Empirical studies of students' conception of Area-measurement, and their implications for Mathematics Education

A Synopsis of the PhD Thesis

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Abstract

The studies reported in this thesis are aimed at understanding student's conception of area measurement (AM), particularly in terms of understanding the cognitive processes (rather than outcomes), and their implications for mathematics education research (MER). The thesis consists of three main sets of studies, which adopt varied theoretical and methodological frameworks, broadly following three major trends of research in MER – constructivism, social constructivism, and enactivism.

The first set of studies were inspired by Piagetian theory of constructivism or individual construction, and aims to understand the status of students' conception of area through naturalistic methods (Moschkovich, 2019). Since naturalistic method does not focus exclusively on the individual learner, but also considers external environmental factors, with minimal to no external interference, the initial studies were conducted in-situ, to understand the pedagogy of AM through classroom observation, students' interviews and textbook analysis. Later, however, structured tasks, based on interviews, were conducted with students in a research setup. The studies highlighted a range of issues with respect to AM conception, and led to a network model of AM, as a way of consolidating the results.

The second set of studies involved a teaching design experiment, where tasks were designed and developed based on insights gained from the previous studies, and applied in a classroom. Inspired by Vygotskian social interaction theories and social constructivism, the lessons were aimed at encouraging collective construction of concepts within a classroom, through the process of argumentation. The analysis of classroom interactions was based on the argumentation framework (Toulmin, 2003; Krummheuer, 2007), to examine the argumentation structure in the classroom. The study highlighted students' conceptual difficulties in connecting spatial and numerical aspects of AM, and the way students engage in the meaning making process through collective argumentation in the classroom.

The third set of studies were inspired by recent advancements in enactivist theories of cognition, and their applications to mathematics education. The studies sought to understand the role played by physical manipulations while solving AM tasks. The study was based on the eye-tracking method, and found significant differences between the eye-movement patterns of students who used

manipulations and those who did not. The eye-movement patterns of the group of students who did specific geometric manipulations, based on tangrams, indicate the use of more efficient strategies to solve the AM task, compared to the group who did not do any manipulation, and also those who did an unrelated manipulation using clay.

The final discussion brings together these diverse results, and discusses multiple conceptual, curricular, and pedagogical implications of these results for the learning of AM.

Graphic overview of the thesis

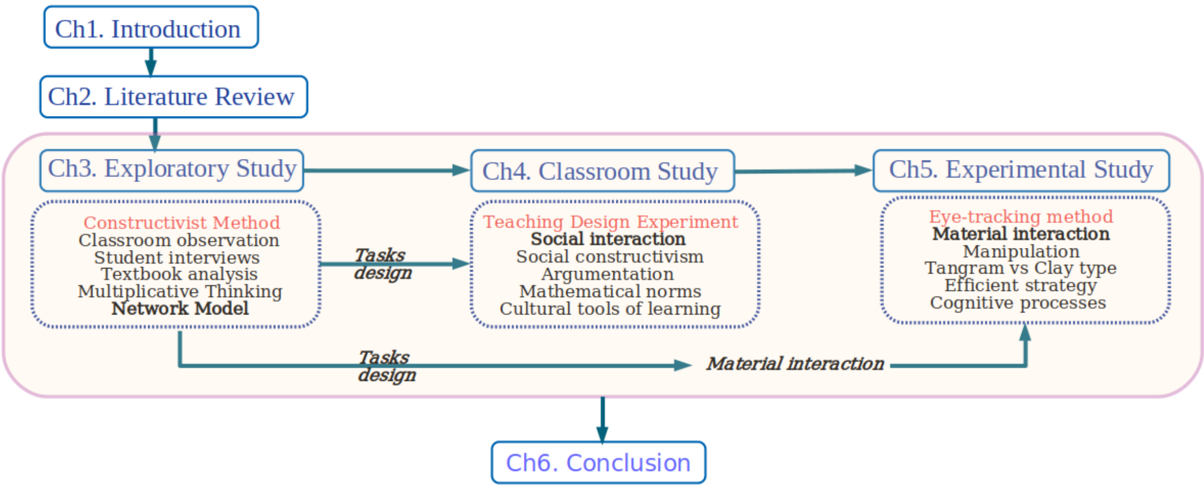


Figure 1. Thesis at a glance

Chapter 1: Introduction

Measurement is one of the most important tools to understand the world around us. It is very prominent in our everyday life (and communication) because of its practical utility, which in turn helps children engage with measurement more intuitively (Smith, Males, & Gonulates, 2016). Measurement requires connecting the domain of geometry with number (Sarama & Clements, 2009). In fact, measurement can provide the foundational basis for geometry, which occupies a significant portion in school mathematics, and can, in turn, influence students' overall mathematical performance. While, historically measurement and geometry were dealt together, most curricula present them as separate topics in school, with more focus given to geometry, which generally precedes measurement (de Freitas & Sinclair, 2020). Even most studies reporting students' poor performance in geometry and measurement, deals with them separately (Battista, 2007), which can further deepen the gap between the geometrical and numerical aspects of measurement.

The Russian mathematics educator Davydov (1975) argued for measurement to be considered a root topic to learn mathematics, contrary to the traditional curriculum that starts with numbers, which he argued was more abstract and psychologically inappropriate for the learner. He recommended aspects of practical measurement to be the basis for a primary arithmetic course. He tested an experimental curriculum that starts with measuring quantities (including comparison) with elementary children. He argued that measurement can bridge the gap between whole numbers and real numbers, by bringing in the need for fractions (or rational numbers) in a more organic way, rather than being dealt as a separate topic – the way it is done in the conventional curriculum. He has further argued that this can eventually bridge the gap between algebra and analysis. Davydov's work shows how measurement can act as a foundational topic, providing the connecting link for several important topics of mathematics. The network model presented in Chapter 3 supports this view, by showing how area measurement connects multiple topics in mathematics learning.

In Piagetian theory, a child is considered to pass through several cognitive stages, to finally understand the abstractions in measurement (Piaget, Inhelder & Szeminska, 1960). However, Vygotskian theories introduce the idea of tools that help attain higher mental functions. For measuring length, rulers are the cultural tools that a child can use to build her own mental tools (Clements, 1999, p.5). While length measurement uses ruler as a tool, no such tool is readily available to measure area, making it difficult for students to engage with area.

The work reported in this thesis focuses on understanding AM as a topic, because of this additional complexity involved in AM, compared to length measurement. Further, in AM, apart from measuring the different dimensions, students also needs to know the multiplicative operation on the dimensions. Here, AM acts as a crucial transition point across other measurements, to open up the ground for numerical computations through formulae, which has further applications in higher mathematics and science e.g., force = mass × acceleration (Smith, Males, & Gonulates, 2016). Area-measurement not just enhances the spatial understanding of measurement, but further integrates and enriches students' mathematical scope, as the area model has application in several important topics of mathematics, including multiplication, fractions, algebraic multiplication, scaling, geometry, functions, and probability (Ron, Dreyfus, & Hershkowitz, 2017; Sisman, & Aksu, 2016; Sarama & Clements, 2009; Outhred & Mitchelmore, 2000). AM can also be extended to engage with other higher mathematical topics of measure theories (de Freitas & Sinclair, 2020). Thus area-measurement can serve as a foundational basis to broaden students' mathematical learning, and lead to an integrated, interconnected, and interdisciplinary understanding of mathematics.

Chapter 2: Literature Review

This chapter covers a range of studies that have tried to understand or address the issues related to students' conception of area-measurement. The literature review is broadly divided into four themes (see Figure 2): Conceptual, Curricular, Material Use and Multiplicative Thinking. The first theme presents different gaps or errors in students' conceptual understanding of area and the gaps found in the AM curriculum. For example, confusing AM with the perimeter of a shape (Cavanagh, 2007; Kanhere, Gupta, & Shah, 2013), an issue found to be present even among teachers (Ma, 1999), not being able to identify the unit of area (Lehrer, Jenkins, & Osana, 1998; Kamii and Kysh, 2006), difficulty in abstracting or applying the understanding of area to different shapes, such as L-shaped figures (Cavanagh, 2007; Zacharos, 2006) etc. These studies indicate that students lack the understanding of area as a measure of a two dimensional plane or space. The AM curriculum mainly emphasis procedural understanding, rather than conceptual understanding, by relying on formula, mainly for rectangles, without trying to build on the conceptual connection between the formula (of multiplication of lengths) and AM (Smith, Males, & Gonulates, 2016). This indicates the need for a radical restructuring of the AM curriculum.

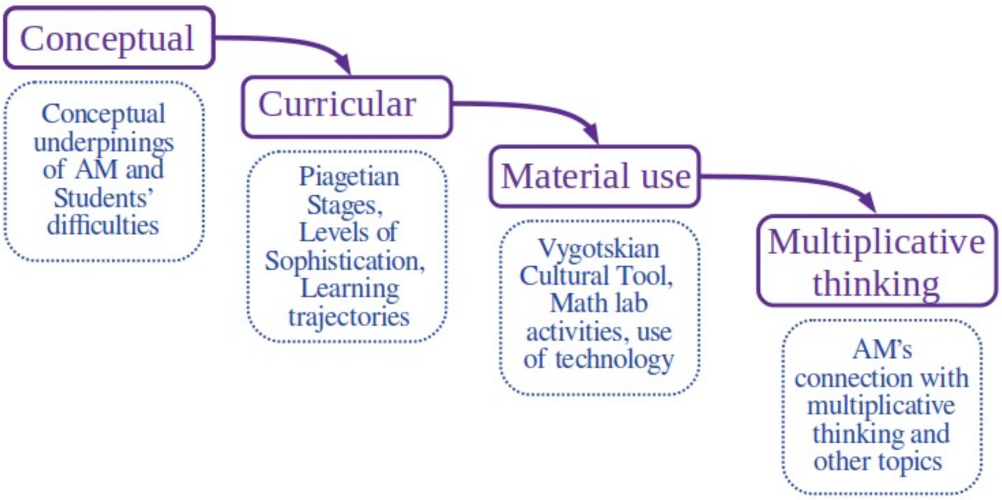


Figure 2. Four themes of the literature review

However, the studies under this theme fall under a deficit perspective, as they aim to find what students don't know with respect to AM, or the gaps in the AM curriculum, rather than acknowledging what a student knows or how the curriculum could be revised. This leads us to the second theme, which was inspired by the Piagetian tradition of looking at what children know about a particular concept (in the present context, about the area concept) at a particular developmental stage, and how the concept develops over time. The development is described variously by different researchers, in terms of Stages of learning (Piaget, Inhelder & Szeminska, 1960), Levels of sophistication (Battista, 2007), Learning Trajectories (Sarama & Clements, 2009), etc. Thus the broad approach under this theme helps identify students' prior knowledge and how one can build on it, providing a possible road-map for what the AM curriculum should look like.

The third theme is aligned with Vygotsky's philosophy around the use of tools. The Vygotskian approach did not consider the absolute stage of a child, but the potential of any child to move to higher psychological processes, mediated through tool use and interactions (Vygotsky, 1980). Culturally developed tools play a significant role (Nunes, Light, & Mason, 1993 as cited in Clements, 1999) in facilitating students' move to higher levels of thinking, and thus may bring about qualitative differences in the developmental paths followed by learners. Piagetian studies did not highlight the uplifting role of the material and other interactions in the making of different schema (or knowledge structures) in the child's cognition, and that the schema are dynamic in nature and constantly growing, even while the child is probed through mediated interaction. Materials also play a significant role in acting as a mediator or a common tool or language, to engage and understand the child's thinking. Thus, to move from understanding a child's thinking to the process of knowledge construction by a child, we need to move our attention to the material factors of the interaction.

Nunes et al. (1993) found that the traditional ruler supports children's reasoning more effectively than a thread (cited in Clements, 1999). Building on the Vygotskian perspective, a ruler acts as a culturally developed instrument that can be appropriated by a child for length measurement through its use, which through further use can be abstracted as a mental tool for the child. Subramaniam & Bose (2012) have highlighted the significance of culturally and historically developed measuring tools (and units) in making the formal learning of measurement more meaningful for students. One must therefore take into account the potential of materials in pushing students to higher stages or higher levels of thinking.

However, one needs to be also careful and conscious about the proper use of materials. Outhred & Mitchelmore (2000) in their study caution that using concrete units might pre-structure the task, rather than disclose students' actual understanding of area. Also, the different conceptual ideas (i.e., iteration, identical units, covering, etc.) of measurement packed within a tool or an instrument may get hidden inside them (or the material). The measurement tools used today are developed through a process of social mediation (Vygotsky, 1980), and for students to adopt them, there is a need to deliver the necessary cultural tools through a proper planned teaching effort (Zacharos, 2006). The materials or tools used for area measurement are mostly covering and counting units, completely skipping the relational aspects hidden between material quantities (de

Freitas, & Sinclair, 2020). To address these objections, we need to focus on the design and use of the material that provides the learner opportunities to engage with higher levels of thinking. That is, allowing them to reason multiplicatively, and not just perform additive counting.

This brings us to the fourth theme, multiplicative thinking or multiplicativity. The connection (or relational understanding) between measurement and multiplicative thinking (having bearing in topics such as fraction, rational number, ratio, percentage etc.) in linear measurements is evident (Mitchell & Horne, 2008), but abstracting the same for higher dimensions like area is still not directly apparent. In AM, multiplicativity arises in ways that do not occur in the case of length measurement, such as the array structuring of units in the case of rectangles, leading to area as the product of length and breadth. Further, there is a multiplicative relation between the area of the rectangle and the unit, between the area and length, and between the area and breadth. Correspondingly, there is an inverse relation between the area measure and the magnitude of the area unit, which is itself dependent on the length and breadth of the unit. Further, the passage to non-rectangular polygons involves triangulation, starting from the area of a right triangle obtained by dividing a rectangle in half, which involves a multiplicative relation.

We thus find that multiplicative relationships are involved in complex ways in AM, indicating a need to design studies to explore different ways in which multiplicative thinking can support AM. Specifically, we need to develop tasks that can elicit or make the connection between these domains more visible. While developing such tasks, we need to move beyond the discrete counting exercise of unit covering, to tasks that allow students to view the continuous nature of area and to see its measure as a continuous composition of lengths (Kobiela, & Lehrer, 2019; de Freitas & Sinclair, 2020).This requires us to re-imagine the tasks – from additive counting of units to multiplicative composition of dimensions. This also aligns with the classic work of Davydov, which argues for the significance of measurement and the need for integrating it with different foundation topics of mathematics, to build a coherent math curriculum (Davydov, 1975).

Drawing from these ideas, the thesis tries to build an integrated model of area, by pulling together various conceptual understandings involved in area, and by connecting them with other foundational topics of mathematics, along with multiplicative thinking.

Chapter 3: Study 1: Exploratory Studies

This chapter outlines the initial exploratory studies to probe students' conception of AM. The broad objective was to characterize the existing scenario of students' understanding of AM in the Indian context, as the literature mainly covers work done in the Western context. A further problem is that many studies end up following a deficit perspective, in labeling students' conceptions as misconceptions, through written tests, rather than meaningfully engaging with students' conceptions (as in Piagetian approaches). Moreover, to plan interventions (reported in the subsequent chapters of the thesis), it was important to study the existing situation beforehand.

Considering the need for a fresh exploration of students' AM conception in the Indian context, observational studies inspired from naturalistic paradigm deemed suitable for this purpose. The naturalistic paradigm, developed mainly from the anthropological or sociological disciplines studies the context with minimal to no interference (Moschkovich, 2019). Drawing from the naturalistic paradigm, students' AM conception are not independent of the curriculum, the material-use, and the aspects of multiplicative thinking that a student learn in other contexts. Thus, the initial studies were done in a naturalistic setting, in the classroom within the regular school schedule. This was supplemented using a mix of approaches. Classroom observations, semi-structured interviews, textbook analysis, planned and structured task-based interviews, and written questionnaires were used. This series of studies are broadly divided into three categories (see Figure 3).

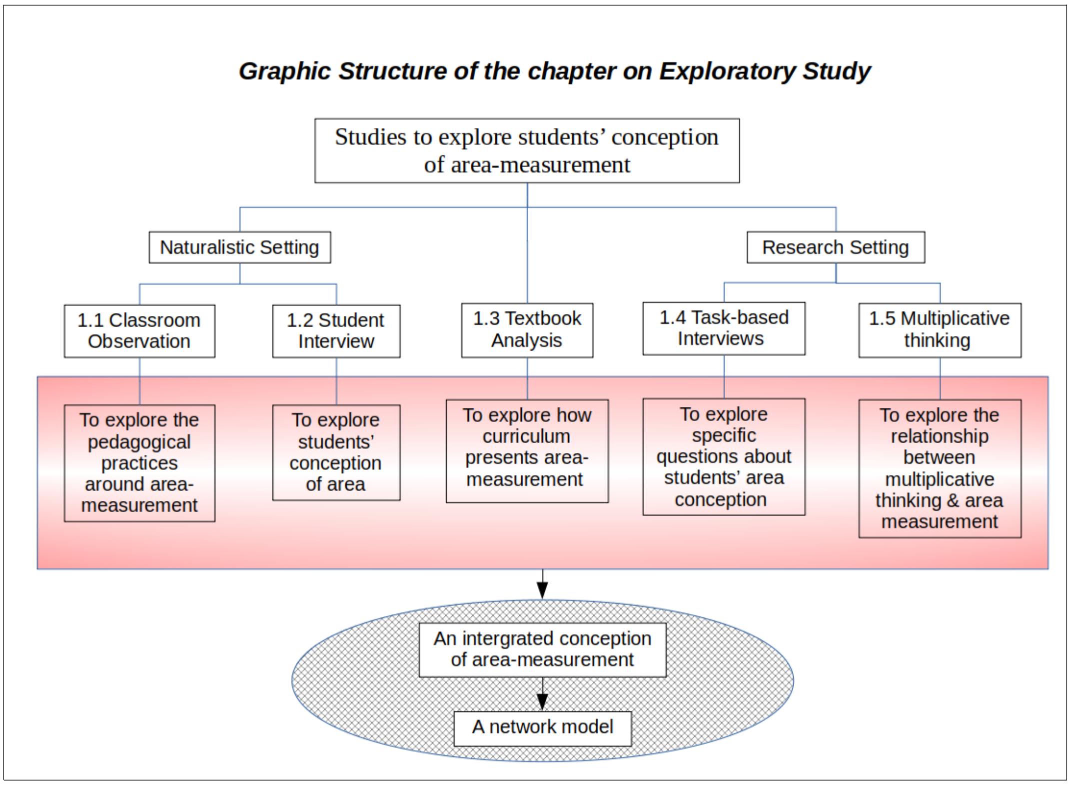


Figure 3.1 : Graphic summary of the chapter

The guiding question for each of these studies are mentioned in Table 3.1. For the first setup (study 1.1 and 1.2), the researcher went to six nearby schools and attended the math classes from 4th to 10th grade, specifically the lessons on geometry and measurement. Since the school setup was generally noisy, the researcher relied mainly on hand-written notes for data-collection, instead of a recording device. The textbook analysis was done by the researcher in her institute. The second setup consisted of task-based interviews with students, either in the researcher's institute or some quiet room provided by the school for using audio recorder.

Table 3.1

Study	Guiding Questions
1.1	What are students' conceptions of "area" ?
1.2	What is the conception of "area" reflected in classroom practices?
1.3	How does the curriculum (or textbook) handle the "area" conception?
1.4	What are students' conceptions of "conservation" of area and perimeter ?
	What are students' representations for area and perimeter?
	How do students interpret area and perimeter for unfamiliar figures?
	What are students' conceptions of unit structuring in area-measurement ?
1.5	What is the connection between area-measurement and multiplicative thinking?
1.6	What is a good model of learning area-measurement? Why should the proposed network model of learning 'area' be adopted?

1.1 Classroom observation findings

Some broad findings drawn about the pedagogy of AM from the classroom observations are: 1) the main focus was on solving the exercise problems in the textbook at the end of each chapter, with an emphasis on the prescribed syllabus and the examination. 2) There was extensive use of numerical calculations in all the classes, with great emphasis on the formula or rule to be used for the given exercise, without delving into the logic for it. 3) There was no discussion on any alternative ways of solving a particular problem. 4) There seemed to be a race among a few students to give the answer fast, and the teacher only looked for the correct answer in the classroom. 5) The discussion on AM was mostly around typical conventional shapes – regular polygons like rectangle, square, triangle etc. 6) The grid was used in a limited way to measure area, without explanation or discussion on why the different sized units or parts are counted in a particular way.

1.2 Student Interview findings

Some of the broad findings from this study context are: 1) when asked about area, students responded with some formula (e.g., l × b, side × side, 2l + b, l+b etc.), which provided a symbolic representation, with no reference to the 2-dimensional (or plane) space. 2) On conservation, most students could calculate the remaining area, but they applied a similar calculation for perimeter, indicating a lack of spatial understanding of these measurements. 3) Most students could not explain why l × b works. 5) While some called it a rule or convention, very few could draw square units along the dimensions. 6) For irregular and curved closed shapes, most students could not say if it had area, indicating that their understanding of area was fixated to typical shapes like square, rectangle, etc.

1.3 Textbook Analysis findings

Six math textbooks were analysed – three from Maharashtra[1] state board (MSB) books for Grades 5, 6, and, 7 and three from National Council of Educational Research and Training (NCERT[2]) for the same grades. Some of the broad observations are: 1) Geometry and measurement related content are dealt separately, with the former referring to purely mathematical objects, while the later referring to real life and practical contexts. 2) Regarding integration of concepts, it was found that the context of area is used in several other math topics, indicating that area has application in broad range of topics. However, except one, none of the books emphasized this integration. 3) All the textbooks mainly showed conventional geometric shapes, with very few references to irregular or curved shapes or real life objects. 4) The unit of measurement for area seemed to be missing in these books, except in the Grade 5 books. Here too, references were few and mainly to standard square units. 5) In order to connect AM with the students' real life contexts, NCERT books have used "size" to refer to area, which seem to be completely missing in the MSB books. 6) Regarding the nature of tasks on measurement, textbooks have exercises that require students to find the measure (or numerical value) for a given line or angle, and also the reverse, where students are required to draw a line or angle of given measure. However for AM tasks, the exercises are mainly of the former kind, with "find" and calculation based questions, rather rather than drawing or visual tasks. 7) The use of grid shown in the textbooks is very procedural, with an emphasis on counting of squares. Other rules were also given, such as: 'consider a square that is more than half-filled as 1 unit and less than half-filled as 0 units', with not much explanation provided for such procedures.

1.4 Task-based interview findings (in research setup)

Task-based interviews were done in the research setup with ten students, from Grade 5 (~11 yr old) followed by a trial with four other students. Students were identified by their math teacher as above average (AA), below average (BA) or average (A) scorer respectively. The guiding questions for the

1 Maharashtra is one of the states in India, out of 28 states and 8 union territories. Maharashtra like most other states has its own state education department and its own curriculum.

2 NCERT is an autonomous body of the Government of India to improve the quality of school education that includes publishing books followed in several school system in India.

interviews are provided in the fourth row in Table 3.1. Some broad observations are: 1) Above-average (AA) students tended to rely heavily on numerical procedures and formulas for area, even for tasks where this was inappropriate. In contrast, the average (A) and below-average (BA) students seemed quite open and flexible in using other simple non-formal strategies, or techniques like estimation, comparison, etc. 2) For conservation tasks, inspired from Piagetian tasks, when students were shown two different arrangements (with equal area) and asked about the available space, students tended to reason based on the convenience of spatial arrangements for particular uses, rather than being perceptually misled towards ignoring conservation. 3) When students were asked to measure the boundary, they could do it. But when asked for perimeter, they tended to apply inappropriate formulas or mathematical operations. 4) When students were asked to highlight (mark or shade) the area and perimeter of two given shapes, almost half of the students looked puzzled and asked for numbers or the dimensions of the figure. 5) For an L-shaped figure, only 2 out of 10 students could find the area.

For the unit-structuring tasks, students were given a rectangle (either in the form of a sheet or by specifying the dimensions) and a unit (rectangle or right-triangle). They were asked if the given unit can cover the given rectangle. Mainly two strategies were used by the students in completing the task: the procedure of dividing the area of the rectangle with the area of the unit, or checking the units along the two dimensions of the rectangular sheet. The students using the former strategy did not care about the dimensions of the unit or the rectangle. Students using the later strategy could recognize the cases where the unit would not fit along one of the dimensions of the rectangular sheet. Some even suggested that the unit needs to be divided into pieces to cover the remaining sheet. For the right-triangular unit, all the ten students thought it could not completely cover the given rectangle, with two students later realising that the triangles can be joined to form a rectangle, which in turn can fill the given rectangle. Thus, the results indicate that it is not enough to just know the procedure or the unit covering aspect for AM, but more connection needs to be established between the two aspects.

1.5 Multiplicative Thinking

These studies attempted to integrate AM and multiplicative thinking, to build a richer understanding of AM. Students' responses to some tasks – developed to elicit the connection between AM and multiplicative thinking -- were examined. The responses of eight Grade 5 students are reported. The tasks were first tried with other students and refined. There were four tasks: Comparison Task, Card Task, Measuring Task, and Unit of units Task (see figure 3.2). In the comparison task, students were given two rectangular sheets to compare. Most students tended to compare the sheets either by length or breadth, only later overlapping the sheets to compare, indicating a tendency to compare the dimensions rather than the space covered. In the card task, students were given some number of cards and were asked to make a rectangle with them. Half the students recognized that they had used multiplication factors to do the task. The remaining students could not explicitly express their strategy or thinking. In the third measuring task, students were asked to compare a square sheet (7inch × 7inch) and a rectangular sheet (8inch × 6inch), with a given small square card (1inch × 1inch). All the students marked the adjacent sides of the rectangular sheet using the given square card. However only three of them multiplied the number of cards fitting along the edges. The rest used repetitive addition to get the total number of cards.

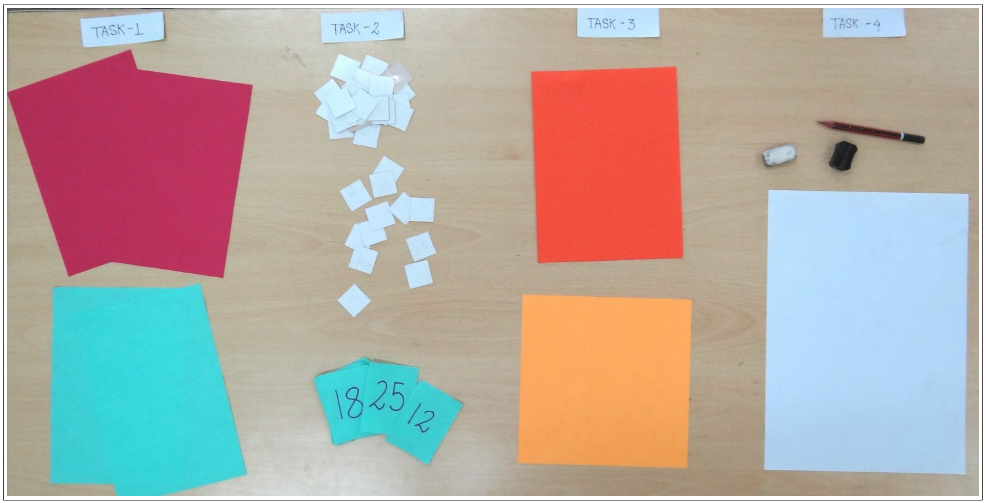


Figure 3.2 Materials used for the task

The fourth unit of units task required students to get the measure of a given A4-sheet, and then to get the measure of a table in terms of the previously used square unit. Six students were able to do this unit of units task. But in this case also three used the multiplicative relation, while the other three used the repetitive addition relation.

1.6 A Network Model of AM

One of the objectives of the work reported in this chapter was to propose a integrated model of AM, consolidating all the findings from the studies. Based on all the findings, we argue that the area concept is best understood as a network concept, requiring the coming together of the four ideas discussed in the previous studies– unit, array, multiplication, and unit of units.

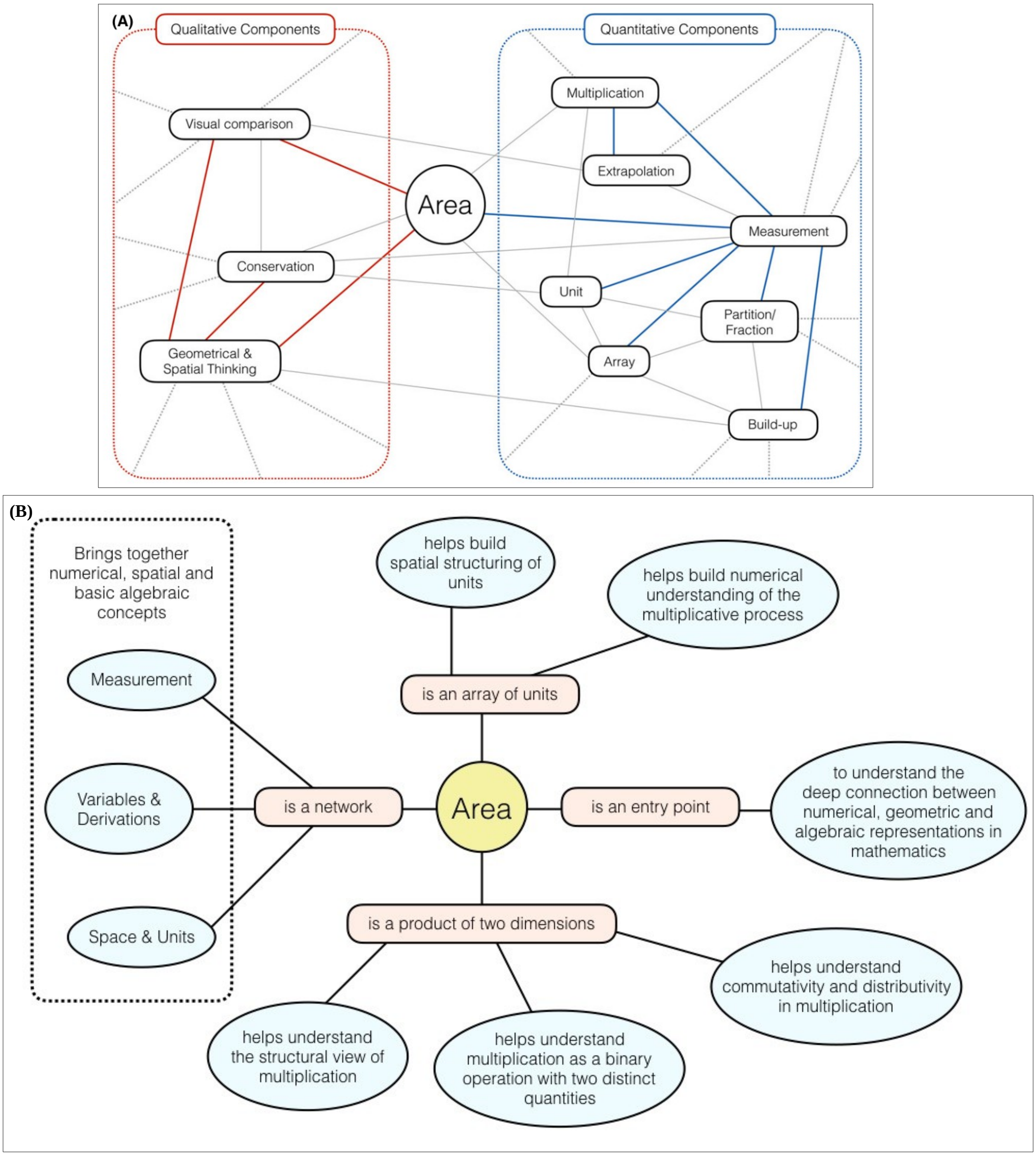


Figure 3. 3 The network model of area. The top image presents the area concept connected with the conceptual components and processes involved in it. The bottom image shows the connection of area with other math topics.

This network model – in contrast to the linear learning models studied in theme 3 of the literature review – aims to support the construction of a more coherent curriculum, by connecting different conceptual content and processes together into an integrated whole. The next chapter pulls together all the insights drawn from these studies into a classroom intervention study.

Chapter 4: Study 2: Classroom Study

In this chapter, we move beyond individual construction, covered in the previous chapter, to study social construction inside a classroom, using a AM lesson designed using insights from the work reported in the previous chapter. The two main processes incorporated in the design of the AM lessons are: a) argumentation, and b) opportunities to integrate spatial and numerical representation in AM, through collective social construction. Toulmin's argumentation framework is used both as a conceptual and analytical framework (Toulmin, 2003; Toulmin, Rieke, & Janik, 1979; Krummheuer, 2007; Reid & Knipping, 2010).

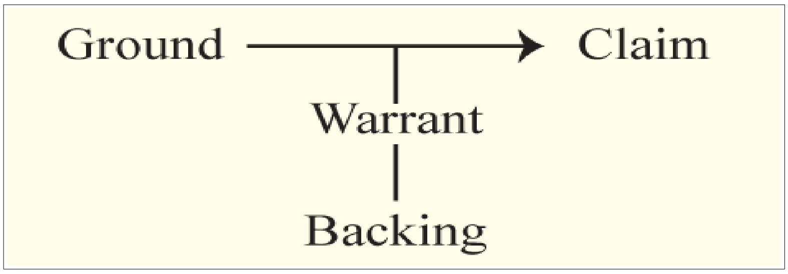


Figure 4.1: Toulmin's framework for Argumentation

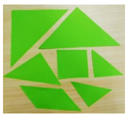
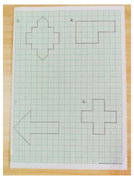
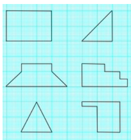
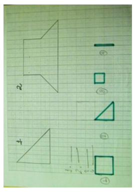
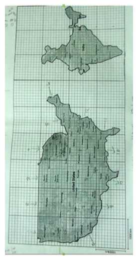
The framework consists of three main components (see figure 4.1): (i) claim whose truth is to be established, (ii) ground consisting of a set of facts, which provides the foundation for the claim and (iii) warrant, which provides the basis to arrive at the claim from the ground. The credential of the warrant comes from the backing. Backing is usually field (or discipline or topic) dependent and likewise warrant also varies with different fields of argumentation. The present study discusses episodes, which are selected events or segments in the classroom lesson, having the two main components of the above framework, i.e., claim and warrant.

In the context of the classroom, claim is recognized as a statement or a solution presented to the whole class or a doubt expressed by any member in the class. Warrant is the explicit rationale provided by the actors in the classroom, either in terms of verbal justification or with objects, or with drawing or symbolic manipulation. Since the study adopts a teaching design experiment approach, a main component is the design of the lesson, reflected mainly in the careful design of the tasks used in the lesson. The next section gives brief descriptions of the tasks.

Task design and description

The tasks were designed based on the insights from the previous studies, with each task aiming to integrate each of the concepts or processes recognised in the network of AM (figure 3.3). The teaching-learning sequence, designed to collaboratively construct an understanding of AM, can be broadly classified into nine segments, which were enacted over 12 days. Table 4.1 captures the details of this design, where the first row is the name of the sessions and the second row gives a glimpse of the activity or the artifacts used in that session.

Table 4.1: Task description

1. Stamp making activity	2. Tangram Activity	3. Graph paper Activity-I	4. Graph paper Activity-II	5. Different units	6. Number to shapes	7. Extra-polation to bigger measure	8.Measur-ing objects around	9. Scaling & curved shapes
					Graph paper, 12 can be drawn as 3×4, 2×6, 1×12	Graph sheet, A4 and A3 size sheets	Door, Black-board, Windows	

The first session allowed students to engage with the ideas of unit, covering, iteration and array structure. Also, the relation between the total number of units required for a given sheet, and the multiplicative relation between the measure of area and the number of units along the length and breadth. Using different units also reveals the variable nature of the measure, in terms of the unit's size. The second session was on the Tangram activity, which allowed students to connect geometrical, numerical and also the algebraic or symbolic abstraction, in terms of the relation between the different Tangram pieces (units) and the whole. Session 3 onward, the graph paper is used extensively, to develop it as a cultural tool for AM in the classroom micro-culture, just like we use the ruler for measuring length.

The task in session 3 can be done by additive counting of the units in the given shapes. The shapes in session 4 were carefully designed, to bring in fractional value (or measure). Session 5 used a nonstandard graph sheet, with four different shaped (fractional) units, to allow students to engage with the multiplicative relation between different units and the resulting value (or measure) of the given shapes. Session 6 took an inverse route, where students were asked to *geometrize* a given number i.e., to generate various possible geometrical shapes on the graph paper, for a given numerical value (or measure). Sessions 7 and 8 were inspired by the component of "extrapolation" in the network model on AM, which requires students to measure bigger spaces. Session 9 is about zooming out into larger spaces on a world map.

About the study: Setting, Data, Analysis, Results

The teaching was done by the researcher and her colleague (7 days by me and 5 days by my colleague) for approximately 2 hours every day, for over a period of 2 weeks (i.e., 12 days in total). Data collection was through video recordings of each lesson, and a fellow researcher writing the lesson log every day, with some other researchers observing, followed by a debriefing session with fellow researchers.

Figure 4.2 The classroom setup

The present study focuses on the interaction among students, and between the students and the teacher, during episodes of argumentation. The episodes particularly bring forth the complexity involved in spatial and numerical aspects of area-measurement. Following the argumentation framework, those episodes were focused on, where varying claims were put forth, challenged and justified. In the interest of keeping the synopsis document concise, only one of the four episodes will be briefly discussed here. Pseudonyms are used for students.

Episode 1: For one of the tasks (Task-6, Table 4.1), students were asked to make different possible rectangles on a graph paper for a given size (numerical value), and then write the numerical multiplication facts. One of the sizes given was 15 units. For this size, students came up with various facts like 3×5=15, 2×7.5=15, 1×15=15. Sajaad came up with 30×1/2=15. He came to the black board and made a 6×5 rectangle, and divided it vertically into two halves, to show that there are 15 units in each half. The teacher then asked the students to come up with more ways to divide the 6×5 rectangle into two equal parts. Many students suggested horizontal division and Sajaad suggested the diagonal division as well. Most of the students agreed that the rectangle can be divided vertically, horizontally or diagonally into two halves. But when Sajaad tried to divide his 6×5 rectangle diagonally into two halves, he was just looking at the rectangle and appeared to be stuck. On asking, he said that he could not count 15 units in each of the triangular parts. To convince Sajaad that a diagonal division of the rectangle will produce halves, the teacher prompted Sajaad to check if the triangular parts are congruent, and gave him a pair of scissors to to cut the rectangle along the diagonal and check whether the two pieces are equal. But Sajaad was not convinced that the two triangular pieces are equal. The teacher guided him to superimpose the pieces to infer that they are congruent halves, but he was still unsure as he was not getting 15 units. Though he agreed later that they are two equal halves, he did not agree that they have 15 units each as he could not count 15 in each piece. The teacher tried to convince the students that even if 15 units cannot be counted in each of the triangular halves, since the two triangular halves of the 6×5 rectangle are equal halves, it must be half of 30. However, the students did not appear to be fully convinced. The argument structure for the teacher was different from the argument structure for the student, as indicated in Figures 4.3 and 4.4.

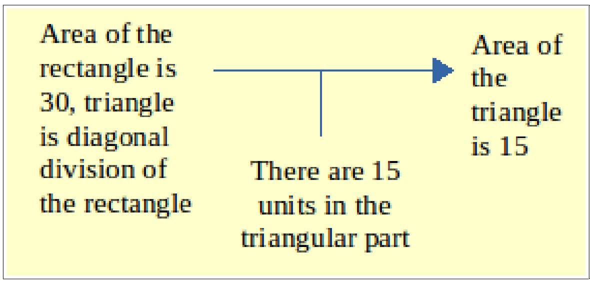


Figure 4.3: Student's argument structure

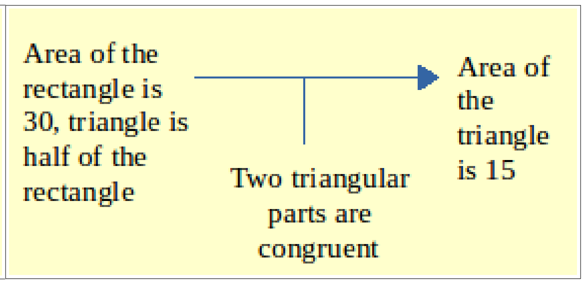


Figure 4.4: Teacher's argument structure

Later Raziya came to the board and facing the class, challenged the teacher’s claim that the units could not be counted. She showed that the parts could be counted and provided support to Sajaad's argumentation framework, by providing the same warrant that he and other students were looking for (see figure 4.5).

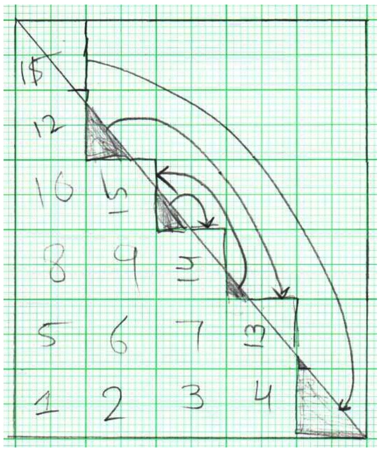


Figure 4.5: Raziya showing that the triangular part contains 15 units

Thus, the present episode demonstrates, how important it is to recognise the argumentation framework of the student, and align one’s reasoning to fit into them, which is why Raziya was more successful in convincing students in comparison to the teacher. Also, the episode highlights the gap between the spatial and numerical aspects of area, where even though students felt that the two pieces appeared to be equal halves spatially, they were not convinced that the number of units in each will also be half of the whole.

Reflections

The study highlights the disconnect between the geometrical and numerical understanding of students. The argumentation framework reveals the different argumentation structure of different actors in the episode, and the need for the structures to be aligned for the collective social construction to happen in the classroom. The study also highlights the need to establish the socio-mathematical norms more meaningfully. Otherwise it may lead to only some rudimentary actions, of the norms getting reflected, rather serving any meaning making exercise in the classroom.

Chapter 5: Experimental study: Analysing the effect of material interaction

The previous chapter was inspired from the social constructivism paradigm, and focused mainly on the social interaction during AM tasks, and could not pay much attention to the material interactions (or manipulations) that were occurring simultaneously. The present chapter, inspired and guided by the recent enactivist perspective, tries to understand the role of material interaction in shaping students’ AM conception. Though the argumentation framework could be used to analyse the process of social interaction, no such framework was readily available to capture the nuanced processes of material interaction. The present study provides a method to track the role of material interactions while learning the AM concept, and also ways to analyse and interpret this data. The chapter presents two studies, which explore the two research questions in table 5.1. The design of the study is shown in figure 5.1.

Table 5.1

Studies	Corresponding research questions
Study 5.1	1. What cognitive process changes happen as a result of manipulation of materials?
Study 5.2	2. How does manipulation transform the process of solving an area-problem?

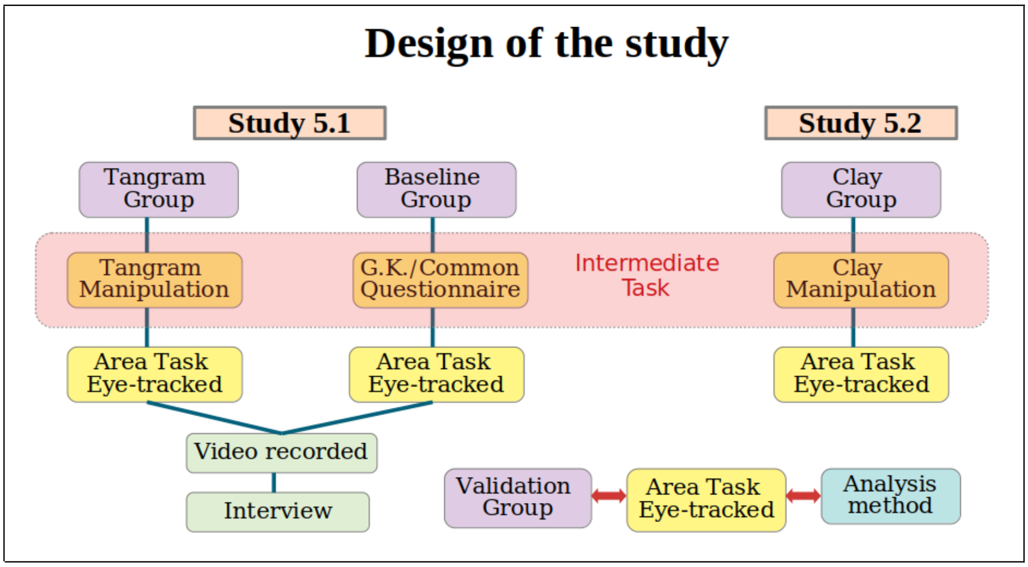


Fig 5.1. Design of the study

As can be seen from the above figure, data for the two studies (5.1 and 5.2) were based on eye-tracking, video recording, and students' interview. Students from Grade 6 (~11-13 year old) were assigned to one of the three groups: Tangram (11 students), Baseline (11 students), or Clay (10 students). Students from the Tangram group were asked to do a simplified Tangram type task, students belonging to the Baseline group were given some simple general knowledge questions, and students from the Clay group were asked to make some simple animal out of clay dough. After finishing their respective intermediate tasks, the primary task given to each student was to calculate the area of the two given non-standard figures (see right of Figure 5.2) drawn on graph paper. A unit was shown in one corner of the graph paper, and students were asked the following area-problem question: *A full cake is shown in the figure. A piece of this cake is shown at the right corner of the graph paper. This piece costs Rupees 1/-. What will be the cost of the entire cake?* The next section gives a brief summary of each of the two studies 5.1 and 5.2.

Study 5.1: The setup of study 5.1 is shown in figure 5.2. When students were doing the primary task, a static eye-tracker and video recorder captured the eye and hand movement of the student respectively.

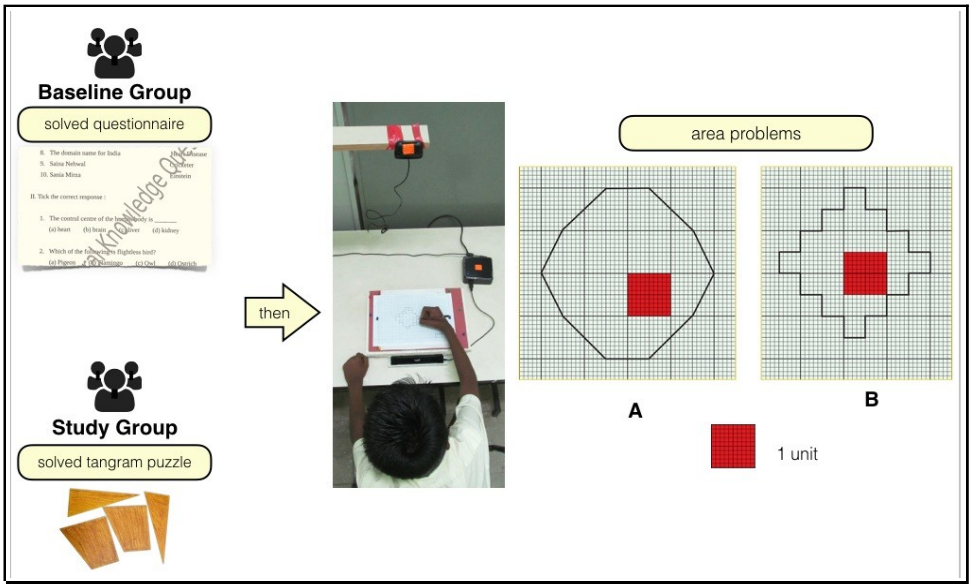


Figure 5.2. The set up of study 5.1

The qualitative analysis of the video data indicated ten strategies that were used by students while solving the area problem (figure 5.3). However, analysis of this data did not clearly show overall differences between the groups. Even though limited in terms of identifying significant trends, this analysis helped highlight three strategies that showed differences across the two groups. This pattern guided the analysis of the eye-movement, which looked for markers of spatial chunking and counting strategies (indicated by the qualitative analysis of videos and interviews) in the different groups.

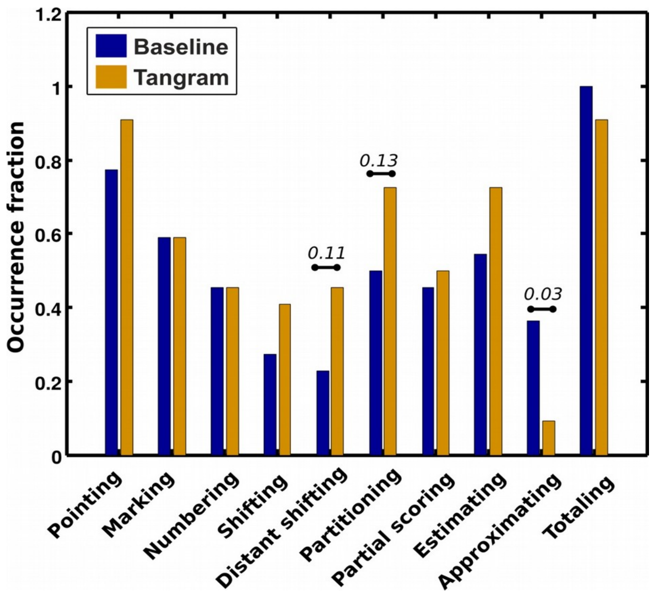


Figure 5.3. Various strategies used by students in the experiment. Number above bar pairs represent p-values for corresponding two sample t tests.]

The analysis of the eye-movement (see figure 5.4) showed significant difference in the eye movement pattern between the baseline and the tangram group, with the tangram group showing significantly less mean pattern change (Task A: t 20 = 2.88, p = .0094; Task B: t 20 = 3.33, p = .0033, p < .01) and high maximum pattern change (Task A: t 20 = 1.63, p = .14; Task B: t 20 = 2.95, p = .0078). Here, the lack of significant difference in maximum pattern chance for Task A can be accounted for by the affordances provided by the task itself that compels students even from the baseline group to rely on partial chunking, which is not the case for Task B.

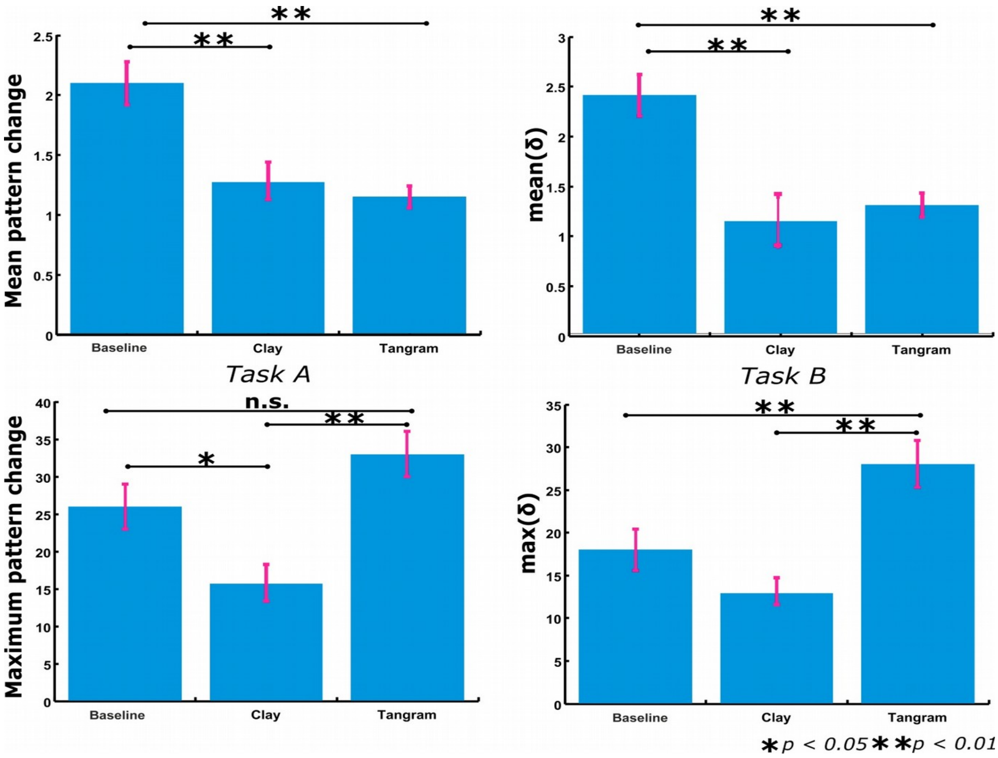


Figure 5.3. Combined result from both the studies or all the three groups

We interpreted the greater mean pattern change to be aligned more with the counting strategy, where one needs to focus on parts or smaller sections. The maximum pattern change was interpreted to be aligned with looking at the task space more holistically, and at the corners, as well as doing larger shifting to move parts from different edges or sections. This pattern aligned with a chunking strategy. This result was further validated by the validation group, where adult participants who were asked to follow the chunking strategy showed their eye movement pattern aligned with the Tangram group in terms of having maximum pattern change and less mean pattern change. The group that was instructed to follow counting strategy had eye movement pattern similar to the baseline group.

Study 5.2: Figure 5.3 indicates that the clay group followed a strategy different from the baseline and tangram groups. The reduced minimum pattern movement indicated they were not focusing on smaller parts as the baseline group, but the reduced maximum pattern movement indicates that they were also not aligned to chunking or longer shifting that we saw in the tangram group. A possible interpretation is that they were at an advantage over the baseline group in not getting stuck to counting, but the clay manipulation was not as effective as the tangram in helping achieve the longer shifting or chunking. However, it is challenging to infer anything conclusive about the specific strategies, as no further video or interview data was gathered for this group.

Overall the studies in the chapter indicate that the group of students exposed to the manipulation of tangram shapes use the relatively more efficient strategy of chunking while solving the area problem, compared to the baseline group, which used the more tedious counting strategy. Further, the second study indicates that the tangram type manipulation had an advantage over clay manipulation, suggesting an effect of the manipulation of a geometric shape (tangram).

Chapter 6: Conclusion

This chapter summarises the overall reflections from the studies and discusses their implications. As discussed, the studies were based on theoretical approaches ranging from Piagetian individual constructivism to social constructivism to enactivist approach. Some of the major contributions of the thesis are outlined below:

Curricular contributions: The findings suggest specific changes that can address the several gaps identified in the treatment of AM in the curriculum. To mention a few: the thesis brings in the need to use local terms for area, to connect the formal knowledge of area with the real life or known experiences of the student. The textbook analysis shows the use of mainly typical shapes, which narrows students' association of area with just these limited shapes. The analysis of AM topic in textbook reveals the presence of mainly "find" or "solve" questions rather any draw or design tasks to tap students' creative potential. The thesis provides a better and richer way of using grid paper, compared to its present use in the textbooks. The thesis also uncovers the potential of activities and materials like Tangram in enhancing students mathematical abilities. In contrast to a topic-wise, disconnected, and linear curriculum, the thesis argues for a network model of AM, and hence a spiral curriculum that can integrate different topics, taking into account the need of students with different sets of prior knowledge. The network model opens up pathways to explore the foundational role of AM in connecting several other important topics of mathematics, especially multiplicative thinking, thereby contributing to the idea of a more coherent math curriculum.

Pedagogical contributions: The thesis provides a new model for the pedagogy of AM. In addition to the usual use of the argumentation framework, in the context of justifying or proving claims, Chapter 4 presents a case where argumentation supports the collective construction of a concept. The classroom study in Chapter 4 also provides a road-map to apply the ideas of social constructivism in the math classroom. The thesis also highlights the aspects of socio-mathematical norms that need to be attended to in the classroom teaching practice.

Research contributions: One of the significant contribution of the thesis is the operational re-imagination of AM, as illustrated by the detailed design of new tasks, to address the links in the network model. The thesis addresses the gaps in the earlier literature, that focused mainly on the additive counting of discrete units, to highlight the continuous nature of the AM quantity through bringing in the aspects of multiplicative composition of units, measure and dimensions. As the thesis investigates students' conception of AM following three different research paradigms, the findings highlight the strengths and weaknesses of the different paradigms. The thesis thus provides a case of adapting the three paradigms, especially the emerging enactivist paradigm, which is under-represented in the field of math education. The thesis indicates that material-interaction and manipulation contributes significantly to the development of the AM concept, and extending this result, material-based designs could lead to better ways to teach and learn mathematics.

Future Implications: The clear operationalization of ideas, the insights from the studies, and the new methods and tasks developed in the thesis could be extended further, to systematically study other mathematical concepts. The integrated and interdisciplinary approach, illustrated by the network model, would be fruitful while investigating the learning of other mathematics content as well. The use of transparent grid as a tool for students to measure area can be recommended in schools. Thus the thesis also promotes ideas for the design and development of proper materials and tasks to address the gaps in MER.

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