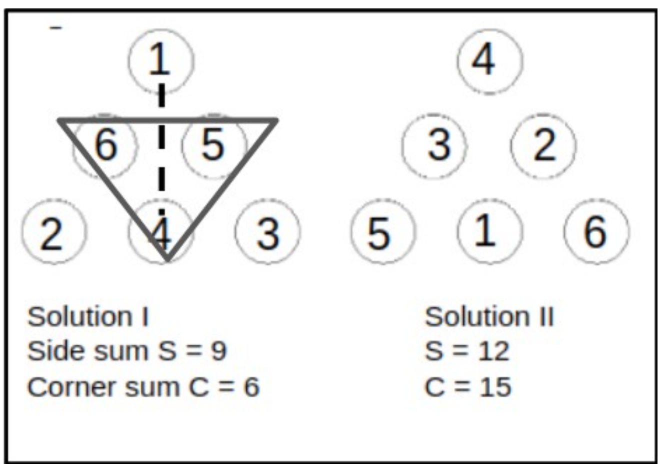


Figure 6.2: Magic triangle: Interchanging the inner and outer triangles

Fixated in my way of thinking of this as a median swap, I could not connect to this student's articulation, nor make sense of it until much later while listening to the recording and reflecting on the class. My immediate response to the student was to ask her to clarify her statement further - which three and what she means by interchange. Not being able to add to what she already said, she did not pursue the idea, and I missed an opportunity to nudge this student to a clearer articulation. Perhaps my push for clarification was not the appropriate response in the situation. In attending to the form of speech, perhaps I lost sight of the student's conceptual focus.

Issues of comprehension apart, the teacher needs to be wary of her own preconceived ideas and judgements coming in the way of hearing what students say. Often what we "hear" is dependent on what we "listen for", or what we are anticipating. Guided by the textbook discourse there is a certain way of talking that the teacher expects from the students in a mathematics class. What the teacher hears in imprecisely and incompletely articulated student formulations may deviate from her expectations and therefore lead to a deficit perspective of students. Listening becomes that much harder when what one hears is unlike what one is tuned to and expects in such situations. Also language carries markers of class, caste, community, region. In a situation where the teacher's socio-cultural background differs markedly from that of students, the teacher needs to be sensitive to her own biases coming from her background and where she is *listening from* (Davis, 1994). Listening from the position of authority or of the custodian of formal mathematics may only lead to further marginalisation.

6.1.5 Absence of prescribed assessment criteria

Yet another challenge that the teacher faces is the lack of prescribed assessment criteria. Assessment criteria give the teacher a sense of what is to be valued and encouraged in a class. Traditional assessments privilege the correct answer and consequently obtaining the right answer becomes central to teaching-learning as well. In the case of an exploration, there is no specified solution or defined end point to be reached, there could be multiple trajectories and a solution arrived at could give rise to more questions. This gives rise to the question if some trajectories/questions are to be privileged over others and the basis on which such decisions could be taken. Similarly in the case of individual students, there is an implicit sense of the nature of mathematical engagement that is valued, but there is no clearly articulated criteria that defines "progress". A teacher who is used to externally defined curriculum, assessment formats and rubrics for evaluation, may find this absence of criteria confusing and may feel the need to have some sense of direction.

Given this lack of well-defined criteria and limited life-span of an exploration in a class, we felt the need to define some points of conclusion in an exploration which we would expect every student in the class to reach. For example, in the case of the Magic triangle exploration, we had defined, finding the four solutions and coming up with an argument that there are no more solutions as such a point of conclusion where we expected every student to reach. From this point, we expected interested students to move ahead and explore further on their own, seeking help when needed. Similarly in the case of Leapfrogs, we had minimally expected every student to find one way of making the transposition and be able to repeat the set of moves. These benchmarks are very much context dependent and need to be defined by the teacher and standardised assessment criteria may not help.

Beyond defining such “must reach benchmarks”, we had a sense of “richness” of an exploration in some implementations compared to others (gauged by such factors as the multiplicity of approaches seen, questions raised, insights offered, etc.), without having a clear articulation of what made those sessions rich. For example, how does one compare a session in which students investigate multiple conditions under which the Magic triangle puzzle could have a solution, without reaching any conclusion with another where they use methods of algebra to investigate the extensions and generalisations of the problem? Without a clear articulation of what needs to be valued in a session, what “progress” of an exploration means, the teacher may find it difficult to choose between the multiple trajectories that open up in her class.

The end goal of explorations is that students become better explorers, but what it means to be a better explorer is open to interpretation. In addition to arriving at expected answers or intermediate landmark points, there are other equally important aspects that need to be taken into account. Posing pertinent and fruitful questions are crucial to further advancing an exploration. Elegant solution approaches and those approaches that open up further questions and opportunities for extensions and generalisations are preferred to brute force solutions. Also, the goal itself is a long-term goal. So it is difficult to gauge, in the course of one or a few explorations, if the student has moved towards the desired goal or not. The very first exploratory task a student engages may very well be tightly scaffolded or largely teacher led, but what matters is how much of the ways of thinking and practices that were suggested to solve the problem has been internalised by the student and available to draw on at a later stage. One needs to have markers for the stages in the progression of mathematical thinking of the student as also for the progress of an exploration itself. These need to be abstracted from multiple explorations which may have very different surface features. This study highlighted the need for such criteria and the complexity involved in coming up with them. While we did see some indicators of students “becoming better explorers” like their coming up with task variations and thinking of generalisations without being prompted, critically evaluating others’ contributions and responding to them, etc., and marked some elements that made for a “rich implementation” the task of clarifying and articulating these criteria is not addressed in this thesis and remains to be taken up in our future work.

Having identified some challenges that explorations bring, some that specifically stem from language-use at the margins I now look at ways of supporting the teacher to overcome these challenges. In the following section, I propose guidemaps as reference material to support teachers to meet the demands on mathematical content knowledge and in the subsequent section provide some pointers to listening and responding to students' mathematics and language in marginalised contexts.

6.2 Guidemaps as teacher support for explorations

In the previous section, we discussed the challenges in designing an exploratory task or adapting available material to be used in a class. We also noted the increased demands on content knowledge posed by the likelihood of an exploration progressing to content domains unfamiliar to the teacher and illustrated through instances where, as the teacher, my content knowledge was inadequate to interpret and respond to students' mathematics. I considered the possibility of a guidemap that lays out possible trajectories in an exploration and the relevant content knowledge and asked what features should such a guidemap incorporate so as to help the teacher facilitate explorations. First we look at the appropriateness of a well chalked-out plan similar to the many "extended tasks" available online, as teacher support for explorations.

6.2.1 What a well-chalked out plan misses out

Figure 6.3 shows a series of guided prompts for the Leapfrogs exploration, that could be used "as is" in class. The prompts incorporate a trajectory of solving the task out for a particular small number, trying out for larger numbers and generalising and further suggesting variations. I developed these prompts modelling them on those for similar tasks that I found elsewhere (Burkhardt, 2009, pp. 10, Figure 3), which offer a plan for a teacher to follow. Similar plans are available for many tasks.

While providing a starting point, a path to generalisation and suggesting other possibilities to explore, the design follows the Data-Patterns-Generalise (DPG) structure. Scholars have found that the DPG structure limits expectations about student work and the richness of mathematics that could be derived from a task (Blanc, 1997; Hewitt, 1994; Morgan, 1997). Morgan (1997) draws attention to the tendency of such tasks to "stereotype investigations" to the DPG structure, which she describes as problems that involve "generating numerical data from several examples arising from the given starting point, spotting a pattern in this numerical data and forming a generalised description of the pattern, preferably using algebraic symbols to express the relationship between the variables. (p57) "Hewitt (1994) also marks the tendency of such task formulations to focus on spotting number patterns and extending them, often disregarding the mathematical situation they came from. According to Hewitt, it is important to stay with a particular situation and learn about the mathematics inherent in it rather than "learning about numbers in a table" (p 51) and ask different questions of different situations, rather than focus on generalisation alone. From this perspective, exploring questions as what if the blank space were at the end instead of the centre? What if we allow jumping over more than one token? What if the arrangement were circular instead of linear?

may be more rewarding than generalising the Leapfrogs problem to larger numbers (and for any number) of tokens. The task design should ideally provide some pointers in this direction.

Six tokens, three each of two colours are laid out in a line of 7 spaces as shown. I want to interchange the black and white tokens, but I am only allowed to move tokens into an adjacent empty space or to jump over one token into an empty space. Can I make the interchange?

- Could you make the interchange? In how many moves?
- What is the minimum number of moves in which you can do the interchange?
- Convince your friend that this is indeed the minimum number of moves
- Make the interchange with 4, 5, 6 tokens of each colour, arranged in a line with an empty space at the centre, as above. Denote a Jump by J and a slide by S and write out the sequence of moves in a sequence of S and J. How many moves did you need in each case? You may want to tabulate your results.

No: of tokens per side	Sequence of moves	Total No: of moves	Number of Jumps	Number of Slides
3	SJSJJSJJJSJJSJS	15	9	6

- Predict the sequence and number of moves you would require if there were 7 tokens each of both colours? How many jumps and how many slides? Verify if your guess is correct.
- Write an expression for the number of moves that is required to make the interchange when there are n tokens of each colour. Convince your friend that the expression works for all values on n.
- Investigate other possibilities, for example other configurations and rules of movement.

Figure 6.3: Leapfrogs: A plan

Representing and recording each move is crucial to solving the problem. The plan above suggests a representation and that way makes it easier for the teacher and students. We have seen students coming up with different representations, making it necessary for the teacher to discuss and choose between these representations. Though demanding on the teacher, this is an important practice of doing mathematics and the task offers an opportunity to engage in this practice. But in privileging a particular representation without discussing other possibilities, the design takes away the flexibility that we intend explorations to bring.

Though suggesting a generalisation through small numbers to larger numbers on to thinking about "any number" and on to an expression that gives the number of moves for any number, the task formulation does not give a hint as to how a student might solve this task. We have seen students pass through some "stages" before they arrive at the minimum number of moves to make this interchange. They usually start by moving the tokens around until they chance upon a way to make the interchange. They need to "get a sense of the sequence of moves" (Mason, 1989) before they can try to optimise the sequence. So repeating the sequence of moves and if possible articulating any strategy they may have to make the moves becomes important. Very often we have seen students getting a sense of when they are making redundant moves – marked by arriving at a configuration where two adjacent tokens are of the same colour, and avoiding this situation by choosing an appropriate move. These insights are derived from observing multiple groups of students engaging with the task. In a curricular context, the teacher may be teaching something which she has taught multiple times, or at least learnt herself, and can anticipate such trajectories. It is unlikely that she would have engaged with explorations sufficiently frequently to come to know these oft taken approaches and milestones on the path to the solution. However, knowing these stages will help the teacher be better prepared and provide appropriate scaffolding to students struggling with the task - for example asking them to repeat a series of moves through which they made the interchange, asking them to record the moves as a means of doing this, studying these to see if some moves are redundant, drawing attention to backtracking moves that they make and the particular configurations that necessitate such moves and so on. So I suggest that the teacher needs to have much more than a series of guiding prompts to enable her to facilitate an exploration in the class. She needs to know the rationale for those prompts, what she could possibly do if the suggested prompts don't work and how she could customise the tasks if she needs to and pointers to how the task might evolve in the class. Thus, supporting students to explore mathematics cannot happen through a scripted sequence of prompts. Drawing on Simon (1995) and Gravemeijer (2004) I suggest guidemaps prepared by mathematicians, seasoned explorers who are familiar with the lay of the land, as means of teacher support for explorations.

6.2.2 Hypothetical Learning Trajectories and Local Instruction Theories as teacher support in reform teaching contexts

Simon (1995) suggested the idea of Hypothetical Learning Trajectories (HLT) that involves anticipating how student thinking might develop and envisioning activities that help them develop the mathematical insights one is aiming for. Gravemeijer (2004) suggests Local Instruction Theories (LIT) conceptualised and experimentally verified by educators as the framework of reference to base their HLTs on. Similarly responding to student mathematical thinking in the context of explorations requires the teacher to think through or be aware of the potential trajectories along which an exploration could evolve. I suggest that guidemaps for explorations prepared by research mathematicians and maths educators can function as reference points for developing “hypothetical exploratory trajectories” (HETs).

Simon suggests that despite the idiosyncrasies of the learning paths of individual students, the students of the same class demonstrate an “expected tendency” to follow similar paths. This assumes that an individual’s learning has some regularity to it and hence many students of the same class can benefit from the same mathematical task. The HLT characterises this “expected tendency” and provides the teacher with a reason for choosing a particular instructional design. Drawing on her mathematical knowledge and knowledge of students, the teacher hypothesises such a path by which learning might proceed. Coming up with an HLT also involves designing learning tasks and activities grounded in what students know, and being able to provoke the kind of thinking that would lead to the desired learning for them. Thus, the generation of a HLT prior to classroom instruction is the process by which the teacher plans for a classroom activity.

As pointed out in Section 6.2.1, we have observed that in spite of the inherent flexibility, most explorations also have an “expected path” of evolution, branching out in different directions with different groups of students. This makes it possible to extend the notion of HLTs to explorations as well. However, unlike the HLT for a particular class in the curricular context, for which the teacher chooses a well-defined goal which directs the design of the learning tasks, an exploratory trajectory has multiple goals and corresponding branching trajectories, which the students choose to pursue and for which the teacher needs to provide the necessary support. I term these potential trajectories “hypothetical exploratory trajectories” (HETs)

The teacher observes and communicates with students as they engage with the planned activities and uses the understanding derived from this to adapt the initial plan to the emerging student conceptions. This would call for a knowledge of the way students’ thought process related to a particular concept evolves, the stages in this evolution and means or prompts to enable movement between these stages, in addition to understanding student thinking itself. Gravemeijer (2004) argues that it is unfair to expect teachers to invent hypothetical learning trajectories without any support and proposes externally developed “Local Instruction Theories” (LIT) as support for teachers in coming up with HLTs. He uses the term HLT for planning of instructional activities in a given classroom on a day-to-day basis, and the term LIT to refer to the description of, and rationale for the envisioned learning route as it relates to a set of instructional activities for a specific topic. As opposed to providing the teacher with pre-designed learning activities which do not give much room for responsiveness to students, externally conceptualised LITs offer the teacher the knowledge base required to come up with her own plan of well- reasoned activities appropriate for her class and to adapt these, factoring in student responses.

An LIT consists of a conjectured learning path and possible means of supporting movement along this path. It spells out a sequence of stages or milestones in terms of the key insights in understanding a mathematical idea, a series of instructional activities to progress along these stages and the role of the teacher in facilitating these (Gravemeijer, 2004). It is intended to help the teacher envision the thinking and learning students might engage in as they participate in the instructional activities. Analogous to LITs in the curricular context, I suggest guidemaps for explorations incorporating these very features and functions.

Using a travel metaphor, an LIT can be considered a “travel plan” that specifies the starting and ending point with possible route(s) between them with major milestones along the route marked and the means of travelling between them. The teacher has to transpose this into an actual “journey”, by choosing appropriate means of travel (instructional activity) along a route which may deviate from the mapped out routes. But the expectation is that having the map in hand, with the milestones marked and suggested means of travelling between them, the teacher would be able to choose an appropriate path between the endpoints through alternate means if required. By providing an externally prepared travel plan or guidemap of the terrain of a specific topic, an LIT supports the teacher in adapting her teaching to the current and evolving understanding of her students and at the same time planning instructional activities in advance, thereby addressing the need to “plan on-the-fly”.

Anticipating students’ thinking around an exploration and customising it to a particular group of students, taking into account their mathematical background requires the teacher to be aware of the various possibilities and what it takes to follow the various trajectories , more so because there are no curricular reference points or benchmarks. Analogous to the LITs, I propose externally prepared guidemaps as support for teachers to come up with hypothetical exploratory trajectories (HETs analogous to HLTs) and ways to provide appropriate scaffolding as students take these trajectories. I look at what elements the guidemap needs to have to enable the teacher to come up with HETs for her class. Guidemaps need to minimally include the pre-requisite content knowledge to engage with an exploration, potential directions an exploration could take off in and the stages of progress in each of these and the mathematical practices that the exploration calls for.

6.2.3 Elements of guidemaps

Useful Prerequisites: One of the task features we identified to make the task accessible at the margins was limiting prerequisite content knowledge. Nevertheless, some explorations might build on a particular theorem or a procedure or familiarity with a certain representation. Some of these may be crucial even to get started on the exploration, while others may help progress in the exploration. For example, the Magic triangle exploration requires no special prior knowledge to get started. However progress is constrained by a facility for algebraisation. The Polygons exploration crucially depends on knowing the angle sum property of polygons and there is very little that can be done on this exploration without this. Knowledge of factors, multiples and highest common factors helps in working on the Clapping game exploration, but one can engage with the exploration even without knowing these concepts and build them as needed. It may be possible to side-step some requirement by simply pursuing an alternate approach to the problem or reframing it alternately. For example, the need for an algebraic formalisation to prove the existence of four and only four solutions for the Magic triangle problem can be side-stepped through proof by systematic and exhaustive counting or other proof strategies. The guidemap needs to spell out the prior knowledge required to engage with an exploration, and ways of side-stepping the need for some knowledge if this becomes necessary.

Also, one needs to be aware that the “prior-knowledge required” need not be a make or break requirement in the case of explorations. For example, the Polygons exploration could start with trying to arrive at angle sum property, or if the requirement is just to use the theorem, it may very well be “looked up” or even “handed down” to the students depending on the centrality of the theorem to the exploration at hand and the mathematical maturity of the students. Also unlike in a curricular case where every child needs to have the necessary prior knowledge in an exploratory context it suffices if one child knows the result and together the group can draw on this knowledge to solve the problem. Thus in an exploratory context, recognizing what theorem or procedure needs to be drawn on and where or how to obtain and use it is more important than knowing the theorem itself.

Key Insights: Almost every exploration relies on a moment of key insight at which the explorer experiences a transition from being muddled to being able to see a route to the goal. This might be a glimpse of the underlying structure of the problem, a particular representation/ formalisation that will aid arriving at a solution or a particular theorem which could be applied. For example, the key to the Magic triangle exploration is the particular formalisation of the numbers suggested in Section 3.6, that leads to a generalisable relation between the corner-sum and side-sums. For the Leapfrogs exploration recognising redundant moves and ways of eliminating them is the turning point that leads to the solution. These key insights may not occur to students as a matter of course. In such cases, the teacher should be able to provide clues or nudges that lead students on to these insights unobtrusively. It is also important to do it at the right time, when students have tried out multiple things unsuccessfully, but are not yet frustrated enough to give up.

***Landmark points**:* Other than the key insight that points to the goal, there could be other landmark points from which further branching is possible. Even with all the variations possible, we have seen some well marked trajectories lined with a series of realisations that come up repeatedly in re-runs of an exploration. These are like the "stages" or milestones described by an LIT in understanding a concept. Some of these may be helpful insights which lead on to the solution, while some may be misleading conclusions arrived at without sufficient thought and some may be potential branch points. Such points would also be termed landmark points and knowing these in advance helps the teacher provide helpful nudges or course corrections as need be.

In the Clapping game exploration, one of the first conjectures that come up is that not everyone claps only when the interval at which clapping happens is a factor of the number of persons in the circle. Knowing that this incorrect conjecture is likely to arise, the teacher can be ready with a counterexample that can refute this, and ask the class how they would modify the conjecture in the light of this example. In the Magic triangle exploration, the realisation that there are 4 and only 4 solutions is a landmark point from which the exploration could branch off into multiple directions. It is a significant step in the progress of the exploration. Having found the four solutions, one could look at transformations of solutions - what transformations lead to other solutions and what ones lead to non-solutions, or proof that there exist four and only four solutions or examine patterns in solutions (say parity rules) or look for solutions with other sets of numbers etc.

Potential and Likely Trajectories: As pointed to earlier multiple trajectories are a key feature of explorations. The multiple trajectories ensure that students with varying mathematical backgrounds may still engage with the exploration at their own level. Some possible variations of the Magic triangle exploration are discussed in Section 4.2.3. Some of the trajectories may require advanced mathematics and some of them may even be yet unsolved problems by the community of mathematicians. Pursuing such tracks may lead to frustration for students. The teacher may also want to plan for variations based on perceived strengths and weaknesses of different students. So it would do well for the teacher to know the possibilities, and what it takes to engage with each of them and their appropriateness for her group of students.

Practices in focus*:* As discussed in Section 6.1.5, the end goal of an exploration is becoming better explorers and this calls for a shift of focus from content to practices and a deeper engagement with the practices of the discipline. Exploratory trajectories should commit to a variety of mathematical processes: reformulating questions to make them clearer and thereby easier to address; experimentation with approaches, notions, techniques; movement from performance of activity to prediction; observing and recording patterns; use of symbols; argumentation; and so on. The teacher needs to anticipate, observe and draw attention to the practices that are being engaged in to enable deeper engagement. So the guidemap needs to spell out the practices to be anticipated in the course of an exploration.

Affordances of task specific choices: Many of the exploratory tasks are generalisable and any number could be the starting point in principle. However, the task specific choices - or the specific numbers or figures that are chosen to launch a task - need to be chosen with due consideration for the group of students and the affordances of specific numbers. For example, starting the Magic triangle exploration with a triangle makes it much more accessible than starting with a square or pentagon. While 3-5 tokens a side may be an optimal starting point for the Leapfrogs exploration, starting with 5 tokens a side gives the teacher an opportunity to draw attention to the heuristic of “solving a simpler problem as a first step to solve a complex problem”. But for students who are not used to the sustained effort required to solve a mathematical challenge starting with three or even two tokens a side may be a more accessible option than the five. In the Clapping game exploration, a number less than 10 in the circle may not be an effective start as these numbers have a limited number of factors. 20 has many factors, is next to 19 a prime and 21 with fewer factors. These properties of 20 make it a good start for Clapping game exploration compared to other numbers in the same range. The teacher needs to be made aware of these specifics so as to enable her to customise the task herself.

I now present an illustrative guidemap for the Leapfrogs exploration incorporating these elements. Thereby, I hope to convey a sense of the level of detail needed for the guidemap to function as an adequate support for the teacher.

6.2.4 Illustrative guidemap for the Leapfrogs exploration

Overview: The starting point is the game shown in Figure 6.4. The possible objectives could be a) minimise the number of moves to effect the interchange and come to an optimal sequence, b) Justify optimality, c) Generalise the game to *n* tokens, d) vary the rules of movement and other parameters and analyse the resulting games.

LEAPFROGS

Ten tokens of two colours are laid out in a line of 11 spaces as shown, I want to interchange the black and white tokens, but I am only allowed to move tokens into an adjacent empty space or to jump over one token into an empty space. Can I make the interchange?

Figure 6.4: Leapfrogs exploration

Useful prerequisites: Being a game situation that can be actually played out with tokens, the game requires no specific prior mathematical knowledge. However, facility with algebra will be useful if solving the general case.

Key Insights: Recognising redundant moves and the felt need to figure out ways of avoiding them is the key insight that points to a solution path. In this case tokens of the same colour coming in adjacent spots (other than in the initial or final configurations) is a pointer to redundant moves.

Landmark Points: The stages that we have usually observed as students engage with the problem are

- Making the transformation in whatever number of moves
- Repeating the transformation with the same moves
- Recognising situations that lead to “wasteful moves” and avoiding them
- Making the transformation in fewer number of moves
- Making the transformation in minimal number of moves
- Justifying optimality
- Generalisation
- Formal proof for optimality, expression for optimal moves in terms of the number of tokens
- Variations

While this list is not a rigid sequence of steps and two (or more) steps could overlap, at least a few of these steps can be seen in nearly every run of this exploration.

One also needs to make a judgement on how far along the sequence a given group of students would go and tailor the exploration accordingly. Students being able to make the transformation in the optimal number of moves with say 4 tokens-a-side; or students being able to make the transformation with any number of tokens-a-side optimally, without being able to articulate their strategy or the number of moves required; being able to inductively obtain the number of moves for any number of tokens, could all be reasonable stopping points depending on the background of students.

This exploration allows for use of physical tokens and moving them around. In this case, the act of moving around these tokens may provide clues to a procedural solution to the generalised problem of interchanging any number of tokens as well. This can be leveraged by the teacher to nudge the students towards a solution. Perhaps, the repeated cycles of the movement is easily noticed or “felt” by students and we have had students sensing an error when the “rhythm” of this movement is broken. It is also possible that students are able to extend this “rhythmic” movement, extending the cycle when there are more tokens, without explicitly articulating what they are doing. (This was articulated by a student as “ma’am the hand knows”) The “physicality of the moves” could be a way to nudge a solution in case students are unable to come up with an optimal solution for the problem, with the teacher demonstrating a solution perhaps in quick succession and students repeating it later.

Repeating a sequence of moves and examining it for redundant moves is an essential step here. So, while it is good to start with physical tokens, one needs to move to some ways of representing/recording a move, which may either be teacher suggested, or student produced. If multiple representations come up one may need to evaluate the pros and cons and choose one to enable easy communication within the group. An appropriate representation also brings to light certain patterns in the moves, which eventually lead to a solution. Also representation allows one to move away from physical tokens and move towards a generalisation.

Possible Trajectories: The initial question only asks if an interchange is possible at all. But it begs other questions - If possible in how many moves? Can it be done differently? Can it be done in fewer moves? Is there an optimal number of moves and so on.

Variations of the game could be created in many ways

- changing the position of the blank space by perhaps having unequal tokens on either side of the blank space, or having the empty space at an extreme,

- Changing the rules of movement by say allowing for say jumping two tokens, swapping tokens, allowing placement of one token on another, or “wrap-around” movement from one extreme to another or disallowing sliding movement
- Arranging the tokens differently - like a circular arrangement, or a triangular arrangement
- Adding tokens of more colours
- Making the tokens “distinguishable”

While it may not be possible to “solve” many of these, even the exercise of coming up with a variation clearly spelling out the initial configuration, final configuration and the set of allowed moves is a worthwhile exercise. Some analysis of these configurations could lead to statements like “given the rules of the game it is NOT possible to make the transformation, or configuration x,y,z will not arise” etc.

At the other end of the spectrum, for a group of first-time explorers even finding the optimal sequence may be a challenge. With this group one track that could be taken up is to suggest a representation of the sequence of moves in terms of the slides and jumps in a sequence of say S and Js to get a sequence SJSJJSJJJSJJSJS of moves for three tokens. Examining these sequences for 1, 2, and 3 tokens and perhaps chunking them as S, JS, JJS, JJJ, SJJ, SJ, S breaking at points when a different colour token is moved could suggest some patterns which can be used to predict and verify the sequences for more tokens. Also coming up with variations of the game is a task that is accessible to all and in our experience something that is greatly enjoyed as well.

The expression for the optimal number of moves can be expressed in many forms - in closed form $n(n + 2)$, or “one less than the next perfect square”, or “keep adding odd numbers” etc. Similarly there are multiple approaches to proving optimality as well. An algebraic approach could look at the “total shift in positions” that needs to happen and the total “jumps” that needs to happen (every token should jump over every token of the other colour and shift position by $(n + 1)$ positions) and use this to calculate the number of jumps and slides and arrive at the number of moves. Informal arguments centering on maximising the number of jumps without allowing for redundant moves, or considering options available at each move and choosing the best available (one that does not lead to redundant moves in the next or subsequent turns) at each turn (search for an optional path across a graph) may also come up. There may be unstated assumptions - for example the existence of a unique best choice for each move above - which may need to be clarified/explicated and some which may be obvious. Whether to insist on proving these assumptions is a choice the teacher has to exercise in the context of her class, but those assumptions which are critical for the argument to hold should at least be made explicit.

Practices in focus: In addition to observing and articulating patterns, conjecturing, proving, justifying, predicting and verifying, specialising and generalising, etc., the practices that are salient here are optimisation and representation.

Optimisation: The driving question of this exploration, “can it be done in fewer moves?” and the heuristic to approach such problems, “avoiding wasteful moves” are ideas that are frequently encountered in mathematics.

Representation: Most probably, one needs to play the game a few times before hitting on an optimal solution. Also one might want to take a careful look at the sequence of moves that one has made, both to observe patterns and to avoid redundant moves. This means there has to be a way of recording the moves. The following representation of the position of tokens at every step of the game, where B stands for a Black token, W stands for a white token and O stands for the blank space, shows up some symmetries like the rows equidistant from the central row above and below it being mirror images of each other.

BBOWW

BOBWW

BWBOW

BWBWO

BWOWB

OWBWB

WOBWB

WWBOB

WWOBB

One can also see a pattern in the way the blank space moves. Not all the patterns that are seen may be of significance. Labelling a slide as S and a jump as J, and representing the sequence of moves in a sequence of S and Js, as mentioned above makes visible other patterns. The representation chosen can help or hinder the solution in this case. Also by inviting students to come up with their own representations, the teacher can create an opportunity to talk of properties of a good representation and criteria for choosing between them. Figure 6.5 shows some representations that we have seen.

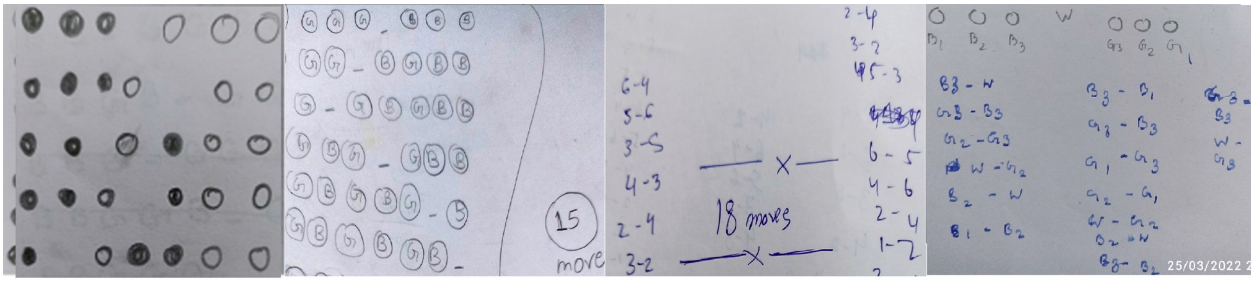


Figure 6.5: Leapfrogs: Multiple representations

The first representation the student has represented the black and white tokens by shaded circles of the corresponding colour. The second representation is similar with the only difference that the tokens have been labelled - B for blue and G for green. The tokens are considered indistinguishable. Both these are visual representations. In the third representation the student had numbered the positions 1-6. The first move, 6 - 4 means that the token at position 6 moves to position 4. The next move 5 - 6 means that the token at position 5 moves to position 6 and so on. In the 4th representation, the student has labelled the places B1, B2, B3, G1, G2, G3 and the blank space W. In this representation B3-W means that the token at position B3 moves to position W.

Affordances of task specific choices: Five tokens to start with could be challenging, but allows one to draw attention to the heuristic of solving a simpler problem as a means to solve a complex one. For a group that is not used to figuring things out on their own, starting with 3 may be ideal. Starting with physical tokens to play with provides an easy entry point.

The features presented in the guidemap above could be expanded to offer more details. Also, I do not claim comprehensiveness in the features that have been listed as components of the guidemaps, nor do I claim that all of them are relevant for *all* explorations. Nevertheless, I do believe that having a grasp of these essential characteristics considerably eases the path of a teacher setting out on an exploration and addresses some of the challenges identified in Section 6.1. A collection of such guidemaps functions as a source and reference material for explorations. The mapping out of potential trajectories and practices in focus helps by equipping teachers with the required content knowledge. The identification of key insights, landmark points and the trajectories supports the teacher to anticipate student trajectories and be prepared with ways of responding to them. I intend to study the usefulness to, and resourcefulness of teachers to use these guidemaps effectively and a deeper analysis of the demands explorations place on the teacher in my future work.

6.3 Listening and responding to students' mathematics

Even if a teacher is sufficiently familiar with the exploration through her own preparatory explorations and the study of guidemaps, an exploratory classroom is replete with contingent moments (Rowland et al., 2015; Rowland & Zazkis, 2013). Responding to such contingencies requires something more than content knowledge acquired from formal courses in mathematics or gleaned from guidemaps. The teacher needs to listen to and respond to students' mathematics in-the-moment and this requires what Mason and Davis (2013) call “a vital connective tissue between mathematical awareness and in-the-moment pedagogy” so as to have a sensible pedagogical action come to mind in response to the mathematics noticed in a student contribution. In Section 6.3.1, I summarise relevant discussion from literature on what in-the-moment pedagogy entails and how teachers can be supported in this.

The criterion of coherent formalisability widens the scope of the discourse that a teacher may admit in class, making it that much more challenging for her to notice and respond to students’ mathematics as seen in the examples described in Section 6.1.3. In Section 6.3.2, I discuss what is entailed in listening for and noticing coherent formalisability and its potential violations in students’ mathematical talk. The teachers’ responsiveness in-the-moment depends on what she listens for and notices. Being aware of potential violations of coherent formalisability, I hope will sensitise the teacher to listen for emerging incoherence in the discourse rather than for expected responses. This can also enrich the teacher’s own repertoire of learning trajectories by completion of student attempts.

6.3.1 In-the-moment pedagogy

Mason and Davis (2013) argue that the most important aspect of “mathematics needed for teaching” is what comes to mind moment-by-moment when teachers are planning or leading a lesson. The major factor influencing this is the scope and range of mathematical thinking a teacher has access to, and the repertoire of pedagogical strategies and didactic tactics that are available to her to come-to-mind in the moment. Going beyond the content knowledge that comes from courses in formal mathematics or “specialised content knowledge” needed for teaching and the knowledge that is available in-practice, Mason and Davis suggest that what matters the most is “knowing-to-act, that is, having knowing-how, perhaps informed by knowing why, come to mind” (p. 191). Responding in-the-moment involves being “*with* mathematics, in relation *to* mathematics” (p. 186, emphasis in original). They call this the teacher’s “mathematical being” since it orients awareness and is the basis for conscious and unconscious choices made by the teacher and allows her to *be* mathematical with and in front of their students. This is similar to Watson and Barton’s (2011) identification of the need for the teacher to “enact mathematics” and “work as a mathematician”.

Developing the “mathematical being” involves teachers nurturing their mathematical awareness and engaging in mathematical thinking for themselves and with like-minded colleagues. The teacher needs to experience for herself and therefore be sensitised to the psychological and socio-cultural aspects of being a learner and doer of mathematics, and to what it is like to encounter epistemological and pedagogical obstacles. It also involves “knowing more deeply and richly in the sense of having possible actions – mathematical, pedagogic and didactic — come to mind when they are needed, whether when planning or in the midst of activity with students" (Mason & Davis, 2013, p. 192). Rather than foreseeing what might happen, preparing for the unexpected involves having access to a rich collection of pedagogic actions together with a narrative for elaborating and justifying the choice in any particular situation. Developing a repertoire of such actions or practices embedded in personal experiences makes it possible to have these actions come to mind, notice opportunities to act freshly and to exercise choice and respond to a situation rather than react habitually. Engaging in the tasks planned for students or those that provide parallel experiences to those of the students to generate relevant experience of what needs to be attended to, how mathematical themes are instantiated, what practises are drawn on, obstacles that come up and action taken to overcome the obstacles. etc., is one way of developing such a repertoire of actions (Mason, 2015).

“Knowing-to-act” requires sensitising oneself to notice opportunities to act, to be aware of situations developing before an action is actually required. It requires that something relevant “comes to mind” or “comes into action” so as to direct attention and inform choices. Mason (2001) suggests a set of four interconnected actions which he terms “The Discipline of Noticing”, to support and enhance sensitivity to notice, and to make it possible to act upon that noticing as events unfold.

> *Systematic reflection:* collecting brief-but-vivid accounts of salient incidents, working on them so that others recognise something from their own experience; developing sensitivities by seeking threads among those accounts, and preparing oneself to notice more detail in the future.
>
> *Preparing and noticing:* imagining oneself acting in some desired manner, using the power of mental imagery to direct and harness emotions, and gradually noticing more and more opportunities; reflecting on the past by reentering situations as vividly as possible and preparing to notice in the future by imagining oneself choosing to act.
>
> *Recognising choices* by accumulating alternative actions and by working at bringing the moment of noticing into the present; being on the lookout to notice alternative behaviours or acts (in other people’s accounts, in texts and articles, while observing others in practice), which you would like to incorporate into your practice;
>
> *Labelling* salient incidents and alternative acts so that they begin to form a rich web of

> interconnected experiences associated with particular collections of incidents, and linking these labels with specific incidents so as both to enrich the moments and to empower the labels to act as triggers to notice fresh opportunities to act in the future. (p 87)

Systematically engaging with these actions enables one to prepare for noticing and acting in the moment, thereby supporting teachers to listen to and respond to students' mathematics.

As discussed in Section 6.1.4, one of the challenges that must be met while having students at the margins engage with explorations is listening. Together with the discipline of noticing mathematics, teachers need to develop a "discipline of listening" to students' mathematics that is different from their own. Such listening encourages students to listen to other students' mathematical talk as well. Acknowledging the need for hermeneutic listening as suggested by Davis (1997), I discuss how this applies to listening for coherent formalisability and point to ways in which coherent formalsability could be violated.

6.3.2 Listening for and noticing coherent formalisability

Listening means broadly an orientation to eliciting and making sense of children's actions and comments. It involves "reaching" the students and is interactive and participatory. While acknowledging the difficulty in listening and responding to student work, Empson and Jacobs (2008) also suggest that how a teacher listens can transform how students talk and what they learn. Davis (1997) suggests that attentiveness to how mathematics teachers listen may be a worthwhile route to pursue as we seek to understand and help teachers better understand their practice. Empson and Jacobs (2008) suggested benchmarks for teacher listening to support teachers learning to listen – a pathway by which children's mathematics becomes progressively more central. They distinguish three kinds of listening: directive listening where the teacher listens to a student's thinking to evaluate its correctness as compared to a preconceived standard, observational listening with an attempt to hear and understand the sense that that the student is making, and responsive listening in which the teacher not only intends to listen carefully to child's thinking but also actively works to support and extend that thinking. Parallel to this, Davis (1997) suggests evaluative listening, interpretive listening and hermeneutic listening. Hermeneutic listening also implies an attentiveness to the social, historical and contextual situations of one's interactions and a willingness to question the biases that frame our perceptions and actions. A diverse classroom, where talk is privileged and has a more encompassing criterion for what constitutes mathematical discourse, calls for hermeneutic listening.

The criterion of coherent formalisability forces a shift in perspective for the teacher as to what to listen for. The teacher needs to go beyond listening for the correct answer, evaluating a student contribution, or even understanding it and listen for coherence. That is going beyond evaluative and interpretive listening, the teacher needs to engage in reflective or hermeneutic listening. She needs to let the discourse evolve as long as it hangs together and seek ways of building on it. For example, as discussed in Section 5.3.2, some students defined “same figures” as “ two match-stick shapes as ‘same’ if they have the same number of matchsticks along each side”. The matchstick shapes that students were considering when they made this definition were triangles, squares and rectangles. The definition is acceptable in this limited domain, if “same” is being used in the sense of congruent. However, two quadrilaterals that satisfy this definition need not be the “same” – either congruent or similar. The distinction between congruence and similarity needs to be made even in the case of triangles, squares and rectangles. Highlighting these distinctions and insisting on correct usage poses the risk of students disengaging from what they find mathematically interesting, especially at the beginning of an exploration, and especially in a low-resource context. As the teacher, I often chose to “go with the flow” getting students to articulate their ideas, in whatever language, informal and imprecise if need be, before going on a corrective mode. Doing so requires an alertness to the imprecision that is being allowed and potential conflicts that this could lead to.

Just as the teacher needs to listen for coherence and formalisability, she also needs to be aware of and listen for potential violations of CF. In the above example, the definition had a limited scope of validity, beyond which it would not hold and contradict other established results. The definition itself was not sufficiently precise - it was not clear whether the students considered “similar” figures as “same” too. Talk encourages vagueness and imprecision. Since the intended audience for the communication is limited to fellow students and the teacher in the class, who share certain ways of talking and have a shared understanding, students may not feel the need to explicate all assumptions – for example, the scope of validity of a definition or generalisation – whether a conjecture holds for whole numbers, or integers or rationals. Among the conjectures discussed in Section 5.2.1, Conjecture S - 3, that the possible side-sums are consecutive numbers, took for granted that consecutive numbers are being used to fill the Magic triangle. We unpacked the assumptions underlying the conjecture S-4 on the minimum and maximum side-sums possible with numbers 3-8. Such implicit assumptions need to be made explicit and evaluated for incoherence or ambiguity.

Assumptions implicit in diagrams and representations and incompatibility in definitions may also go unexamined and be potential sources of incoherence. We saw an instance of this in the different ways that students chose to represent the moves in the Leapfrogs exploration (see Section 6.2.4) one of them assumed the tokens to be distinguishable, while the others did not. Such assumptions if not unearthed and spelt out could lead to contradictions. We saw students redefining polygons (Section 4.2.1), “same shape” (earlier in this section and in Section 5.3.2), with their definitions being different from the accepted textbook definitions. In one implementation of the Polygons exploration, students defined the exterior angle of a polygon as “360^o - the measure of the corresponding interior angle”. While it is acceptable to work with such definitions in the context of the explorations, the teacher needs to be aware that they lead to results that may also differ from the accepted results. For example, the sum of exterior angles of a polygon defined as above will no longer be 360^o.

Ill-defined and inappropriate scope of generalisation is another potential source of incoherence. Amongst the examples discussed in various sections, we see an instance of this when students counted exterior right angles of a polygon as well and concluded that “there can be as many right angles in a polygon as there are number of sides” (Section 4.2.1). They arrived at this generalisation based on experimenting with only even-sided polygons and I suggested they experiment with some odd-sided polygons as well. So the statement, while a valid statement for even-sided concave polygons, including exterior right angles, is not universally valid. While arriving at such results is a practice that is to be encouraged in the context of explorations the scope of validity of the statements also needs to be spelt out.

Literature also points to some sources of incoherence. Bardelle (2013) suggests that the interpretation of verbal statements in a mathematical setting may happen based on everyday context and some sentences involving logical connectives evoke meanings that contradict the mathematical interpretation. Student difficulties with regard to differentiating between an proposition and its converse (Hoyles & Küchemann, 2002), understanding logical implications (Durand-Guerrier, 2003), tendency to think in terms of whole numbers (for example not considering non-integer solutions to equations, extending properties like “multiplication makes bigger” or “quotient is less than the dividend” beyond their range of applicability) could all be potential sources of mis-communication if underlying assumptions are not interrogated.

To sum up, the teacher needs to be aware of these different ways in which CF could be violated and be alert to any evolving incoherence. Some ways in which this could happen are:

a) taken for granted and implicit assumptions b) Incompatible definitions, representations, notations and assumptions underlying these c) ill defined and inappropriate scope of generalisation d) erroneous use of conditionals and other logical connectives e) Implicit and common sense assumptions that are conflicting.

In Section 6.1.4 we noted that the challenges in listening and responding to students’ mathematics expressed in “formalisable” language, are further exacerbated by the teachers’ own biases and judgements. Language carries markers of class, caste and community and a language different from her own may prejudice the teacher leading her to take a deficit view of the imprecise and incompletely articulated student formulations. What the teacher listens to and notices, and how the teachers and students respond in-the-moment in the classroom is shaped by the relationship between the teacher, the student and mathematics. The teachers' deficit views of students and self-assumed lack of relationship between students and mathematics (what Herheim, 2020, calls an I-It relationship) are important factors that shape the way students and teachers respond in class. In Chapter 2, Sections 2.2 and 2.1, we discussed the harmful effects of deficit discourses and the marginalising effects of mathematics. The overarching goal of this study was to find ways of mitigating these marginalising effects. To this end, I suggested mathematical explorations which offer opportunities to move away from the rigid ways of teaching-learning mathematics induced by the textbook culture, privileging talk as a means to do and communicate mathematics in the context of explorations and adopting a more accommodating acceptability criterion of coherent formalisability for mathematical discourses in such contexts. For a movement away from the margins, the hold of deficit discourses must also be disrupted. In the following Section, I examine how explorations and the criterion of coherent formalisability contribute to this end.

6.4 Disrupting deficit discourses

As discussed in Section 2.3 deficit discourses focus on students' shortcomings, disregarding their strengths. Literature reviewed in the section points to the existence, persistence and the harm caused by deficit discourses. The prevalence of such discourses lead to deficit frames being adopted. Researchers have suggested that teachers' framing of classroom activities and student work — for example, either as something that needs to be corrected and moulded/enhanced to normative levels or as self-correcting and self-enhancing progression — drives much of their practice (Russ & Luna, 2013). Despite the power that culturally dominant frames draw from institutionalised social practices and policy documents, scholars have contended that "intentional reframing" is possible with "substantial and ongoing work, including work at the level of individual teachers and work at the level of systems and institutions" (Louie et al., 2021). As discussed in Section 2.4.3, scholars have suggested such reframing - strength-based framing (Scheiner, 2023), anti-deficit framing (Louie et al., 2021) - to deliberately highlight the abundant and varied strengths of marginalised students and a multidimensional framing of mathematical activity that includes practices such as sense-making, connection-seeking, experimentation, collaboration and argumentation to disrupt deficit discourses. This expands the meaning of mathematical competence and who can be seen as capable.

Aligned with this orientation, I suggest that flexibility offered by explorations and the more encompassing nature of coherent formalisability as an acceptability criterion, support a non-deficit framing of what it means to do mathematics and what counts as mathematical language. By privileging mathematical practices over the right answer, mathematical explorations enable a multidimensional framing of mathematical activity as described above. In the many instances discussed in the previous chapters, we saw the potential of explorations to create opportunities for students to demonstrate their mathematical competence by engaging in disciplinary practices, generating counterstories to the idea that the marginalised (the “others”) are not capable of acquiring mathematics at a “normal level”. thus disrupting deficit discourses. Going further, I also suggest reframing the notion of “deficit” in terms of a “distance” rather than a “gap”.

6.4.1 Disrupting deficit discourse by reframing “gap” as a “distance”

Gutierrez (2008) draws attention to a phenomenon related to the prevalence of deficit discourses: the “gap-gazing” fetish in mathematics education and suggests that the studies of “achievement gap” offer a static picture of inequities in schools and rely on one-time responses from teachers and students. Moving away from “gap-gazing”, I suggest reframing the notion of “deficit” in terms of a “distance” rather than a “gap”. In contrast to a gap, distance suggests something transient or variable, something that would be traversed or covered with the passage of time. Accordingly, recentering the margin would entail negotiating and traversing this distance. Further, in place of the "deficit - anti-deficit" binary, distance provides a spectrum, the possibility that a teacher who has a deficit perspective may yet listen and alter her perspective. Moreover, the distance metaphor also suggests that traversal may happen in both directions, of the teacher towards the student and her mathematics, as much as of the student towards the mathematics of the teacher.

The understanding of the separation of the margin from the centre as distance offers the potential of traversal towards the centre, at the same time illustrating the difficulty of such traversal. For the teacher in the classroom, this provides a spectrum of possibility and multiple traversal. This encourages listening to children, for their mathematical thinking per se, rather than whether it conforms to formal standards, even while being aware of distances to be traversed. Moreover, success in this enterprise helps the teacher overcome her own deficit perspective.

There is also implicit in this discussion a notion of potential that orients the move, from the margin to the centre, in terms of the effort needed. Listening to children and noticing their own mathematical expression are conscious epistemic acts by the teacher placed in a social norm that privileges formal mathematical language (typically as used by the textbook), and therefore require effort. Further, the teacher needs to articulate what is gained by formalisation and generalisation and share this insight with students. If students are made aware that their ideas and expressions can be “formalised” or aligned to the “accepted ways”, it may give them confidence to go further. This offers students an epistemology of effort, lacking which, they too only see the alienness of their own language to that in the textbook.

How is such distance framed in the interaction in the classroom? We suggest that distances could be framed in deficit terms, leading to the foregrounding of “deficit distances”, or alternatively in non-deficit terms, leading to noticing and giving importance to “potential distances”. An example of deficit distance is the gap between grade level expectation of mathematical knowledge and students’ knowledge as elicited through examinations. In contrast, the distance between what students are mathematically capable of and what is acknowledged by the institution (teachers, schools, exams, etc.), would be an example of potential distance. In non-deficit listening and responding, teachers struggle with the tension between deficit and potential distances and must negotiate this tension. We now look at the potential of an acceptability criterion like coherent formalisability to disrupt deficit discourses in the light of this distance metaphor.

6.4.2 Coherent formalisability as indicator of potential distance

I suggest that coherent formalisability contributes to a broader framing of what constitutes mathematical talk. It enables one to see “potential distances” between what students can actually do and what is expected from them by assessments and teachers and thus supports a non-deficit perspective. The extent of missing elements - terminology, definitions, reasoning - that need to be supplied to map the discourse to a formal one is indicative of the distance between students’ mathematics and the mathematics that is expected of them. Thus criterion of coherent formalisability provides an indicator of the distance to be traversed. Moreover, it is the formalisability of the students’ mathematics that is kept in sight; thus the distance to be traversed is a potential distance. By drawing attention to the core value of formalisability in discourses that differ from the dominant conceptualisation of mathematical discourse, the criterion brings these within the fold of acceptability. I revisit the analysis of the proof attempts in Section 5.5.4 to substantiate this point.

The proof attempts discussed in the section may be judged to be inadequate from a deficit perspective. Krithi’s started with a particular arrangement of numbers on the base of the triangle and tried to complete it so as to get a side-sum of 8. She concluded that this is not possible when she did not succeed in getting it. It can be interpreted as an unwarranted conclusion based on trying out a single possibility. V2 articulated his algorithm for finding the maximum and minimum side-sums through one particular case. One may raise questions about its generalisability and if V2 was even aware of it. Both V2’s and Maran’s proof were articulated in informal terms, and in colloquial language.

Krithi’s proof becomes a valid proof with some gap-filling in the form of extending her proof-scheme, from “look for combinations of numbers that make a desired side-sum” to “look for a*ll possible* combinations of numbers that make a particular side-sum”. V2 used a representative example to state the generality that he has arrived at. Maran "specialised usefully" (Polya, 1954, p. 17) when considering those combinations of numbers that would have 6, the largest available option, as one of the numbers. All three proofs can be rearticulated in more formal terms as can be seen from Section 5.5.4. Krithi's proof needed some augmentation, and V2's and Maran's proof could be mapped on to a formal proof step by step. The only missing element was symbolisation. Examining the proofs for formalisability brings out what is mathematical in them, creating the possibility of countering a deficit view. It is worth noting that the heuristics of "choosing a representative example" and "specialising usefully" that V2 and Maran adopted are powerful problem solving approaches. These can be considered as potential distances. I thus suggest that framing the difference between students' informal articulations and the corresponding formalised versions as a potential distance allows one to see the extent of the distance that needs to be traversed to formalise it.

All three students believe that they have proved their claims. On their own, they do not know how much effort would go into proceeding further and arriving at an "acceptable proof". When teachers show that their proof idea is formalisable, it may give them confidence and motivation to go further. Similarly, when students make inconsistent assertions or use ambiguous terms, rather than rejecting such use (from a deficit perspective) pointing out the distance to consistent assertions (if possible) and precise terms would help student effort in subsequent discussions.

In Section 6.1.4, on the other hand, we see an example where I missed out on "hearing" what Sumi was trying to say, because her way of expressing the transformation as "interchanging the numbers in the inner and outer triangles" was different from my way - a median swap. Rather than focus on the "formalisability" of her utterance, and how I could help her clarify it, I was perhaps fixated on my own way of thinking about the transformation and missed an opportunity. This could be interpreted as the reassertion of deficit framing.

The examples of students' engagement with mathematics discussed thus far are taken from exploratory tasks. Explorations inherently offers some flexibility to the teacher in enacting it in the classroom, as compared to regular school lessons. The teacher has the freedom to define what she expects from her class based on the requirements for the exploration at hand and how far she expects her students to progress in it given their mathematical background. So it may be easier for a teacher to take a more accepting view of students' mathematics and language. However, in a curricular context, the expectations are set by externally defined curriculum and assessment schemes and the teacher has very little choice. Although these might induce a deficit perspective, the possibility of unearthing potential distances to and from students' mathematics do exist.

6.5 Mathematical engagement in curricular context

In this section I address Research Question 4:

What *could* mathematical engagement look like in curricular context? (see Section 3.1)

As noted in Section 3.1. I tried to bring in the flexibility enabled by explorations and coherent formalisabilty to the curricular sessions as well. Drawing on some instances from these classes that remained salient in my memory, I point to what mathematical engagement *could be* like in non-exploratory contexts when the teacher is more accepting of students' mathematics and languages. Reflecting on these instances, I draw attention to the potential distances and the possibilities of traversing such distances. A study of curricular contexts was not part of the initial design and happened because of the schools' request to engage in some curricular teaching as well. I think it is of value to share these examples as indicators of the potential of flexible pedagogies to disrupt deficit discourses.

6.5.1 Potential distance in the curricular context: An example

This instance, which happened when I was solving some Mensuration problems, also serves to illustrate "potential distances" - the distance between what students are mathematically capable of and what is acknowledged by the institution (teachers, schools, exams, etc.). The square root algorithm is something which I have usually seen students following without knowing why it works. Even teachers tend to reproduce the algorithm blindly and stumble in explaining the rationale behind it. I did not expect students to raise the questions that they did in the instance described here.

In one of the classes with Grade 9 students we were practising problems on area and perimeter. We were on a problem that asked to find the area of a rhombus given its perimeter and one diagonal. This implied their finding the length of the other diagonal using Pythagoras theorem and for this they had to find a square-root. The calculation involved was $\sqrt{(40^2-24^2)}$ *, which they evaluated as* $\sqrt{1600-576}$ = $\sqrt{1024}$ *and used the division algorithm to find the square-root of 1024 as in Figure 6.6. As they went through the calculation, a student asked for the rationale for the step of doubling the number currently "at the top" as we move to the next iteration in the algorithm (The circled numbers in Figure 6.6) and why one should put the same number at the "top and side" (the digit 2 in the figure). I suggested that they use the identity by which the calculation becomes that of finding the* $\sqrt{(40+24)(40-24)}$ *, i.e.,* $\sqrt{64\times16}$ *, which is easily evaluated as 8 × 4 = 32. They had not considered this approach to the problem and were surprised at the ease with which the answer came out and started clapping spontaneously.*

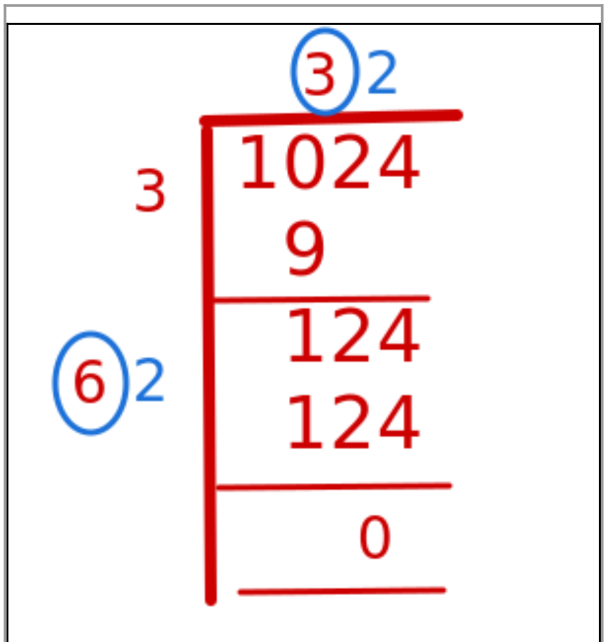


Figure 6.6: Square-root of 1024

This square root algorithm was something which the students had learnt in earlier grades and could execute correctly. Most likely these students had not been told the rationale for the algorithm, and they had not tried to make sense of it nor questioned why it works. In this instance they were trying to make sense of it and raised pertinent questions.

While being accepting of the students' approach to the problem using the square root algorithm, I offered an alternate suggestion which simplifies the calculation and is a method that can be used in other contexts as well. My suggestion also points out how the algebraic identities could be used to do calculations in the context of geometry as well, an attempt to draw connections between the seemingly disparate content areas of school mathematics - arithmetic, algebra and geometry. The students were quick to notice and appreciate this and adopt it in subsequent problems. We see in this the students' openness to alternate strategies and willingness to adopt a strategy different from theirs when they are convinced of the benefits and to appreciate the connection between "algebra and perimeter problems" as they said. I now describe another instance which brings out some potential distances.

In a subsequent class when we were solving problems on volume of solids, I assigned them the problem of finding the side-length of a cube whose volume is 3125 cc. When I realised that this would lead to an irrational number as an answer I offered to change the numbers, but they wanted to go ahead with the same number and did not want me to make it easier for them. Having found that 3125 = 5^5*, they said they would have a 5 in the answer corresponding to* 5^3 *and asked what they should do with the remaining two*

5s. I suggested that they write the answer as 5 $\times$. $\sqrt[3]{25}$*. They then wanted to work back and check their answer by multiplying this number by itself three times to see if they got 3125 as the result. To multiply* $\sqrt[3]{25}$ *by itself, they asked me how to multiply when they did not know the last digit and the last digit did not even exist for such a number.*

In this instance, giving them the number 3125 was a calculation error on my part, I had intended to give them 5^6 and gave them 5^5 instead. When I realised my mistake, I offered to correct it, but they wanted to go ahead with what they saw as the more difficult problem. This points to their persistence and willingness to engage with difficult problems. Another point that is striking in this description is that having found an answer, they wanted to back calculate and check if their answer is correct. Their question related to what they should do with the "remaining two 5s" might give the impression that they are responding mechanically, but it is clear that attempts at sensemaking are also being made alongside execution of procedures.

The representation $\sqrt[3]{25}$ can be considered an algebraic expression and as a number that corresponds to a point on the number line. While I was working with the algebraic representation $\sqrt[3]{25}$, the students were working with the number that $\sqrt[3]{25}$ stands for. So they extended the familiar schema of the multiplication algorithm to this number as well. The discomfort that students seem to be experiencing here is the lack of an operational meaning for $\sqrt[3]{25}$ and their question on how they would carry out the multiplication $\sqrt[3]{25} \times \sqrt[3]{25}$ can be interpreted as a demand for an operational semantics of product terms such as these in terms of a procedure for multiplication. The interpretation of $\sqrt[3]{25} \times \sqrt[3]{25}$ as a number that can be used in calculations, rather than as an algebraic expression, is one which is seen among *all* students and not just those from marginalised contexts. The students here were trying to untangle the two representations encoded in one symbol $\sqrt[3]{25}$.

In both these instances the students used their agency to ask critical questions to the teacher to understand content that I have seen even some teachers accept unquestioningly. The students' attempt at seeking and negotiating meaning rather than blindly following procedures is an instance of what I termed potential distance. The teacher's attempt at creating an atmosphere that builds the confidence to question, challenge each other and the teacher, and assert themselves can be seen as an effort to highlight the potential distance.

In Section 6.4.1, we noted that traversal of the distance may happen in both directions, of the teacher towards the student and her mathematics, and the student towards the mathematics of the teacher. The above instance can be considered an example of effort by the students in traversing the distance to the teacher's mathematics. In the following section, I illustrate how a movement on the teacher's part towards the student's mathematics opened up the possibility of a potential future movement on the student's part towards more formal mathematics. In analysing the instance, I also highlight how coherent formalisability enables us to see the distance between the student's approach to the problem and the textbook defined approach.

6.5.2 Traversing the distances to students' mathematics in curricular contexts: An example

In this section I present and analyse an instance from a session where the class was doing some problems on percentages and discounts. In this instance a student, Selvam, came up with his own way of calculating a percentage and insisted on going ahead with it, despite my initial response of suggesting an "easier" (and textbook given!) method.

In response to the task of finding out "what percentage of 150 is 50?" student Selvam in Grade 9 stated that dividing a number by 100 gives one percent of the number. He calculated 150/100 as 3/2 and I helped him interpret it as "one and a half". He then added one and a half and one and a half to say that 2 percent of 150 is 3. He then continued as 2% of 150 is 3, 4% is 6, 6% is 9, 8% is 12 and so on, intending to keep adding 2 and 3 successively till he gets x% is 50. I noted his adding in "two lots of one-and-a-half" at each step and tried to nudge him into "successive doubling", i.e., from 4% is 6 to 8% is 12 and 16% is 24 and so on, to reduce the tedium of calculation, but he insisted on doing it his way and refused to take my cue, saying "Porummaya calculate pannaren miss" (let me calculate patiently/slowly). He went on adding in steps of 2% till he reached 32% is 48 and 34% is 51, and hence concluded that 50 is a little less than 34% of 150.

When viewed strictly with an expectation defined by the curriculum, one might take a deficit view of this response and note the gaps in the student's knowledge. Having been introduced to percentages in Grade 7, one might expect a Grade 9 student to answer this question as a matter of course. In comparison, this student's struggle to interpret 3/2 (which he initially said was 3 rupees and 20 paise), later understood as one-and-a half, when he saw that there were as many 100s in 150; his reluctance to divide by a fraction (50 ÷ 3/2) and instead attempting to repeatedly add one-and-a halves (in lots of 2) to find out "how many one-and-a-halves are there in 50?" may seem "deficient." The expected way of solving the problem using the school taught formula would be as follows:

$$Percentage = \frac{part}{whole} \times 100. \qquad \frac{50}{150} \times 100 = 33\tfrac{1}{3}\%$$

If this is used as a reference, then we may not be able to make any inference about Selvam's understanding of the concepts. But the method adopted by Selvam left no doubt about his conceptual understanding. He found out 1% of 150, and then calculated out how many lots of this 1% makes 50. It is clearly formalisable as the unitary method, a often used procedure in school mathematics. What he does differently is that he formalised the division problem as a repeated addition problem and used a "counting based strategy" (Venkat et al., 2021) to arrive at the answer. Also, he preferred the everyday representation of one-and-a-half over the "school taught" representations of 3/2 or 1.5. This use of everyday representation enabled him to solve the problem that would have otherwise been inaccessible to him. The distance that needs to be traversed is not conceptual, but in terms of the *representations* and the *algorithms* that he chooses. The question is one of "efficiency" of Selvam's algorithms and representations but not their correctness. Analysing the instance from the perspective of "distances" , we see that the apparent "deficiency" or gap from "grade appropriate content knowledge" is only superficial, based on an arbitrary definition of what constitutes grade appropriate content knowledge, and not a deficiency. The distance here stems from the narrow conceptualisation of school mathematics, what are considered acceptable ways of doing it and the constraints, including that of allotted time, imposed by prevailing assessment schemes.

Noting that Selvam initially used a doubling strategy, perhaps in order to avoid the inconvenience of adding a fraction, one-and-a-half, multiple times and work with a whole number 3 instead, I suggested using it repeatedly to reduce the number of calculations. But Selvam insisted on continuing in his own way which I accepted. This can be interpreted as Selvam using his agency to hold on to the mathematics that he owned, and by doing so, creating a possibility for me as teacher to traverse the distance to this "other" mathematics. I interpret my acceptance of his approach as a willingness on my part to accept Selvam's implied invitation to his mathematics. This creates the possibility of a potential future movement on Selvam's part towards a more formal approach to carrying out the division operation.

After this instance I noted a change in Selvam's willingness to attend my classes (which were optional and during after school hours) and his engagement in class. In the subsequent academic year, contrary to the regular schoolteacher's marking Selvam as a potential "disrupter" of the class, he continued to be an enthusiastic participant in my class, eager to complete his work and help others as well.

I followed a pedagogy similar to that occasioned by explorations in the curricular sessions as well, and allowed space for students' ways of doing mathematics as described in the above episode. Other pedagogical moves like not being particular about classroom organisation and allowing students to sit where they liked, being accepting of their unwillingness to write in notebooks, consciously adopting their language, enlisting their support in the data collection process by assigning them to operate the audio recorders (see Section 3.8), letting them take the role of the teacher occasionally, and over and above all creating an atmosphere of care and respect for each other contributed to building a relationship between me and the students. This might have been a factor that influenced their classroom response and engagement. Some of these were feasible because I did not have the mandate of completing the syllabus and may not be possible for a regular teacher. But this study makes evident the need to listen and understand students' mathematics without taking a deficit perspective to enable better engagement with mathematics.

6.6 Summary

Based on my experience of facilitating explorations in marginalised contexts, my reflections and discussions of these experiences with collaborators, I identified challenges that a teacher could face as she sets out to do explorations in similar contexts. Key among these are the absence of ready-to-use material that could be drawn on, the demands on content knowledge arising from the need to encounter unfamiliar mathematics, the need to recognise and respond to students' mathematics expressed in informal and possibly unexpected ways.

Building on the idea of Hypothetical Learning Trajectories, I proposed guidemaps prepared by research mathematicians as support for teachers to facilitate explorations. I identified the desirable features of such guidemaps. They should ideally spell out the key insight that can lead to a resolution of a problem, the various intermediate results leading to a solution, the multiple trajectories along which an exploration could progress, the mathematical practices that may be salient in a particular exploration, the prerequisite knowledge required and the task specific choices possible and their implications. These features provide the teacher with the content and pedagogical knowledge required to facilitate an exploration.

Explorations bring about many contingent situations and the teacher needs to be prepared to respond to these in-the-moment. I draw on literature related to the "Discipline of Noticing" that suggests cycles of reflection, preparation, noticing and recognising salient events, and labelling them as ways of strengthening the teachers responsiveness. I also illustrate some ways in which coherent formalisability could be violated, so as to strengthen teacher sensitivity in this regard.

Recognising the need to overcome deficit perspectives to enable responsive listening to students' mathematics, and the need to disrupt deficit discourses, I further suggested re-framing perceived gaps as potential distances to be traversed. Potential distances are the distance between what students are mathematically capable of and what is acknowledged institutionally. I also examined how privileging the formalisable over the formal facilitates attention to potential distances.

An exploratory context with the potential to create opportunities for students to bring out their mathematical competence is rich with examples of potential distances. I suggest that it is possible to unearth potential distances in a curricular context too and illustrate this through an instance. Through another example, I also illustrate actual and potential traversal of this distance - both from the student to mathematics and between the teacher and the student. These instances also point to what mathematical engagement *could* be like in a curricular context given a non-judgemental and inclusive environment.

7 Conclusions, Limitations, and Further Work

In this concluding chapter, I summarise the thesis and discuss some implications that follow from the study. In Section 7.1, I recapitulate the key points from the thesis. In Section 7.2, I discuss some implications of the study and in Section 7.3, I point to some limitations and directions of future work.

7.1 Thesis summary

In this section, I briefly spell out the background of the study and its motivation, the questions that I attempted to answer, the methods adopted, and the findings.

7.1.1 Background

Starting from the well-acknowledged fact that mathematics contributes to the marginalisation of some students, we identified three dimensions - performance dimension, disciplinary dimension, and language dimension - along which this happens. The performative dimension arises from the "overvaluing" of mathematics in popular culture, making mathematics performance critical to access jobs and opportunities, even where such performance is not relevant. Also, the personal experience of learning mathematics may negatively impact the confidence and self-perception of many students. The disciplinary dimension is rooted in what is perceived as the "right way" of doing mathematics, especially in a school context. A focus on adherence to taught procedures and the expected responses puts those who deviate from these at a disadvantage. The formal language of mathematics with its predominance of symbols and specialised linguistic structures that condense meaning into precisely articulated phrases also poses an entry barrier to the discipline. The broad concern of this study was to come up with ways of mitigating the marginalising effect of mathematics. While changing public perception of mathematics and the importance accorded to it in society requires systemic measures, I focussed attention on the measures a teacher could take in addressing the problem.

Research in mathematics education points to the exercise paradigm prevalent in schools, where students solve problems to gain mastery over prescribed concepts and procedures and are penalised for mistakes, as one of the reasons leading to the marginalisation of some students. Also, it is widely acknowledged that many students have difficulty handling the predominantly symbolic language of mathematics. Scholars suggest a broader conceptualisation of what it means to do mathematics and building on the resources that students bring - be it their lived experience or their language - to help them go further, as ways to address these difficulties. Based on these, explorations or open tasks that allow for multiple approaches, open up possibilities for multiple questions and answers, and create opportunities for students to engage with mathematics in their own language, drawing on what they know, seemed to have the potential to present an alternative to the school mathematics paradigm. Explorations shift focus from the one right answer to the practices of mathematics and mathematical thinking and allow for a broader conceptualisation of *doing mathematics* than that frequently encountered in schools. They also allow for engagement at multiple levels in terms of the mathematical content knowledge and facility with the mathematical language required. While explorations are generally believed to be useful for those who are proficient in mathematics, their potential to support mathematical thinking of students who are mathematically marginalised has not been sufficiently explored, especially in the Indian context. The study aimed to investigate the potential of mathematical explorations to support mathematical thinking at the margins and to recentre the margins wedged by mathematics.

7.1.2 Questions and methods

The key questions that I attempted to answer are

- RQ. 1. What task features support mathematical thinking at the margins?
- RQ. 2a. What is the nature of mathematical thinking seen as students at the margins engage with explorations?
- RQ. 2b. How do they communicate their mathematical thinking?
- RQ. 2c. How does language support or hinder mathematical communication?
- RQ. 2d. What counts as mathematical discourse in such contexts?
- RQ. 3. What does facilitating explorations at the margins entail for the teacher?
- RQ. 4. What could mathematical engagement look like in curricular contexts?

Answering these questions required that I observe students at the margins engaging in mathematical explorations on a sustained basis, understand the nature of thinking that it gives rise to, and the nature of demands that it places on the students and teacher. Given the rarity of such a situation, I decided to create the context for the study. Supported and mentored by a mathematician and an educational researcher, I designed and implemented mathematical explorations in two schools catering to socio-economically disadvantaged students. I facilitated explorations in these schools on a weekly basis and pitched these as optional after-school enrichment classes. I adopted the stance of a researcher-teacher and investigated my own classes to understand what it entails for students at the margins to engage with exploration. I audio-recorded the sessions, maintained a teacher diary, and had the class observed by an independent observer whenever possible. Ongoing discussion with the research team of the day-to-day evolution of the classes, in the light of the audio recordings and the events that stood out for me, constituted a form of in-situ analysis. The audio recordings were listened to and discussed multiple times and student response to the module was understood, interpreted, and described. This post-facto analysis led to the identification of the instances discussed in this study.

7.1.3 Findings

Given the goals of the study to address the disciplinary and language dimensions of the margins, it was important to create opportunities for students to make their mathematical thinking visible without the formal language being a barrier to do so. The tasks needed to be sufficiently open to accommodate multiple goals, methods, and answers. These requirements suggested flexibility and accessibility as the design principles guiding task development. I further sought to elaborate and operationalise these principles by identifying specific features to be incorporated in the tasks. I addressed this question in Chapter 4 of the thesis and described task features that enable flexibility, and accessibility; and the student engagement with the tasks consequent to these features.

Task formulations that balance openness and specificity, affordances to function at multiple levels of formalisation, and branch out along multiple trajectories are features that allow for flexibility. Flexibility in tasks makes it possible for teachers to suitably tune the activity and tasks so that students are encouraged to engage. However, without some specificity with respect to the goals to be pursued or direction to be taken, students may be at a loss as to how to proceed. Therefore the task formulation should be such that these aspects are balanced. When students have limited access to the formal language of mathematics, affordances to function at multiple levels of formalisation becomes important. Also, The starting points for an exploration should ideally offer students a choice of goals or multiple trajectories to pursue and be generative of further questions.

Dependence on minimal prerequisite content knowledge, affordances to work with physical material - hands-on or imagined, and incorporation of multiple entry points were identified as features that contribute to the accessibility of tasks. An important feature that allows students from different mathematical backgrounds to engage with tasks is a minimal dependence on specialised prerequisite content knowledge. When such dependence is unavoidable, one needs to explore the possibility of unobtrusively passing on the required prerequisite knowledge to students or helping them arrive at it. I observed that when the starting point for an exploration involves working with physical material, it allows for a solution in terms of the material and does not rely on symbol manipulation. This serves as a first step in the transition to a more formal framing and solution. Having multiple entry points or several easily approachable trajectories ensures that students who are unable to solve a particular problem or follow a particular trajectory have an alternate path to pursue. Problems that have more than one solution like the Magic triangle puzzle open up the possibility that students can find at least some of these solutions if not all, and make some progress. It should also be stated that making tasks accessible does not imply reducing the intellectual challenge to the students.

In Chapter 5, adopting Burton's framework of mathematical thinking, I discussed the nature of mathematical thinking seen as students engaged with explorations and the means they adopted to communicate their thinking. Students performed mathematical operations such as comparing, classifying, making correspondences, studying relationships, engaged in mathematical processes like specialising and generalising, conjecturing and convincing, and built on already found results in the process of problem-solving. A high level of engagement was seen and elements of mathematical thinking described in literature were discernible in the way students explored. This points to the feasibility of mathematical explorations in a marginalised context and their potential to support mathematical thinking.

The means of communication students adopted were marked by the use of multiple languages (English and Tamil), multiple registers (everyday register and mathematics register), interspersing of formal and informal language, and use of means of communication such as diagrams and gestures. The communication was primarily oral. Writing was used as a means to support their thought processes rather than to communicate their work to others; perhaps because of this students preferred to write on impermanent surfaces like the blackboard or classroom floor and not in notebooks. The discourse differed from the characterisations of mathematical discourse in the research literature in that word- use was not reified, there was limited use of symbols and students pointed to multiple examples to establish the truth of a proposition (inductive means) rather than prove it deductively. However, students also offered mathematically convincing justifications for some of their stated conjectures. Their talk included half-formed sentences, frequent use of pointing words, and was imprecise and vague at times. Though the talk deviated from accepted characterisations of mathematical discourse, it was rich in elements of mathematical thinking.

In order to avoid deficit perspectives of mathematical thinking expressed through “unconventional means” there is a need to define a more accommodating acceptability criterion for what counts as mathematical discourse. However, the role of formalisation in teaching-learning cannot be overlooked because it enables access to opportunities for higher education and professions as well as to make progress in mathematical learning. I also noted that progress in an exploration was also made easier through formalisation. So there is a need to balance the insistence on the formal with the need to be accepting of students’ mathematics. Inspired by the practice of research mathematicians who draw on informal means during the process of discovery, keeping in sight the f*ormalisability* of their thought process to ensure consistency, I proposed *coherent formalisability* as such a criterion. I defined coherent formalisability as the potential of a section of discourse to be mapped to a formal one by supplying missing terminology, definitions, and reasoning, in a uniform manner.

The flexibility afforded by explorations and the more lenient acceptability criterion of coherent formalisability places demands on teachers. In Chapter 6, I discussed the additional challenges that this brings for the teacher based on the challenges I faced in the process. These include the non-availability of ready-to-use reference material, the need to know mathematics content and practices beyond what is learned through courses, the need to respond to students' mathematics in-the-moment, the need to listen and understand the ways of communicating mathematics adopted by marginalised students from a non-deficit perspective and absence of prescribed assessment criteria.

I suggested guidemaps prepared by research mathematicians and educators as teacher support to facilitate explorations. I also identified desirable features of such guidemaps: they should spell out the key insights that could lead to the solution of a problem and the intermediate results, map out the different possible trajectories through which the exploration could progress, describe the ones that students are likely to take and the mathematical practices that are salient in an exploration. Drawing on literature, I discussed the practices of the *Discipline of noticing* and *hermeneutic listening* which could help the teacher listen and respond to students' mathematics in-the-moment. I also discussed some ways in which coherent formalisability could be violated so as to give the teacher some pointers to be alert to.

Key to helping the teacher listen to and understand students at the margins is overcoming her own deficit perspectives. I suggested reframing perceived deficits as a "distance" rather than a "gap" as a step in this direction. I argued that the extent of missing elements and reasoning that need to be filled in to map students' discourse to a formal one is indicative of the distance between their mathematics and the mathematics that is expected of them. By focussing on this distance, the coherent formalisability criterion facilitates attention to what students are capable of and ways of improving on that, rather than on the deficits. In the final section of the chapter, I argue that coherent formalisability as an acceptability criterion is also applicable in curricular contexts. Looking at students' mathematics through this lens reveals that they are indeed capable of doing much more than what is expected of them by the institutions (school, assessments). I termed this distance between their capabilities and institutional expectations as a "potential distance". I also noted that attention to coherent formalisability enables traversal of distances between the teacher's and students' mathematics.

7.2 Conclusions and Implications

The study demonstrates the feasibility of enabling access to mathematics at the margins and the potential to mitigate the marginalising effects of mathematics through enacting well-chosen explorations bringing in flexibility and the move of privileging talk. Having a more accommodating acceptability criterion - coherent formalisability in talk rather than formal mathematical language - and adopting a non-deficit perspective reveals that students at the margins are capable of doing much more than what their performance in standardised assessments indicates. Though the study itself was done in a school catering to students from socio-economically disadvantaged backgrounds, I suggest that the conclusions also apply to ‘mathematically marginalised students’ from any background. Based on this the study has the following implications:

The curriculum needs to be reorganised to allow time and space for explorations and talk.: An often-heard reason for opportunities not being created for students to engage with explorations is that they take away from the limited time available to “cover the syllabus” for the examination. The structured nature of assessments, which are predominantly written, influences the nature of teaching that happens in the classroom. Time is devoted to solving specific problem types and helping students refine their writing in the organised way that is expected in the year-end evaluations. When the focus is on producing a solution written out in a preferred style, talk (and even thought!) gets relegated to the background. In the course of this study, I noted that students may be unwilling to write and offer the written work for scrutiny even when inclined to talk freely. In this study, writing was done to the extent that it was required to support the thought process. But they were eager to talk and share their insights and solutions. In a marginalised context, where proficiency in written language may be limited, an insistence on writing hinders students from expressing their ideas and demonstrating their mathematical competence. Therefore, privileging talk emerged as a way of supporting students to do this. The focus on end-of-chapter exercise problems leaves little room for students to find things for themselves. This study demonstrated that explorations along with pedagogies that support explorations, could be a countermeasure that brings students’ talk, discussions, and discoveries to centerstage. This implies that there should be committed effort to allot time for explorations in the school schedules and to create opportunities for and validate oral communication of mathematics. This may involve conceptualising different classroom activities that draw on different discourse practices.

An important learning from the study is that contrary to the popular belief that one requires a certain level of mathematical maturity and facility with formal mathematics to be able to engage in mathematical explorations, students in marginalised contexts, who are supposedly “behind” their peers in grade-appropriate content knowledge *can* explore, solve problems, build on solutions and engage in mathematical practices. The many instances described in the thesis attest to the extent of mathematical thinking they are capable of. Therefore the recommendation that space and time be allotted for explorations applies to schools across contexts. The flexibility afforded by explorations and talk enables even students to whom the formal language of mathematics proves to be a barrier to engage in mathematical thinking.

Assessments need to be broad-based and not rely entirely on facility with formal language: It is well acknowledged that assessments drive the content being covered in class and the pedagogy being adopted. Therefore for explorations and talk to become a part of the school context, it is also necessary that assessment formats be reconsidered. The dominant means of assessment, written exams, privilege the formal. This puts at a disadvantage students like Selvam whose approach to a percentage problem was discussed in Section 6.5.2. Though based on a sound understanding of percentages and the unitary method, there is no way that he can show his understanding within the prevalent assessment formats. Also, he will have to factor in the time restrictions of the exam, before going through the kind of lengthy calculation that he did. While it is necessary for him to learn more efficient or “better” ways of solving the problem, the first step to this is acknowledging what he knows and building from there. So I suggest that the assessments need to be designed that take a favourable view of informal approaches like Selvam’s. Also, there should be scope to communicate understanding through means other than the formal language of mathematics - orally or through gestures or diagrams or other means that are comfortable for the student.

Teacher education needs to include relevant elements for a pedagogy for explorations: In addition to content knowledge demands over and above the school curriculum, the key challenges that are likely to be faced while facilitating exploration are listening and responding to students' mathematics and refraining from taking a deficit perspective. Making explorations part of a regular school schedule would also imply equipping teachers with the necessary means to meet these challenges. One part of this involves the creation of material in the form of guidemaps as discussed in Chapter 6 of the thesis accompanied by some illustrative student work. In addition, teacher education programmes should allow time and space for doing mathematical explorations for the prospective teacher. More importantly, the teacher needs to be enabled to create a classroom culture where the students can express themselves and share their solutions or findings without the fear of being ridiculed or judged. Such a classroom culture based on care and respect for each other and for mathematics itself can be a resource that enables explorations.

Teacher education programs should incorporate elements that explicitly hone the teacher’s expertise in noticing students' mathematical thinking and significant moments (significant for their potential to enhance and advance students' mathematical thinking) in the complex scenario in a class. There is also a need for teachers to reflect on their practice and be aware of what they notice and the frames that they adopt when interacting with students and mathematics. When framed in terms of a disability, a deficit perspective dominates and this has implications for inclusive pedagogy. Teacher education programmes should have a component that enables such reflection and a deliberate focussing of attention to the resources students bring thus enabling a shift to anti-deficit and strength-based framings.

I suggested that focussing on *formalisability* instead of *formalisation* of students' mathematics is an enabler of anti-deficit framing and is a way of highlighting what students are capable of doing as compared to what is expected of them. The experience of trying to formalise students' mathematics by augmenting it as necessary will enhance the teachers' sensitivity to formalisability. The teacher also needs to be sensitised to ways in which coherence could break. In the preceding paragraphs, I highlighted some pedagogical elements that support an exploratory pedagogy. There is a need to identify such elements and ways of providing teacher support to facilitate explorations and incorporate these in teacher education programmes.

7.3 Limitations and further work

This study focussed on student mathematical thinking at the margins, ways that they express their thinking, and the influence of flexibility on their mathematical engagement. The study was not designed to investigate in depth the role of the teacher in enabling this flexibility. The insights that I offered on the challenges a teacher might face in this and the suggested workaround of guidemaps are based on the reflections and experience of the research team. Being a first-person research conducted outside of the normal class timetable, the efficacy and the day-to-day challenges a teacher would face in implementing explorations have not been looked into. Moreover, teaching operates within a community of practice, and recentering the margins would be possible only if teaching communities internalise anti-deficit perspectives and learn from each other. This requires a different line of research than the one undertaken here.

The guidemaps that I suggested were limited to the mathematical and pedagogical aspects of facilitating an exploration. Crucially, the guidemap also needs to inform the teacher of ways of working with the informal language. Questions like ways of structuring student talk, building bridges between students' ways of talking and the formal language, and moving to more formal ways of talking need to be addressed in the guidemap. The current experiential basis for the guidemaps needs to be further strengthened by drawing on teaching theories and principles of task design. We need to work further with teachers, involving them in developing such guidemaps, and gain a deeper understanding of both the usefulness of our guidemaps and teachers' resourcefulness in using them effectively.

In addition to guidemaps, there is also a need to develop assessment rubrics for explorations. What does it mean to "progress" in an exploration? What are pointers that one can look for to make sure that students are making mathematical gains, for instance, progressively moving towards more formal means of communication? Such rubrics are important both from the perspective of student evaluation and providing teacher support to facilitate explorations.

Writing is an important aspect of learning mathematics. Our primary focus has been on talk and we relegated writing to the background in the interest of keeping up student engagement. However there is a need to study the nature of writing that an exploratory context calls for, the kind of writing that students produce in such contexts, how their writing relates to their talk, and what may be considered acceptable writing. Further, the differences between written and spoken natural language (such as in Tamil) may also have an impact on students writing mathematics.

With the students who were part of this study, we found that insistence on writing was hampering engagement, and allowing talk was an enabler. We also noted the use of other resources such as manipulatives, gestures, pictures, and diagrams in communicating mathematics. One of the future points of investigation is the potential of these other resources to communicate mathematics, the specific occasions when students choose these means, the affordances that they offer, and what the formalisabilty of mathematics expressed through these involves. It is also of relevance to ask if there is a hierarchy among these modes of communication in terms of formalisability.

As mentioned in the thesis, the pedagogy adopted was largely whole-class teaching with some work in small groups. A key aspect that was noticed, but was not studied was the interaction and collaboration between students. In an exploratory context, the group dynamics is important. There have been instances when the group overcame the need for some prerequisite knowledge by drawing on the knowledge of one person in the class, who knew this. I have also seen instances where the discovery of one student becomes shared knowledge of the whole class and another member builds on it. This aspect makes a study of collective explorations by a large group markedly different from an individual student pursuing it. Understanding the inter-group interactions and how these affect the mathematics that students work on as well as the development of their mathematical discourse a worthwhile point to investigate.

Recentering the mathematical margins would require commitment and action at a larger systemic level and a great deal of research. I hope that this study contributes some initial steps in this direction.

Appendix

Consent form

Sub: Invitation to Participate in a Research Study on Mathematical Explorations

Dear Madam,

I request your permission to conduct a research study on ‘Mathematical Explorations’ in your school.

I, Jayasree Subramanian a Research Scholar, at the Homi Bhabha Centre for Science Education, Mumbai will be the Principal Investigator in this study. I will be mentored by Prof. K. Subramaniam, Homi Bhabha Centre for Science Education, Mumbai and Prof. R Ramanujam, Institute of Mathematical Sciences, Chennai in this study. This study aims to provide opportunities for mathematical explorations to class IX students and study how they respond to these opportunities.

If you agree to be a part of this study, I would request your consent to teach once a week in the grade IX class of your school. During this time, I would be doing some exploratory tasks in mathematics, with the students. I would also be audio recording the class and the interactions with students that happen therein and collecting the work that they do in these classes. I request your permission for audio recording and to analyse the interactions and student work to understand how students engage in mathematical explorations.

The data may be stored for the analysis of encounters and used for further research studies.

I plan to publish the results of this study. I will not include any information that would identify you, the school or any of the students involved. Their privacy will be protected and the research records will be confidential. The data will not be seen by anyone other than the researchers engaged with the research issue.

Looking forward to your cooperation.

Consent

I permit Ms. Jayasree Subramanian to teach grade IX and to audio record the proceedings. I understand that these records will be only used for the purpose of the research. I also understand that the researcher will take care that the identity of the school and students will not be attached to any results, findings, discussions, or academic papers produced by this project.

____

Participant’s Name Signature Date

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Researcher’s Name Signature Date

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