Empirical studies of students' conception of Area-measurement, and their implications for Mathematics Education

A Thesis

Submitted to the
Tata Institute of Fundamental Research, Mumbai
for the degree of Doctor of Philosophy
in Subject Board of Science Education

by

Jeenath Rahaman

Homi Bhabha Centre for Science Education,
Tata Institute of Fundamental Research
Mumbai

January, 2023

DECLARATION

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions.

The work was done under the guidance of Professor K. Subramaniam and Professor Sanjay Chandrasekharan, at the Tata Institute of Fundamental Research, Mumbai.

[Jeenath Rahaman]

In my capacity as supervisor of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.

[K. Subramaniam]

Date: 29-01-2023

[Sanjay Chandrasekharan]

Date: 30/01/2023

Acknowledgement

I acknowledge my mentors, colleagues, friends, and family for kindling the quest of research in me. I deeply thank my supervisors K. Subramaniam and Sanjay Chandrasekharan for their constant support and engagement with my work.

I am indebted to my guide K. Subramaniam for introducing me to Mathematics Education. His conscientious approach has pushed me to learn about quality research. His feedback has helped me immensely to improvise my work. His sound reading and wide understanding has helped in building criticality in my research practice.

I am grateful to my co-guide Sanjay Chandrasekharan for introducing me to the field of Learning Sciences. His wide interest and methodical approaches has helped me to get in-depth knowledge about the field. The conversations and chats with him provided interesting perspectives to make the field and the work more exciting.

I express my gratitude to all the students and the teachers, who participated in my study, for giving their valuable time generously and playing a central role in the thesis. Engaging and interacting with them helped me get a real picture of the field and learn from them.

The HBCSE math cell has a huge role to play in shaping my research work. The math cell members worked as a team for every event. The debriefing and feedback sessions helped me gain criticality and learn about the practice better. I sincerely acknowledge Tuba for her profound support and involvement in conducting the teaching camp for my study, Aaloka for comprehending difficult research papers with me in the beginning helped me understand academic writing better, Shweta for introducing me to the principles behind math lab, Arindam for providing sound academic advices and other members Harita, Reena, Arati, Saritha, Arshiya, Rajendra and Sushant who helped in many other ways.

The LSR (Learning Science Research) Lab provided novel perspectives to my work and also helped in making sense of my data. I heartily acknowledge Harshit for his contribution in designing a study setup and collecting the precise data, Nisheeth for his help in the quantitative analysis, Geetanjali for helping me organise my studies, Durga Prasad for providing valuable ideas for my study, and others including Prajakt and Deborah for their inputs.

The HBCSE research scholars have provided me with the strength to persevere throughout the PhD journey. I acknowledge Shikha for giving useful inputs on different occasions, Rossi for helping in articulating complex problems and Anveshna, Shraddha, Rafikh, Gurinder for helping me immensely on innumerable occasions. My interactions with Ishan helped in noticing new ideas. Discussions with

Arul, Himanshu, Meenakshi, Charudatta, Chaitanya, Saurabh helped me gain new perspectives in my work. I am grateful to my seniors Amit, Aisha, Aswathy, Farhat, Saurav, Imran, Mashood for their guidance.

This work was possible due to the support of several HBCSE staff members. Manoj contributed in design aspects and supported in resolving technical issues. Anil, Ashish, Rajkumar and Soham provided support in troubleshooting computer related problems. On many occasions my interactions with Johnson, Jaikishan and Ravi Sinha were helpful.

I learned a lot from the HBCSE faculty members including Jayashree Ramadas, G. Nagarjuna, Sugra Chunawala, Ankush Gupta. My formal and informal discussions with Karen Haydock were immensely helpful in refining my ideas. Prithwijit De being my TAC member encouraged me in my work. Dean Savita Ladge and Director Arnab Bhattacharya gave their support in handling the procedures.

I thank my colleagues at TISS for the support and the exposure I received working with them. I acknowledge Padma Sarangapani for her support. My interactions with Suchi Srinivas and Ruchi Kumar helped in sharpening my ideas.

I offer my appreciation to the spaces that inspired me to work. Math-lab, HBCSE’s canteen, TISS-library, and several other open public spaces provided a stimulating environment that helped generate new ideas.

I acknowledge NBHM (National Board of Higher Mathematics) for supporting my conference visits. The feedback received from researchers and reviewers from different institutes helped in broadening my landscape. I acknowledge Jayashree Subramanian for conducting MES in India that helped in gaining fresh perspectives.

Finally, I would like to thank my mother and brother for their unconditional love and trust in me.

We acknowledge the support of the Department of Atomic Energy, Govt. Of India, under Project Identification No. RTI4001.

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Abstract

The studies reported in this thesis are aimed at understanding student's conception of area measurement (AM), particularly in terms of understanding the cognitive processes (rather than outcomes), and their implications for mathematics education research (MER). The thesis consists of three main sets of studies, which adopt varied theoretical and methodological frameworks, broadly following three major trends of research in MER – constructivism, social constructivism, and enactivism.

The first set of studies were inspired by Piagetian theory of constructivism or individual construction, and aims to understand the status of students' conception of area through naturalistic methods (Moschkovich, 2019). Since naturalistic method does not focus exclusively on the individual learner, but also considers external environmental factors, with minimal to no external interference, the initial studies were conducted in-situ, to understand the pedagogy of AM through classroom observation, students' interviews and textbook analysis. Later, however, structured tasks, based on interviews, were conducted with students in a research setup. The studies highlighted a range of issues with respect to AM conception, and led to a network model of AM, as a way of consolidating the results.

The second set of studies involved a teaching design experiment, where tasks were designed and developed based on insights gained from the previous studies, and applied in a classroom. Inspired by Vygotskian social interaction theories and social constructivism, the lessons were aimed at encouraging collective construction of concepts within a classroom, through the process of argumentation. The analysis of classroom interactions was based on the argumentation framework (Toulmin, 2003; Krummheuer, 2007), to examine the argumentation structure in the classroom. The study highlighted students' conceptual difficulties in connecting spatial and numerical aspects of AM, and the way students engage in the meaning making process through collective argumentation in the classroom.

The third set of studies were inspired by recent advancements in enactivist theories of cognition, and their applications to mathematics education. The studies sought to understand the role played by physical manipulations while solving AM tasks. The study was based on the eye-tracking method, and found significant differences between the eye-movement patterns of students who used manipulations and those who did not. The eye-movement patterns of the group of students who did specific geometric manipulations, based on tangrams, indicate the use of more efficient strategies to solve the AM task, compared to the group who did not do any manipulation, and also those who did an unrelated manipulation using clay.

The final discussion brings together these diverse results, and discusses multiple conceptual, curricular, and pedagogical implications of these results for the learning of AM.

1

Introduction

In this thesis, I report a journey to understand students' conception of area-measurement and the process by which students construct the conception of area-measurement.

1.1 Personal Motivation

During the late 1980's, when I was very young, my father, who was in the armed forces, and other elders used to give me sweets or money to buy them. I used to get coins of only 5 paise, 10 paise, 20 paise or at most 25 paise. Irrespective of the value of the coins I got, I always spent the money on biscuits and candies from a nearby store. Of course, currency notes for Re 1 and Rs 2 and even higher denominations existed, but I was never given that much money. In the year 1989, I was about 2 and a half years old and my father came home after a very long time. Following our tradition, he gave me a Rs 2 currency note. I was thrilled to receive that red-coloured note, I knew that the note was special and was more valuable than the coins I used to get or even the blue Re 1 note I had seen. I went to the store to buy my favourite items and I handed over the currency note to the shopkeeper and asked him to give me candies and biscuits. Looking at the higher denomination note, he asked me, "*For the whole note?*" This question was unfamiliar to me so I thought about it for a while and then I tore the currency note in half and handed him one of the halves. The shop-keeper got angry, he refused to take the piece of the note and give me what I had asked for. His reaction made me feel as if I have done something wrong. I came back home with a sad and heavy heart and shared this embarrassing incident with my parents and showed them the torn Rs 2 note. But to my surprise, my parents felt happy and proud. They told me that a note loses its value if it is torn but they also made it a point to praise my inner reasoning to tear a Rs 2 note. They appreciated my reason that by tearing the note I could give the shop-keeper a part of the full amount and not the full Rs 2 note. This incident is one of most powerful memories of my childhood for my parents and me, so much so that the characteristics of the note, the size, the colour, are still vivid in my memory. I still ponder about the reasoning used by that young child, the reasoning that comes to a child much before any formal intervention (or formal mathematical learning). That is also a reason why the Piagetian way of probing students' intuitive reasoning initially inspired me to choose this topic of study. To me this instance indicates the intuitive connection between quantity and space that a young child makes before any sort of formal school intervention.

1.2 Academic Motivation

1.2.1 Practical motivation

Measurement is one of the most important tools to understand the world around us. Some of the most basic everyday measurements that we come across are time, distance, weight, volume, temperature and length. The reason measurement is so prominent in our everyday life (and communication) is its practical utility. The knowledge and use of measurement is also essential to be an informed individual in the present advanced society. These factors make measurement an important topic in a child's education.

1.2.2 Curricular motivation

The need to measure comes intuitively to children (Smith, Males, & Gonulates, 2016). However, in formal learning contexts (e.g. school) Indian students mainly (or at least initially) encounter measurement in mathematics. A major basis of measurement learning thus gets formed through the school mathematics curriculum. While the school curriculum gives attention to different measurements, the major focus is given to geometric measurement. Most curricula, including the Indian curricula, present measurement as an important content area, but keep it separate from geometry. The word geometry literally means "earth measurement", indicating that even geometry might have been conceptualized through measurement. Also, the general understanding is that geometry was developed to measure and mark fields in Egypt, because every year, the boundaries would be erased when the Nile flooded (Lloyd, 2008; de Freitas & Sinclair, 2020). However, contemporary curricular geometry is stripped of its measurement roots, and is kept separate from the topic of measurement, with measurement mostly following geometry, rather than the other way round, which might be closer to history. Several math educators have established the importance of connecting geometry and measurement, to support the conceptual understanding of measurements involving spatial components such as length and area (e.g. Huang, 2017; Owens & Outhred, 2006). However, this research has not changed curricula, which still present measurement and geometry as separate topics in the school curriculum, with more focus on geometry, which generally precedes measurement (de Freitas & Sinclair, 2020).

Studies of cultural anthropology and symbolism suggest that stages in the child's cognitive development follow a progression similar to evolutionary stages of human development (Borchert & Zihlman 1990, Bates 1979, Wynn 1979, as cited in Foster, Mary LeCron, 1994). That is, children might be able to cognize many of the concepts better if presented in the same sequence as they have evolved historically (naturally) in human society. Contrary to this idea, school curricula present the final abstracted (or idealized) forms, such as the Pythagoras theorem, using formal statements or deductive proofs, completely stripped of its historical human roots.

Typically, the curriculum presents high school geometry as a collection of abstractions, in terms of definitions, proofs, axioms, and postulates while primary school geometry focuses more on vocabulary (Sinclair & Bruce, 2015). A major part of school geometry is mainly Euclidean geometry (Sinclair & Bruce, 2015), with an emphasis on deductive reasoning rather than building on students' experience and intuition, which are important for learning. Perhaps it might be because of the influence of Platonism on Euclidean geometry (and Mathematics), which considers geometrical (and mathematical) objects as independent of human practices, making mathematics a thinking activity (or a thought experiment) done in an abstract world rather considering mathematics as something that people in the real world do. Geometry occupies a significant portion of school mathematics curricula compared to measurement, with a prime focus on the Euclidean plane (not the real plane) with imaginary or idealized mathematical objects (e.g., dimensionless point, line without any thickness). This detaches geometry from real world objects, which makes it difficult for students to understand geometry. Moreover, not being able to deal with geometry impacts students' overall mathematical performance, and their ability to understand mathematics. This eventually can be a reason for a student to give up or hate mathematics (Clements & Sarama, 2011). As measurement has practical applications, it can bridge the gap between the real-life roots of geometry and its abstractions, which in turn can help students to better access mathematics in general.

Measurement was seen as a root topic of learning mathematics by the classic mathematics educator Davydov (1975). He challenged the age-old traditional curriculum that starts with the most abstract concept of numbers, which may be logically and psychologically inappropriate for the learner. He proposed and tested an experimental curriculum (with elementary/primary children), which started with measuring quantities based on comparing the structure and the relationship between quantities. He recommended using aspects of practical measurement to the basis for a primary arithmetic course. He argued that measurement can bridge the gap between whole numbers and real numbers, by bringing in the need for fractions (or rational numbers) in a more organic way, rather than being dealt with as separate topics, which is the way it is done in the conventional curriculum. Further, he has argued that this can eventually bridge the gap between algebra and analysis. In his experimental curriculum, he worked out an extensive course of how the foundation of mathematics can be laid by allowing learners to handle and compare different quantities, to come up with different relationships between the quantities, For example, relationships of "less than", "greater than" or "equal to" could be represented symbolically using {<, >, =}. After this, he proposed that students move to counting, and actually measuring quantities, which can in some cases bring in the idea of the "remainder" and can set the ground for fractions.

Drawing on Davydov's work, we see that measurement can act as a foundational connecting link for several important topics of mathematics. This approach can provide an alternative to the current school curriculum, which presents measurement mainly as a geometric measurement, starting with length measurement on a single dimension. The final goal is to make the student learn to use a ruler, followed by area and volume in two and three dimensions respectively, and the eventual goal is the use of formulae.

Considering the crucial role played by measurement in the learning of many basic topics of mathematics, it could be given prime attention in our school mathematics curriculum. However, student's performance on measurement tasks is not encouraging. Battista (2007) cites several studies showing the poor performance of students on measurement tasks. In 2000, the National Assessment of Educational Progress (NAEP) showed that around 75% of 4th graders and 40% of 8th graders could not find the correct length of an object when it is placed above a ruler, with its end not aligned to the end of the ruler (Kloosterman et al., 2004; Sowder et al., 2004, as cited in Battista, 2007, p. 892). Similar findings were also reported in five metropolitan cities of India, where it was found that a large fraction of students (49% in 4th, 42% in 6th, 25% in 8th grade) made an error in using the scale, despite length measurement being introduced in the second grade (Kanhere, Gupta, & Shah, 2013; Educational Initiatives, 2006, p. 10). The situation worsens when it comes to measuring area. In a study with eighth grade students, only 25% could find the surface area of a rectangular solid, and only 14 % could find the number of tiles required to cover a region with given dimensions (Sowder et al., 2004 as cited in Battista, 2007). This poor performance is attributed to the lack of conceptual understanding of area, and the disconnect between spatial and measure-based numerical reasoning. These have been discussed as the main factors for students' poor performance in the use of scale for length measurement and formula for area measurement (Kanhere, et al., 2013; Educational Initiatives, 2006; Battista, 2007; Sarama & Clements, 2009).

1.2.3 Theoretical motivation

Though the tasks used in the above studies were quite successful, and insightful in bringing out the gaps in students' understanding with respect to measurement, the objective of some of these tasks was mainly finding students' misconceptions. On the other hand, the Piagetian approach (and the tasks there) evolved with the objective to explore what children know, by opening up the door for children's intuitive understanding in general and their understanding of measurement in particular (Piaget, Inhelder, & Szeminska, 1960). Piaget, et al., (1960) reports extensively how young children move along different stages of learning of length measurement, starting from reconstructing the relations of distance to understanding conservation with respect to change of position, to measuring length to measuring subdivisions of length. Piagetian studies have been highly influential for decades. The longitudinal, detailed and extensive nature of the study has led several educators to develop curricular sequences inspired by this model (Clements, 1999; Sarama & Clements, 2009). However, such sequential curricula, where students move from comparison to using non-standard units to standard manipulative units to the use of scale[1], have also been challenged by studies. For example, in a study where 6-8 year old children were given a sheet with a line drawn on it, and the task was to communicate (over a telephone) whether the line is equal, longer, or shorter compared to their partner's, children with the standard ruler or even a broken ruler performed or reasoned better than when they were using a thread (Nunes, Light, & Mason, 1993). Further, Clements (1999) observes:

> "The Piagetian-based argument, that children must conserve length before they can make sense of ready-made systems such as rulers (or computer tools, such as those discussed in the following section), may be an overstatement. These findings support a Vygotskian perspective, in which rulers are viewed as cultural instruments children can appropriate. "That is children can use rulers, make them their own, and so build new mental tools." (Clements, 1999, p.5)

Here the scale is proposed as a cultural instrument that can be adopted and appropriated for length measurement, just like clocks are used for time measurement or thermometers for temperature. Despite having a culturally developed tool or instrument to work with, in the form of scale or ruler, students seem to perform poorly.

The situation becomes more challenging while measuring areas, where there are no such commonly available or culturally developed instruments or tools. The abstraction becomes two-fold when children are asked to measure the area of a rectangle, as they not only have to measure the dimensions (length and breadth) of the rectangle but also have to use the formula of multiplying the dimensions to get the area of the rectangle.

The lack of conceptual understanding in students' use of scale/ ruler could be because of the hidden conceptual processes packed within the scale. Studies have identified the processes involved in length measurement as identifying the attribute, conservation, transitivity, equal partitioning, unit and iterations, accumulation of distance and addition, starting point or origin, and numerical representation (Sarama & Clements, 2009; Piaget et al., 1960). Linear measurement can thus be seen as an assemblage of several such conceptual processes or components (or units), which suggests that the structure of the measurement concept is a network, formed from several interconnected sub-concepts.

1 Marked ruler is referred to as scale here. Transparent Scales are commonly available in the market.

Related to this view, the scale can be seen as a culturally developed instrument, structured as an assemblage of the above mentioned conceptual processes. Through its wide everyday use, it gains the status of a culturally appropriated instrument for the learner (Nunes, et al., 1993). Ruler or scale as an instrument forms a graspable learning resource, which holds together different conceptual processes in a compact, accessible form, allowing learners to play around with this form, and eventually abstract out the conceptual elements that are assembled in its construction (Zacharos & Chassapis, 2012). Through appropriate and meaningful use of the ruler as a resource, a learner may abstract out the mental model of the ruler, along with the different conceptual components involved in it in a compact accessible form (Schwartz & Holton, 2000). Developing a mental ruler, through the successful use of a physical ruler, allows a learner to see lengths in terms of a scale. In the case of area, such a compact, accessible instrument or tool is not available. This makes area measurement almost impossible for students to abstract out through activity, and form a mental model for area measurement (Smith, Males, & Gonulates, 2016).

Similar to the way conceptual processes or components are assembled within a scale or ruler, there is a need to develop an instrument/ tool where the conceptual processes involved in area measurement could be assembled. This design problem requires visualizing the concept of area as a network of several interrelated conceptual components. Further, since the most common area tasks with rectangular areas require students to measure the linear dimension, it becomes hard for students to distinguish or abstract out the area attribute. That is, since students end up using the same instrument (scale or ruler), for finding the area, they might end up thinking linearly or only along one dimension, even though area is a two-dimensional attribute. Thus, drawing from this need, we tried to develop and use a tool or an instrument for students to engage with area measurement.

1.2.4 The case of area measurement

Area measurement is an important topic in school mathematics education, as it poses many challenges for the learner. It also brings in the next higher level of abstraction in the domain of measurement. To elaborate further, for measuring area, the learner not only has to measure the linear dimensions of a given two-dimensional space, but also has to do a multiplicative operation on the dimensions. That is, for measuring area, students are required to move from using physical tools like rulers to doing numerical operations to the use of algebraic abstractions like formulae. Thus, area measurement acts as a crucial transition point in the topic of measurement in general, by opening up the ground for numerical computations through formulae, which have further applications in higher mathematics and science, for e.g., force = mass × acceleration (Smith, Males, & Gonulates, 2016). Area measurement provides affordances even for other very important and advanced mathematical topics, like fractions and calculus and thus plays a foundational role in school mathematics.

Understanding of area is integral to understanding calculus as it provides the geometric meaning of the integration operation in calculus. While the derivative of a function is the rate of change of that function, the integral of a function is graphically defined as the sum of the areas under the curve. For example, the following integral expression as presented below can also be expressed through the graph as shown in Figure 1.1.

$\int_{x=4}^{x=14} f(x)\,dx =$ Area under the curve f(x) from point C to point D.

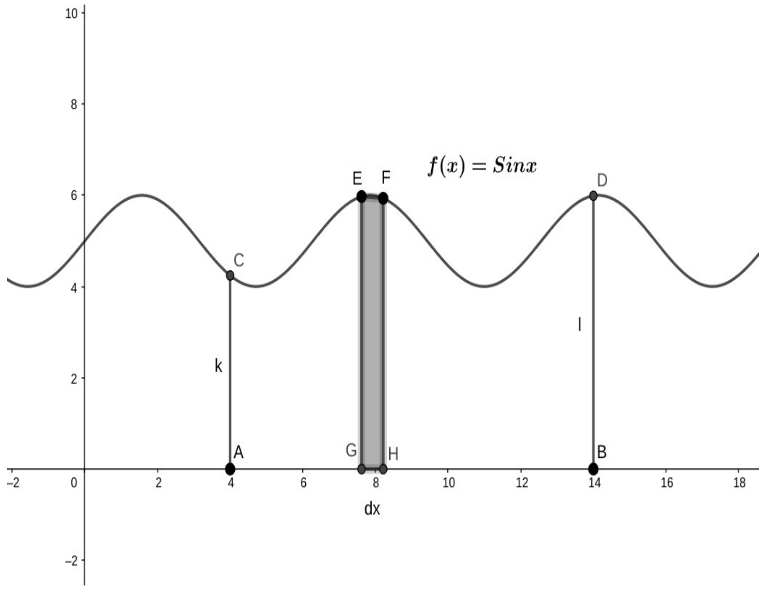


Figure 1.1: Integration of a function as the area under the curve

Area-measurement thus enhances the spatial understanding of measurement. It also integrates and enriches the scope of mathematics as the area-model has application in several other topics, including multiplication, fractions, algebraic multiplication, scaling, geometry, functions, and probability (Ron, Dreyfus, & Hershkowitz, 2017; Sisman, & Aksu, 2016; Sarama & Clements, 2009; Outhred & Mitchelmore, 2000). It can be further extended to engage with other higher mathematical topics such as measure theory (de Freitas & Sinclair, 2020). Area-measurement thus serves as a foundational basis to broaden students' mathematical learning. This further highlights the integrated, interconnected, and the interdisciplinary nature of area measurement. This thesis explores this complex structure of the area concept, and the associated learning difficulties. The following outline captures the structure of the thesis.

1.3 Thesis outline

1.3.1 Graphic outline of the thesis structure

The thesis has six chapters (see Figure 1.2), starting with this introduction, followed by the literature review chapter. This leads to the first study of the thesis, which is reported in the third chapter. Two

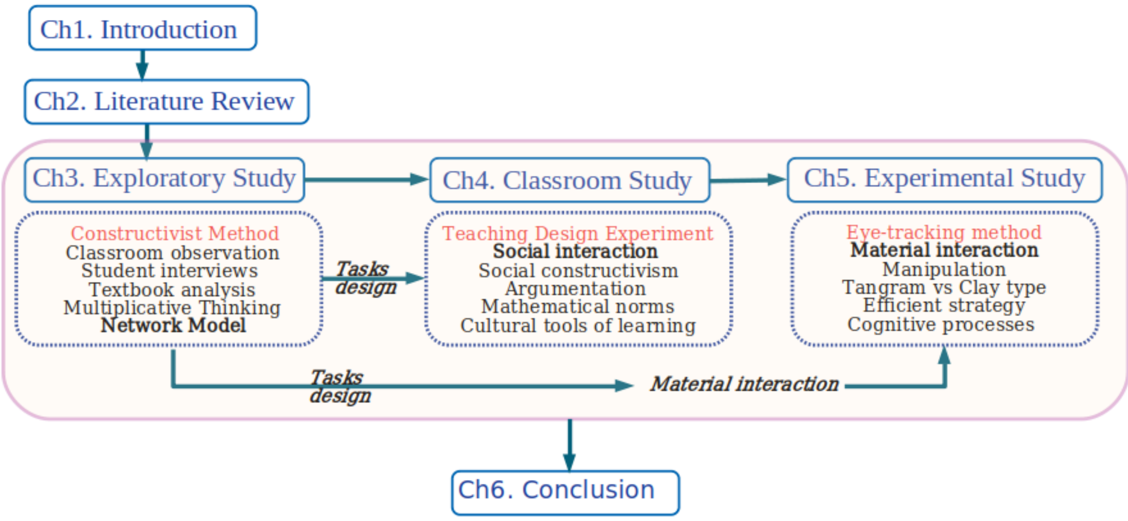


Figure 1.2: Graphic outline of the thesis structure

more studies are reported, one covering aspects of social interaction (fourth chapter) and one covering aspects of material interaction (fifth chapter). The final conclusion chapter summarizes the overall contribution of the thesis.

1.3.2 Overview of studies done in this thesis

1.3.2.1 Study 1 (Chapter 3. Exploratory study)

The first study chapter reports studies inspired by earlier studies that have explored students' nuanced understanding of area measurement. In these studies, I have explored students' conceptual understanding of the concept of area and the nature of the gap between spatial and numerical understanding, mainly using observational methodologies.

1.3.2.2 Study 2 (Chapter 4. Classroom study)

The second study adopts an active participant approach, where I examined how students construct the concept of area-measurement in the classroom. Drawing from the idea of scale or ruler as a culturally developed instrument (or tool) for length measurement, we incorporated graph paper sheets in our area lessons, and tried to develop it as an instrument or tool for area measurement, to be appropriated through social interaction in the classroom context.

1.3.2.3 Study 3 (Chapter 5. Experimental study)

In the first and the second studies, the focus was on students' explicit actions and interactions, which provided data points to understand student's conception and construction of area respectively. In the third study, we have looked at how the materials used in the second study function at the cognitive level, thus exploring development of the area concept in a way that is independent of social or cultural roots. This study also examines the role of material interaction in students' engagement with the area problem. For this, a comparative experimental study was conducted, to understand how physical manipulation of materials changed students' cognitive processes related to problem solving. This relationship was explored by an eye-tracker.

It should be noted that the student samples in all the three sets of studies were different and were convenient samples that were made accessible to the researcher.

1.3.2.4 Conclusions

Overall, the findings from the studies provide more insight into the gap between students' spatial and numerical understanding, particularly with respect to area measurement. Further, this work provides insight into the role of social and material interaction in the process of learning area measurement. The thesis also tried to address the question: "Can the gap between spatial and numerical understanding be bridged by highlighting the role of the integrated and interdisciplinary nature of the area concept or topics?"

The thesis aims to address the above question by exploring and extracting the essential aspects of social and material interaction in constructing meaning. In social interaction, the thesis highlights the role of argumentation and students' warrant as focal components in the process of knowledge construction and thus adds to the literature of social construction of knowledge. Another outcome of the thesis is the design and development of several tasks on AM after a reasonable ground work on identifying the gaps in AM. Extending the tasks further, the thesis investigates material interaction by studying the effect of specific manipulation on students' solution strategies. The thesis presents novel designs and methodologies to conduct such investigations integrating the disciplines of mathematics education research and the learning sciences to advocate and add to new paradigms of research.

2

Review of Literature

Measurement is the act of quantifying an attribute or aspect of physical material or process (e.g., weight of a pineapple, length of a piece of cloth, time taken to cover a given distance, etc.,), while geometric measurement is the act of quantifying an attribute of a geometric or spatial object (e.g., length of a line segment, area of a closed figure) with reference to an appropriate unit. Geometric measurement not only requires one to identify the geometric or spatial feature of the given object, but to also visualise the spatial structuring of the appropriate unit within the given object, in order to quantify or carry out any numerical operation on it (Battista, 2007). Thus, geometric measurement requires integration of geometry, spatial, and number based reasoning, which connects and enriches the two critical domains of mathematics: geometry and number. However, formal instruction often fails to make use of this important conceptual connection between these two mathematical domains while teaching geometric measurement (Sarama & Clements, 2009).

The low performance of students in measurement tasks has been reported in several studies as mentioned in the previous chapter. An important reason for the low performance, according to several studies, is the disconnect between spatial reasoning and measure-based numerical reasoning in students, i.e., students making improper connections between the process of unit-measure iteration and numerical measurements (Barrett & Clements, 2003; Battista, 2001; Clements, Battista, Sarama, Swaminathan, & McMillen, 1997, as cited in Battista, 2007).

Sarama & Clements (2009) also found that children face difficulty in finding an appropriate unit for measuring the attribute e.g., length or area. They recommended that connecting the curriculum with student's out-of-school measurement experiences and spatial abilities could tlead to better performance on measurement. Most earlier studies have seen area-measurement (AM) as part of a continuum of geometric measurement that includes length and volume measurement (Curry, Mitchelmore, & Outhred, 2006; Battista, 2007). Such studies have tried to highlight the common problems and common solutions for all geometric measurements. For example, Curry, Mitchelmore, & Outhred, (2006) identified the five principles of geometric measurement: need of congruent/ identical units, use of an appropriate unit, using the same unit for comparing objects, inverse relation between the unit-size and the measure, and structuring of repeated units on the measure. Later, Battista (2007) emphasised the importance of connecting and inter-relating various measurement concepts in the curriculum, as it was found that students have difficulty in relating and separating the concept of length, area, and volume.

However, looking at AM as a continuum of geometric measurement may deprive us of insights into the specific issues related to the nature of AM. Thus, in this chapter, I will review the literature focusing specifically on area-measurement (AM).

As we saw in the previous chapter, students' performance on AM tasks is worse compared to length measurement tasks. One of the main reasons for this could be that in the AM curriculum the major emphasis is on procedural understanding rather than conceptual understanding, compared to the length measurement curriculum, as found in the curricular analysis done by Smith, Males, & Gonulates (2016). They found that most curriculum handles AM by relying on formulas, mainly for rectangles. However, none of the textbooks they analysed did much to build the conceptual connection between the formula (of multiplication of lengths) and area measures. Huang & Witz (2011), citing the works of Strutchens, Harris, & Martin (2001) and Tan (1995, 1999), argue that curriculum and instruction of school mathematics most likely result in children's inflexibility in dealing with AM problems. They also state that with respect to AM, teachers most often adopt an algorithmic or numerical-calculation approach, stressing on procedures and formulas rather than allowing children to explain the functioning of such formulas, or explore the rationale behind using these formulas. Furthermore, overemphasizing formulas restrains children from having the required time and experience to visualise the geometric figures, their properties, and how the formulas for area measurement work (Fuys, Geddes, & Tischler, 1988, cited in Huang & Witz, 2011). This is despite the fact that AM is considered to be a significant topic of school mathematics in the curriculum guidelines of several countries (NCTM, 2000 & TME, 2003 as cited in Huang & Witz, 2011). Thus, there is an immediate need to critically review and analyse the present AM curriculum, and to also restructure it to provide a way forward, towards a more conceptually sound approach to present AM.

Thus, the aim of this literature review is not just to understand the existing conceptual difficulties or gaps around the learning of AM, but to also look into various curricular efforts made towards overcoming them. Further, I have also tried to extend and connect this literature with recent approaches/themes that have been developed in mathematics education research in other topics. To do this, I have organised the literature related to the learning of area measurement under four themes. I have put forth the possible research questions arising from each of these themes, which lays the ground for the three studies that follow.

2.1 Themes in the Literature Review

To provide a conceptual background to the thesis studies, I have divided the vast literature I have drawn inspiration and insights from, into four themes. These four themes are not mutually exclusive, but overlap somewhat, as some studies belong to more than one theme. The four themes are: Conceptual Studies, Curricular Studies, Tool Use and Multiplicative Thinking. In the first theme, I have categorized the different gaps or errors found in students' understanding related to area measurement. From these studies, I have identified the errors in students' conceptual understanding of area and the gaps found in AM curriculum. However, these studies were carried out from a deficit perspective, with an aim to find what students don't know with respect to AM, or the gaps in the AM curriculum, rather than acknowledging what a student knows or how the curriculum could be revised. This leads to the second theme, which was inspired by the Piagetian tradition of looking at what children know about a particular concept (in the present context, the concept of area or AM) at a particular developmental stage, and how it develops over time. This also leads to a possible road-map of what the AM curriculum should look like. The third theme is aligned with Vygotsky's philosophy around the use of tools. As argued in the introduction chapter, culturally appropriate tools can help the learner jump to higher developmental stages rather than following a prior stated path discussed in the previous (second) theme. Lastly, as mentioned in the beginning, measurement requires connecting the domain of geometry with number. Area-measurement specifically requires one to connect geometry with the multiplicative operations. Multiplicative thinking has already evolved as a broad domain in mathematics education, to support students in handling multiplicative operations. Thus, to support the connection between AM and multiplicative operations, there is a need to take into account the developments in the field of multiplicative thinking, and further explore the connections between multiplicative thinking and AM.

I elaborate on each of the four themes below.

2.1.1 Conceptual Studies

In this theme, I have gathered those studies which have discussed specific gaps in students' understanding with respect to AM. Several studies have reported that students confuse AM of a given shape with that of the perimeter or the measure of the shape's boundary (Cavanagh, 2007; Kanhere, Gupta, & Shah, 2013; Education Initiatives, 2006, p.16). Even teachers probably end up feeling that the two measures -- area and perimeter -- are connected, as they were found to agree that as perimeter increases, the area will also increase (Ma, 1999). This indicates the conceptual difficulty, and hence the need to pay attention to identifying and understanding the attribute of area and how it is different from the other measures. Lehrer, Jenkins, & Osana (1998) also found that a majority of students could not identify the unit of area as a unit of cover. Rather they tend to focus on units of length, with 1/3rd of students unable to measure area when given card-board cut-outs of square and right-triangular units. This happens because the most fundamental aspects of measurement, like the relationship between the unit of measure and the attribute to be measured, remain unaddressed in the conventional curriculum (Lehrer et. al., 1998).

Cavanagh (2007) probed Grade 7 students with a questionnaire asking: define area, find the area of a given rectangle and a right triangle and explain the method. He identified three misconceptions among students: confusion between area and perimeter, using slant height instead of perpendicular height for calculating the area (e.g., of a parallelogram) and not being able to see the relationship, that a right triangle is half of a rectangle. The school curriculum mostly introduces area with simple shapes like rectangles, triangles and their area measure is introduced as formulas of some operations of the shape's linear dimensions, like length multiplied by breadth for a rectangle, and half of base multiplied by height for a triangle. So perhaps students end up associating or focusing on the linear dimensions or the boundary measures for the area attribute of the given shape, rather than engaging or experiencing the two-dimensionality or covering aspect of the space. Thus, Cavanagh (2007) emphasised the need for appropriate activities, and sparing enough time for students to develop a sound conceptual understanding of array structure, before using numerical formulas. The premature use of formulas leads to misconceptions about area-measurement, with students skipping the physical meaning (behind the numerical representation) of AM (Zacharos, 2006), and several students not able to identify the number of unit areas fitting into a rectangle, even after calculating the area of that rectangle by applying the area formula.

These studies show that while the school curriculum extensively uses formulas such as length×breadth (or $l \times b$) for getting the area of a rectangular space, this abstraction may hide the sense of unit of area, and the continuous covering nature of area (Kobiela and Lehrer, 2019). Conversely, the tiling tasks might prestructure the activity (Outhred & Mitchelmore, 2000) as a counting activity, which might hide the $l \times b$ abstraction. Thus, simplifying the area activity as a tiling task has its own limitations, specially to move to a multiplicatively structured abstraction (elaborated further in the fourth theme of this chapter). Furthermore, when a Geoboard was used by Kamii and Kysh (2006) to show fourth grade students a 3×3 and a 2×4 rectangle on two Geoboards respectively and were asked, "If these were chocolate bars, which one would be bigger and have more to eat?". It was found that only a very few students counted the unit squares, most often student counted Geoboard pegs rather the square spaces between them, showing that a majority of students don't consider square as a unit of area for comparing two-dimensional space even on a Geoboard, indicating a serious conceptual gap among students in such comparison tasks. This is also connected to the third theme of the literature review, about the use of tools or materials, in raising the caution that it is not enough to just use any tool or material, they need to be properly and carefully grounded in the concept.

Students face difficulty in abstracting or applying the understanding of area to different shapes other than the conventional shapes, such as L-shaped figures (Cavanagh, 2007; Zacharos, 2006). This difficulty could be due to various factors, like students' understanding of area fixated only with conventional shapes like rectangle, square, triangle, or limited to specific formulas, thus lacking the general understanding of area as a measure of a two dimensional plane or space. Students may not be able to deduce the area of a given different shape or figure with their existing knowledge of area. To understand the underlying cause behind the students' difficulty, we need to explore what students mean by the term "area".

The discussion under this theme may not be exhaustive, covering every study reporting difficulty or gaps in students' understanding with respect to area measurement. It is aimed at giving a flavour of the nature of such studies, which have tried to elaborate and highlight the gaps through students' interviews or students' ways of attempting different area specific tasks. For my first study, I have drawn heavily from this theme, to understand students' existing understanding with respect to AM and also how curriculum and teaching may contribute to such understanding. What I gathered from this strand of studies is the methodology of student interviews, and the design of specific tasks. However, as I mentioned earlier, some of these studies are based on a deficit perspective, where the main aim is to uncover the students' misconception or the students' errors (through testing). The studies reported in this theme are generally based on short duration contact with students, and thus lack a longer engagement with their learning process. Hence, the role of teaching or instruction remains unattended in this theme. In my first study, I have followed a holistic approach in understanding the present state of AM conception, by taking into account students' understanding, teaching and the curriculum around AM.

This brings us to the next strand or theme of research studies that adopt a developmental perspective and are generally longitudinal in nature.

2.1.2 Curricular Studies (Stages of learning, Levels of Sophistication, and Learning Trajectories)

In this section, I explore studies which are longitudinal in nature, and follow a developmental model of learning area measurement. The development is described variously by different researchers, in terms of Stages of learning (Piaget, Inhelder & Szeminska, 1960), Levels of sophistication (Battista, 2007), Learning Trajectories (Sarama & Clements, 2009) etc. These studies typically report and recommend a step-wise development in understanding a topic, and in teaching. The progression presented is generally hierarchical. Other approaches (e.g. Izsak, 2005) take a contrasting view.

Starting with the classic work of Piaget, Inhelder & Szeminska (1960), developmentally oriented studies have found that children's reasoning about various measurement concepts like length, area, etc., develop in sequential stages of learning. They used several tasks to understand children's reasoning with AM, especially conservation and operational thinking. In one of these tasks, children were shown two identical green cardboards as grass meadows, and a small wooden cow was placed on each meadow. Subsequently an equal number of identical wooden house blocks were placed on each "meadow" in different configurations, and children were asked whether the same amount of grass is available for the two cows to graze.

The authors identified roughly four stages of understanding area among children, based on their articulation of reasoning. starting with stage I, where a child hardly engages with the given context. In stage II, the child is interested and engaged in the task, but her reasoning is mainly perceptual. Stage II is divided into two sub-stages IIA and IIB. In stage IIA, the child may accept that the two given meadows have the same space, even with the introduction of first few pair of house (or blocks), so far as the arrangement in the two meadows are same, but will disagree immediately if the arrangement of houses are changed in the two meadows. In stage IIB, there are a range of intermediate responses, where children may agree with the complementary space (or area) being equal (even with different arrangement) after a few pairs of blocks, but deny the remaining space to be equal after a certain number of blocks. Stage III is also divided into two sub-stages, IIIA and IIIB. At this point, newer tasks were introduced to explore further stages. At stage III A, the child is able to acknowledge that transforming a shape doesn't change the space that is conserved by it. However, deducing that the complementary area will also be equal only comes at stage IIIB. Stage IV is identified as that stage where the child is able to abstract the formal operation through unit iteration, and understand the relation between length and area, which is not covered in this study.

As we go into the details of the interaction between the child and the interviewer, we come across several cases where a child almost falls back to stage II of perceptual reasoning. With some prompting, they move to stage III of acknowledging operational conservation, when there is little variation in the task. Thus, Piagetian studies develop this rich methodology of uncovering children's intuitive reasoning, with respect to area conservation and measurement. The insights that come out of the studies is not just what children do know about area conservation and measurement, but also the rich context designed by the researchers and the rich facilitating interaction provided by the interviewer. Another interesting point that the work reported in "The Child's Conception of Geometry" is the way the authors conceptualized geometry. It was not in terms of the conventional geometry curriculum of introducing shapes as some abstract mathematical objects, but more in terms of concrete, actionable or accessible forms of measurement. Though, for the current theme, I am highlighting the "stages" from the Piagetian theory, but Piagetian studies are much more than that, and a major chunk of the studies mentioned in the literature review draws heavily from Piagetian studies. Further, the studies reported in the next chapter are mainly guided by Piagetian methods of clinical interviews, task design, and framing of questions.

Moving to the level-wise development of AM, a key resource is the review of research done by Battista (2007) in the chapter on “The development of geometric and spatial thinking” in the book “Second Handbook of Research on Mathematics Teaching and Learning”. There has been some remarkable and influential work in defining the level-wise growth of geometric reasoning among children, of which the most influential one is Van Hiele levels of geometric reasoning. Both level-wise and Piagetian’s stage-wise development mark qualitatively distinct types of cognition, occurring in a hierarchy at each level and stage respectively. However, levels are defined for a specific domain (or a particular concept), but stages are defined across different domains (Clements and Battista, 2001, cited by Battista, 2007). Moreover, despite Battista’s acknowledgment that historically geometric measurement is intertwined with the conception of geometry, in his book chapter, the literature review of geometric measurement is kept separate from geometry, i.e., different theories on geometry learning are covered first, followed by theories on geometric measurement. Though one may present them separately in the interest of a simplified and systematic discussion, I caution against assuming a deep divide between the domains of geometry and measurement.

In the above mentioned chapter reviewing “The development of geometric and spatial thinking”, Battista (2007) characterised students' construction of length into two fundamentally different types of reasoning: measurement and non-measurement. The author also elaborated different levels of sophistication for each of these types of reasoning. However, for area and volume measurement, the author proposed integrated developmental models. Overall, the author recommended that the low performance of students in the measurement tasks could be due to the disconnect between spatial reasoning (e.g., units iteration process) and measure-based numerical reasoning. He suggested that children often face difficulty in making the transition from filling a space with concrete units to visualizing and using the unit structure, which is needed for conceptualizing and measuring area or volume. Battista (2007) explained that this transition happens in different levels of sophistication, and have defined the following seven levels:

1. Absence of the processes of Units-Locating and Organizing- by-Composites (the latter refers to collecting the units in a row or column),

2. Beginning Use of the Units-Locating and the Organizing-by-Composites Processes,

3. Units-Locating Process Sufficiently Coordinated to Eliminate Double-Counting,

4. Use of Maximal Composites, But Insufficient Coordination for Iteration,

5. Use of Units-Locating Process Sufficient to Correctly Locate All Units, But Less-Than-Maximal

Composites Employed,

6. Complete Development and Coordination of Both the Units-Locating and the Organizing-by-Composites Processes,

7. Numerical Procedures Connected to Spatial Structurings, Generalization

These levels, start from having no sense of unit or organising-by-composites and go up to the development of numerical procedures with spatial structuring (Battista, 1999; Battista & Clements, 1996; Battista, Clements, Arnoff, Battista, & Van Auken Borrow, 1998, as cited in Battista, 2007). These levels are hierarchical in nature, and are drawn from the student’s unit structuring tasks, which could be very narrow, considering the wide range of tasks and tools available for measuring. Again the initial levels defined here convey a deficit view about students, and it is assumed that students are supposed to reach to the highest level (level 7) before being able to measure area or volume.

However, apart from the levels, Battista (2007) also state that there are five basic cognitive processes that are essential for meaningful enumeration of arrays of squares and cubes: abstraction, forming and using mental models, spatial structuring, units locating, and organizing-by-composites. He suggests that the cognitive processes underlying geometric measurement could consist of two types of abstraction. One is abstracting the attribute, where the student initially abstracts the attribute to be measured (e.g., length) from the other spatial attributes of the object, through experience, and the other is abstracting unit iterations in structuring. He emphasised that structuring is a reflective abstraction, and not an empirical one. According to him, the critical components required in tiling a shape are constructing a mental model of the shape, which includes critical features of the shape's geometry that can be mentally manipulated (e.g., being able to visualise tiling of a shape even without having the physical materials to do the tiling) and developing an appropriate structured mental model of the array of shapes in the tiling. Students need proper orientation with iteration of appropriate units rather than directly using formulas for AM and standard measuring tools (e.g., Rulers) for length in a traditional way. Again the developmental levels defined here, for area and volume measurement, involve only the measurement (numerical) reasoning having to do with enumerating units. This is different from the case of length, where non-measurement ideas (like conservation) were also explored, like in the Piagetian approach, which could be based purely on spatial inputs.

Several other papers have also organised students’ thinking through levels of sophistication. For example, Outhred & Mitchelmore (2000) have defined levels of sophistication among elementary students' drawing of unit-square coverings of rectangles. Clements, Wilson, and Sarama (2004) have described levels of sophistication in young children's ability to make a larger shape with pattern blocks (cited in Battista, 2007).

Another significant model of developmental progression is “learning trajectories” proposed by

Sarama & Clements (2009). According to them, during development and learning, children follow a natural developmental progression, and when instruction was mapped according to this, with a sequence of activities, it led to developmentally appropriate and efficient learning environments. The main focus of a learning trajectory are the developmental paths followed by children. Two books were published by Clements and Sarama in this regard (2009a; 2009b); one book synthesizes various developmental research studies on the foundational domains of mathematics, and the other book highlights the corresponding instructional activities for these domains. Several foundational domains were characterised by learning trajectories, starting from counting and arithmetic to geometry and measurement. Three major components of learning trajectories are identified: a mathematical goal, a developmental path that a child follows to reach that goal, and a set of instructional activities or tasks to match with the developmental path of children, to facilitate their movement to higher levels of thinking. Further, researchers suggest that professional development based on learning trajectories increases teachers' professional knowledge, and students' motivation and achievement, by facilitating developmentally appropriate teaching and learning for all children (Clements & Sarama, 2009). However, they have also discussed the domain of geometry and measurement in different chapters, with the chapter on measurement following the chapter on geometry, rather than the reverse (while the reverse sequence is argued to be more logical and based on historical progression). Despite acknowledging that measurement can act as a bridge between the foundational domains of geometry and number, the authors keep the chapter on measurement separate with no connections made with the chapters or learning trajectories on number & geometry.

As can be seen from the preceding discussion, the developmental models offer rich guidance for curriculum and instruction on AM. However, the literature discussed under this theme overlooks some important issues with respect to AM, which will be addressed in the next themes. Most of the above developmental models of measurement dealt with measurement separately from geometry or spatial thinking, with development in the former domain generally following the latter, unlike Piagetian work, where measurement is seen as integral to geometry. Although most of the studies acknowledge the significance of measurement as a bridge topic between geometry or spatial thinking with quantification or numerical thinking, such integration is not given prominence. I will explore the interconnected and integrated nature of measurement in the fourth theme of this chapter.

The above developmental models generally follow a linear growth, unlike some other learning models, such as the knowledge in pieces perspective (Izsak, 2005). The work of Izsak draws extensively on DiSessa's work on knowledge in pieces perspective and on coordination and refinement of concepts or knowledge. Izsák has adopted and extended diSessa's work specifically to the context of AM, as diSessa's work mostly covers topics in physics and higher mathematics. We will further explore such learning models in the fourth theme, and also in the subsequent chapters of this thesis. Thus, it may be misleading to interpret all learning progressions as a sequence of stable levels. The thesis does not intend to label or generalise all learning progressions as having a linear model.

As argued in the introduction chapter, culturally developed tools play a significant role (Nunes, Light, & Mason, 1993 as cited in Clements, 1999) in facilitating students' move to higher levels of thinking, and thus may bring qualitative differences in the developmental paths followed by learners. Apart from the global standard tools (e.g ruler), there are local, cultural measuring tools, based on usually non-standard length units, developed or used by local communities over a long time. These can also have great significance in the learning of measurement (Subramaniam & Bose, 2012). While the studies mentioned in this theme do guide us in the design of curricular activities that facilitate learner's AM conception, the next theme specifically focuses on the nature of different tools and how they support or facilitate students' conception with respect to AM. We will try to delve deeper into the aspect of different tools, activities and other operational factors in the learning of AM.

2.1.3 Use of material interaction (tools, instrument, and gestures) in measurement

The main motivation for this theme came from the mathematics lab[2] (math-lab) of the institute, where my research is based. The math-lab was developed by a team of mathematics educators and collaborators. Apart from the hands-on nature of the activities in the math-lab, the pedagogical approach has also evolved as a way that allows children to explore different mathematical patterns or results through a process of collaborative argumentation and reasoning. The lab consists of several materials, artifacts and activities like different kinds of Abacus, Napier's bones, Tower of Hanoi, Magic number cards, tic-tac-toe multiplication cards, Nim-game, jigsaw puzzles, origami papers, etc. Of these, the activities that are particularly relevant to AM are Tangram, Missing-area, and Geoboard. I will briefly explain how the three activities are done in the math-lab, to highlight the important educational and pedagogical insights drawn for this theme, before moving to the published literature in this regard.

2.1.3.1 Tangram

Tangram is a widely used and known activity (see Figure 2.1). In the math-lab, the activity typically starts with an A4 sheet. The first action is to get the largest square out of the sheet by removing the remaining rectangular part out of it. And then the seven Tangram pieces are created out of the square sheet, by folding and cutting. Here, the action of folding is used to understand the equivalence of pieces or shapes. Throughout this process, there will be instruction and discussion, about the actions

2 https://mathedu.hbcse.tifr.res.in/mathematics-laboratory/

to be taken on the pieces, by focusing on the different spatial characteristics and attributes of the shapes that are created.

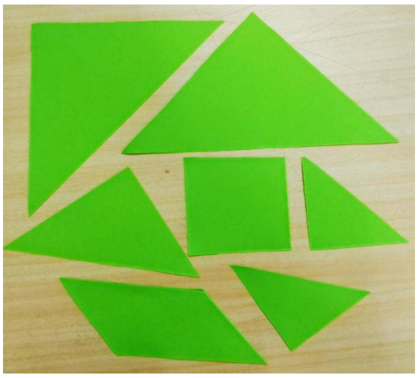


Figure 2.1: Tangram Pieces

Once everyone completes making the seven pieces, they are asked to join them together to get back the square they started with. Later they are also asked to create rectangular and triangular shapes out of the same seven pieces. By creating different shapes out of the same seven pieces, students are exposed to a material experience of having different spatial arrangements out of the same seven pieces. That is, the seven pieces are constant or conserved in this activity. Thus, the activity allows students to engage with the spatial features and attributes of the shapes, through physical manipulation of the Tangram pieces. Afterwards, the activity moves from the spatial discourse to numerical discourse, by identifying or quantifying one piece as a unit and quantifying the total of the all pieces together based on that. For example, students are asked, if the size of the smallest piece is assigned to be a value of 1 or 2 or ½ units, what will be the value of the whole square? Students then compare different pieces with respect to the smallest piece, and find their value, and subsequently give the value of the whole, and derive the multiplicative relation between the smallest unit and the whole. Likewise, even with assigning different values to the smallest unit, students readily find the value of the whole. And then, by generalizing, even if the value of the smallest unit is an unknown x, they always predict that the whole would be 16x. Thus, the activity very organically connects the spatial with the numerical, and then later with algebraic generalization to some extent.

2.1.3.2 Geoboard

Geoboard is a very simple and elegant material resource to introduce area as a count of unit squares, even to young children. As can be seen in Figure 2.2, the geoboard consists of a square board with an array of nails at uniform distance from each other. Different polygons (regular, irregular, convex, concave) can be made on it with an elastic rubber band. The best part is to find the area of the polygon without using any formula, but by mere counting of the small square boxes (units), and some simple arithmetic.

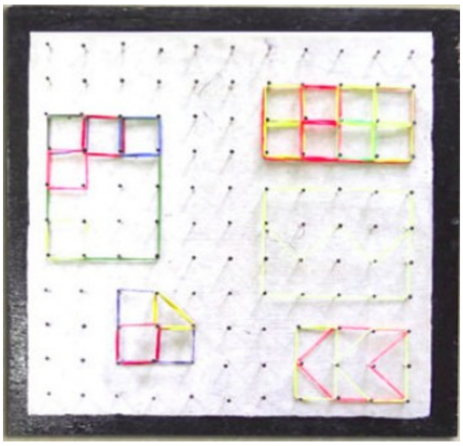


Figure 2.2: Geoboard

By using even the visual fact that if a rectangle is divided along the diagonal, it divides the space into two equal symmetric parts, by knowing the number of square units in the rectangle, we can predict the value of its half. This is also known as completing the rectangle strategy. For example, to find the area of the polygon ABCDE in Figure 2.3, completing the rectangle strategy is used to envelope different triangular parts in order to find the area.

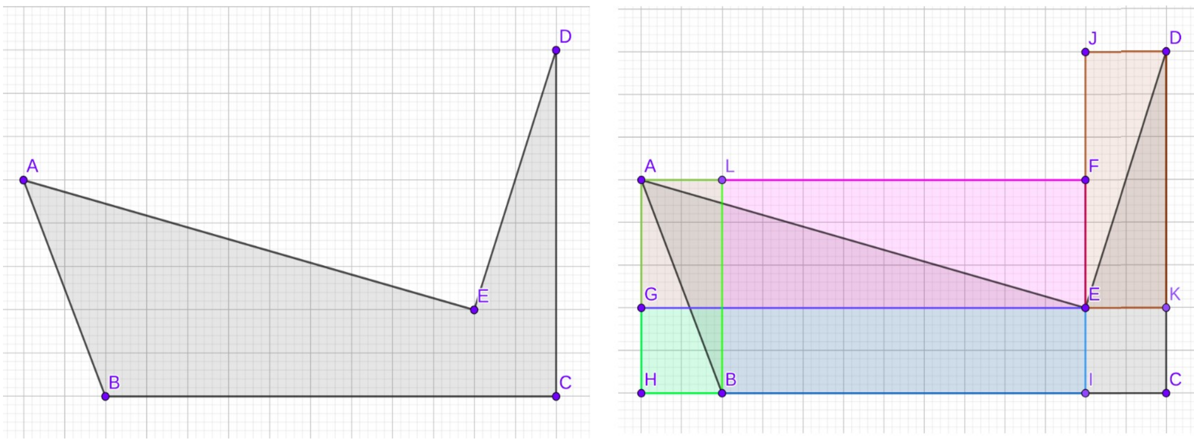


Figure 2.3: Computing the area of the polygon ABCDE by completing rectangles

Area of polygon (ABCDE) = Area of Square (EKCI)

+ ½ × {Area of Rectangle (JDKE)}

+ Area of Rectangle (AFIH)

– ½ × {Area of Rectangle (AFEG)}

– ½ × {Area of Rectangle (ALBH)}

Thus, we come across this innovative way of computing the area of a polygon which even very young children can attempt, while different polygons made on it appear challenging for adults and teachers. Further possibilities of exploration exist that make this activity an example of a low-floor high-ceiling activity.

2.1.3.3 Missing Area (or missing square) Activity

Missing square (or area) is an activity involving a square sheet of paper with a side of length, say 8 units, and divided into four parts, as shown in Figure 2.4(a) and 2.4(b). On rearranging the four pieces, the area of the new shape appears to have increased or decreased by 1 unit. For example, the area of the square as shown in Figure 2.4(b) appears to have increased its area to 65 sq units when its pieces are rearranged in the configuration as shown in Figure 2.5(a), and decreases its area to 63 sq units when rearranged as per the configuration as depicted in Figure 2.5(b). Therefore, when the same four pieces are arranged into a square, a rectangle or the third shape, they appear to have the total number unit squares in each of them to be different. Thus the activity appears to challenge the concept of area conservation by having a difference of one unit square with a rearrangement. The activity creates a situation which can lead to a very rich mathematical discussion.

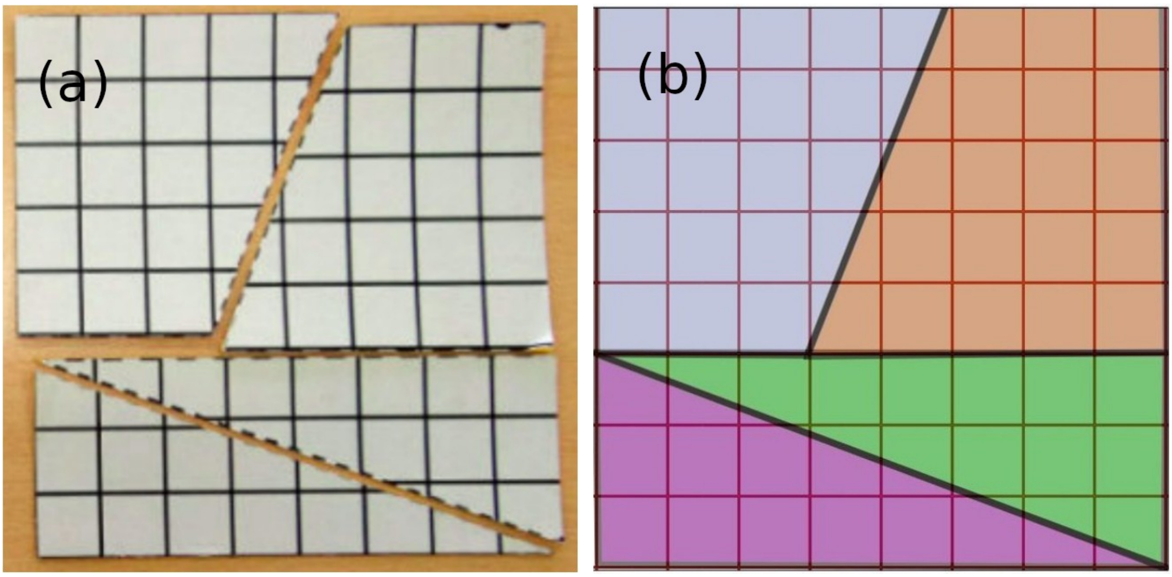


Figure 2.4: (a) "Missing Area" activity using square of side 8 units, with area 64 sq units, (b) "Missing Area" activity with each piece represented using different colors

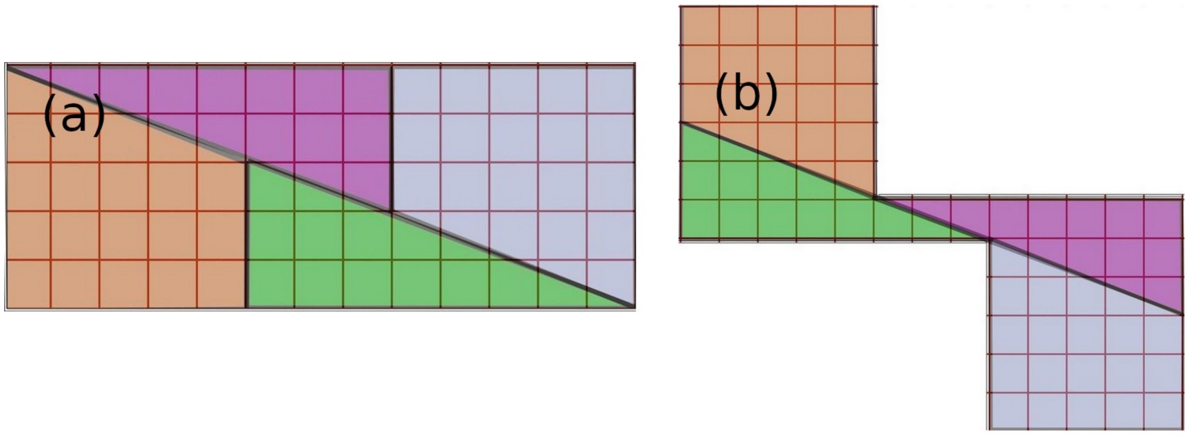


Figure 2.5: (a) "Missing Area" activity sheet after rearranging pieces into a rectangle with apparent area 13×5 = 65 sq units, (b) "Missing Area" activity sheet after rearranging pieces with apparent area 63 sq units

The activity can be designed using squares of sides 5 units, 13 units, and in fact any number (≥5) that occurs in a Fibonacci sequence. However, if the side is of odd length, the rectangle formed from the square will appear to have decreased in area, as opposed to the case of a square of even length whose area will appear to increase in area when rearranged into a rectangle.

The apparent paradox of the activity involves the fact that, if F_n, F_{n+1}, F_{n+2} are three consecutive numbers in the Fibonacci number sequence, then the following result holds true:

$F_{n+1}{}^2 – F_n×F_{n+2} = ± 1$

For example, in the fig. 2.6, the parts of the square sheet with side 5 units can be rearranged into a rectangle with sides 3 and 8 units respectively.

As we go further right in the series, the ratio of any two consecutive numbers converges to 1.62 (also known as the golden ratio). Thus cutting along the diagonal of a rectangle whose sides are in Fibonacci numbers of the form F_n and $F_{n+2,}$ the slope of the diagonal line will give the impression that they can fit together as the slope of the diagonal line would be close to the square of the golden ratio ($F_{n+2} / F_n = [F_{n+2} / F_{n+1}] / [F_n / F_{n+1}] \sim 1.62^2$)

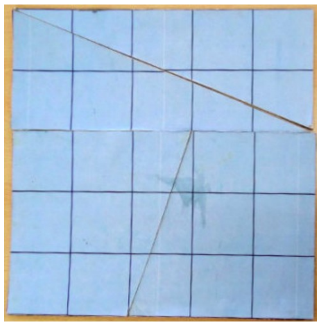


Figure 2.6: "Missing Area" activity with side 5 units can be rearranged into a 3×8 rectangle

The main insights from the above mentioned activities are the common pedagogical characteristics in each of them, that is, they are accessible, hands-on, have low-floor high- ceiling nature, allow multiple arrangements and possibilities, thus allowing the learner to explore different facets of AM. Another important characteristic they possess is that they allow or provide a very integrated approach to learning a particular concept.

2.1.3.4 Theoretical support for material interaction

Connecting to the classic Piagetian work, in the previous two themes, I focussed on the theoretical and methodological standpoints in Piagetian studies. However, there are many factors and their minute details that are present in the Piagetian tasks that lets children meaningfully engage with them, and thus allows researchers to explore and understand a child's thinking. The factors can be classified under the nature of material, context, actions and interactions involved in the tasks used in Piagetian studies.

Piaget et. al. (1960), studied children's conception of measurement by asking children to build a tower of blocks, on the floor, of the same height as the tower built on a nearby table. As children engaged with the task, it allowed Piaget et. al. (1960) to characterise different stages of development in the logic of measurement among children. The first stage was on Perceptual Comparison. The second stage was characterised by Manual Transfer (i.e., placing objects side-by-side and comparing them directly), Body Transfer (i.e., grasping gesture, like opening your hand to hold the object) or, Object Imitation (i.e., comparing with one's body e.g., measuring the object against one's arm). The third stage involved the use of a symbolic object to imitate the size of the measured object, and treat them as a unit of measurement. This developmental sequence was characterised in terms of the children's progressive ability to use a symbolic object as the intermediary term in measurement (i.e., understanding of the general logical principle of transitivity). The main point to highlight here is the material (or tool) used for investigation. Also, the actions and gestures of the child on the material were as much a unit of analysis as the verbal reasoning of the child, in the characterisation of the stages.

Piagetian studies used materials to understand what children know at a particular stage (i.e., mainly to define a child's stage), indicating a static nature of knowledge held by the learner at a particular point of time. Piagetian theory believes that a child is born with a certain knowledge form, considering that a child is not “a blank slate,” and that the child keeps accumulating new knowledge forms from the environment that fits with their previous existing forms and also incrementally modifies them. However, the Vygotskian approach did not consider the absolute stage of a child, but the potential of any child to move to higher psychological processes, mediated through tool use and interactions (Vygotsky, 1980). Piagetian studies do not highlight the uplifting role of the material and the interaction in the making of different schema (or knowledge structure) in the child’s cognition, and that the schema are dynamic in nature and constantly growing, even while the child is probed through mediated interaction. Here, materials play a significant role in acting as a mediator or a common tool or language, to engage and understand the child’s thinking. Thus, to move from knowing a child’s thinking to the process of knowledge construction by a child, we need to move our attention to the material factors of the interaction.

The instructional sequence recommended in several publications adopted from Piagetian theory (stage-wise development of length conservation) are: gross comparisons of length, measurement with nonstandard units such as paper clips, measurement with manipulative standard units, and finally measurement with standard instruments e.g., rulers (Clements, 1999). However, this stage-wise development has been challenged by several studies, which have claimed that Piagetian reasoning abilities do not necessarily determine measurement concepts (Clements, 1999). For instance, children use intermediate measurements to compare two lengths without any explicit transitivity question and move a unit to measure the length of an object without worrying about the length conservation. Boulton-Lewis, Wilss, & Mutch (1996) found in one of their tasks that the strategies used by children support the claim that non-standard units do not necessarily help children understand the need for standardized units in the length measuring process. As mentioned in the introduction chapter, Nunes et al. (1993) found that the traditional ruler supports children's reasoning more effectively than a thread (cited in Clements, 1999). And building on a Vygotskian perspective, a ruler acts as a culturally developed instrument that can be appropriated by a child for length measurement through its use, which through further use can be abstracted as a mental tool for the child. Subramaniam & Bose (2012) have also highlighted the significance of culturally and historically developed measuring tools (and units) in making the formal learning of measurement more meaningful for students. One must therefore take into account the potential of materials in pushing students to higher stages or higher levels of thinking.

In the context of area measurement, even among elementary students, when given a rectangular figure and a cardboard unit, those who used the strategy of concrete covering – that is moving the concrete unit along the figure – were much more successful (75% of cases) in determining the number of units (Outhred & Mitchelmore, 2000). In the same study, in other situations where the concrete unit was not given, but only a drawing or measure of the unit was given, only few students could determine the units. However, the study acknowledges that a concrete unit might pre-structure the task. Also, we often see that a tool or instrument may hide different foundational ideas (i.e., iteration, identical units, covering etc.) of measurement assembled in that material object. But despite such objections, we cannot deny the significant role of tools in developing and exploring students' conception of measurement, and we should let children construct their conception through interacting with tools or by making meaningful connections with the real world. The measurement tools used today are developed through a process of social mediation (Vygotsky, 1980), and for students to adopt them, there is a need to deliver the necessary cultural tools through a proper planned teaching effort (Zacharos, 2006; Zacharos & Chassapis, 2012). The materials or tools used for area measurement are mostly covering and counting units, completely skipping the relational aspects hidden between material quantities (de Freitas, & Sinclair, 2020). Thus, there is a need to re-conceptualize the use of tools, which can potentially integrate both the computational and conceptual aspect of area-measurement, by moving beyond the discrete counting exercise to accommodate the aspect of area as a continuous quantity. For example, using the context of painting or sweeping with physical tools and material interactions to access the continuous nature of AM (Kobiela, & Lehrer, 2019).

However to address these objections, we need to focus on the design and use of the material that provides the learner opportunities to engage with higher levels of thinking, that is allow them to reason multiplicatively, and not just perform additive counting. I elaborate more on this in the next theme.

2.1.4 Role of Multiplicative thinking in area-measurement

As I argued in the introduction chapter, area acts as the first measuring quantity that a student encounters in school mathematics, which is defined (or dealt with) as a multiplication of two dimensions (or quantities). Thus, abstracting the $l \times b$ formula – that is the product of two quantities (length and breadth) for a rectangular area – requires one to reason multiplicatively (Stephan and Clements 2003, p. 10, as cited in Huang, 2014). However, students' persistent use of additive counting methods (of units) might hinder the development of multiplicative thinking in them, which becomes a basic requirement to understand area-measurement (Cavanagh, 2007). This points to the need to see the connection between multiplicative thinking and area-measurement (AM). Even though several research studies have recognised the importance of the multiplication operation in area formula, and hence AM, there is dearth of research that explicitly acknowledges the role of multiplicative thinking in AM. (Huang, 2014).

Multiplicative thinking is a well researched area in mathematics education. Multiplicative thinking leads to a multiplicative response to a situation, by identifying or constructing the multiplicand, the multiplier and their simultaneous coordination in that situation (Jacob and Willis, 2003). It involves attending to the multiplicative relation between quantities and magnitudes, and the capacity to mathematically deal with such situations (Subramaniam, 2011). Multiplicative thinking has application in a broad range of mathematical topics, like understanding the inverse relation between multiplication and division, part-whole relation, fractions, proportion, etc. In contrast, the domain of measurement is relatively less researched, with even fewer studies that explicitly discuss the connection between measurement and multiplicative thinking. More recent studies have also argued for using measurement-based meaning of multiplication, as it is found to encompass diverse multiplication situations, and thus pedagogically provides more coherence in the school mathematics curriculum (Izsák & Beckmann, 2019).

Geometric measurement involves deriving a new quantity, "the number of units", from the known quantities – magnitude of the unit and magnitude of the space to be measured -- between which there is a multiplicative relation, namely, that the target magnitude is "so many times" the unit. Thus, unlike in the case of direct counting of discrete quantities, multiplicative thinking lies at the heart of the concept of measurement. Lamon (2007) and several others have argued that the way measurement is handled in the elementary curriculum leads students to just do an act of measuring, rather than developing the concept of measurement. She reports that very few students could understand that the unit of measure could be further broken into smaller subunits, to make the measurement more precise. That is, the unit of measure has a multiplicative relation with other smaller units, which are formed by subdividing or partitioning the initial unit. Petitto (1990) found that children shifted from sequential reasoning to proportional reasoning when given a set of number-line estimation problems during their first three elementary grades, and reported a connection between students' performance on number-lines and measurement tasks. Further, Mitchell & Horne (2008) argues and establishes the connection (or relational understanding) existing between fraction, rational numbers and measurement through a study with Grade 6 children using number line tasks. Thus, the connection between measurement and multiplicative thinking in linear measurements is not hard to see, but abstracting the multiplicative relation in higher dimensions is still not directly apparent like in area-measurement.

In the literature on multiplicative thinking, most of the situations and contexts examine proportionality, and involve a linear relation between two single dimensional measures (for example, the relation between cost and weight, time and wage, speed and distance, etc.,). Each such single dimensional measure is analogous to length; so, many of the concepts explored in the proportionality context have their analogues in the case of the geometric measurement of length. For example, unitization, the process of mentally chunking discrete units into either a larger convenient unit (chunked unit), or breaking a unit into smaller units, plays an important role in proportional reasoning (Lamon, 2007). Unitization is also the basis of measurement, and flexible unitization is involved in tasks that require construction of a “unit of units” (Reynolds & Wheatley, 1996). The number line, which is a direct representation of length, is useful in reasoning in proportionality contexts. The double number line in particular is a convenient representation of proportional relationships (Subramaniam, 2008), which affords the structuring and co-ordination of subunits and chunked units. Battista (2007) has recommended the use of fractional-units to help children understand the principle of unit structuring and unit iteration in measurement, which is similar to the process of unitizing in multiplicative thinking. Thus, both measurement and multiplicative thinking, involves comparison of quantities, understanding of unit, inverse relation, part-whole relation, fractions (partitioning), proportion, etc.,

The five measurement principles stated by Curry, Mitchelmore, & Outhred (2006) are: need of congruent units, use of an appropriate unit, using the same unit for comparing objects, relation between the unit and the measure, and structuring of unit iteration. Each of the above five principles requires appreciating the multiplicative relations that arise in the context of geometric measurement in various ways. Some measurement tasks require general logical reasoning. One is to know the inverse relationship between the size of the unit and the number of those units required to cover any fixed space, and the other is the need of equal-length or equal-sized units for measuring (Clements, 1999). Such logical reasoning involves the multiplicative relation between the size of the unit and the number of those units required to cover any space. It also involves the part-whole relation, when a given whole space is divided into equal parts of units, or units are subdivided into smaller units. Again, an understanding of the multiplicative relationship is required to get the area of a rectangular surface using its length and breadth, and to get the volume of a solid using its height and cross-sectional area (Battista, 2007).

In the case of area measurement, multiplicative thinking arises first in ways similar to length measurement, such as: (i) the use of sub-units and chunked units (unit of units) in determining area (ii) inverse relation between size of unit and the measure. It also arises in ways that do not occur in the case of length measurement, such as the array structuring of units in the case of rectangles, leading to area as the product of length and breadth. Further, there is a multiplicative relation between the area of the rectangle and the unit, between the area and length, and between the area and breadth. Correspondingly, there is an inverse relation between the area measure and the magnitude of the area unit, which is itself dependent on the length and breadth of the unit. Further, the passage to non-rectangular polygons involves triangulation, starting from the area of a right triangle obtained by dividing a rectangle in half, which involves a multiplicative relation. Thus we find that multiplicative relationships are involved in complex ways in area measurement. An instance of this can be seen in a study by Reynolds & Wheatley (1996), where a fourth grader solved the problem of finding the number of 3-by-5 cards required to cover a 15-by-30 rectangle. She solved it by dividing the area (450 divided by 15), despite being skeptical of the calculation, as she wanted to verify it by drawing, in an attempt to connect the spatial and numerical form of the problem. Thus, in this case, it was not determined if the student knew that the number obtained after dividing the areas would be correct only if the two dimensions (length and breadth) of the small card completely divide the two dimensions of the large rectangle respectively (cited in Battista, 2007). This gives an instance where the unit is related to the target area, not only in terms of the multiplicative relation between the magnitude of the unit and the magnitude of the target area but also in terms of the multiplicative relation between the dimensions of the unit with the dimensions (length and breadth) of the target area (i.e., area of the space to be measured).

The above discussion shows that an understanding of the area concept requires connecting multiplication to geometry. Multiplicative thinking is a well-researched foundational topic in mathematics education and has application in a broad range of domains. Thus, there is a need to design studies to explore different ways in which multiplicative thinking can support the geometric measurement of area. Specifically, we needed to develop tasks that can elicit or make the connection between these domains more visible. Again, while developing tasks we need to move beyond the discrete counting exercise of unit covering, to tasks that allow students to view the continuous nature of area and to see it’s measure as a continuous composition of lengths (Kobiela, & Lehrer, 2019; de Freitas & Sinclair, 2020).This requires us to re-imagine the tasks from additive counting of units to multiplicative composition of dimensions.

Referring back to the curriculum proposed by Davydov (1975), he argued that measurement connects the gap between whole numbers and real numbers, by bringing out the need of fraction (or rational numbers), and it further connects algebra and analysis in a very organic way. Thus, the significance of measurement, and the need of integrating different foundation topics of mathematics with measurement, was realised in the classic work of Davydov, to build a coherent curriculum for mathematics.

Drawing on this understanding, I will try to build an integrated model of area, by pulling together various conceptual understanding involved in area, and by connecting it with other foundational topics of mathematics, along with multiplicative thinking. Thus, through my studies done in the following chapters I will propose a curriculum for learning area measurement which can integrate or connect different conceptual topics in a network form.

2.2 Reflections and need for further studies

Drawing from the first theme, Chapter 3 describes the first set of studies of the thesis, which will explore and understand students' conception of area in their existing setting through observational studies. The research design and methodology is mostly inspired from the studies reported in the first and second themes of the present chapter. Thus, Chapter 3 will explore the students' existing understanding of AM or *area* through studying the existing curriculum, pedagogy and students' task-based interviews. Drawing from the insights of theme 3 of the present chapter and building on the findings from Chapter 3, in the 2nd study (reported in Chapter 4), I will design and develop a teaching sequence on AM, and implement the same in a classroom context. However, the intention is not just to explore the success of the lesson, or the effect of different materials used, but to also focus on the complex forms of social interaction involved in the construction of the area concept, and how it can be analysed. To further explore the role of material interaction separately, without the noise of complex social interaction, I will present the 3rd study (reported in Chapter 5), which will explore this aspect in a controlled lab set-up.

3

Study 1: Exploratory Studies

This chapter outlines the initial exploratory studies to probe students' conception of area measurement. These studies were done in a naturalistic setting as far as possible, in the classroom and within the regular school schedule. As the studies were exploratory in nature, I have used a mix of approaches, ranging from classroom observations to semi-structured interviews, textbook analysis, planned and structured task-based interviews, and written questionnaires. The broad objective was characterizing the existing scenario of students' understanding of area-measurement in the Indian context, as the literature review we saw in the previous chapter mostly covers work done in other countries, especially in the Western context. As I progressed with the exploratory studies, our observations and data collection became more and more refined, to a more focused and structured study, which explored specific aspects of students' understanding of the concept of area-measurement.

3.1 Overview

The main objective of my thesis in general and the present chapter in particular is to understand and explore students' conception of area-measurement (AM). The literature review already throws some light on students' conception of AM and it also covers some of the major issues faced by students while learning AM. While a majority of such issues were covered under the conceptual theme of the literature review, we saw their bearing/connection with the other three themes of the literature review, that is, the curriculum, the material-use, and multiplicative thinking. Thus, students' conception of AM, or the issues faced by the student while learning AM, are not independent of the curriculum, the material-use, or the aspects of multiplicative thinking involved in the pedagogy of AM. Thus, to properly understand students' conception of AM, one needs to engage holistically with all the above mentioned aspects of AM instruction. Most of the studies mentioned in the literature review on AM, except a few, are not from the Indian context. Unfortunately, these few Indian studies also tend to be inspired by international tests and studies e.g., PISA, TIMSS, and end up strengthening and verifying some of the already reported misconceptions related to area and perimeter found in such tests (Kanhere, Gupta, & Shah, 2013; Educational Initiatives, 2006). Moreover, such tests follow a deficit perspective, in labeling students' conceptions identified through written tests as misconceptions, rather than meaningfully engaging with students' conceptions (like in the Piagetian approach). They thus provide a narrow understanding of the underlying factors affecting students' conceptions. Thus, to bring out aspects of Indian students' AM conception, there is a need for further exploration. Moreover, to understand students' conception of AM in the Indian context, it is equally important to understand it within the context of Indian school, that includes the practiced Indian curricula, and the pedagogy that is followed in shaping students' conceptions. Also, to plan any intervention in a given situation, it is important to know the existing situation. Since the subsequent studies reported in the thesis are intervention studies, it was important to study the existing issues in the present setting. Considering the need for a fresh exploration of the issues at hand in the Indian context, the most suitable methodology was Naturalistic methodology. I elaborate this in detail in the next section.

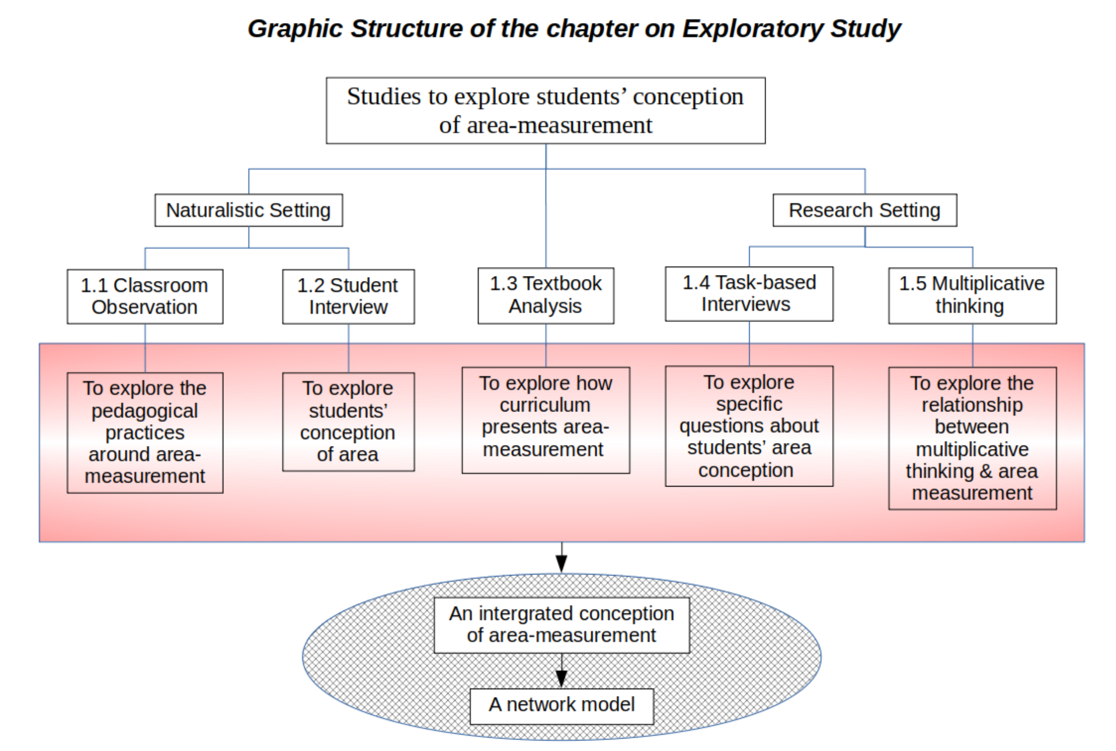


Figure 3.1: Graphic summary of the chapter

This chapter presents a series of studies, which can be divided broadly into three categories. The first set of studies were done in students' existing natural (or regular) setting, that is the school setting of the students. The second category included the analysis of the curriculum or textbook. The third set of studies were done using one-on-one, task based interviews, conducted in the research institute or an isolated lab provided to me in the school itself (e.g., school's computer lab). A concise graphic picture of the forms of studies covered in this chapter are shown below in Figure 3.1. The chapter ends with a proposed model for learning the concept of area-measurement.

3.2 Methodology

In this study we have used the Naturalistic research methodology that falls under the Naturalistic paradigm, and it integrates several data collection methods (Moschkovich, & Brenner, 2000). Naturalistic methodology has evolved from the paradigms of sociology and anthropology, as a challenge to the positivist trend of investigation, and thus acknowledges the role of the observer along with the participants in the construction of meaning (Moschkovich, & Brenner, 2000). The naturalistic paradigm has three main principles, elaborated as below:

The first principle requires one to consider multiple points of views/data. In this study, we have addressed this by exploring students' conception of AM through three data sources: classroom observation, textbook (curriculum) analysis and students' interviews. This method requires one to view students' ideas purely from their own cultural positioning, with a more open and original outlook to listen to students' description and definitions, rather than looking for pre-existing definitions or issues or problems mentioned in the existing literature from different contexts. Thus the aim is not to attain objectivity by compromising on the natural factors, but rather giving subjectivity its due by acknowledging it and describing it completely as much as possible.

The second principle of the methodology demands one to not just verify existing theories, but to help create new ones. Thus in this chapter, we not only try to characterise the overall picture of AM learning in the Indian context, but also generate and propose a new theory based on the data collected. This "network model" of AM pedagogy, elaborated at the end of this chapter, is an outcome of several studies. Thus this chapter not only contributes to forming a better-picture of AM conception in the Indian context, but it also proposes a theory (network model of AM) to address the specific yet complex character of AM pedagogy.

The third principle discusses the significance of context in the study of cognitive activity or learning. Thus the methodology requires one to study and include aspects of the setting or context of the student, as the student is not an individual learner separate from her or his context, but draws meaning out of the place and its practices. Also it is important to elaborate how the existing setting is affecting the learning and cognition, instead of just describing or qualifying the "natural" setting. The initial set of studies of the present chapter are done in the regular natural setting of the students, while the later set of studies, which were planned to be more structured cognitive studies, were done in a research setting.

The chapter starts with the following three broad research questions:

RQ 1. What are students' conceptions of "area" ?

RQ 2. What is the conception of "area" reflected in classroom practices?

RQ 3. How does the curriculum deal with the "area" conception?

However these broad research questions become more specific as we move through the different studies covered in this chapter and go deeper into each of these studies. The next sections delve into the studies done to explore each of the above three research questions. In the process of exploring these three broad research questions, further specific research questions emerge, leading to further structured studies done later on. The end of the chapter presents an attempt to consolidate all the studies, in order to re-imagine the pedagogy and curriculum for AM.

3.2.1 Classroom Observation on Area-measurement

In order to understand and probe the existing pedagogy around AM, and to address the second research question (conception of "area" reflected in the classroom practices), classroom observations in regular schools were carried out. The very first study in this venture started with visiting schools located in an urban setting. Close to our Research Institute are six schools, which are part of a chain of federally funded schools distributed across multiple locations in India. This school chain follows a central government curriculum, and is different from the other school systems which are either private or state government schools. The schools cater to students from middle-to-upper-middle-class families, and mostly to students whose parents work in central government jobs. The schools follow the books produced by the National Council of Education Research and Training (NCERT), which is an autonomous body of the Government of India to improve the quality of school education.

I went to all the six schools, took permission from the principal and then asked the mathematics teacher there if I could attend their class on geometry and measurement. All of them gave me slots to observe the classes. These visits and observations were made during November-December, 2010. For over a month I observed 30 lessons, and made hand-written notes for 20 of them. I attended and observed a wide range of classes, from Grade 4 to 10, based on whichever class the mathematics teacher allowed me to attend. The schools, and the classes I attended, were thus chosen based on convenience. The teacher taught in those classes quite confidently, and did not see me as a threat or an obstacle in her teaching, possibly because I was open to observations and I was not there to evaluate or judge the teaching. I myself have grown up studying in a similar school system and learning from a similar syllabus, though the textbooks have been revised since my time as a student. I have repeated some of these observations in other kinds of schools as well, for example in government aided, private, and state government schools. Although I will not be reporting them in this section, they are broadly similar to the present set of observations. Some broad observations that could be drawn through the notes about the pedagogy of AM are as follows:

1. In almost all the classes I observed, across grades, the main focus was on solving the exercise problems in the textbook at the end of each chapter. Sometimes the teacher also mentioned how many marks a particular exercise question would carry in the exam. Thus it was literally an exercise for exams.

   In one class of 7th grade, the teacher allowed me to observe her class. She told me that she generally tries to have a discussion with her whole class and that day she was planning to teach area-measurement. The teacher started with exercise questions, and helped children recognise the use of the formula for area and perimeter in different question situations. After a few questions she talked about the distinct contexts for area and perimeter. Thus, the teacher mentioned that perimeter will be used for contexts of lace, track, decorating, etc., and area will be used in the contexts of polish, design, distributing land, sowing seeds, tiling, etc. The teacher also said that for making a door, we want an area, but we strictly need length and breadth.

   In another school, where the teacher started the topic of area and perimeter for her Grade 7 students, the class started by asking students about perimeter. While one student said that it's the length of the boundary, the teacher started asking about specific conventional shapes like squares and rectangles. She did a few example problems and then moved to discuss about area. Following is an excerpt of the interaction that happened afterwards in the classroom (since these were hand written by me in real time, there is scope of human error or imperfection or omission, so the excerpt may not capture the exact utterances):

   > Teacher: What is area ?
   >
   > Student 1: It is the total space occupied, no I am not getting the right term
   >
   > Student 2: It's the total region
   >
   > Teacher: It is measured in unit square. Where 1 square centimeter means area of the square is 1 cm^2
   >
   > [Teacher draws a rectangle on the board, marking it's length as l and breadth as b]
   >
   > Teacher: How can you prove that area is length × breadth ?
   >
   > … [some chattering] ...

Student 1: It's divided into unit squares. It's the multiplication of the squares in the length multiplied by the number of squares in the breadth.

[As the student explains, the teacher draws some lines inside the previously drawn rectangle, some parallel to the lengths and some parallel to the breadth]

Student 2: Is there any other way to prove it?

Student 3: How is this a proof?

Student 4: So why ?

Teacher: When we say area, what we say are units. What does square-cm mean? How?

Teacher: 1 square-meter is the area of a square of side 1 meter.

Teacher: Now you will have to practice questions. This syllabus was done in October to December.

Teacher: I think all of you got the answer.

As can be seen from the above interaction, the teacher asked for the proof of area as equal to length × breadth, which led to students responding and raising questions to each other or asking for proof. But the teacher did not address those questions, and switched the interaction focus to the syllabus. This may be because the teacher wanted to focus on things which are more relevant for the exam. The episode can also be seen as students trying to make sense of area by wanting a materialistic or realistic understanding of area, but the teacher tried to switch the interaction to a more formalized understanding of area, and did not try to connect students' experiences about area with the formal way of finding it.

2. There was extensive use of numerical calculations in all the classes, with great emphasis on the formula or rule to be used for the given exercise or numerical problem. The use of formulas were seen as conventions or rules in math, without delving into the logic or reason for it. For example, the formula for area and perimeter of shapes like Rectangle, Square, etc., and formula for surface area for cone, cylinder etc., were told to students, for them to use during solving different numerical problems.

3. There was no discussion on any alternative ways of solving a particular problem or exercise question. So, when a teacher posed a particular exercise question, and students gave their responses, students only got the feedback of whether the response is correct or wrong, with no discussion on why it is correct or incorrect.

4. Students rush into the race to give the answer first, and the teacher looks for only the correct

answer. Even if different answers come from students, teachers don't engage with these, considering them to be wrong.

5. In the classroom discussion on area-measurement mostly typical shapes, which are regular polygons like rectangle, square, triangle etc., appear. In Class 6, the teacher introduces how to find the area of an irregular shape as shown in Figure 3.2 and Table 3.1 (similar examples can be seen in the textbook analysis in the next section).

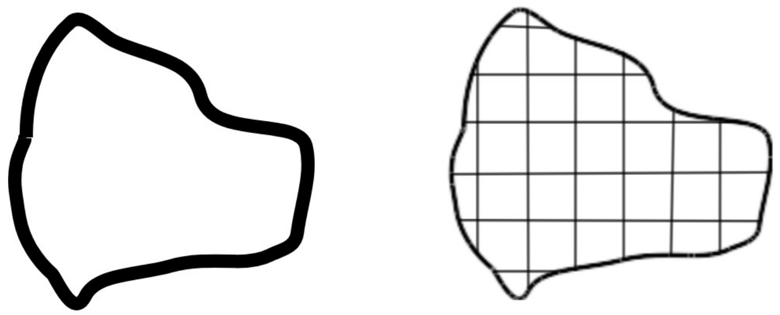


Figure 3.2: A classroom example of finding the area of an irregular shape

Table 3.1: Procedure followed in finding the area of an irregular shape

Covered Area	Number	Estimated area (in sq cm)
Full square	13	13 sq cm
More than half	8	8 sq cm
Half square	3	3/2 sq cm
Less than half	7	0

Thus a particular rule is presented to find the area of irregular shapes. As can be seen from the first and second row of the above table (Table 3.1), squares which are covered, fully or more than half were counted as whole, squares which are covered half are considered half, while squares which are covered less than half are considered or counted as zero. However, there was no discussion beforehand on why such a rule was being used, or why it works to find the area of an irregular shape. Though this rule could provide a powerful strategy of partitioning a given area into smaller square units, and noting the counts of different sized units into a tabular form, it lacks proper grounding for using such strategies. Thus the pedagogy missed any discussion on why the different sized units or parts are counted in this particular way, for example, why the estimated area of parts that were less than a half square were taken as zero. Also the instruction of numbering the units restricted the students to count each unit, rather than coming up with some alternative optimal way of chunking the units into some bigger shape.

Thus some general and specific observations arose regarding the pedagogical practices around measurement, from several classroom observations. These observations and insights helped me in reflecting about designing my own teaching sequence on learning AM. This is presented in the next chapter.

3.2.2 Students' interviews inside school

In this study, I extended my observation from looking at the pedagogy around measurement to what the students' conceptions of area-measurement are. I planned to interview a few students from each class that I observed. This is a general practice that I followed for most of my observations, because after doing the classroom observations, students noticed me and became familiar with me to some extent. Generally, I requested the mathematics teacher to identify six students for me, two who scored above average, two with average scores, and two with below average scores, from each class, to have some fair representation of students to some extent. I interviewed some 20 students, and made notes. Students were in the age group of 10-12 years. The nature of the interview was open-ended, with no rigid structure or set of questions. The interviews were fully guided by a curiosity to know what students understand by area or area-measurement, maintaining an informal tone. The setup was also kept informal, and the interviews were done in the school premises, either in the playground, or the stairs or corridors of the school. The interviews usually started with some general question about the student's name, age, etc., and then they were asked what do they know about area, or understand about area measurement, or just "what is area?". To facilitate the discussion further, I also asked them about perimeter. A casual tone was maintained in the interview, to make students feel comfortable. They were also assured that their identity will be kept anonymous. Considering the noise in such an informal (and unstructured) setup, I relied completely on my hand written notes, instead of any recording device. Some of the broad observations in this context are as below:

1. Interestingly, almost all the students who could respond to the area question spoke about area as $l \times b$, side × side, $2l + b$, $l + b$ etc., and for perimeter they said, side+side+side+side, $2\times(l+b)$, $2+l+b$. Though I went with not much expectation, I still thought they might indicate the 2-dimensional space or show me the plane surface when I asked them about area. But to my surprise their association with area was not with any space but with some symbolic and numerical representation.
2. In further discussion with the students, they were shown a rectangle drawn in a notebook with given dimensions (measures of length and breadth) and were asked about its area and

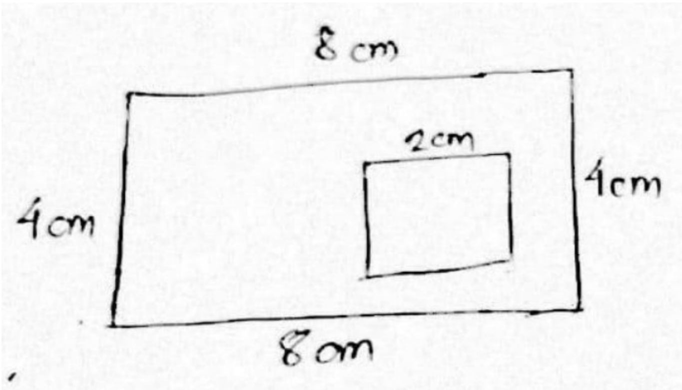


Figure 3.3: Finding area when a square part is removed from the inside of a rectangle

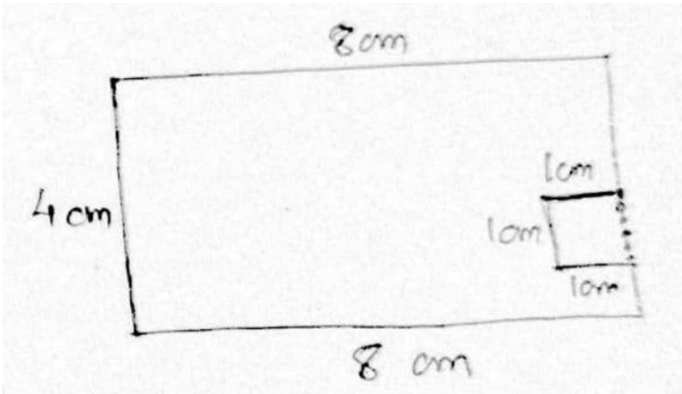


Figure 3.4: Finding the area when a square part is removed from the edge of a rectangle

perimeter. Again, most students tended to mention some numerical operation, rather than trying to highlight the attributes that are measured for area and perimeter respectively. After that students were asked if a square part is removed from a rectangle what will be the area and perimeter of the resulting shape (see Figure 3.3 and 3.4). Some students (around four) subtracted the perimeter of the square from the perimeter of the rectangle e.g., students calculated 24cm-4cm=20cm as the perimeter of the resulting shape in Figure 3.4, rather than finding the measure of the resulting boundary. A few students could not say how to find the perimeter of the resulting figure as also reported in several other studies (Cavanagh, 2007; Kanhere, Gupta, & Shah, 2013; Educational Initiatives, 2006). Thus, students' understanding of perimeter is confused with that of area, where students apply the same operation of (subtracting) area for perimeter too. This also highlights the complete lack of spatial understanding of the attributes and their measure among several students.

3. When students were asked why area is $l \times b$ and perimeter is $2\times(l+b)$, most students could not explain, or said that they see it as a rule or convention. However, two students connected

it with the unit squares on the length multiplied by the units on the breadth, and they also explained the perimeter as the measure of the boundary. Two students drew both horizontal and vertical strips on the rectangle and said that's why it is length multiplied by breadth. One of these students explained it as the length is getting repeated along the breadth and that's why it is length into breadth (i.e., length times breadth).

4. When I drew an irregular closed shape and asked them about its area, some students (around 7) either could not respond, or with some prompt they said it would not have any area. For them, area was associated with only typical regular shapes like square, rectangle, triangle and in some very few cases circle. For example, two students responded like this

   I: *aur circle ka area*? [and circle's area?]

   S': *ma'am circle ka area nahi hota* [ma'am circle doesn't have area]

   S: *Kyuki circle means squares or rectangles ko number likh sakte hain…circle ko side nahi hote number nahi likh sakte…* [because in circle means for squares and rectangles number can be written… circle doesn't have side so number can't be written…]

   Thus, the student's meaning of area is just associated with some ideal mathematical object (e.g., square, rectangle) completely disconnected to any realistic material or physical object. However, some other students (around 5) tried doing it similar to the way explained in Figure 3.2, that is, partitioning the given irregular shape roughly into squares and counting the squares in the given irregular shape. Here, again students were conveniently leaving out the parts around the corner of irregular shape after partitioning, without any explanation for it. So, either they are just implicitly following the rules that they had learned in the class, or they didn't consider that part as belonging to the area of the region. But this could not be ascertained. Thus these observations indicate that students' understanding of area is just limited to some symbolic representation or some numerical operation or numerical value, and is totally devoid of any spatial or material or physical understanding of what the "area" concept represents.

5. Another observation that came up in students' interviews is that the moment I utter the word *area*, students try to reproduce the formal knowledge that they had learned in school, with no reference to their out of school context. It could be due to students having not much exposure to area context outside of their schooling, as most of these students were from urban areas. Or it could be due to students having different references (or contexts or words) for area in their own local language or local context, which I was not aware of at that time. Since English was the second or third language for most of the students, students may have some other term for

area in their own language and thus asking questions on the English term "area" might have hindered them in connecting it with their own experiential or cultural meaning of area.

These observations also indicated some gap in student's understanding of area. This motivated me to design some structured interviews, to probe some of the observations further in a more systematic manner, after doing a textbook analysis.

3.2.3 Curricular Material: Textbook Analysis

As argued earlier, the above aspects of conceptual understanding are not independent of the remaining three themes of the literature review i.e., the curriculum, the material use, and the role of multiplicative thinking (MT). Here I report a textbook analysis, using an analytical framework based on the insights drawn from the last three themes of the literature review. In the last theme of the literature review, I argued for the role of MT in learning AM, through several studies that have established the presence of MT in AM. However, the connection between AM and MT cannot be made one way (by just bringing MT to AM) but has to be established both ways, by also having the contexts of area or AM within different MT contexts. For example, several math educators have extensively used graphic area models of multiplication to provide visual and concrete representations for students learning two-digit multiplication (Englert and Sinicrope, 1994; Izsák, & Beckmann, 2019). Extending the same model, several educators have used the base-10 materials (as in graph paper) as a model, not just for multiplication of whole numbers but also decimal fractions (Rathouz, 2011). Thus, through this textbook analysis, it is also important to explore and report such instances where the area model is used for multiplication, or to illustrate decimal fractions and how the concept of area appears in such multiple topics.

To get a deeper understanding of area measurement (AM) in the curriculum, I have analysed the math textbooks as a curriculum material. For the present section, I have analysed the math textbooks of Grades 5, 6, and, 7 of Maharashtra[3] state board (MSB) books and similarly the math textbooks of the corresponding grades of National Council of Educational Research and Training (NCERT) books respectively. The MSB books are followed mainly in one state of India in schools that come under the state board, while NCERT books are followed across India in several schools that come under the Central Board of secondary Education (CBSE). From here on, I will be referring to the math textbooks of MSB and NCERT of Grades 5, 6, and 7 as MSB 5, MSB 6, MSB 7 and NCERT 5, NCERT 6, NCERT 7 respectively.

3 Maharashtra is one of the states in India, out of 28 states and 8 union territories. Maharashtra, like most other states, has its own state education department and its own curriculum.

3.2.3.1 Geometry vs geometric measurement

One of the main arguments in the previous two chapters is how over time geometry has become separated from geometric measurement, with more weightage given to the former over the latter. An impression of the same can be seen in the content distribution of the textbooks analysed here. In Table 3.2[4], the chapters on geometry are highlighted in yellow, while the chapters on geometric measurement are highlighted in blue. It can be clearly inferred from Table 3.2 that geometric measurement is kept separate from geometry[5] in almost all textbooks. Again as we move to higher grades, the proportion of geometry-related content increases over measurement-related content. After much deliberation, the topics (or chapters) on "practical geometry" in NCERT books and the one on "constructions" in MSB books are identified under the category of measurement because they are mainly based on the principles of measuring using the geometry toolbox. However, the analysis indicates that these chapters only have a few exercises on measurement, and most of the exercises in them are based on the application of results and proofs from the chapters on geometry. Thus, the chapters on "practical geometry" and "constructions" act as application or verification of the already existing geometric results and proofs rather than a space for students to construct their own authentic geometric results and proofs.

Table 3.2: Distribution of Geometry and Measurement related topic in textbooks

MSB 5	MSB 6	MSB 7
• Geometry: Basic Concepts • Angle and Triangle • Measurement • Segment: Measurement and Construction • Properties and Rectangles and Squares • Circle • Perimeter • Area	• Point, Line, Plane • Angle • Pair of Angles • Perimeter • Triangles and Types of Triangles • Properties of Triangles • Geometric Constructions • Area • Volume • Circle	• Properties of Triangles • Theorem of Pythagoras • Construction of Triangles • Quadrilaterals • Congruence • Types of Quadrilaterals • Area • Volume and Surface Area • Circle • Construction of Quadrilaterals
Geometry based = 4	Geometry based = 6	Geometry based = 6
Measurement based = 4	Measurement based = 4	Measurement based = 4

4 The list of all the chapters in different textbooks are mentioned in Appendix II.

5 With regard to the distinction created between geometry and measurement related content, the content on geometry deals purely with mathematical objects or entities with very little reference to real life objects. The content on measurement will have more real life contexts and exercises, and an aspect of the practical or physical measurement using tools like ruler, protractor, etc.

Total chapters = 22	Total chapters = 23	Total chapters = 23
NCERT 5	NCERT 6	NCERT 7
• The Fish Tale • Shapes and Angles • How Many Squares? • Does it Look the Same? • Mapping Your Way • Boxes and Sketches • Area and its Boundary	• Basic Geometrical Ideas • Understanding Elementary Shapes • Mensuration • Symmetry • Practical Geometry	• Lines and Angles • The Triangle and its Properties • Congruence of Triangles • Practical Geometry • Perimeter and Area • Symmetry • Visualising Solid Shapes
Geometry based = 3	Geometry based = 4	Geometry based = 6
Measurement based = 4	Measurement based = 2	Measurement based = 2
Total chapters = 14	Total chapters = 14	Total chapters = 15

It is evident from the above table (Table 3.2) that the topic of geometry and measurement together covers around 35-55% of the total mathematics content in all the given textbooks. It can be clearly noticed that except NCERT 5, in most of the other textbooks, geometry related topics majorly precede measurement related topics and have more number of chapters devoted to geometry than measurement.

3.2.3.2 Use of area contexts in other topics of mathematics (Integration of concepts)

In addition to the chapters on geometry and measurement, I also explored the presence or absence of area context in other topics of mathematics. In this part of the analysis, I searched for the term "area", or the context of area, in all the chapters of the selected books. Table 3.3 gives the list of all chapters (their name and their chapter number from the respective textbook) which have used the term area or the context of area. The number of instances such contexts are used are mentioned in brackets.

Table 3.3: Titles of chapters using some form of area representation in different textbooks (number of instances in brackets

MSB 5	MSB 6	MSB 7	NCERT 5	NCERT 6	NCERT 7
Profit and loss (1)	Squares and square roots (1)	Theorem of Pythagoras (1)	Parts and Wholes (2)	Knowing our numbers (1)	Fraction and Decimals (2)
Equivalent fractions (1)	Volume (1)	Quadrilaterals (1)	Mapping your way (7)	Algebra (1)	Data Handling (2)
Multiplication and division of		Identity (1)	Boxes and Sketches (1)		The triangle and its properties (1)

fractions (3)					
		Volume and surface area (1)	Ways to multiply and divide (2)		Congruence of triangle (2)
		Circle (1)	How big? How heavy? (3)		Comparing quantities (2)
					Algebraic expression (1)

From the above table (Table 3.3), it's clear that the context of area is used in several topics in both NCERT and MSB books[6]. The use of area context in this wide range of topics (or chapters) shows that the context of area has application in a broad range of topics. However, it is also seen that the use of area in other chapters is not given its due importance in these set of books, except for the NCERT Grade 5 book, where each of these chapters consists of a few notes (at the bottom of the page) for teachers about how the context of area can be consciously used with students to build understanding of the topics in those chapters. The notes emphasise the idea of integration of concepts. None of the other books have any such notes which emphasise or highlight the use of area in those chapters.

The integration of area concept with other topics is comparatively poor in MSB books. In MSB 5, the sixth chapter on “Profit and loss” has the following exercise question,

> *Makarand purchased a plot for Rs 81,450. After a few years, he sold it to Ajit at a profit of Rs 1,750. What was the price for which Ajit purchased that plot ? (p.32)*

Although the present framing of the question uses the notion of plot just as an object with some price, devoid of any discussion, it does assume the social knowledge of buying and selling of plots of land, which involves the 2-dimensional quantification of such plots. Thus, even though the context of a plot is kept immaterial, and the emphasis is merely on the price and the profit, the context does have underlying it the notion of quantification of a 2-dimensional space in terms of price. This could provide a rich context to integrate AM with the topic of profit and loss. The potential of integrating AM with other topics could have been made more explicit by providing a teacher's note or footnote for such contexts.

The chapter on “congruence” in MSB 7 book even has congruence of triangles and quadrilaterals. But

6 The books were analysed in the year 2012, however the very next year itself the MSB books were revised and thus they are no longer followed by the state boards after the year 2013. Broadly speaking, those revisions have some elements of the NCERT books, as some of the members involved in the revision of MSB books were also involved in the NCERT book revision process.

it doesn't have any mention of area, even though it is so imperative to the understanding of congruence, as one of the main implications of congruence of shapes is that they will have equal area. So, while congruent shapes may visually look different due to translation and rotation, all their corresponding measurements (of dimensions) will remain the same. Thus the spatial or numerical representation of area can be more explicitly used in each of these chapters, either as a context or to broaden the understanding of each of these concepts.

3.2.3.3 Kind of shapes and "unit" representations

As can be seen from Table 3.4, except in NCERT 5, and in very few instances in NCERT 6, most other books use very conventional geometric shapes in the chapters on geometric measurement, causing children to associate area or AM only with conventional geometric shapes (as reported earlier in interviews with students). This also results in students developing a very limiting view or meaning of AM or area, as a mathematical concept cut off from the real world, or having no meaning in the world outside.

Table 3.4: Shapes shown in the chapters on Geometric Measurement

MSB 5	MSB 6	MSB 7	NCERT 5	NCERT 6	NCERT 7
triangle, circle, hexagon, rectangle, irregular curved shapes (on grid/graph), polygons (unit-square shown)	triangle, rectangle, circle, square, pentagon	rectangle	grid, triangle, rectangle, real-life examples (e.g., animal skin, footprint, etc.), parallelogram, curved closed shape, quadrilaterals, design patterns. (Square unit, triangular unit)	circle, rectangle, star shape, leaf, irregular polygons, cloud and curved close shape on grid	rectangle, irregular polygon, house, triangle, quadrilateral, parallelogram, circle, racing track, circular objects.

Most of the curricular developmental models on measurement cite the importance of units, and the need for moving from informal units to standard units (Piaget, Inhelder & Szeminska, 1960; Battista, 2007; Sarama & Clements, 2009). However, in the textbook analysis, we see that the idea of unit is there only in the fifth grade of both MSB and NCERT books, but not much in later grades. Again, except NCERT 5 that does bring in triangular units, most other books talk about or assume only standard square units. While the use of different fractional units are strongly recommended in the literature (Battista, 2007), and these are considered very important for understanding the conceptual spatial structuring and unitization involved in AM.

3.2.3.4 Connection with real life terms like "size"

As we saw in the section on students' interviews, most students did not connect with, or give any real life contexts, for "area". So I tried to find what the general term used for area or AM in regular everyday language is by asking a few lay people. What I realised is that, in most daily life contexts the word "size" is used to refer to area or AM. The last curricular reform recommends connecting the school's content with students' real life in the National Curriculum Framework (NCERT, 2005). Thus, in this part, I explored the uses of the word "size" in these textbooks. Referring to Table 3.5, I checked for occurrences of the word "size" in each of the books. We see significant uses of the word "size" throughout the NCERT books. However it occurs in only one instance in MSB 5, and there are no instances in MSB 6 and MSB 7. Though the word "size" is majorly used to refer to AM in these books, on some occasions it is also used to refer to linear or volume measure, or some other quantity. Thus, the word "size" is, in general, ambiguous and derives its specific meaning from the context where it is used.

Table 3.5: Uses of the word "size" in the following books

MSB 5	MSB 6	MSB 7	NCERT 5	NCERT 6	NCERT 7
1	0	0	18	20	34

Although the word "size" seems non-mathematical, it does bring in a rich and integrated generic understanding of measurement, which is also related to students' regular language. This will be further explored and established in our later study, where most of the students seem to be comfortable using the term "size" for area.

3.2.3.5 Kind of AM tasks or exercises (Numerical to spatial representation conversion for AM missing)

In the topics on length or angle measurement, or even in the chapters on practical geometry, students are required to draw or construct a line segment of given length or given angle. That is, with given numerical values, students were expected to draw the corresponding spatial representation. There are also a few reverse situations, where students are required to measure the length and the angle (i.e., getting the numerical value) given some spatial representation like shapes, etc. However, in the case of AM, except for the NCERT 5 book, most other textbooks do not have exercises or tasks, where students could actively engage with the visual representation of AM. Most of the exercise questions on AM are "find" questions which are calculation based, rather than drawing, creation or construction based tasks.

Some researchers (e.g., Boaler, 2015) have emphasised the use of open and creative tasks, such as finding multiple visual representations for the product 18×5. That is, tasks, where given some numerical measure for area, students are asked to create (or draw or visually represent) different figures having the same numerical area measure. Such tasks encourage conversion from numerical to spatial representation for area. Tasks that allow movement in both forms (moving from numerical to spatial and vice versa) can strengthen the connections between different representations. Such tasks may further help in building connections with different contexts, and help students to transfer them to real-life contexts. One reason for the lack of such tasks in AM could be due to the absence of any instrument or tool in the case of AM, unlike other measures like length and angle, where students have ruler and protractor respectively. However, this absence also needs to be addressed through connecting or building new “cultural tools” for AM within the classroom.

3.2.3.6 Area of irregular or curved shapes

In students’ interviews we also saw that for irregular or curved shapes, some students said that it does not have area; for them area was something associated only with a few conventional shapes like square and rectangle. Thus it was important to explore how textbooks present areas of irregular shapes. It was found that there are several mentions of irregular curved shapes having area in Grade 5 books of both NCERT and MSB. However, as we dig deeper, we see that the measurement of irregular shapes is introduced as some procedure or calculations, with counting and some formula, with less attention paid to the rationale for such procedures. This was also corroborated by the classroom observations, where students found the areas for irregular shapes.

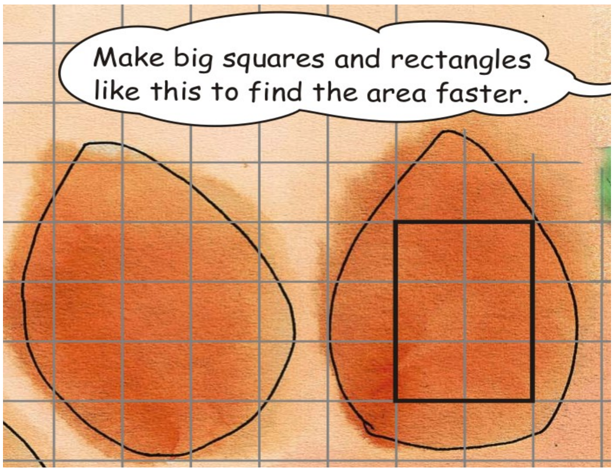


Figure 3.5: NCERT 5 (NCERT Textbook, p. 39)

For example, NCERT 5 introduces areas for irregular shapes as shown in Figure 3.5, and requires students to count the number of squares (complete squares) to get the area. Again, in NCERT 6, area for irregular shapes is handled through making a table to count or categorize the fully-filled, half-filled, more than half-filled and less than half-filled squares (Figure 3.6[7], Figure 3.7).

Covered area	Number	Area estimate (sq units)
(i) Fully-filled squares	1	11
(ii) Half-filled squares	–	–
(iii) More than half-filled squares	7	7
(iv) Less than half-filled squares	9	0

Total area = 1 + 7 = 8 sq units.

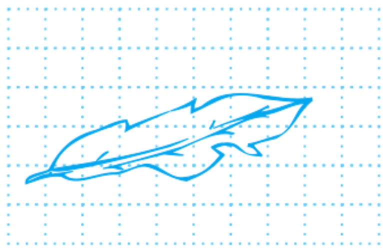


Fig 10.12

Figure 3.6: NCERT 6 (NCERT Textbook1, p. 216)

Though the textbooks mention that it is an estimate of the area, they do not elaborate further why such categories appear in the table. Or why is such counting used? Also, why are we considering more than half-filled square as 1 unit and less than half-filled square as 0 units? Why is such a formula used at the end? What is the actual area? How close is this estimate to the actual area? Can we move further in getting the precise area?

7 The number of fully-filled squares, as given as “1” in the first row of the table, could be a printing error.

Example 11 : By counting squares, estimate the area of the figure 10.9 b.

Soultion : Make an outline of the figure on a graph sheet. (Fig 10.11)

Covered area	Number	Area estimate (sq units)
(i) Fully-filled squares	11	11
(ii) Half-filled squares	3	$3 \times \frac{1}{2}$
(iii) More than half-filled squares	7	7
(iv) Less than half-filled squares	5	0

Total area $= 11 + 3 \times \frac{1}{2} + 7 = 19\frac{1}{2}$ **sq units.**

How do the squares cover it?

Fig 10.11

Try These

1. Draw any circle on a graph sheet. Count the squares and use them to estimate the area of the

Figure 3.7: NCERT 6 (NCERT Textbook, p. 215)

The emphasis on counting of squares and rectangles (within the curved shape), could be a reason that some of the students in the interview did not associate area with irregular curved shapes.

In MSB books, irregular curved shapes are attended to in Grade 5, but only as shown in Figure 3.8. Here a different procedure for calculating the area of irregular shapes is provided.

Here again no rationale is provided for this procedure, and the emphasis is on counting the squares. However, the use (or representation) of the graph sheet in MSB 5 book, instead of grid in NCERT books, does seem to have more potential for increasing the precision, that is, in addressing the questions of reaching more precise measures in finding the area for irregular curved shapes. Thus, the graph sheet can be used as a tool for measuring area, just like the ruler is used in measuring length.

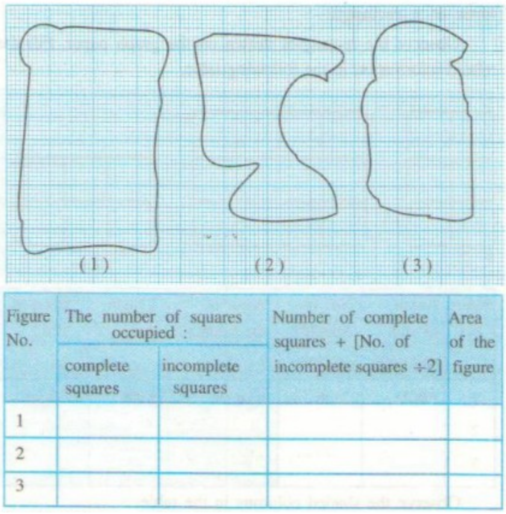


Figure No.	The number of squares occupied :		Number of complete squares + [No. of incomplete squares ÷2]	Area of the figure
	complete squares	incomplete squares		
1				
2				
3				

Figure 3.8: MSB 5, Maharashtra State Board, mathematics textbook for class 5

3.2.4 Structured studies

To develop a more structured cognitive understanding of students' conception of area, I designed and developed a set of tasks and contexts for students, inspired by Piagetian clinical studies, and adapting the approach adopted by Piaget. Before the task-based interviews, I did a written test with some of the tasks with a few students, to assess the accessibility of these tasks without interference from the interviewer. This helped determine how the tasks can be further clarified, and what further probing questions the interviewer can ask. Unlike the interviews mentioned in the earlier section, the ones conducted in these structured studies were based on materials prepared beforehand, and are done in a more structured set-up, to avoid any noise present in a natural setup. The details of these structured studies are described in the next section.

As elaborated in the initial part of the present chapter, we started with three broad research questions, leading to three broad studies in the naturalistic setting: classroom observation, students' interviews, and curriculum analysis. One of the main insights drawn from these broad studies was that asking direct questions around the formal term "area" can be limiting and misleading in accessing students' true conception about AM. Thus, here I move beyond the broad research questions, to specific research questions studied in a research setting, guided by the Piagetian way of investigation. The questions aimed at assessing students' conception grounded in meaningful contexts were:

1. What are students' conceptions of "conservation" of area and perimeter ?
2. What are students' representations for area and perimeter? Or, Which attributes of the figure are measured through area and perimeter?

3. How do students interpret area and perimeter for unfamiliar figures?
4. What are students' conceptions of unit structuring in area-measurement ?

The next part elaborates the design and development of the structured studies, to investigate the specific research questions mentioned above.

3.2.4.1 Sample and setting

From the previous unstructured observations and informal interactions with students from Grades 4 to 7, it was evident that students hold widely different understanding of measurements of area and perimeter. That is, different students construct the meaning of area and perimeter differently even with common instructional practices.

In this study (see Table 3.6), students from Grade 5 were called, and before carrying out the task-based-interviews, a pilot test was conducted with four students from the school Sc0 using a written questionnaire, to check the comprehensibility and accessibility of some of the tasks, and get an estimate of the rough time duration. This was followed by a task-based interview with students from two different schools (Sc1 and Sc 2), from the same chain of schools catering to students' whose parents serve in government jobs. Students were selected by their respective math teacher and were identified by the teacher as above or below average or average scorer in math. This information was requested from the teacher, in order to select students representing the variation in the population in terms of math ability.

Table 3.6: Sample set from three different schools

Sample Set	1st group (School: Sc0)	2nd group (School: Sc1)	3rd group (School: Sc2)
Students selected by the Math teacher	Pilot study (4 pupils): 2 AA, 2 A	Main study (5 pupils): 2 AA (S1,S2), 2 A (S3,S4) and 1 BA (S5)	Main study (5 pupils): 3 AA(S1',S2',S3'), 1 A (S4'), and 1 BA (S5')
AA: above average scorer, A: average scorer, BA: below average scorer S1, S2,… and S1', S2',… are students belonging to school Sc1 and Sc2 respectively.			

Task-based interviews of students were done and were audio recorded after getting the consent from (Appendix I) both students and parents. Students' drawings and written notes were also collected as data. The next section gives a description and rationale of the tasks used for this study. The actual tasks are mentioned in Appendix III.

3.2.4.2 Task description

The tasks used in this study are inspired from the research reported in the literature review and the studies reported in the earlier sections. Broadly, there were nine tasks, along with some sub tasks in the interview (Appendix III). The first task and subtasks within the first task were inspired from Piagetian conservation tasks with children (Piaget, Inhelder & Szeminska, 1960). As reported in the previous literature review chapter, Piaget et al., (1960) used the context of grass meadows, wooden house blocks and small wooden cows to explore children's conception of conservation of area and operational thinking with respect to area. Children were asked questions around the amount of grass available for the cows to graze. Thus, drawing from the nature of Piagetian tasks, instead of asking direct questions on area, the tasks were contextualized in some way that required students to think in terms of area without using the term "area". Since most of the students I was interviewing were from the urban setup and are familiar with school playgrounds, the first task was around whether same amount of space is available in two different set-ups (or arrangements, see Fig 3.9) of a playground, when a small square part is used up (or, taken away) from two different locations of the same playground respectively. While the first two sub-tasks of the first task were on area, the third sub-task was on perimeter, but here again, I used the term boundary rather than perimeter. Thus, for this task, the formal terms "area" and "perimeter" were deliberately avoided, to avert students from falling back to formulas.

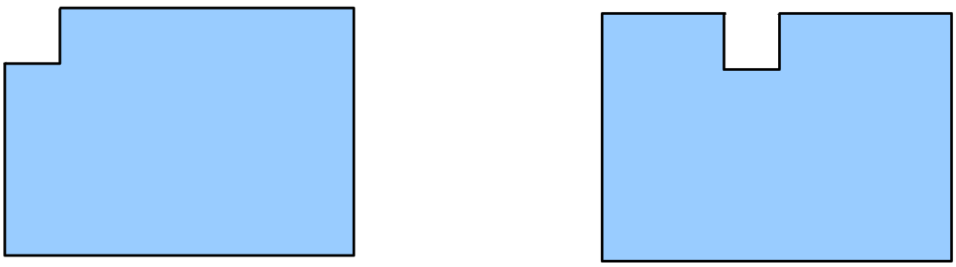


Figure 3.9: Two arrangements

The second task was directly related to the research question of how students represent or highlight the area and perimeter for given figures. The rationale for this task and the research question came from the literature that reported the challenges in connecting the numerical and geometrical aspects of area-measure (Sarama & Clements, 2009; Battista, 2007). That is, area, apart from having a numerical measure, also refers to a geometric or a geographic region. That is, it has real or material existence, apart from having the formal abstract or constructed existence. Some tasks have tried connecting these two through covering tasks, where students were asked to find the number of units which fit in a particular given rectangular region or draw the units that cover a given particular area (Clements &

Stephan, 2004; Outhred & Mitchelmore; 2000). These tasks still rely on discrete counting, and lack the aspect of area as a continuous quantity. More recently, Kobiela & Lehrer (2019) used sweeping or painting to bring in the geometrical or spatial aspect of area. Drawing inspiration from these tasks, here I asked students to highlight the area and perimeter for given shapes through coloring, shading, drawing, or darkening.

The third task (task 3) was inspired from several studies that used L-shaped figures (Cavanagh, 2007; Zacharos, 2006), to explore whether students could extend their understanding or computation of AM beyond rectangles, to other rectilinear shapes. The fourth task was seeded from the textbook analysis, which showed a dearth of tasks or activities that requires students to construct or draw figures (geometrically or spatially) given the numerical value of the area. The nature of the fifth task was almost like the first task, with the exception that students were explicitly asked to find the area and perimeter of the remaining rectangular part when a square part is used up (taken away) for some other purpose. Thus, unlike the first one, where students were asked to qualitatively compare the remaining space, for this task students were expected to compute the remaining space quantitatively. This task was also to explore whether students could distinguish between the two measures i.e., area and perimeter. The sixth task was also very similar to the fifth one, but used a triangular base instead of a rectangular one.

7. The student is given a rectangular sheet of sides 21cm and 12cm, and three different paper tiles of dimensions 2×2, 3×4, and 6×2, given one at a time.
Can this tile, if pasted repeatedly, cover the sheet? If Yes, ask the student: If the same tile is pasted repeatedly, how many of such tiles will be required to cover the sheet.
Ask the student to write it on the sheet provided.

8. You want to cover a rectangular floor with a length of 19m and a breadth of 6m using tiles. Can you cover the floor with a rectangular tile having sides of 3m and 2m? What other tiles can you use to cover this floor?

9. You want to cover a rectangular floor with a length of 15m and a breadth of 8m using tiles. Can you cover the floor with tiles that are right triangles of height 2m and base 5m?
Optional question: What other tiles can you use to cover this floor?

Figure 3.10: Tasks on units or tiles of different dimensions

The last three tasks (Tasks 7, 8, 9, see Figure 3.10) were of the same nature, where given the dimensions of the rectangular space (sheet or floor) and the dimensions of the units (rectangular and triangular one), students were asked whether such units can be used for covering the given space, and how many such units will be required to cover the given space. Thus, these last three tasks were not just to explore student's handling of unit-structuring, but to also explore how they deal with the multiplicative relation between the dimensions of the unit and the measure.

3.2.4.3 Results and discussion

One broad impression that emerged from the interviews across tasks was that above-average (AA) students tend to rely heavily on numerical procedures and formulas, even incorrect ones, without having proper justification for their application. This was seen on many occasions, despite having no direct reference to formal terms like area or perimeter for some question items. On the other hand, average (A) and below-average (BA) students seem quite open and flexible in using other simple non-formal strategies, or techniques like estimation, comparison, etc. Although, this might be a small sample to say this conclusively, this trend does indicate that the criteria for the identification of students by the teacher was based on their performance in the use of numerical procedure. Apart from this broad observation across tasks, I discuss below some of the main results of this structured study task-wise.

1. For the first sub-task of task-1 (see Figure 3.9), four out of ten students said there will be the same space left to play in the two arrangements. Among these, one of them justified it through numerical calculation i.e., computing the area of the remaining space for each case. However, the other four mentioned that the arrangement with the square table at a corner (i.e., arrangement on the left of Fig 3.9) has more space to play than the one where it is placed along an edge. On further probing, it was found that their reasoning was based on the convenience aspect, as the play could be better if the table is at one corner than at an edge. This was different from a purely perceptual justification of one appearing to have more space than the other.

   However, for the second sub-task of task-1, which required students to decide on whether making a lawn in the two cases will cost the same or different, eight out of ten concluded that both arrangements will cost the same. Thus, an important insight that I could draw from these two sub-tasks, is that, in the first sub-task, some students tend to think more from the practically accessible remaining land (of a rectangular space) when something is at a corner than something is at an edge. This is probably because the context in the first sub-task becomes more qualitative, and thus more removed from the operational thinking. Thus, even in the Piagetian task, probably not all students were judging the available space perceptually. They may have considered the practical accessibility of space (grass for cows). In contrast, the second sub-task had attributes like value of the land or space in terms of money, and this could be the reason most students (eight out of ten) promptly responded that both arrangements will cost the same.

2. The third sub-task of task-1, was about the conservation of perimeter, but the word "perimeter" was not here, instead students were asked about the length of the boundary in the two arrangements. All the students, after some probing, correctly concluded, either by computing or without it, that the second arrangement will have a slightly longer boundary.

   However, in task-6, most students tended to deduct the perimeter of the piece to be eliminated from the perimeter of the initial figure, incorrectly extending the rule for area to perimeter. This further supports the observation that students tend to fall back to the superfluous use of rules and formulas when formal words are used, probably more so when a meaningful material context is not present.

3. In task-2, students were asked to highlight the spatial aspects of area and perimeter. That is, students were asked to represent the area (i.e., shade or color the region measured by area) and perimeter (i.e., darkening the boundaries) of two given shapes respectively. Only 5 out of 10 students could shade the full interior region for the area, and only 7 out of 10 students could darken the boundary for the perimeter. Probably because students were not asked to calculate the numerical values for area, and they were not provided with any dimensions for the given figures, most students looked puzzled with the task, and asked for some number, as can be seen in the following excerpts of the interviews:

   [The language of communication for the interview was Hindi. 'I' and 'S' stand for the interviewer and the student respectively. Different instances of interview with different students are separated by horizontal lines]

Excerpt 1

I: Can you show me the area here with pencil or can you colour the area of the first two figures with pencil ?

S_1: Area means? *Number nahi hain!* [Area means? There are no numbers here!]

Excerpt 2

I: *Inn dono figure ke area dikha sakti hon?* [Can you show me the area of these two figures?]

S_2: *Ma'am length aur breadth toh pata nahi.* [Ma'am length and breadth are not known.]

I: *Inn dono ke area ko colour ker sakti hon?* [Can you colour the area of these two?]

S_2: *...length ka number nahi diya toh isme length hi nahi diya toh area kaha se hoga.* [length's number is not given so it has no length given, then how can there be area.]

Excerpt 3

I: *Iss figure ke area ko pencil se colour ker ke dikhao...dikha sakti hon?* [Show me the area of this figure by colouring with pencil? Can you show?]

S_3: *Ma'am nahi.* [No ma'am.]

I: *Iss figure ke kis portion ko area kehte hain pata hain?* [Do you know which portion of this figure is known as area?]

S_3: Ma'am length into breadth.

It can be seen from the above three excerpts that the students are expecting some numerical value for the dimensions of the given shapes. A few students could articulate their thinking and asked for numerical values, while some others looked stuck with the task, probably because they had not encountered such tasks on area and have only dealt with the numerical aspect of area.

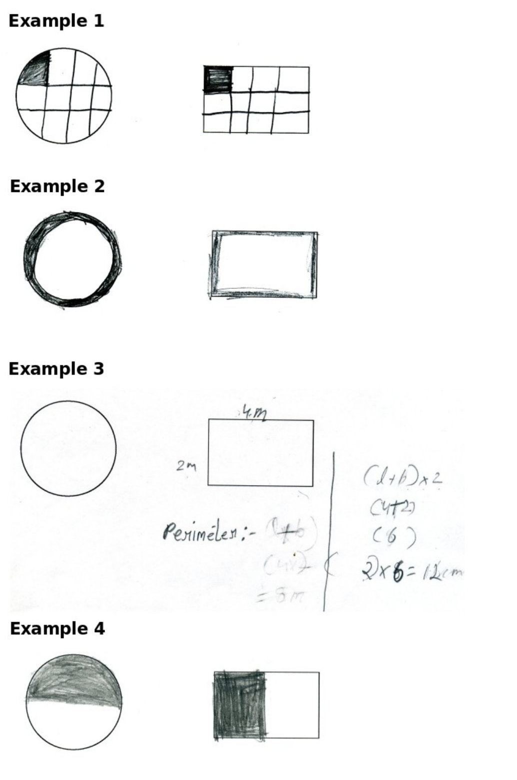


Figure 3.11: Examples of students' representation of area

Some of the examples of incorrect shading of the region considered for area are shown in

Figure 3.11. These errors could also be a result of the framing of the questions, which students could not relate to, based on their prior experience and understanding. Since textbooks talk of region for area, I could have asked students to shade the region considered for area than asking them to shade area, which for the students and the curriculum is a numerical measure and not the material attribute it is associated with.

4. For the L-shaped figure (non-conventional shape), only two students out of 10 could find the area correctly. Furthermore, two other students could correctly find the perimeter. The rest of the students expressed varied forms of numerical operations. For example, Figure 3.12 shows the improper use of formulas for finding the area (e.g., multiplying all the given dimensions). When asked for reasons, most students could not respond why they chose to multiply the dimensions for the area. Probably they just extended the area formula of multiplying the dimensions for a rectangle to multiplying all the given dimensions for an L-shaped figure. Likewise, they extended the formula for the perimeter to add all the given dimensions. Extending the perimeter formula still turns out to be correct, which is not the case for the area. Though both of these students could correctly determine the perimeter, they still had difficulty explaining the rationale behind it.

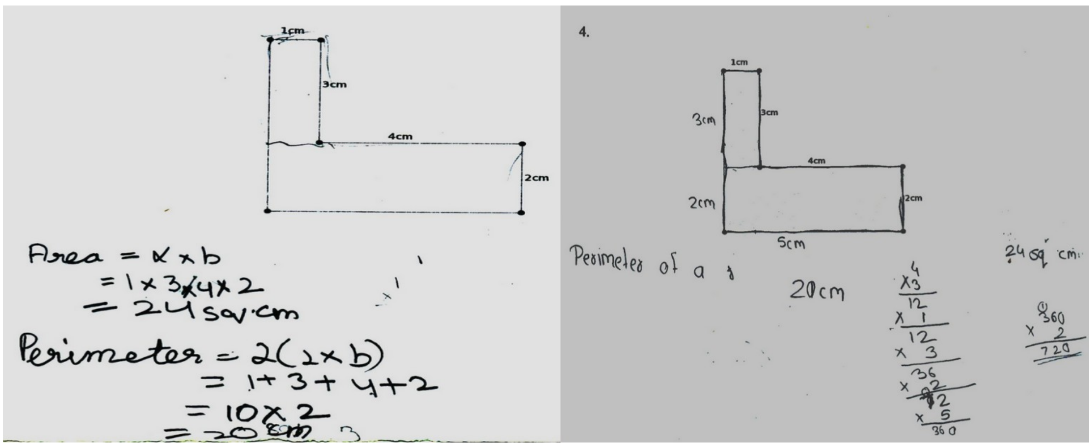


Figure 3.12: Student's computation of area and perimeter of an L-shaped figure

5. The last three tasks explored students’ understanding of unit-structuring (Battista, 2007) of area. That is, given a rectangular sheet or a rectangle with given dimensions and given units (rectangular or triangular), whether students can predict if such units can be used for covering the given rectangular space, and if yes, how many such units will be required to cover the same. Two kinds of strategies emerged among students (while doing task-7): Close to the formal procedural strategy, where four out of ten students divided the area of the rectangular sheet (calculated by multiplying the dimensions) with that of the area of the unit (rectangular

cards) to decide whether the unit can cover. Further, they calculated the number of units required to cover the given area. The rest of the six students did the other strategy which was a more concrete actionable form, of checking the tiles along the two dimensions of the rectangular sheet. Students using the former strategy were not able to make a correct judgment about the cases where the dimension of the unit was not a factor of the dimension of the rectangle (they missed to check if the dimensions of the unit divided the dimensions of the rectangle separately or not). The group of students using the second strategy could recognize the cases where the unit would not fit along any dimension of the rectangular sheet, and some of them also suggested that the unit needs to be divided into pieces to cover the remaining sheet. These responses suggest that students doing the action of tiling along the edges have an advantage in engaging and understanding unit-structuring than the group which used the formula. This also highlights that it is not enough to know the multiplicative relation between the area of the unit and the area of the rectangle. The multiplicative relation between dimension of the unit and dimension of the rectangular surface also needs to be understood. Similar observations were found for task-8 as well.

For the last task (task-9) on unit-structuring, where students were given the dimensions of a right-triangular tile, every student first said that such a tile cannot completely cover a given rectangular space. Two students changed their response later, realizing that two such triangles can be joined to obtain a rectangular unit. This is either because students expected the measurement unit to be of the same shape as that of the shape to be measured (Heraud, 1987, as cited in Zacharos, 2006) or probably because they have not used triangular units much in the formal curriculum.

Overall, these tasks indicate some gaps in students' understanding of unit-structuring. Broadly, this gap is due to lack of engagement with the relation between the unit and the measure. More specifically, it is due to lack of exposure in terms of different multiplicative and geometric structures within the unit-structuring. Thus, we could infer that conceptual understanding of area-measurement requires strengthening of the unit-structuring through different fractional units (Battista, 2007), to provide a variety of scaffolding for students to abstract and connect different multiplication structures with geometrical notions. This study provided the ground for our next study, where we explored ways to facilitate and further explore the connection between multiplicative thinking and area-measurement.

3.2.4.4 Conclusion

The findings suggest that there is a disconnect between students' formal understanding of measurement concepts and the way area is widely understood. There is thus a need to move beyond formal learning, to connect area with other meaningful contexts. The term “area” mostly restricts students to think of area only in terms of what they have learned in school, and limits them from seeing area in other shapes. Instead of using the term “area” maybe we should use some other more accessible term with students. The above findings suggest that most students lack the geometrical or spatial understanding of area and perimeter (2-D measurements). The studies indicate a lack of connection between the geometrical, spatial, numerical and even algebraic (area formula) aspects in students’ understanding of area measurement. Despite the immense potential for application of measurement in mathematics and in everyday life, students continue to have difficulty in meaningful learning of the measurement concepts. More research is needed to develop an exhaustive list of students' difficulties in area, and the effects and effectiveness of alternative instructional approaches. This may require moving beyond the child psychology perspective, to think in broader terms that includes the curriculum and the pedagogy. A careful analysis of students’ responses for the last unit-structuring (or tiling) tasks, indicates a missing link between the number of units (or tiles) that could cover a rectangle and its relation with the dimensions of the tile and the rectangle. There’s a need to rethink the learning progression for understanding area as covering a space with units, to include the aspect of multiplicative thinking between the unit and the measure.

3.2.5 Exploring the connection between Multiplicative thinking and area-measurement

Since the last study suggests the need for further exploration on the connection between multiplicative thinking and the measurement of area, in this study I tried developing tasks that could elicit such forms of thinking, by including more action based and construction based practices of unit structuring. Task-based interviews of a different set of students, from the same cohort, were done and video recorded after getting consent from students and their parents. Interviews were done either in school or in the research institute. Video recordings of the interviews were used for the analysis. As the name of the section suggests, the research question explored here was:

RQ: What is the connection between area-measurement and multiplicative thinking?

As elaborated in the literature review chapter, multiplicative thinking is a well researched area in mathematics education, and has application in a broad range of topics. However, in the present section of the study, I explored different ways in which multiplicative thinking is involved in the geometric measurement of area. Specifically, the focus is on developing tasks that facilitate and elicit the use of multiplicative thinking in finding the area of geometric figures. I report the tasks that were designed and developed for this purpose, and explore the connections between numerical and geometrical aspects of area-measurement using multiplicative thinking.

3.2.5.1 Sample

The sample consisted of a convenient sampling of students studying in Grade 5 from two different schools. Ten students came from a school serving mostly middle income families (from the same school as mentioned in the earlier studies), and nine students come from another school serving mostly low to middle income families (from a different school than the ones mentioned earlier). The tasks used were not identical across all these students, as they were progressively adapted in the course of the study. However, the approaches taken by the students to do the tasks were broadly comparable. The final set of identical tasks was given to eight students, which constitutes the final sample set of the study, where four students were picked from each of the two schools respectively.

3.2.5.2 Tasks

Unlike the tasks used in the previous studies, which explored students' existing knowledge about a particular concept, the tasks used in the present study explored how students engaged with the task and what they drew out of the tasks. As these tasks also follow a progression, I also tried to explore how one task interferes with or facilitates the next task. The use of actual physical or concrete materials provided students the flexibility to work with the given objects in the way they wanted. There were four tasks, which are explained below along with the materials shown in Figure 3.13.

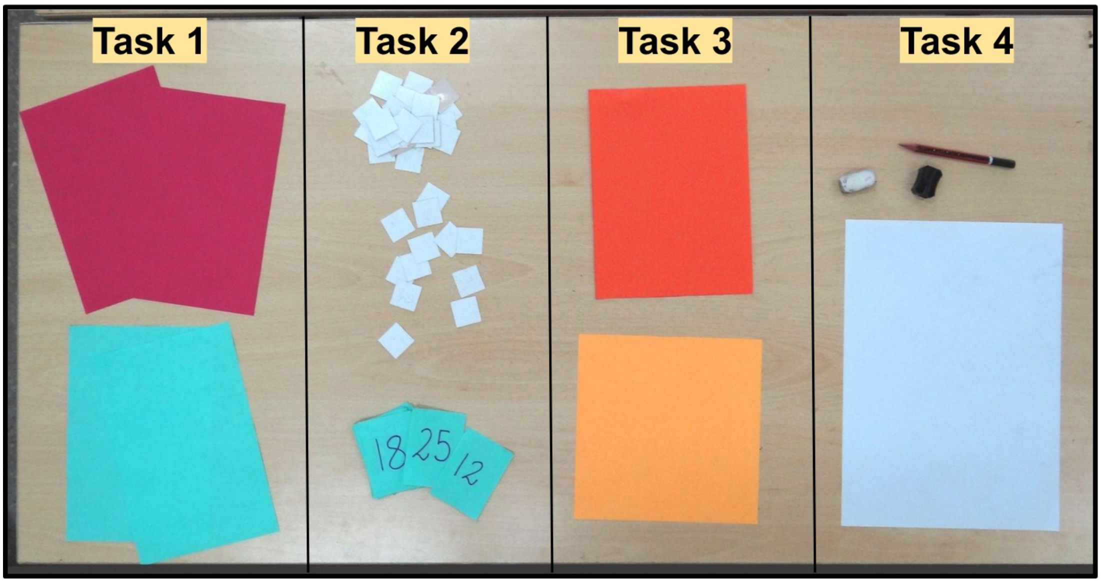


Figure 3.13: Materials used for the tasks

- **Task 1. Comparison Task:** This task required students to compare two given pairs of rectangular sheets with very small differences, either in length or breadth but not both (see leftmost in Figure 3.13). The aim of this task was to prime students with rectangular sheets and to explore whether the students compare area by overlap or by comparing only attributes like length or breadth.

- **Task 2. Card Task:** This task required students to construct a rectangle with a given number of unit square cards (1inch×1inch). In this task students were first shown a number and were then allowed to take those many square cards from a box to make a rectangle (see second left in Figure 3.13). This task provides the possibility to connect the number of cards and the resulting rectangular array, and also seeks to exploit the flexibility of physical manipulation. This task implicitly requires one to notice the multiplicative relation between the given number and its factors along the length and breadth. Since the task requires overt action on the part of students, it allows one to see whether students are implicitly attending to the multiplicative relation involved, even if they do not overtly express this relation. For the card task they were shown a specific number written on small sheets. In the first set of trials, students were shown one of the composite numbers such as: 10, 12, 14, 15, 16, etc. They were then asked to take those many cards from a given collection of cards to make a rectangle. In the next set of trials, the students were given a composite number and were asked to respond verbally about the rectangle that could be made from the cards. Finally students were randomly shown either a prime (e.g., 11, 13, 17, etc.) or a composite number, and then were asked whether they could make a rectangle with the given number and how many cards would be there along its length and breadth.
- **Task 3. Measuring Task:** This task required students to compare two sheets to decide which is larger (see Figure 3.13) – a square sheet (7 inch × 7 inch) and a rectangular sheet (8 inch × 6 inch). The difference in area between these sheets is small, and cannot be determined directly by overlap. Students were also given a small square card (1 inch × 1 inch) and asked to use it if they needed to. After the card task, the measuring task allowed us to explore the various strategies (e.g. array structuring, complete covering, multiplication, etc.) used by students while measuring the sheets. Further, this task allowed us to look into whether the students apply the ideas abstracted from the previous tasks, i.e., whether they use the multiplicative relation or repetitive addition to get the measure of the two areas.
- **Task 4. Unit of units Task:** For this task students needed to get the measure of a given A4-sheet, and then get the measure of a table in terms of the previous square unit (see the rightmost in Figure 3.13). Students were given an A4-sheet and they were free to use the materials used in the previous task. This task was developed to explore whether students could extend their understanding of area-measurement to bigger shapes. Further, this task may create the need to optimize the number of operations, and thus use the nested multiplicative relation.

3.2.5.3 Findings

I report findings from a detailed analysis of the responses of eight students, who were presented with stable versions of the tasks. A few instances of the earlier additional interviews (of the initial set of 20 students) will also be presented, to give a picture of some specific strategies students used during the study.

1. In the first comparison task, there were only minute differences in either length or breadth between a pair of sheets, and these could not be determined by merely looking at them. Among the eight students, all except one tended to compare the rectangular sheets either by length or breadth, when the sheets were placed flat on the table next to each other. However, later they overlapped the sheets to compare them. This suggests a natural tendency to compare the sides of rectangles when students were asked to compare the sizes of two rectangular sheets, and this indicates an implicit understanding of the relation between the sides and size.

2. For the second (card) task, four students (after a few trials) understood the connection between the factors of a given number and the resulting rectangular shape. Of which, one student could explicitly say that he is looking at the factors of the number for making the rectangle, while another three students showed the use of multiplication tables for the card task. For the remaining four students, this connection was either implicit or unstable. It appeared that in some of the trials, they were implicitly using the multiplicative relation between the number of cards and the resulting arrangement. For example, in several instances students created the first row of cards using a number that was a factor of the given number. However, they were not able to explain why they chose that number. The connection was unstable for some students, who were not consistent with their strategy. For instance, one student made 4 × 3 and 6 × 2 rectangles with 12 cards and said 15 can be made into a rectangle as "3-5 za[8] 15" (i.e., 3 times 5 is 15). Later, when asked about the sides of the rectangle that can be made with 10 and 13 cards respectively, the same student said 3, 7 and 3, 10. Another student who said "7-4 za 28" for the number 28, also said 8 squares in length and 6 squares in breadth for the same number and wrote 8+6=14 and 14 × 6 = 64 on paper. This showed that students shift between the additive and multiplicative relations between numbers while doing this task.

   These last four students, although not expressing their idea or thought process, indicated the implicit use of the relation between the given number and its factors through their actions.

8 "3, 5 za 15" is a corrupted English form of "three fives are fifteen". Most Indian school students and teachers recite the multiplication tables in this way, without often being aware of the original expression.

These students started with a factor of the given number, but were not able to say why they started with that number of cards. In fact, one student said the number came to her by own without thinking. In another instance, the student said squares can be made with numbers which will come double e.g. 6-6 za, 1-1 za, 10-10 za, 3-3 za. This showed that the student had an idea of how a number is related to a square arrangement. Students with more awareness of the multiplication facts did get the measurement-multiplication connection sooner than those without, or those relying more on addition facts rather than multiplicative ones. We infer this because all the four students who decomposed the number into factors were fluent with the multiplication tables, while the other four were unsure about the multiplication tables.

In some instances, when the students were not able to find the factors for a number, they tried to arrange the cards along the perimeter of a rectangle leaving a gap in the centre. In such cases, students were asked to make a complete rectangle that is fully covered. However this move indicates that the task does not necessarily constrain students from making a figure by filling a rectangular space with cards. The perimeter arrangement is interesting because students then decompose a given number using both additive and multiplicative relations, as seen earlier in the instance where a student decomposed 28 as $2 \times (8+6)$. This also indicates that a rectangle is imagined in two ways, one as an array (or filled space) and the other as a border (with empty space in the middle). An interesting question is how these two ways of conceptualising the geometric figure influences the learning of the area concept.

The card task also allowed some students to explore rectangles with fractional lengths. Two of the eight students cut the cards into half to get rectangles: one student made a 7.5×2 rectangle from 15 cards by cutting one card into half. Another student suggested a $5\frac{1}{4} \times 4$ rectangle with 21 cards and made a $6\frac{1}{2} \times 2$ rectangle with 13 cards.

3. For the measurement task, one student compared the extra space that was left on both the square and the rectangle once they were overlapped or placed one above the other, and saw that the width of the space left was one unit in each case. So the student said both the sheets have equal space. But the student missed the fact that the extra space of the square can hold 7 cards, but the extra space in the rectangle can hold only 6 cards.

   All the eight students, when asked to find the number of cards that can be made out of the rectangular sheet, marked the adjacent sides of the rectangle using the given square card. Only three students multiplied the number of cards that can fit along the adjacent sides of the rectangle, to get the total number of cards. The other five did repetitive addition to get the total number of cards. This was found even with the students interviewed earlier. Thus it appears that the most common method was to add the number of cards in one row repeatedly as they counted the card marks along the adjacent side.

4. Six students were able to do the unit of units task, but in this case also three used the multiplicative relation while the other three used the repetitive addition relation. For example, for illustration purpose, if we assume that a student could find that an an A4-sheet can have 100 cards, and a table can be filled with 10 such A4-sheets, then the number of cards for the table was arrived at by adding 100 ten times rather than multiplying 10 with 100 to get ten 100 cards.

   Some students initially tended to find out the number of times a rectangle placed lengthwise covers the length and breadth of the table respectively. They orient the rectangle lengthwise even when they place the rectangle along the breadth of the table. The students then multiply the numbers they get, to obtain a wrong result for the number of rectangles that can cover the table. But the students were not consistent in this strategy, and changed their strategy when asked to explain how they got the total number of (the given square) cards in the table.

   It is worth noting here that for the unit of units task, the unit was not a standard square unit, but some multiplicative (or chunked) unit. In such instances it is not enough to see the multiplicative relation in measurement. One also needs to see the geometrical division of the measure in terms of this new multiplicative unit. In other words, the students need to coordinate both the numerical and geometric aspects to perform this task. One way to interpret the above strategy (of measuring lengthwise) is to consider the students as having an implicit understanding of the need for coordinating the two aspects, but not understanding the nature of the array structure. Perhaps the numerical aspect dominates, and the measurement is done to get values for multiplication, while a proper understanding requires keeping the multiplication and geometric structure in mind simultaneously.

3.2.5.4 Discussion

The tasks used in the present study provide students with the possibility of directly connecting the measurement unit with the number. The five important insights from the present study are:

1. Students were inclined to focus on the sides of the rectangle rather than space (or area) covered by it in the comparison task. Even for the card task, some students often missed to fill the inside of the rectangle, and placed the cards either along the length, breadth or the boundary.

2. Students often used the additive relation between the numbers in the card task, rather than the multiplicative one, while splitting up the given number for constructing a rectangle.

3. Even when students use the connection between multiplication and array structure in the card task, this strategy is not stable.

4. From the method perspective, the tasks used in the present study gives students flexibility to explore various structures and their connections to numbers, and allows us to explore students' understanding about multiplicative (chunked) units, even when they are unable to articulate their understanding.

   Students have an implicit understanding of the link between numerical properties and area but they are unable to express this understanding. The tasks used in the present study allows children to manipulate many structures, giving us insights into their implicit thinking.

   The tasks thus allowed students to manipulate and explore many spatial structures and their connections with numbers and different sub-concepts like unit, array, multiplication. Considering the wide presence of multiplicative thinking in other topics of mathematics education, we tried connecting it with area-measurement. The present study not only attempts to establish this connection but also opens new ways of looking into the problems of understanding of area-measurement.

   Thus, the card task requires students to decompose a given number into a pair of factors that represent arrays (or chunks) of units in rows or columns. This model or representation connects multiplication facts with unit structuring of a rectangular area and is inspired by studies connecting measurement with multiplication (Izsák, 2005; Izsák & Beckmann, 2019). The task allowed students to predict different possible multiplication facts for a given number and see different possible resulting rectangles and the respective dimensions as nothing but the numbers that appear as factors. Additionally, keeping the same area enables students to observe that increasing the number by a factor along one dimension decreases the number along the other dimension by the same factor. This further highlights the connection between area conservation and the "conservation" of the multiplication of the dimensions, even when the dimensions vary. The task, however, in its current form, could not capture this completely, and needs to be extended further to guide students to see the variability and stability involved in AM and explore the multiplicative reasoning involved in it (Kobiela & Lehrer, 2019). Thus, the task in its current form is not sufficient for covering multiplicative thinking. However, it can be seen as an initial building block for multiplicative thinking.

   This perspective enriched our task design, from developing purely informative tasks to track students' understanding of a particular concept, to more constructive tasks that helped connect existing concepts with other mathematical areas. The studies also allowed us to look into students' difficulties with the area concept in detail, as their moves were explicit, particularly the problems they faced while integrating the area with other mathematical concepts and sub-concepts. Based on this detailed understanding, I propose in the next section a new model for learning the concept of area.

3.2.6 A Network Model of the Mathematical Concept of Area

As indicated in the methodology section of this chapter, we[9] propose a learning model for the concept of area, by consolidating all the studies reported in this chapter. Based on the insights drawn from the earlier studies, as well as the literature review in the previous chapter, we imagine and develop a new learning model for the concept of area. Based on this model, we argue that the concept of area can act as an entry point in exploring the deep connection between geometric, multiplicative and even algebraic structures.

Learning about area requires integrating the spatial, numerical and algebraic aspects of the area concept, which indicates that the learning pathway for the concept can be thought of as a network. We thus present, and argue for, a "network model" of area learning. The next section outlines the model, following the two didactical questions below.

DQ 1: What is the network model of learning area-measurement?

DQ 2: Why does a network model of learning the concept of area need to be adopted?

The notion of area as an array of units is central to understanding the area concept. Previous works show that array representations can enhance: 1) learning the spatial structuring of units in the area concept, 2) the understanding of the two-dimensional nature of the multiplicative process, and 3) abstracting it into a general algebraic form (or formula). Integrating these multiple (spatial, numerical and, algebraic) roles played by arrays, we structure the area concept as a network, which requires the student to coagulate spatial, numerical and algebraic concepts. Based on this theoretical model of a network concept, the set of tasks in the previous study can be seen as a spiral of physical manipulations, which allow students to experience and tie together different aspects (i.e., the spatial, numerical, and algebraic aspects) of the area concept in the network.

3.2.6.1 Area: A Network Model

We view the area concept as a network concept, requiring the coming together, or coagulation, of four ideas – unit, array, multiplication, and unit of units. In this view, area could be seen as an array of arrays. We introduce below a broader concept related to this, which we term "extrapolation". When the area concept is introduced, students usually do not have a fully developed understanding of the individual ideas that constitute the network. Also, these concepts could be: partially or implicitly known, partially stable, understood in an intuitive or qualitative fashion, sometimes connected to each other in unstable ways, and sometimes may not be known. Thus, our first objective is to uncover the different elements (ideas/ concepts/ operations) associated with the concept of area and make the

9 The network model discussed in this section and the following sections was developed as a joint work with my advisors to consolidate the results of the studies in this chapter. Hence the plural "we" is used from here.

connection explicit, in order to strengthen them in the pedagogy.

In Figure 3.14, we present a concept map of the network notion of area. The elements (Array, Unit, Multiplication) on the right indicate the standard concepts/operations associated with area. The elements (Conservation, Partition/Fraction) indicate intuitive concepts used to facilitate the understanding of area. The *Extrapolation* element indicates a concept/operation exemplified in the "Unit of unit task" in the previous study, which we will elaborate further in the next section. The "Card task" in the previous study led to the element *Build-Up*.

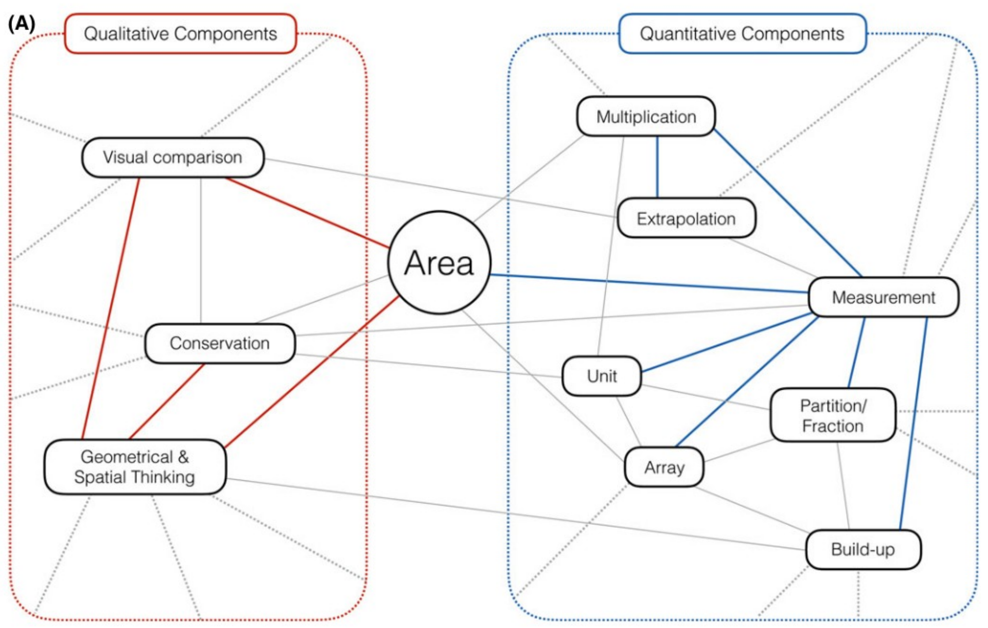


Figure 3.14: A network representation of the area concept

The network in Figure 3.14 roughly represents the cognitive model of these connections within our mind. The red and blue lines indicate the connections (connecting links) with the qualitative and the quantitative components of area respectively. The darker lines represent the explicitly seen connections with area, while the dotted lines represent other implicit connections that may become available in appropriate contexts. Some components are not linked with area, because their connections are not directly evident, but become apparent with the focused/connected context or the task. For e.g., the connection of conservation and extrapolation to area is not directly evident, but emerges in the context of suitably designed tasks.

3.2.6.2 Extrapolation: A new construct

In relation to the network notion of area, we introduce a new construct/operation – extrapolation, which we consider a key part of understanding the area concept. Extrapolation refers to the use of the multiplication algorithm to calculate the area of rectangular spaces that are very large (or very small, folded, etc.), where physical arraying is difficult to achieve, but need to be imagined. This operation, we believe, is one of the key objectives of learning to calculate area, where the action of iteration is replaced by (or embedded within) a structured multiplication algorithm. Further, the ability to do this operation is an indication that the learner has consolidated her understanding of the connection between area, array and multiplication. We would like to note here how this view is different from the standard way of testing the area concept, which usually involves calculating the area of complex figures made up of standard shapes, such as an L-shaped figure made of two rectangles. This task is a variant of the partitioning task, requiring the learner to imagine partitioning the given complex figure, and consider it as being made up of some shapes whose areas can be easily calculated, and then applying the multiplication algorithm to each partition (given some values for the sides), and then adding the results. The extrapolation test, on the other hand, requires the learner to imagine how the area of a large space (such as the floor of a room) could be measured using a known shape (such as a square). The multiplication algorithm is then applied twice, first to the known unit, and then to the larger unit measured by it. This operation requires a deeper understanding of the array structure, where any space is seen as an array of units, and any given unit can be used to build up an array. Further, it requires understanding the relation between multiplication and arrays, as the unit is used to measure only the borders of the larger space. It also requires a deeper understanding of the relation between multiplication, array, and measurement, where the multiplication operation used to calculate the area of the given unit is extrapolated to a wider space through the use of the given unit as a measurement unit.

Ideally, learners who understand the area concept should be able to move quickly to this operation, which provides a good test of whether learners have internalised the whole area concept network, in an integrated way. We believe that the extrapolation operation should thus be one of the key objectives of learning the area concept, and this operation needs to be supported by tasks designed to teach the area concept. The previous study shows how the above network of related concepts involved in area and multiplicative thinking could be checked, and also built up, through a series of tasks based on physical manipulations.

3.2.6.3 Reflections on the previous studies, based on the Network Model

We revisited the previous studies, using the lens of the network model. This gave us better clarity in terms of the rationale and design of the tasks in the study. The four sets of tasks used in the previous study: Comparison Task, Card Task, Measuring Task, and Unit of unit task can now be understood as Qualitative comparison task, Build-up task, Quantitative comparison task and Extrapolation task respectively. We redefined the Comparison Task as a Qualitative Comparison Task because the task requires the student to look for the qualitative differences between the two rectangular sheets, rather than finding a quantitative measure of it. Thus the first task emphasises the spatial or geometrical aspect of area measurement, rather than the numerical aspect. The second task can be seen as a Build-up task, as it seeks to build up a geometrical figure from a set of units (pieces), as opposed to the partitioning task that expects students to unitise (i.e., produce a covering of units of) a given shape (Outhred & Mitchelmore, 2000). The flexibility provided by physical manipulation potentially allowed students to develop rectangular array structures with various dimensions, and thus understand the relationship between a given number, the resulting rectangle and the multiplicative relationship between the given number and the dimensions. Since the task requires overt action on the part of students, it also allowed us to infer their understanding of the multiplicative relation from their actions.

After a few trials, for the second phase of the Build-up task, students were given a number, and they had to verbally respond about the possible rectangle, without physically arranging the cards. They also had to give the number of cards along the length and breadth of the rectangle. Starting with simple composite numbers, students were later shown composite or prime numbers in random order. The aim of this variation was to let students explore the relation between different types of numbers, and whether they could or could not be decomposed into factors yielding the array structure. We also asked them about the number of cards along the length and breadth in these cases. This task could prime the multiplicative relation between the given number (of cards), and the way its factors correspond to the length and breadth of the rectangle.

The third task, the measure task, was again a comparison task like the first one. But this time it was resourced with unit cards, which provides the students with a physical tool to even do a quantitative comparison. The aim of the task was to see whether the previous build-up task (array structuring) helped students in understanding the relation between area and arrays, to the point where they could think of comparison in a numerical/quantitative fashion. The task also allowed us to see whether the students used the multiplicative or additive relation to get the measure of the two areas.

The Unit of units or the Extrapolation task sought to explore whether students could extrapolate their understanding of area-measurement to bigger rectangles, and use an efficient unit for measuring. This task also creates the need to optimize the number of operations, as it is difficult to measure the table using the small square unit. This means the students have to think of the nested multiplicative relation.

The most interesting pattern emerging from the results of these tasks are the various levels of stability in learners' ideas (as in Task 2 in Section 3.2.5.3), as some students were able to arrange those many number of cards that is a factor of the total number of cards, even though they were not able to articulate how they thought of that number along a dimension. This is further brought about when instead of physical cards, students were just shown a number and were asked to guess the number of cards along one dimension if they have to be arranged in a rectangle. Another pattern that emerged here are the inconsistencies in the application of additive and multiplicative thinking to the tasks (see Task 2, 3 and 4 in Section 3.2.5.3). For example, occasionally students switch between additive and multiplicative division of the given number in Task 2. In Task 3 and 4 instead of seeing the multiplicative relation in the resulting array structure or the nested array structure, they carried out the tedious exercise of physically counting all the units or qualitatively comparing as in Task 1 without seeing the space covering quantitatively. This pattern of unstable and partial connection among students is better accounted for by the network model of the area concept (where understanding the concept of area requires a coagulation of the multiple aspects involved), than accounts that treat the understanding of the area concept as involving a (linear) shift from a qualitative to a quantitative notion of space (Piaget et al., 1960; Battista, 2007).

The sequencing of the tasks in a spiral fashion did not entirely achieve the objective of interconnecting different aspects of the area concept, but it helped in revealing some of the issues involved in coagulating the individual concepts involved in the understanding of area. Table 3.7 presents a contrast between network & trajectory model of learning (Sarama & Clements, 2009; Piaget et al., 1960; Battista, 2007), where the trajectory model is the one proposed in the most recent curricular work on area-measurement.

Table 3.7: The table of contrast between network & trajectory models

Network Model	Trajectory Model
Spiral structure	Linear structure
Different concepts may be partially known	Partial knowledge not supported (considered as error or misconception)
Concepts are stable to different degrees	Assume sable concepts
Interconnections could be unstable	Integration of concepts not addressed

Further, from a methodological perspective, the physical and manipulable nature of the tasks used above provided us with a *process understanding* of students' thinking and learning involved in the area concept, particularly the use of multiplication in creating the array structure. The conceptual elements like qualitative comparison, array structuring, etc., were exhibited by students at an intuitive level through physical manipulation, and this also revealed the weak connection with quantitative or numerical thinking. A network model of learning accounts for this partially stable mode of learning better than the linear model which proposes smooth transition from qualitative to quantitative understanding, as seen in the conventional curriculum.

In conclusion, learning the concept of area as an array of units requires "breaking" a given figure into units (partitioning), counting the units, and understanding the notion of conservation – i.e., fixed number of units re-arranged in any spatial pattern would have the same area. At the next level of complexity, area requires understanding these physical/ spatial operations as numerical operation of fractions and multiplication, as both these concepts are involved in calculating area (Outhred and Mitchelmore, 2005). In the other direction, these operations also get strengthened conceptually by an in-depth understanding of the area concept (Ball, Lubienski & Mewborn, 2001). At the third level, a rudimentary notion of algebra is required to calculate area, say of a rectangle using the Length × Breadth ($l \times b$) formula and generalising it for all such rectangles. Thus, learning the concept of area requires bringing together, in an integrated fashion, spatial, numerical, and some rudimentary algebraic ideas, as well as shifting between them.

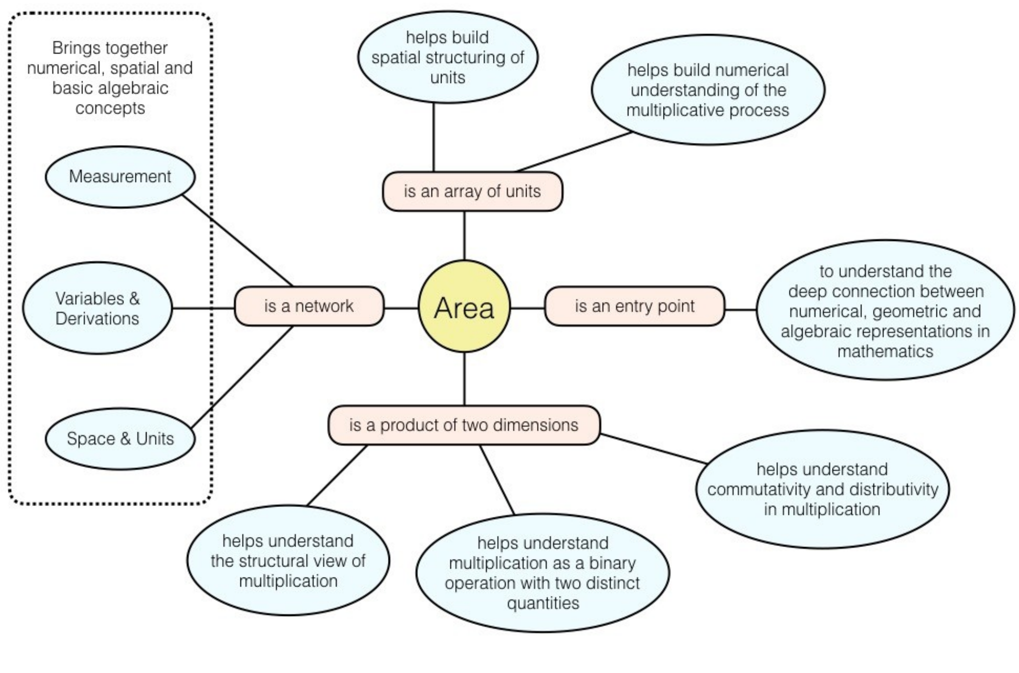


Figure 3.15: A concept map of how area is related to other topics

Figure 3.15 reveals the connection of AM with other topics of math. Here, the study reveals that the area is connected to four different components (i) as an array of units (uncovered mainly during covering or tiling tasks), (ii) as an entry point for the connections between numeric, geometric, and algebraic representations, (iii) as a product of two dimensions (whereby it opens up many multiplicative relations among quantities), and (iv) as a network.

Area can thus be thought of as a network concept that integrates all these elemental concepts. The area network is very rich and complex, and establishing the interconnections between the disparate concepts involved is a central difficulty in learning about area. Our studies show that the area concept, and its connection to other concepts, tend to be patchy and unstable. We also found that students tended to follow algorithmic approaches, such as applying the formula $l \times b$, and did not understand the spatial relation between unit and area. Interview sessions with students also showed that students have difficulty interpreting $l \times b$ abstraction or understanding its connection with the counting of units. The students also found the distinction between perimeter and area difficult. The network relation between these constructs (specific attributes, connection between unit and area, the perimeter/area distinction, and the integration of conceptual elements) is the key feature that needs to be strengthened, in an integrated fashion, by interventions that seek to teach the concept of area.

The studies discussed in this chapter allowed us to consider a range of tasks to identify the multitude of aspects connected as a network in the understanding of AM. The tasks and studies in this chapter were inspired by and evolved from Piagetian studies. The coordination classes and the knowledge in pieces perspective proposed in A. diSessa's work have also evolved from Piagetian studies and they bear close resemblance to the network model (Izsak, 2005). In recent work, AM tasks have been extended from the discrete tiling or counting tasks to integrate the continuous nature of area with the $l \times b$ abstraction. The dynamic sweeping (or painting) task allows students to visualise generating area as the product of length measure, by seeing the multiplicative change in area with the multiplicative change in lengths along the dimensions (Kobiela & Lehrer, 2019, Brady & Lehrer, 2021, Panorkou, 2020, 2021). The distinct nature of the dynamic sweeping task is promising in highlighting the idea of continuity of area. This aspect seemed to be missing among students as also reported in the thesis when students were asked to shade the area of a given shape, they partially shade only the boundary or just a section of the given shape (Figure 3.11, p.74).

Such dynamic tasks are of importance in visualizing or experiencing the rate of generation of area and its relation with the $l \times b$ abstraction for rectangular spaces. However, one challenge noticed in connecting area with the lengths of the sides was that the students miss noticing the unit of area. This was observed during some tasks reported in the thesis (for e.g., where students tend to compare the rectangular sheets either along the length or breadth rather than focusing on the area – point 1 on page 80 of the thesis, or when they use a rectangular unit to measure a larger rectangular area, holding the rectangle lengthwise while measuring both length and breadth, point 4 on page 82 of the thesis.). Thus, to avoid simplified generalisation about the relation between area and sides and also to move beyond rectangular shapes to triangles or, parallelogram, the need of grid and dissection arises. In some studies, these were clearly presented through dynamic media (Kobiela & Lehrer, 2019, Brady & Lehrer, 2021, Panorkou, 2020). However, considering the unavailability of such digital platforms in the Indian classroom, perhaps such richer tasks can be contextualised or localised using concrete materials e.g., graph paper, so as to allow all students to experience the different aspects of area-measurement.

4

Study 2: Constructing the concept of area measurement in a classroom

The previous chapter investigated what students already know about area-measurement (AM) and went further to investigate students' individual construction of AM in a task set-up. However, in this chapter, we move beyond individual construction to social construction happening within a classroom context. The present chapter consists of a research study to model a classroom pedagogy on AM (or a teaching-learning process on AM) guided by insights from the previous chapter and the theories of social construction. The two main processes broadly incorporated in the design of this AM lesson are: 2a) argumentation, and 2b) integrating spatial and numerical representation in the classroom. While the second process (2b) was identified as a major challenge in the learning of AM, the first process (2a) is the most practical or visible expression of social constructivism expected in a mathematics classroom. In order to adopt the social constructivist approach in the classroom, argumentation was used both as a pedagogical practice and also as an analytical framework to analyse the episodes of argumentation in the classroom, which were taken to be the sites of social construction.

This chapter will discuss the motivation for the study, the nature of the intervention, the context or setting, and an analysis of episodes of argumentation among students while engaging in social construction of AM. The episodes of argumentation mainly emerged when students were proposing their varied and contending solutions for a given problem. As mentioned earlier, the analysis will elaborate on the process of social construction of the area concept in the classroom by focusing on two major aspects of the classroom interaction. The first aspect is the nature and structure of students' argumentation, how they defend their claim and what "warrants" they use. The second aspect of the analysis explores various facets of the students' conceptual understanding of area and the tension they face as they move between spatial and numerical representations.

I will elaborate on the motivation and theoretical framework of the study in the introduction section (4.1), followed by the conceptual and analytical framework of the present study, that is, what constructs we used to analyse our data in Section 4.2. Further, in the next section (4.3), I will talk about the design of the study including the design of the activities and tasks used for the teaching design experiment in the classroom. Then I will move on to the analysis and discuss the classroom data or the result of the study in Section 4.4. And finally I will conclude the chapter in Section 4.5 with the insights and reflections drawn from the present study.

4.1 Introduction

The previous chapter focused on student's conceptions of AM in an individual setting; however this one specifically focuses on the role played by classroom interaction in shaping students' conceptions. That is, most of the studies mentioned in the previous chapter were heavily influenced by the Piagetian ways of investigation, where the learner is considered to be an individual constructing knowledge (or scheme) on their own, independent of their social setting. However, the influence of society in shaping a child was brought to notice by the Vygotskian tradition (Saxe, 2015). Thus, in the classroom context, it is important to consider social construction rather than individual construction, as the student engages in the co-construction of knowledge through social interaction and negotiation (Ernest, 1998; Restivo, 2017).

Research in Mathematics Education has shifted its focus from looking at an individual learner to the social process of learning as a product of social interactions (Voigt, 1994; Ernest, 1998; Restivo, 2017). Researchers have emphasised the role of social constructivist and interactionist approaches in the learning of mathematics (Cobb & Bauersfeld, 1995; Ernest, 1998; Restivo, 2017). However, this trend is not reflected in studies focusing on particular conceptual developments. For example, in studies of the development of the "concept of area", the major focus is on reporting individual students' conception or construction of AM rather than what can be conceived or constructed collectively. This may be because most of these studies were inspired by the Piagetian tradition that underplayed the role of social interaction in contrast to Vygotskian tradition (refer to section 2.1.3.4 of the literature review chapter). Thus, regarding the "area-concept", there is a need to move beyond individual student's conception, to the construction of the concept in a classroom setting guided by a teaching design experiment. In school, students come across "area" only as a formula or as a part of some symbolic manipulation in textbook problems. However, the learners need to move away from rote symbol manipulation to collective argumentation in order to meaningfully engage with a particular mathematical concept (Forman, Larreamendy-Joerns, Stein, & Brown, 1998; Krummheuer, 2007).

Argumentation is considered more effective than proofs in convincing students of the validity of mathematical results (Carrascal, 2015). In the context of proving, Reid & Knipping (2010), have extensively used Toulmin's argumentation structure (refer Figure 1) in analysing students' interactions in a classroom, but mainly in the context of proving. In their review, they have indicated the need for more empirical work in the field of argumentation together with sound theoretical groundwork. They argue for the need to study the "global argument" structure (i.e., a gross structure having a complex network or web of several such structures joined together in one whole) emerging within a classroom, which is taken to be more insightful than the simplistic structure (generally either a single or a linear chain of argumentation structure) reported in earlier research (Reid & Knipping, 2010, p.191). Lately, in the context of mathematical proof, their work highlights the importance of structural aspects (gross structures) of argumentation emerging within the classroom unlike the functional (Krummheuer, 2007), finer aspects (Toulmin, 2003) of an argumentation structure. However, considering the potential of Toulmin's argumentation structure, we need to move beyond the context of proving and extend such intensive analysis to the context of mathematical definitions and notations, and concept formation. Again, by focusing on the structural aspect of argumentation structure emerging within a classroom as a whole, by clubbing or merging different argumentation structure into one single gross structure (Krummheuer, 2007; Reid & Knipping, 2010), we tend to overlook the contending views held by each actor or a group of actors in a brief dialogue or an exchange within a classroom, holding different argumentation structures. Thus, in the present study, I have used Toulmin's argumentation structure, not just to extract the structural aspect but to also analyse its role in the co-construction of the "area-concept" in the classroom. Again, since constructing the "concept of area" using collective argumentation will bring out a complex process of negotiation involving socio-mathematical norms within the classroom (Voigt, 1994; Yackel and Cobb, 1996), I have also looked into some other aspects influencing the argumentation in the classroom for e.g., the act of convincing each other, or resolving the conflict in one's warrant.

Toulmin's argument structure has been modeled and re-modeled by several researchers according to their requirement (Krummheuer, 2007; Reid & Knipping, 2010), I have used an adapted and simplified version of Toulmin's structure of argumentation to analyse the selected episodes from the classroom (Toulmin, 2003; Toulmin, Rieke, & Janik, 1979), details of which are elaborated in the next section.

4.2 Conceptual and Analytical Framework

Toulmin's structure of argumentation consists of three main components (see Figure 4.1): one, claim whose truth is to be established, two, ground consisting of a set of facts which provides the foundation for the claim and three, warrant, which provides the basis to arrive at the claim from the ground. The credential of the warrant comes from the backing. Backing is usually field (or discipline or topic) dependent and likewise warrant also varies with different fields of argumentation. The basic structure (or framework) of Toulmin's argumentation layout is shown in Figure 4.1.

In the present study, I have identified an event and segment in the classroom as an episode when I felt the presence of the two main components of the above framework, i.e., claim and warrant. In the context of the classroom, I have recognized an instance as a claim, when there is a statement or a

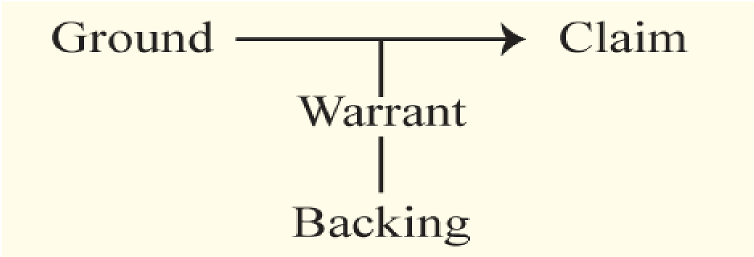


Figure 4.1: Toulmin's framework of Argumentation

solution presented to the whole class or a doubt expressed by any member in the class. In the classroom context, ground is the set of facts that appear in the classroom as a problem statement, a question, an explanation, or an action, which further creates the basis for the emergence of the claim. Ground is not usually expressed explicitly in the conversation but can be inferred from the context. Warrant in our context is the explicit rationale provided by the actors in the classroom either in terms of verbal justification, or with objects, drawings or symbolic manipulation. Though backing refers to the established norms, logic, or rules of a specific field or discipline, in the present mathematics classroom context, some norms will also be co-constructed within the classroom.

4.3 Methodology: Design of the Study

Since we needed a classroom context where we also have some freedom in terms of our lesson plan, we[10] conducted a teaching camp with students in the school during the school vacation time (see Figure 4.20 at the end of the chapter). The research methodology adopted for the present study is a teaching design experiment, where we designed some activities and tasks to engage with the concept of area. Teaching experiments have been adopted as a research methodology for various purposes, one among which is the development of ideas in a classroom environment (Kelly & Lesh, 2000). Teaching design experiments are based on our knowledge of existing research and theory and seek "to trace the evolution of learning in complex, messy classrooms and schools, test and build theories of teaching and learning" (Shavelson, Phillips, Towne & Feuer, 2003, p. 25). Design experiments allow us to connect the concrete realities of a classroom environment with the theoretical concerns of a particular discipline (Cobb, Confrey, diSessa, Lehrer, and Schauble, 2003). In our study, tasks or activities were designed based on the insights gained from research literature and our earlier work reported in the previous chapter, in particular to highlight the distinct numerical and spatial solution strategies used by students (Battista, 2007), and to study the interaction between the two.

The main objective of the study was not to teach a particular concept, but to understand students'

10 Here "we" refers to the team involved in the design and conducting of the teaching camp, and also the team members involved in observation and later providing feedback.

collective construction of the concept of area. In the classroom, we have adopted a naturalistic approach towards the evolution of the construction of the area-concept. The researcher cum teacher and her fellow colleague involved in the teaching deliberately avoided using the formal term "area" throughout the teaching camp to avoid any interference of students' prior formal knowledge about area with the tasks in the camp. Instead the word "size" was used whenever there was a need to refer to area as argued for in the previous chapter (in Section 3.2.3.4).

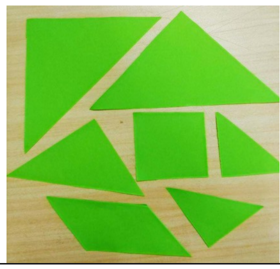
We used the principles of variation in our lesson design; variation allows students to act in powerful and multiple ways while engaging with a problem by seeing the same problem in different ways by attending to its different aspects (Marton, Tsui, Chik, Ko, & Lo, 2004). Further, "variation enables learners to experience the features that are critical for a particular learning as well as for the development of certain capabilities" (Marton, et al. 2004). Considering one of the main challenges in the learning of AM is to connect the spatial and numerical aspects, we designed a variety of area problems to highlight and strengthen this connection (as elaborated in the next Section 4.3.1). Thus, while the textbook problems on area-concept are usually of the same nature focussing on "find" and "solve" type questions, we adopted a variety in the context of area problems that open up multiple pathways for students to connect the spatial and numerical aspects of AM. We also included elements in the tasks that allow students to create and construct multiple solutions. Thus instead of the typical area problems of textbooks that generally expects a single solution, our lessons were designed in a way to allow students to get varied solutions. The variation served two purposes, first it allowed learners to explore the unseen dynamic features of the area concept and it also created conditions for multiple solutions that led to argumentation in the classroom. Inspired by the ideas of variation, several researchers have adopted variation further in their teaching and design of the lesson (Watson, & Mason, 2006; Holmqvist, Gustavsson, & Wernberg, 2008). Variation allows one to have alternative learning outcomes, by making small changes, based on the reflections from the classroom, and thus includes the potential to improvise the learning trajectory. Thus, the approach of variation allowed us to design conducive conditions for students to come up with alternative solutions and it also provided us the readiness to be able to handle varied claims emerging in the classroom through collective argumentation. Before moving into the classroom and the interactions that emerged, I will elaborate on the tasks that were developed and evolved in the study in the following section.

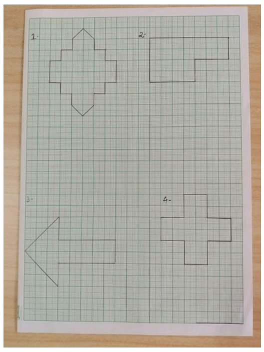
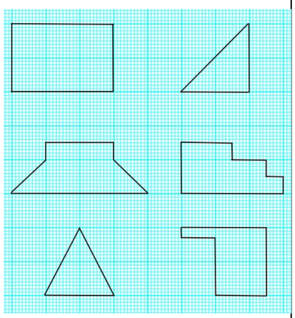
4.3.1 Tasks: design and description

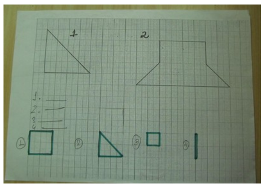
The tasks were developed and designed based on the experiences and insights from the previous studies. As just described, there was an effort to include the principles of variation in the design of the tasks, that allows different students to have varied solutions, which can further create opportunities for argumentation in the classroom. In all the tasks or activities, there is an attempt to achieve an integration of different concepts or processes of mathematics as a network as argued for in the previous chapter. This is done through designing conditions that allow for integration by having well thought materials that are manipulable and by following an open ended approach (that is having an openness to new responses or outcomes in the classroom).

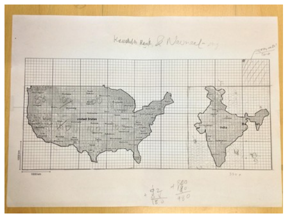
The lesson plan or the teaching-learning sequence can be broadly classified into nine sessions, designed for collaborative construction of AM (refer Table 4.1). The common principle behind the design of each session (including the tasks or activities in each session) was to integrate multiple concepts or processes recognised in the network of AM (as argued for in the previous chapter), for e.g., identifying the attribute for area, conservation, unitization or the concept of unit, array, iteration, quantification, the notion of size or magnitude for a two-dimensional space or measurement, etc. Again, there was an effort to ground the activities with local contexts and goals (refer Table 4.1). Also, the activities or tasks were given to students in pairs or in groups (except in some occasional cases to individual students) to encourage collaborative practices and participation, in order to have conducive conditions for social constructivism.

Table 4.1: Description of tasks used in the classroom

Sl. No.	Session Name	Session (or Task) Description	Learning Objective	Artifacts used
1	Stamp making activity	Given an A4 sheet and four different rectangular photo pieces of different sizes or dimensions as stamps, students were asked how many such photos of each type can be printed on (or made out of) an A4 sheet by a photo shop keeper. Then given the cost of each photo what will be the total cost of all such pictures made out of an A4 sheet.	To introduce different units, covering, arraying, optimized use of the space. Again connecting space to number through quantification using the familiar context of price.	
2	Tangram Activity	Making the seven tangram pieces out of a given sheet (an A4 or a colored rectangular sheet) by folding and cutting, and discussing the shapes and geometric properties of the pieces. Then students were asked to use all the seven tangram pieces to make a square,	To introduce different units, relationship between units, conservation and the sense of magnitude.	Tangram pieces 

		or a rectangle, or a triangle. Students were asked to assume the pieces as *barfi* (an Indian sweet). Then assigning some numerical value (whole or fractional) to one such piece as the price of that piece, students were asked to determine the price of the other pieces and all the pieces together.	Quantification of all the pieces in terms of a unit by assigning whole or fractional numerical value to it. Encouraging multiplicative thinking & multiple representation.	
3	Graph Paper Activity-I	Students were given standard graph paper (sq cm or cm^2) with four different geometric figures drawn on them. Students were introduced to the graph paper; the squares made out of the bold lines were highlighted and they were introduced as "boxes" in the class. Students were asked to determine the total number of boxes in each of the given closed figures.	Quantification of different shapes, moving from geometry to number (whole number quantification). Engaging with unit, exploring the relationship between the unit and the measure of the figure.	Graph paper 
4	Graph Paper Activity-II	An extension of what is being done in the previous activity with Graph paper. However, in this case students were asked how much of the boxes are included in each of the given figures, since here the figures were intentionally designed to give fractional value (or measure)	Same as the above activity with an added component of fractional quantification.	Graph paper 
5	Activity on different	Here, a nonstandard graph paper (sq cm or cm^2 magnified to double its size) is used with the same figures drawn on	Quantification of a figure on a non standard graph	(Non-standard) graph paper

	units	it as in the previous activity and students were asked the same question as above. However, in this case, unlike the previous tasks, four different units were used. Students were shown the drawings of the four units, by drawing it in on the graph paper (Where the first unit consists of 4 boxes, while the second unit is half of the first unit, and the third unit is a quarter (¼) of the first unit that is one box, and the last unit is 1/10th of the first unit.)	sheet in terms of four different units. Seeing the multiplicative relation between units and hence the resulting relation between the measure value of each figure in terms of different units.	
6	Number to shapes	Students were explained with some examples in the classroom that given some number, they can represent it into some geometrical figure, especially a rectangle. For example, given a number 10, which can be seen as 10 square units which can be represented by a 2×5 rectangle or a 1×10 rectangle and also by rearranging the order of the factors. Also one can move to fraction or decimal factors as well. In order to highlight the multiplicative relation, we asked students to draw rectangular shapes.	Moving from numerical to various possible geometrical representations of area on the graph paper. To also explore the inverse connection of numbers to the resulting geometrical shape.	Graph paper
7	Extrapolation to bigger measure	Students were given a standard graph sheet, an A4 sheet and an A3 sheet. Students were asked to use the graph sheet and find the size of an A4 and A3 sheet in terms of the units of the graph sheet (sq inch or $inch^2$).	To quantify a given physical object, e.g., A4, A3 sheet using graph sheet.	Graph sheet, A4 and A3 size sheets
8	Measuring objects around	All the students in the classroom were divided into 3 groups, with each group asked to measure one of these: the Door, the blackboard, and the windows of the classroom. And get the value in terms of units of the graph sheet (sq inch or $inch^2$).	To quantify a given physical space. Following the previous activity, this one expects students to extrapolate the units from a	Graph sheet, A4 and A3 size sheets

			graph sheet to bigger measures like A4 and A3 and further to other physical material or space.	
9	Scaling and Irregular shapes (curved boundaries)	Given a world map printed on a grid/graph paper with the boundary of USA & India clearly highlighted. Also the scale of the graph to be clearly mentioned at the bottom left corner e.g., 4 cm=1000 km marked along one single square unit. Here, students were asked to consider one box as 1000000 km^2 (or sq-km). The task was to compare the land covered by the two countries, and how large the USA is compared to India with regard to land area.	Scaling and measuring (or estimating) the area of irregular curved shapes. Also multiplicative relation between the land areas (e.g., India and USA).	

As can be seen from Table 4.1, in session 1, the activity requires students to find the total number of different sized stamps they can create out of an A4 sheet, a local context of stamp printing is used to provide meaning and motivation for students to engage with this activity. This task allows students to do this activity by physically covering, iterating the picture pieces on the given A4 sheet, to find out the maximum number of photos they can extract out of an A4 sheet, further probably abstracting out the array structure, and seeing the relation between the total number of such pieces and the multiplicative relation it holds with number of such pieces along the length or breadth. Once the majority of students in the class could determine the total for each of the photos, some of them were asked to explain how they arrived at their number to the whole class and others were asked to attend, engage or question, or express their doubts on the explanation. For the later part of this task, if the price of printing one photo is known, students were asked to determine the total cost in printing an array of a particular photo on an A4 sheet. This part of the activity allowed students to engage with the multiplicative relation between the value of one unit and multiple units.

The second session was on the Tangram activity. The various affordances provided by this activity have already been described in the literature review chapter (see Section 2.1.3.1) including the integrating aspect of Tangram that allows students to connect geometrical, numerical and further the algebraic or symbolic abstraction in terms of the relation between the different Tangram pieces (units) and the whole. Further, the physical manipulation aspect of Tangram allows students to engage and explain their solution process with accessible physical units.

From Session 3 onward, we used the graph paper extensively in order to develop the graph sheet as a cultural tool for measuring area or two dimensional plane surfaces in the classroom culture just like we have the ruler (or linear scale) for measuring length. Thus, session 3 onward, there was an attempt to introduce, use, and interact in terms of the graph paper, to redefine it within the classroom as a culturally developed tool for quantifying area without using the term “area” in the class. Session 3 involved introduction of unit in a standard graph sheet and quantification of different shapes drawn on the graph sheet in terms of the unit. Session 4 also had the same activity, except that in session 4 the shapes were carefully designed to bring fractional value (or measure), while the shapes in session 3 were designed to give whole number value. Session 5 uses a nonstandard graph where basically the previous graph sheets were magnified to double their sizes, with four different shaped (fractional) units highlighted on the graph sheets. Thus, session 5 allows students to engage with different fractional units and explore the multiplicative relation between different units and the resulting value (or measure) of the given shapes.

Unlike in the previous sessions, where students were asked to quantify a given space (or find the numerical value out of a given geometrical shape) in terms of a unit, in session 6, we took the inverse route where students were encouraged to *geometrize* a given number i.e., generate various possible geometrical shapes or representations on the graph paper for a given numerical value (or measure). To streamline this task further, students were shown and asked to make rectangular or triangular shapes. For e.g., given the number 12, which can be taken as 12 sq units, one can represent this as 2×6 or 3×4 or other possible rectangles. Thus, the main learning objectives behind this task are to strengthen the connection between the geometrical and numerical representation of area by having an inverse task, and also to allow variations in students’ responses to create the conditions for possible argumentation within the classroom. The primary idea for this task emerged from the Card Task in Section 3.2.5 of the previous chapter.

Session 7 and 8 were inspired by the component of “extrapolation” elaborated in the network model on AM in the previous chapter. Basically, if we go back to the course of tasks followed so far, we can see that we have defined or picked up a smaller unit to work with. However, in extrapolation, we redefine another bigger unit by clustering several smaller units in order to measure bigger objects. Thus, while session 7 required students to get the size of an A4 and an A3 sheet in terms of the units of the graph sheet (sq inch or $inch^2$), session 8 requires students to measure bigger objects around their classroom, for e.g., the blackboard, the door, or the window. The idea was to explore whether they can use their collectively quantified or constructed bigger units (e.g., A4, A3 sheet or part of it) to measure the bigger objects around them.

Session 9 is about zooming out into larger spaces on a world map. The objective was to introduce the idea of scaling and measuring irregular curved shapes or regions. Students were asked to compare the land covered by the two countries, and how large the USA is compared to India in terms of land area. Here again the term *area* was not used, students were asked to consider one box as 1000000 km^2 (or sq-km) and compare the land covered by the two countries. They were also encouraged to find out how many times larger the USA is compared to India, bringing in the multiplicative aspect between the land areas. It was hoped that these tasks would be meaningful and address the integrated and network understanding of area.

4.3.2 Nature of the intervention and the setting

The teaching was done by the researcher and her colleague (i.e., 7 days by me and 5 days by my colleague) for approximately 2 hours every day for over a period of 2 weeks (i.e., 12 days in total). Each day began with a warm-up mathematical game or activity for about 20-30 minutes followed by the tasks based on area-measurement. The study consisted of a convenient sampling of students who voluntarily participated for the study and are different than the one mentioned in the previous chapter. On average, 30 students participated[11] in the study which included both sixth and seventh grade students. Data collection happened through video recordings of each lesson, and a fellow researcher writing the lesson log every day with some other researchers and colleagues observing the lesson on several occasions to provide inputs in the follow-up discussion after each lesson. Each day’s lesson was followed by a debriefing session with a community of researchers, which focused on the conduct of the lesson, the insights gained and the planning for the next day’s lesson. The video recordings of each lesson were transcribed by me and my colleague after the teaching camp was over for further analysis. The teaching camp ended with an informal meetup program, where we chatted with students about their experiences, learning and obtained informal feedback about the teaching camp. The regular teachers and head-mistress of the school also shared their experience about the camp and invited us for more such collaborations in the future.

The school where the intervention was carried out is an Urdu-medium school run by the municipal corporation, located in the city of Mumbai, India, catering to students living in the surrounding slum, reported to be a productive base for house-hold based economy (Bose, & Subramaniam, 2011). Thus, the school[12] was an interesting and rich site for carrying out the intervention, where students could bring in their experiential learning into the classroom, which could further support the collective

11 Around fifty students enrolled for the intervention by filling in the consent form (Appendix IV)

12 The school was mainly catering to the working class population living in the closeby densely populated slum, the students in the study mainly belong to immigrant muslim communities from the northern states of India.

construction of the concept in focus.

The present study focuses on the interaction among students and between the students and the teacher during episodes of argumentation, which we will elaborate further in the next section. These episodes were identified by studying the transcripts and reviewing the video recordings and notes of the sessions. The episodes particularly bring forth the complexity involved in spatial and numerical aspects of area-measurement. The argumentation framework prompted us to focus on those episodes where varying claims are put forth, are challenged and justified.

4.4 Episodes of argumentation from the intervention study

As argued earlier, the section is mainly around the episodes of argumentation happening within the classroom during the teaching camp. For the present study, I have selected four episodes from the entire teaching camp, where I felt the presence of a genuine disruption in the class due to the differences in the argumentation framework or structure by different actors of the classroom. These four episodes were selected because the interactions in the episodes indicated high levels of engagement by the students and suggested the collective process of concept construction in the classroom. The intense engagement by students during these episodes were also independently reported by my colleagues, who observed the class sessions, during the debriefing sessions after each class.

In this section, I will discuss each of the four episodes that are extracted from the present intervention study (or teaching design experiment) to understand the process of construction of area-concept happening in a classroom through collective argumentation. The discussion of each episode consists of an overall identification of the segment of the teaching camp (12 day long intervention study) from where the episode is picked-up, a description of the context including the task or the activity, important excerpts of the transcript representing the episode and an analysis of the episode. The two major aspects covered in the analysis of each episode are: (1) the structure of argumentation in students' discussion in the classroom and (2) the conceptual underpinnings in these discussions. Pseudonyms are used to protect the identity of the students.

4.4.1 Episode-1

In one of the tasks (task-6, Table 4.1), the students were asked to make different possible rectangles for a given size on a graph paper and then write the numerical multiplication facts. One of the sizes given was 15 units. For this size students came up with various facts like $3 \times 5 = 15$, $2 \times 7.5 = 15$, $1 \times 15 = 15$. Sajaad came up with $30 \times ½ = 15$. He came to the black board and made a 6×5 rectangle and divided it vertically into two halves to show that there are 15 units in each half. The teacher then asked the students to come up with more ways to divide a 6 × 5 rectangle into two equal parts. Many students suggested horizontal division and Sajaad suggested diagonal division as well. Most of the students agreed that the rectangle can be divided vertically, horizontally or diagonally into two halves. But when Sajaad tried to divide his 6 × 5 rectangle diagonally into two halves, he was just looking at the rectangle and wondering for quite some time (for the teacher he appeared to be stuck). He appeared very engaged and confused (looking at his diagram doubtfully) as he was not able to identify 15 units as contained in each triangular part. So he thought that the diagonal division was not giving half the area. In an attempt to convince Sajaad that a diagonal division too will produce halves, the teacher prompted Sajaad to check if the diagonal divisions are congruent. The teacher gave him a pair of scissors to check whether the two pieces are equal without actually counting the units in each part. But Sajaad was not convinced that the two pieces are equal, as indicated in the following transcript.

> Teacher: … *Ye ek dusre ke barabar ho raha hai? Ho raha hai na?* [… Are they becoming equal to each other? They are becoming equal, right?]
>
> Sajaad: *Nahi*. [No.]
>
> After cutting the rectangular piece diagonally into two halves, he was unable to superimpose the congruent halves without the teacher's help. Even after the teacher demonstrated that the halves are indeed congruent, he reacted as:
>
> Sajaad: *Lekin ye aa kyun nahin raha hain pandrah?* [But why are we not getting 15 for this?]

Using Toulmin's argument structure, we identify the claim as "When a 6 × 5 rectangle is diagonally divided into two equal halves, the area of each triangle is 15 units". The data or ground for this claim consists in the two halves being congruent to one another. However, the inference from this data to the claim is mediated by other assertions, which can be categorized as "warrant" following Toulmin's scheme. The argument structure for the student shows that even when two parts of a whole seem spatially or geometrically congruent (equal), there is a doubt about the numerical value of the area being exactly half that of the whole. We interpret this as a gap between the spatial understanding and the numerical understanding considering the fact that Sajaad recognizes congruence of the two parts which is clear in his response to another student as follows.

> Merajuddin: *Lambayi aur chaurayi mein farq hai, isiliye aadha nahi katega.* [Length and breadth are different, so it won't get cut into two (equal) halves.]
>
> Sajaad: *Aadha katega lekin ginti mein pura nahi hoga.* [It will get cut into two (equal) halves, but we will not get the total when we count.]

The teacher tried to convince the students that even if we cannot count 15 units in each of the triangular halves, since the two triangular halves of the 6×5 rectangle are equal halves it must be half of 30. After this the teacher moved on to discuss other number fact problems. However, the students did not appear to be fully convinced as evidenced by Raziya’s subsequent intervention. Raziya intervened to bring the focus back on the area of the triangular half. She said that she could make the 15 units and she showed the teacher how this was possible on her graph paper. The teacher then asked her to demonstrate this to the class on a bigger graph paper. Thus two different kinds of warrants emerge in the episode. The argument structure for the teacher was different from the argument structure for the student as indicated in Figures 4.2 and 4.3.

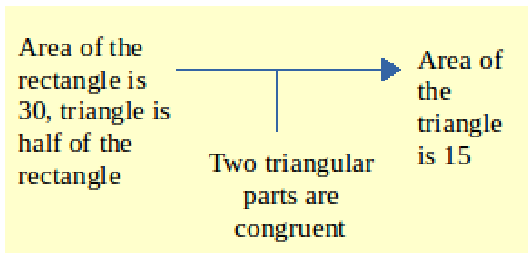


Figure 4.3: Teacher's argument structure

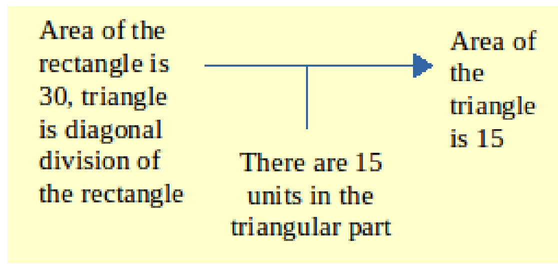


Figure 4.2: Student's argument structure

Later, Raziya came to the board and pointed out that although the teacher had said that we couldn't count the units in the diagonal division of a rectangle, it can actually be counted. She showed her work on a bigger graph paper to explain how diagonally dividing a 6×5 rectangle gives 15 units in one half. As can be seen in Figure 4.4, Raziya is using the strategy of moving parts to complete the units along the diagonal in one triangular half. Raziya supported Sajaad's argument by providing the same warrant that he and other students were looking for.

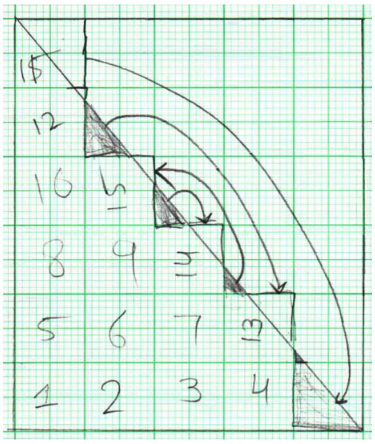


Figure 4.4: Raziya showing that the triangular part contains 15 units

This episode demonstrates that a few students including Raziya were still engaged with the problem of finding 15 units in the diagonal division of the rectangle. The students were seeking a warrant to support the claim through identifying units in the figure, which was different from the teacher's warrant for the claim.

Recognizing that the argumentation structures held by the teacher and the student were different helped us in understanding the reason why the students were not getting convinced. We note that the teacher, rather than engaging with the argumentation structure of the student, came up with her own argumentation structure to convince the student. However, later another student (Raziya) came forward and tried to work with the same argumentation structure that was provided by Sajjad.

Though the argumentation framework helped us here in recognizing the different argumentation structures held by the teacher and the student, identifying the structure was not enough to engage with the underlying conceptual understanding that emerges within the interaction. Thus, the argumentation structure gave us some idea about the process of Sajjad's construction of the area concept in the classroom context, but engaging beyond the structure allows us to see that there was a conflict between the spatial and numerical understanding of area which led to this process of argumentation. Again the teacher's attempt to convince the student brought out the teacher's argumentation structure in the interaction. Thus apart from the different argumentation structures, it is also insightful to see what are factors that led to such argumentations and what information it gives about the conceptual construction.

4.4.2 Episode-2

Unlike the previous episode, which evolved from the planned task, this one emerged out of a given assigned task in the classroom. This episode is relatively more complex (because of the parallel conversations happening at one time). Hence the work on the task was split over two days lasting for more than an hour each day and was accompanied by a very rich discussion. This episode is taken from the fourth and fifth day of the teaching sequence. In this episode, students were asked to find the size of six given shapes outlined on an inch-graph paper in terms of a square-inch unit. One of the figures is shown in Figure 4.5.

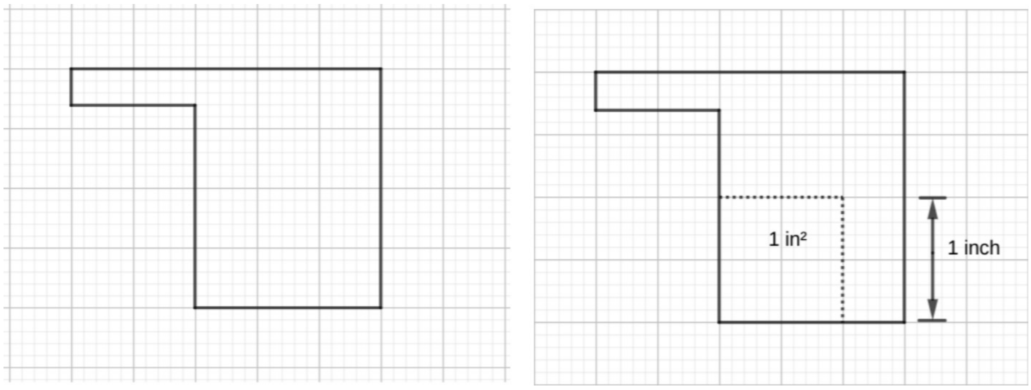


Figure 4.5: The problem

The general solution method was first to find the 3 units on the right of the figure and the remaining part on the left is shown in Figure 4.6. However, multiple ways were elicited in getting the measure of the rectangular part on the left.

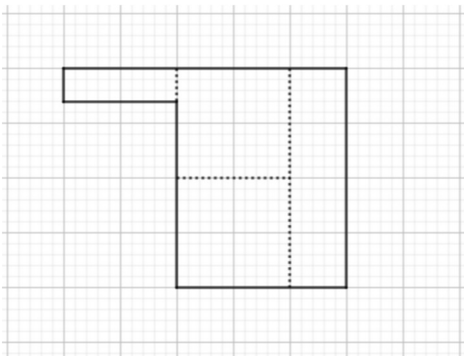


Figure 4.6: General solution method

Thus, in contrast to the previous episode where the student focused only on seeing and counting the unit to get their solution, here they had to find out the value of a small part of the unit. Most students could identify the three complete units, however many of them struggled with the remaining part that extended to the left of the rectangle. There were several discussions among small groups within the class about how to represent the remaining part.

By the end of the lesson, the teacher asked all the students to share their answers. There were multiple answers suggested by different students (see figure 4.7), which included “3 quarter, 3.3, 3.6, 3.60, 3.5 half, 3 3/10, 3.30, 3.1”.

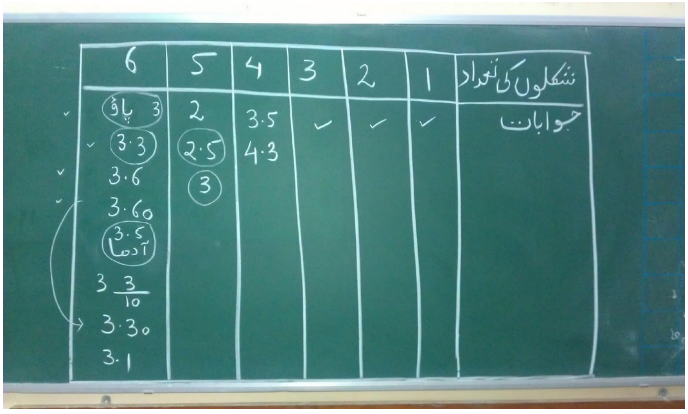


Figure 4.7: Multiple answers suggested by students

The teacher wrote all the answers on the board by the end of the fourth day, and announced that they would be discussed the next day. On the fifth day, the teacher asked each student to defend his or her solution in front of the class. This episode has four different parts where different students were using different units as a backing to support their solution.

4.4.2.1 Part-1

Suhana explained the solution 3.3 given by a boy the previous day, (who happened to be absent on Day 5), as presented in the excerpt below:

> Suhana: *Teen box hai na chote chote wahi ginke likha usne.* [There are these three small boxes, he has counted them]

Thus, in this solution the student has identified the extended part on the left as consisting of three rectangular strips calling them as three small boxes.

4.4.2.2 Part-2

Aliza claimed 3.6 as the value for the given space and justified her claim as below (also refer Figure 4.8 a):

> Aliza: *Teacher agar ye, ek box rehta ye wala, isko hum teen asariya paanch mante, aur isme ek box wo jo ek chota wala tha na wo ek zyada hai, isiliye teen asariya panch mante na, toh usme ek aur box aa gya toh teen cheh manenge na usko.*
>
> [Teacher if there was one box, this one (pointing to the quarter part at the left outlined in

bold in figure), then we would have considered it three point five, here there is one more small box, so three point five after having one more, so it will be considered three six]

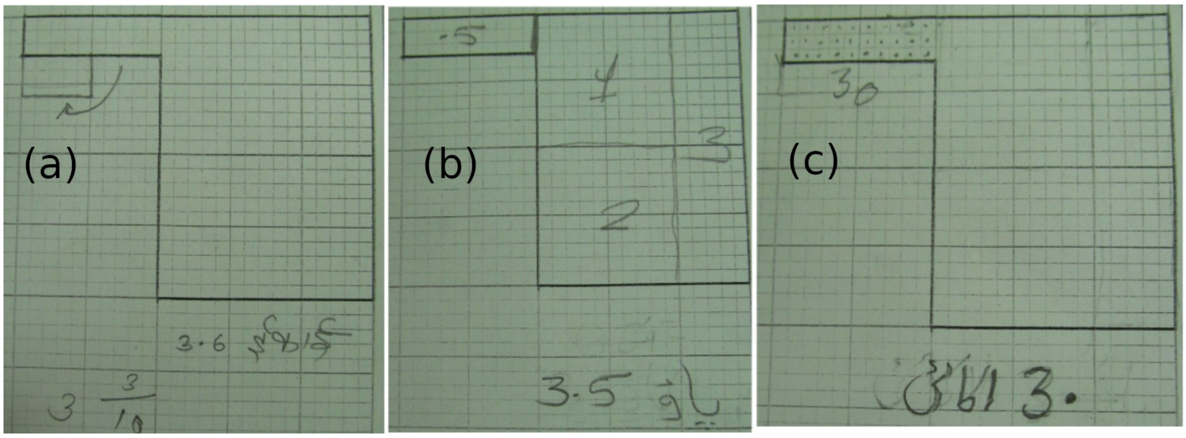


Figure 4.8: (a) Showing 3.6, (b) Showing 3.5, (c) Showing 3.30

The teacher drew the same shape on the blackboard in an enlarged version for everyone to see. Aliza explained her solution to the class by working on the figure made on the blackboard. She erased one-half of the remaining extended part and moved it to the bottom of the other half (similar to the first example as shown in the corresponding Figure). While in the previous case Suhana referred to three strips, in this case Aliza refers to six strips made after moving one half of the remaining part. So in the previous case, the three rectangular strips were recognized as .3 in 3.3. However, in Aliza's case she identified six rectangular strips, which represent to .6 in 3.6. Aliza further said if there were five such strips, they would have been recognized as .5 in 3.5. But her claim was countered by Raziya as below:

Aliza: *Kyu? teen asariya panch, agar ye ek khana nahi rehta toh teen asariya panch bolte na*
[Why? If this one space was not there, then we will call this three point five right.]

Raziya: *Nahi bolte.* [We won't call that.]

Aliza: *Kyu nahi bolte?* [Why not?]

Raziya: *Teen asariya panch yani aadha, woh toh adha nahi hai pau hai.* [Three decimal/point five means half, that is not half, its quarter.]

Thus, Aliza misidentified the quarter part made after moving the smaller parts as point five. But Raziya's comment was aimed at making Aliza notice that the part she is referring to is not half but quarter of the full unit. Thus, here they realise that 3.6 is not correct and came up with a different claim as elaborated in the next part.

4.4.2.3 Part-3

Raziya and Afia came to defend 3 3/10. Raziya referred to the earlier lessons on how to represent fractions and took the example of Roti (Indian bread). Afia drew a circle on the board to represent a Roti, and made ten divisions in it.

> Raziya: *Dus tukre kiye, aur isme se maine teen hisse kha liye, toh phir kaise likhenge.* [Made ten divisions, of which I have eaten up three parts, then how will we write that]
>
> Aliza: *Teen batte dus.* [Three by ten, i.e., 3/10]

Using that context as base, Raziya went back to the main task and justified her claim to the whole class as below:

> Raziya: *Teen batte dus na, toh waise hi ye line mein agar humlog aise aare mein lete hai, toh usme teen line thi, aisi teen line thi, dus line hai aur teen line, toh kya hua, dus batte teen hua na, toh teen box aur dus batte teen.* [Three by ten right, so similarly if we take this line horizontally, then there are three lines, ten lines are there, and three lines, then what will be the value, ten by three, so three boxes and ten by three.]
>
> Aliza: *Dus batte teen nahi cheh batte teen, cheh batte dus hoyega na.* [Not ten by three, six by three, six by ten will be the value.]
>
> Raziya: *Cheh batte tab hoga, tumne khali aari line gini hai.* [Six by will be when you have only counted the horizontal line.]
>
> Afia: *Aisi line aisi, niche nahi hai, aise hi.* [This kind of line, it's not going down.]
>
> Raziya: *Cheh batte agar bolenge na toh apne ko aari aur khari dono leni padhegi, isme aari bhi dus hai, khari bhi dus hai, toh agar hum cheh batte lete hai, toh cheh batte bees lenge, aur agar usko katenge toh phir, do daham dus, do tiya cheh, teen batte dus aaya, toh apka answer aayega teen sahi teen batte dus.* [If we say six by, then we have to take both horizontal and vertical lines, here there is ten horizontal and ten vertical, so if we are taking six by, then we have to take six by twenty, and then if we cancel them out, two tens are ten, two threes are six, three by ten will come, so your answer will come as 3 3/10.]

Thus, Aliza was consistently looking at the extended part as six divisions, So Raziya tried to fit her reasoning in Aliza's argumentation structure by referring that in the case of taking six divisions also, there will be twenty divisions in all. So in that case also Aliza’s solution will come out to be the same as three by ten.

4.4.2.4 Part-4

Sajaad came to defend 3.30 as the answer for the task as shown in the following excerpt:

> Sajaad: *Teen hissa ye bhi hai, teen hissa ye bhi hai, toh ek line main ek khane main ja rha hai*

panch, panch line, idher teen hain na idher se doh lenge, isko ek khana banaenge pau kerenge, aur idher ka ek line bach gya, toh ye pau ka pachees hota hai, toh pachees ka ye, aur panch ye ek line ko manenge toh panch aur pachees, tees ho gya na, teen point thirty.

[This has also three parts and this has also three parts, so in one line (or part) there is five, five line, here there is three (lines), so we will give two, we will make it one quarter, and here one line remained, so here quarter will be 25, and if we consider one line as 5 then 5 and 25 will be 30, 3.30.]

Sajaad first recognised the extended part as consisting of six rectangular strips as was done by Aliza in part-2. He then moved two of these strips at the bottom of the remaining three. He identified it as the quarter and represented it as 25 (perhaps using a money representation of 25 paise) and the remaining strip as 5. Sajad often used the context of money to justify his reasoning in other tasks. His use of the context of money was familiar to most of his classmates. Thus he identified 30 as accounting for the extended part and 3.30 as the size of the given shape. Thus, in this episode, even with the same primary data, students came up with different claims by following different solution (or argumentation) structures.

This episode brings variation in terms of students' responses where they are claiming different answers and presenting their respective warrant. However, here instead of the argumentation structure, what comes out as the taking point are the different language complexities brought out by students, for e.g., using box, line, *patti* (strip), etc. to identify or talk about the part they were finding the measure of. In part-2, Raziya challenges Aliza's warrant as not an acceptable warrant.

We also note the use of analogy by Raziya in part-3 of this episode to justify or more importantly to convince the class of her claim. Thus, this episode is not about the structure but about the different aspects acting at the periphery to lead to argumentation in the classroom, which further lead to the student's construction of their own conception. Again, it also serves as an example of the classroom norm where students brought in different answers, most of which also become acceptable because of the convincing warrant to justify a particular answer.

4.4.3 Episode-3

This episode is simpler to understand in terms of the argumentation structure that emerged. Students had to find the size of the shape highlighted in bold in terms of the unit highlighted in red on the bottom left of the Figure 4.9. The numeral "4" marked at the top left represents the 4th shape of the task given to students on the 7th day of the camp (Figure 4.10). The graph paper on which the shape was drawn was not a standard graph paper to enrich students' experiences with different kinds of grids, with a fractional unit drawn on it for reference.

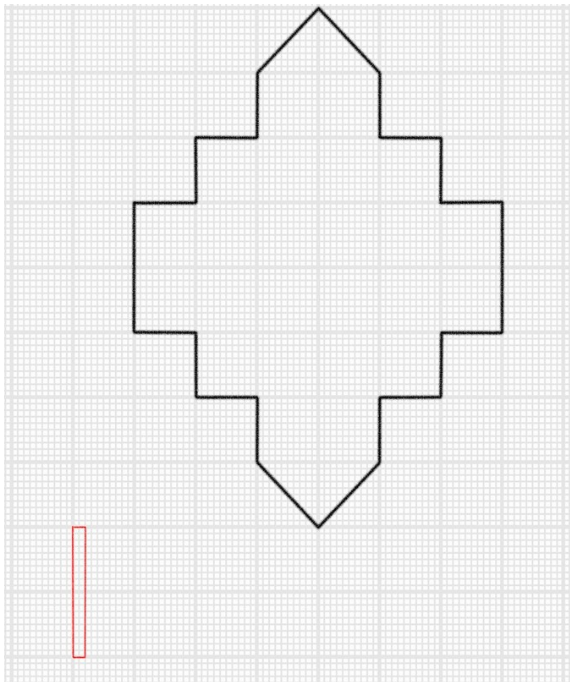


Figure 4.9: The task analysed in Episode 3

The two contending solutions for this problem were 65 or 60.5 for the same shape as shown in Figure 4.10. Initially, the teacher put up the contending solutions on the board. The students (Aliza and Afia) with 60.5 units as their solution came to the board to explain their strategy (shown on the right in Figure 4.10). They initially outlined all the possible big square units in the shape and counted them as 6. Each such big square unit consists of 10 such small rectangular units that the students had to use. Thus, since one such square consists of 10 of those small units, they got 60 for the 6 full units and have written .5 for the part that remained, which was half of the big square units at the top of the shape. Another answer was 65, claimed by Sajaad as a solution, with a similar technique of partitioning the shape and adding the parts. Afia said both 65 and 60.5 are correct answers. Aliza also said both the values will come (as the answer) since both are same. Aliza and Afia were using the decimal point as a separator between two different results of counting and not in the accepted convention of indicating decimal fractions with place value.

Later, Sajaad who had a different result also said that both the values are same. Here, despite getting two different numerical values, initially the students did not see them as contending values. They agreed that both values are correct because they were convinced with each other's warrants for arriving at their particular value as answer.

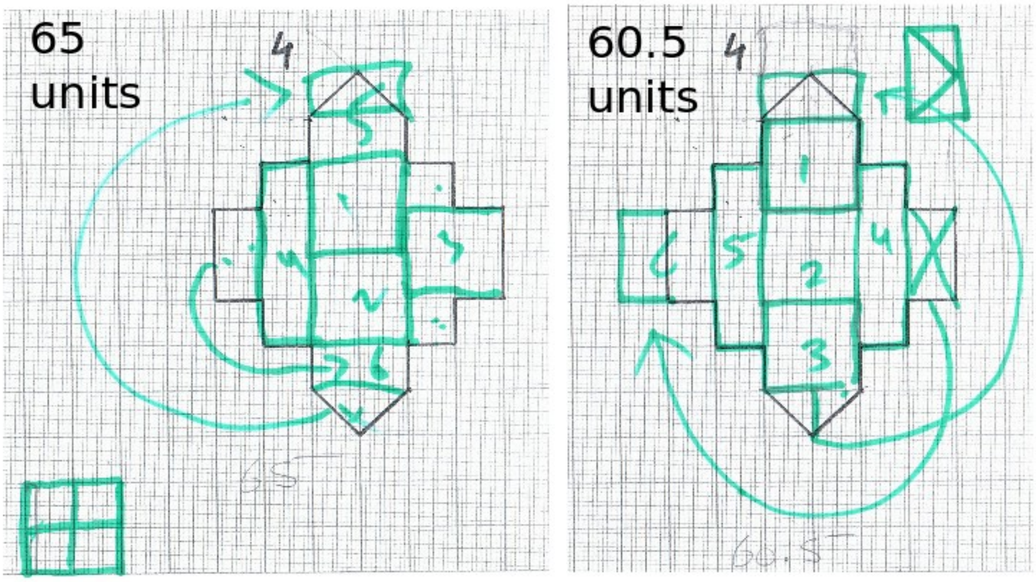


Figure 4.10: The two contending solutions

Accepting two different numerical values may come from the established classroom norm of having a variety of answers as acceptable, provided they are substantiated with proper warrant. However, if the warrants are mathematically not acceptable there is a danger of presenting mathematics as a subjective discipline rather than an objective one. Thus, the acceptable norm of having different answers and giving a suitable warrant is not enough; the warrants must be mathematically acceptable. Unlike in part 2 of episode 2 where Raziya challenges Aliza's warrant, here 60.5 despite being mathematically incorrect was initially acceptable to the students as a correct solution. At this point, the teacher decided to discuss this for some more time rather than directly objecting to the value 60.5, since the teacher wanted the challenge to come from the students. The teacher asked for an explanation for different values, for which Aliza and Afia said that their answer is different because their strategy is different. At this point Sajaad challenged them by saying that the answer cannot be different, it must be the same. Though Sajaad thought that Aliza and Afia's solution was correct, he said that the answer should not be different and that they have written the answer in their own way. Aliza still felt the different answers are acceptable because of their different strategies, especially in the way they have outlined the last half unit while partitioning the shape.

> Aliza: *Haan toh tu aise aadha kiya na, aur humlog hai na aare mai aadha kiye hai, toh alag aayega na humlog ka jawab tere se.* [Yes, so you have halved in this way, and we have diagonally halved it, then there will be a different answer for us than yours.]
>
> Sajaad: *Alag kaise aayega, wahi hai ho rha hai.* [How can a different answer come, the same thing (i.e., count) is coming.]
>
> Afia: *Wahi hai, bus point hai.* [It's the same, only point is there.]

> Aliza: *Sirf humne point laga diye aur tune point nahi lagaya hai.* [We have just attached a point and you have not.]
>
> Sajaad: *Maine kya kiya hai, pure ko jod-jad ke ek baari main likh diya hai...* [What I have done is, I have added all of them and written them together.]

Another student Merajuddin started saying that 65 is the correct answer with no decimal, but Aliza was adamant that 60.5 is the answer. In Sajaad's explanation, one half unit consists of five of the given smaller units and he had added those 5 units in 60 to have 65 units. However, the other group have not added the 5 units in 60 and have got 60.5. When the teacher asked whether 60.5 is bigger or smaller than 65. Merajuddin said chota hai (it’s smaller), but Aliza and Afia said they are the same, while Sajaad said they are the same answers but in different ways.

> Merajuddin: *Teacher ye paisath (65) hai, ye saath (60) hai, yeh point hatke hai, ye number mein nahi aayega, isliye ye saath (60) chota hai, paisath bada hai.* [Teacher this is 65 and this is 60, the point(decimal) is separate, it won’t come in the number, so 60 is smaller and 65 is bigger.]

When Aliza asked why 60.5 is less than 65, Merajuddin even said if between 60 and 5 there was '+', it would still work but since there is a point between them it's less than 65. Merajuddin even tried to explain that .5 and .5 makes one, so 60.5 is even less than 61, so 60.5 is definitely less than 65. Aliza & Afia did not find any contradiction in writing numbers in two forms perhaps because of past experience from earlier lessons that the same number can have different fractional and decimal representations.

Aliza later realised that Merajuddin is trying to say that the value changes with point (decimal point) and among 5 and .5, it's 5 that is bigger, she made the following drawing (see Figure 4.11) on board to show and understand the difference between 5 and .5 .

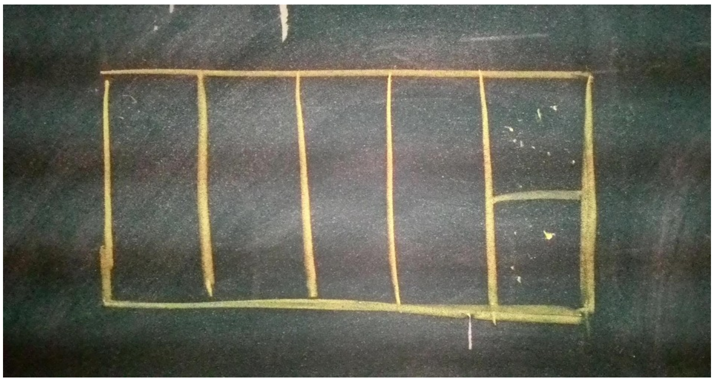


Figure 4.11: Representing 5 and .5 on a blackboard

So the initial justification for writing 60.5 was that after counting the 6 full square units, she represented the remaining half square unit as .5. Since it was half in terms of the full square unit, but in terms of the given unit it was 5. The drawing helped her clarify the difference between 5 and .5. The argumentation structures of Aliza and Merajuddin are shown in Figures 4.12 and Figure 4.13.

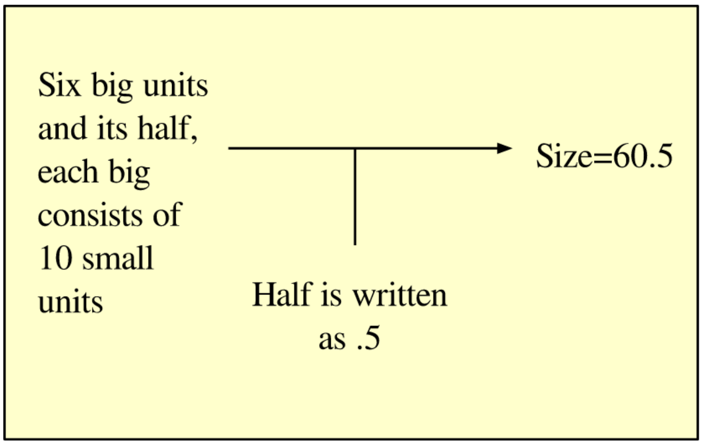


Figure 4.12: Aliza's argument structure

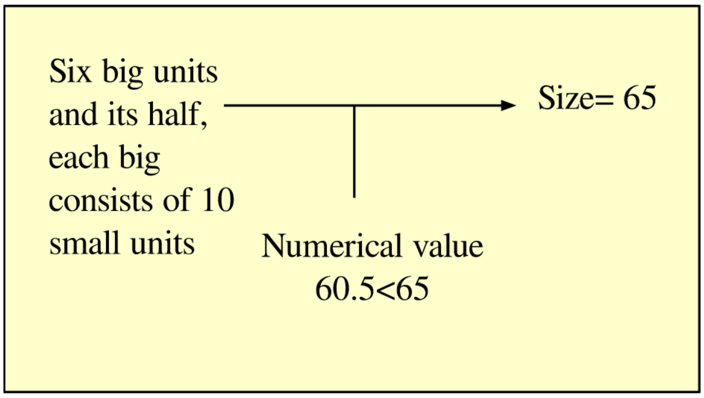


Figure 4.13: Merajuddin's argument structure

Unlike the first episode where the students were actually facing difficulty in integrating the spatial and numerical understanding (or concluding numerical result from spatial justification) here students were not distinguishing two different numbers. This shows that in the beginning either the students did not have a clear understanding of numerical representation or for them expressing the result in terms of two different units did not seem incorrect for them. But finally this was resolved by reasoning about the numerical value being the same or different.

4.4.4 Episode-4 (Day 10, 11, 12)

In this episode, the task (Task 7 in Table 4.1) was to find the size (area-measure) of an A4 and an A3 sheet in terms of the *boxes* (units) of the given graph paper (having square-inch as the unit). All the students in the classroom were divided into three groups and were asked to work in their respective groups by the teacher cum participant researcher[13]. This episode was extended over three days (Day 10, 11 and 12). The three parts of the episode have been designated as Episode 4.1, Episode 4.2 and Episode 4.3 respectively. Even though it is spread across three days, I have collated the events into one episode because of the same common conflict that emerges in the process of co-construction or resolution of differences. However, each day brings in different warrants or arguments to support the conflicting claim as a part of the negotiation or co-construction process.

Broadly speaking, in the first part of the episode, the teacher gave the task and went to each individual group to discuss their solution strategy. In the second part, the teacher asked the students in each group to present their solution or their strategy to the whole class. The third part is an extension of the second part, since no resolution was reached. There were three different groups of students using three different strategies. We focus mainly on one of the groups, which was challenged by the others and

13 Here I, the author, was the teacher, but in this particular context I have analysed and referred my teaching as a third person.

came up with three different warrants leading to different argumentation structures to defend their solution.

4.4.4.1 Episode 4.1 (Day 10)

On Day-10 of the camp, the teacher gave the task and went to each group to understand their strategy and to speak to them. Group-1 consisting of Merajuddin, Hanif and three other students used counting of units as their strategy to get the size of an A4-sheet. First, Merajuddin folded the extra space around the edges of the inch graph paper to get a 10×7.5 square-inch rectangle. He used that folded graph-sheet as a bigger unit for measuring and placed it over the top of the A4 sheet. Merajuddin and Hanif then marked its boundary (as shown in Figure 4.14).

For the blank part, remaining over on the A4 sheet, Merajuddin partitioned it into wholes, halves, quarters, 2-tenth and 3-tenth units (approximately in relation to the square inch unit of the given graph) and made markings of the same. They initially counted 88 units in the rectangular part [(7.5+.5)×(10+1)=8×11=88] on the A4 sheet and added 8 half units to it to get 92 (=88+8×1/2). Further, they joined 8 two-tenth units along the breadth (horizontally) with 8 three-tenth unit at the length (vertically) to make 8 half units (or 16 quarter units), and again counted the total as 96 (=92+16×1/4).

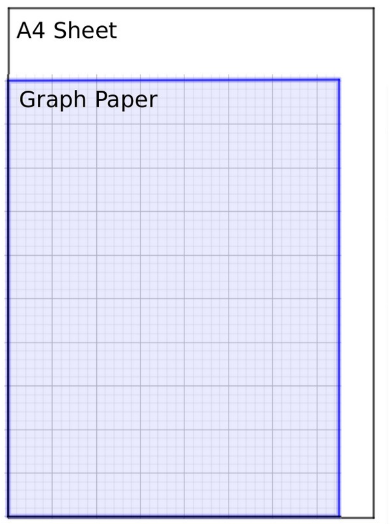


Figure 4.14: Task in episode 4.1

Some part was still left along the length, for which Merajuddin said: “teen pau hoyega, toh itna hi bachega” [this will become three-quarter, then only this much will remain]. Thus Merajuddin computed the last remaining part as *teen-pau* (three-quarter) with a negligibly small part remaining over, which he did not estimate. When the teacher asked him what he would write for the last remaining part, that is, the "*teen pau*" [three-quarter], he initially said "*ek mein teen aayega*" [three will come in one] and wrote it as 1/3, but then said "*teen batte ek aayega*" [three divided by one will come] and changed it to 3/1. At this point, Merajuddin identified the part as *teen pau* (three-quarter) but he was not sure about how to write that part numerically. The teacher initially asked him to write one quarter and add three such quarters together by drawing or writing. When Merajuddin was asked to write a quarter, he could write it as ¼, but he was not able to add the quarters (¼'s), but when Hanif said it is *Pauna* (quarter less than 1), Merajuddin could write the total as ¾, possibly drawing on his familiarity with this representation in out-of-school contexts. Even though Merajuddin was initially not sure how to add the fractions, he finally wrote 96¾ as the value for the size of the A4 sheet.

When asked about the size of the A3 sheet, Hanif said "*uska jama ker denge*" (we will add it), while Merajuddin gestured through rotating his hand that it has to be added again. When the teacher asked them how they knew that it had to be doubled, they placed the A4 sheet on the A3 sheet which was half-folded, to show that A4 can fit in half of an A3. They got the size of A3 by adding 96 twice to get 192. And added ¾ twice by drawing a rectangle with parts and saying "*pauna-pauna milake kitna hoga"* to get *dedh 1½*" (three-fourths and three-fourths make one-and-a-half), and thus got the total area of A3 as 193½. Though this group used a clear and logical strategy of unit structuring to get the size of an A4 sheet, the task took comparatively longer for them to complete, since the process was cumbersome. They used a more naive or intuitive strategy of counting the units and even if they did not fully know fraction representation or fraction operations numerically or symbolically, they could do the same using their local contextual knowledge.

After engaging with Group-1, the teacher went to Group-2, which had Raziya, Aliza and a few other students, to understand their solution and strategy. In contrast to Group-1, Group-2 first measured the length and breadth of the A4 sheet and used the area formula $l \times b$ to get the size of an A4 sheet. When asked how they knew that it would give them the exact size, they responded saying that their teacher had taught them this. So they used the teacher's authority as a warrant to justify their use of the $l \times b$ formula. For measuring the size of the A3 sheet, they doubled both the length and breadth of the A4 sheet, and multiplied them together to get the area of the A3 sheet. Thus, they got the area of an A4 and an A3 sheet as 97.11 and 388.44 inch-sq units respectively. When asked for a reason, Raziya said both the length and breadth of an A3 is double of that of A4. Since the A3 sheet was folded at the middle, she showed the folding of the A3 along the length (longer side) as warrant to show that the length is getting doubled and for the doubling of breadth, she showed one breadth at the edge and another parallel mark of the folding at the middle as another breadth as warrant as seen in the following excerpt:

Raziya: *Toh aapki lambai chaurayi ke baraber hai*. [Then its length is equal to its breadth, showing that the length of A4 is same as the breadth of A3.]

Teacher: *Haan*. [Yes.]

Raziya: *Hai na, toh phir ye do chaurayi ho gayi.* [Then, this will be two breadths.]

Teacher: *Kaise doh chaurayi hui?* [How are there two breadths?]

Raziya: *Ek idher ki aur ek idher ki.* [One from this side and the other from that side, showing one edge of A3 as one breadth and parallel folding mark on the A3 as the other breadth.]

During further discussions, Raziya said that A3 is double of A4. But when she was asked to check the calculation, despite realizing that the area that she obtained for A3 is four times that of A4, she was quite convinced with her algorithm and her reasoning that both length and breadth will be doubled for A3 and that then applying the formula $l \times b$ will give the proper area.

4.4.4.2 Episode 4.2 (Day 11)

On Day-11 of the camp, the groups were to present their solution strategy to the whole class. First Group-1 came and presented their solution, which was the same as they had explained the previous day. Then, Raziya from Group-2 challenged Group-1's method as inefficient as it may leave some space out of the measurement. Hanif and Merajuddin from Group-1 came up to the board to argue:

Hanif: *Teacher yaha itna jagah bacch raha hai na, humne usko fold kiye the.* [Teacher here this much space is getting left, we have folded that part.]

After this Group-2 also presented their solution. Since Group-2's method was very different from Group-1 and since Group-1 were also challenged by Group-2, Group-1 members started focussing on Group-2's solution method more carefully. Hanif showed a deeper concern with the method used by Group-2, as measuring the length is like looking at one edge leaving out the whole space while Group-1 had taken into account the space while measuring. This points to the complexity in moving from one dimension to 2-dimensional measurement. For Hanif, just measuring the linear dimension did not convince him that it could give the area. In his group's method, they had measured it fully in terms of an area unit. Even for the blank part left on the A4, they measured it by either folding or cutting the part and using part of the graph sheet to measure the part.

Raziya responded to Hanif by saying "*wo hum log naap nahi rahe hai*" [we are not measuring that part]. Instead, she showed the length to be 11.7 and breadth as 8.3 and said that to get the area of a rectangle one has to multiply the length and breadth. She showed doubling of length and breadth of A4 by multiplying the dimensions by 2 on the board, and then asked the students to multiply the resulting two numbers to get the area of an A3 sheet. When the teacher asked her to check the length and breadth of an A3 sheet, Raziya said, "*teacher wahi aayega*" [teacher same thing will come]. She was reluctant to check the dimensions of the A3 sheet and was quite convinced that her algorithm was correct. So, even on the 2nd day of doing the same activity, Group-2 was still convinced with their mathematically incorrect warrant for obtaining the dimensions of the A3 sheet.

4.4.4.3 Episode 4.3 (Day-12)

On Day-12, Raziya and Aliza came to the blackboard and drew rectangles on the board to represent A3 and A4-sheets. They did the calculation shown in Figure 4.15 on the board to justify their reasoning.

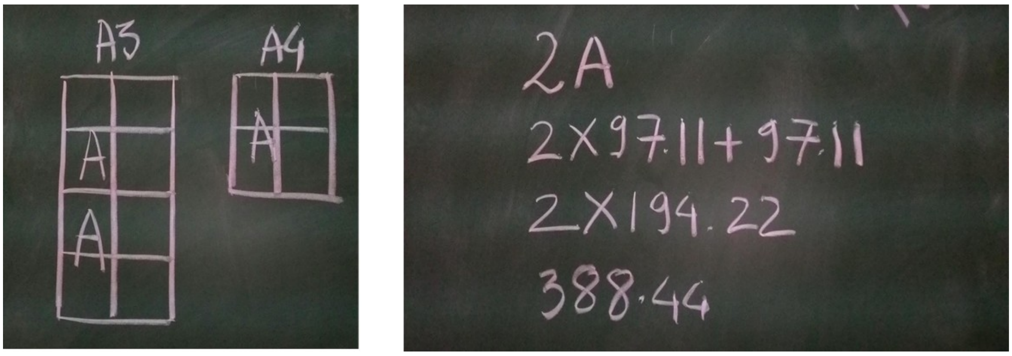


Figure 4.15: Raziya and Aliza's work on board

Previously they had offered a warrant based on the area formula and numerical operations. However, this time they provided a warrant by drawing pictures on the board and using the model of algebraic and numerical operation – I will refer to them together as "symbolic manipulation" – to convince other students that the size of the A3 sheet would be 388.44. Despite the teacher asking them to explain, they were quite convinced with their procedure and just ran through the operations rather than explaining the steps in a mathematically correct way. Hanif and Merajuddin from Group-1 pointed to the discrepancy in the calculation as below:

> Hanif: Teacher area area ko jama kia 2A aaya na toh phir wohi jama kisliye kerenge? [Teacher, adding two areas will give 2A, then why would one add them again?]
>
> Merajuddin: Haan teacher. [Yes teacher.]
>
> Aliza: Toh A, A agar iske badle koi number rehta toh? [So A, if instead of A there was some number, then?]
>
> Hanif: Haan toh jama kar dete, phir waha bhi jama ker de. [Then we would have added, so add there also.]
>
> Aliza: Allah! Wo kai ke liye likha hai tum ye batao usko. [To Raziya: Oh God! Why you

have written it, you tell him that.]

Raziya: A, A ko jama kerenge two A aayega. [Adding A,A together will give 2A]

Hanif: Aa gaya. [It came.]

Meena: Haan. [Yes.]

Raziya: Ye iska double hai, A4 ka ye A3 hai na. [A3 is double of A4, A3 has is made up of A4]

Aliza: Ye area ka hum logo ko na number pata nahi hai islia hum log A dale iske liye. [For this area we don't know the value in number, that's why we wrote A for this.]

Raziya: Ye iska double hai na, phir se humne double ke liye 2 likhe, ab ye dono ko hum logo ne, A A ko jama kiye the na, yahan pe kuch number nahi tha isliye hum A, A liye the, ab A ka value nikla tha 97.11 nikla tha. [A3 is double of A4, so we wrote 2 for double, now since we had added these two, we have added A, A together, there was no number, so we took A, now A's value is 97.11.]

Hanif: Haan. [Ok.]

Raziya: Four A ka rugba 97.11 nikla tha hum logo ka, tum logo ka jo bhi nikla ho, toh phir hum logo ne uske jama kar diye. [Four A's (A4's) area was 97.11 for us, don't know about your value, then we added them.]

Aliza: Dono ko mila dia, ye kiya na jama, idhar. [We added them together, here we have added them.]

Raziya: Dekha? [Did you see?]

Discussion about the l × b formula

Raziya and Aliza were convinced with their algorithm and their explanation. A little later Merajuddin from Group-1 expressed his doubt about using multiplication to find the area: "*jarab kyu ho raha hai?*" (why is multiplication used?). Raziya defended it by saying that it is a formula for area to multiply the length and the breadth. Hanif also expressed similar doubts about the length and breadth being multiplied to get the size. Thus, Merajuddin and Hanif challenged the warrant used by Raziya and Aliza, that is they challenged the use of the pre-existing mathematical formula for area. At this point, the teacher asked the students if the results would be the same in both cases, i.e., counting the total number of squares and using the multiplication strategy. Raziya and Aliza were sure that both the strategies would give the same result. Aliza, to provide further backing for their warrant, went to another room to get a poster with array structuring drawn on it to prove that multiplying the lengths is the same as finding the total number of units. We note however that the array drawing with only whole units is insufficient to abstract the $l \times b$ generalization.

Raziya and Aliza came forward to explain their solution. Raziya said that finding the size of the A4 or A3 sheets is actually a problem of finding the area (*Ragba* in Urdu) and claimed $l \times b$ to be the formula for area. Until this point the word “area” was hardly ever used in any discussion in the classroom. At this point, Raziya and Aliza showed the charts, one with the definition of the area including the $l \times b$ formula and the other with the array structure. Raziya justified the $l \times b$ formula by taking an example with 12 units along the horizontal and 4 units along the vertical, with the total units inside the resulting rectangle amounting to 12×4=48 units. Merajuddin interrupted her asking why she had multiplied 11.7 by 2 (while doubling to find the size for the A3 sheet). At that moment, Raziya’s flow was disrupted and the teacher pitched in to exemplify the area formula using the principle of variation. She took a few more examples starting with whole units like 3×6=18, 6×7=42, 8×7=56, and then moved beyond the whole units to include fractional units as well. With the last whole unit example of 8×7=56, the teacher added another 1/2 or .5 at the end of 8 units (making the length 8.5 units) and asked students the total number of units in that case. Finally, the teacher considered a rectangle with 7.5 units and 8.5 units along the length and breadth respectively. Using these examples, the teacher tried to make the connection between the two strategies explicit, of which one strategy was additive involving counting the total number of units, while the other was multiplicative involving multiplying the units along the length and the breadth.

In the second half of Day-12, Merajuddin asked how many boxes there were in A4, showing that he could not understand how $l \times b$ could give the number of boxes (squares) on the sheet. He was insistent on asking: “*boxes kaise gina ? Gin ke bata do*” (how did you count the boxes? Show the counting.). This again presents an example of a different argumentation structure possessed by the students from Group-1, where the field of argumentation is purely unit structuring (or counting the area units in the given space), while the argumentation structure possessed by Group-2 is of using the multiplication of dimensions (or numerical calculations) and extending it to A3.

Raziya then advanced a different and new argument to counter Group-1 by saying that they have left 0.36 units while computing the size. She did the following calculation on the board (Figure 4.16) to show the same.

At this point, the discussion returned to finding the size of the A3 sheet.

Size of A3 revisited

Merajuddin again asked about A3, saying that A3 should be double (of A4), but Raziya responded saying not double but four times. Merajuddin came to the blackboard, pointed to A3 and asked “*ye char guna hai?*” (is this four times?). The teacher asked Raziya to respond.

This time, she drew another model on the board as below (Figure 4.17a),

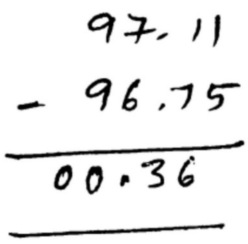


Figure 4.16: Calculation to show the difference in A4 measurements by two groups

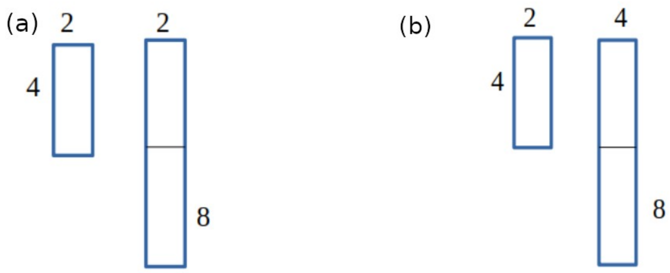


Figure 4.17: (a) Raziya's drawing on the board, (b) Merajuddin representing Raziya's A3 calculations

Merajuddin then jumped in and on the same drawing wrote 4 and 8 for the rectangle on the right as shown above in Figure 4.17 (b) to reveal Raziya's strategy for A3 to the whole class.

Meena (from Group-3) noticed and highlighted the error by asking "*4 kaise aayega?*" (how 4 will come?). Thus, with further questions from Meena and the teacher, Merajuddin rather than extending his argument (that both the dimensions are not getting doubled but only one is getting doubled), he went back to his seat.

The teacher then asked Raziya to complete her explanation. Raziya came to the board and started computing numbers on her own on the board where she doubled the longer side of the A4 and multiplied it with the shorter side of A4. Finally, she got the size of A3 to be double of A4. At this point, Raziya erased and changed her answer and admitted that the area for A3-sheet was coming out to be four times that of A4-sheet earlier, because they had doubled both the length and breadth while finding the area of A3. At this point, the teacher thought that the situation got resolved in the classroom and that finally Group-2 realised their error. But only when we watched the video of the classroom again did we realise that Raziya had doubled the longer side of A4 rather than the smaller side of A4. Though both the calculations give the same value, spatially/geometrically Raziya's representation would be incorrect, since doubling the longer dimension of the A4 does not give the shape of A3.

At the end, there was not much time left, so the teacher asked the third group to come and explain their way of finding the size of A4 and A3. Rehana from Group-3 came to explain their way of solving (or their way of reasoning). Rehana said they have got 11.7 along the length and 8.3 along the breadth (using scale/ruler) and declared the result to be 94.78.

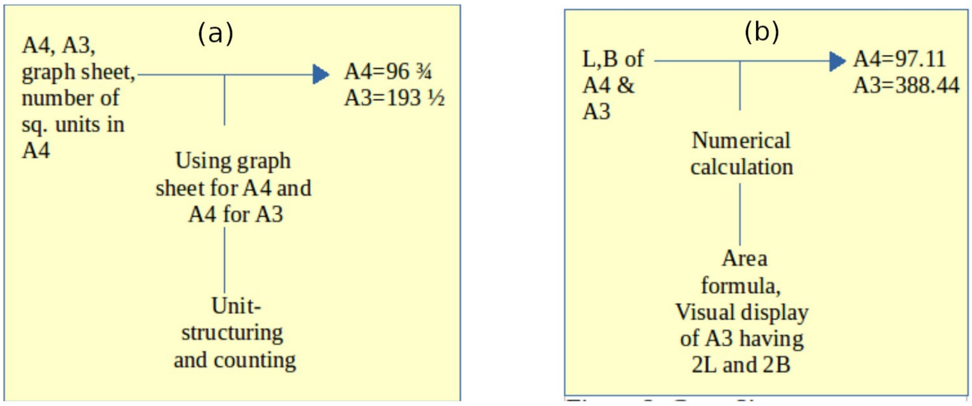


Figure 4.18: (a) Group-1's argument, (b) Group-2's argument

When the teacher asked for an explanation, she said that she had counted the smaller boxes in the remaining part. The sheet of Group-3 on which they had worked also had an array structure drawn on them (but not very precisely drawn). They had used multiplication while counting for the 11×8 units and for the remaining part they had approximated it as halves and quarters and had added them. So, Rehana said that they had got eight halves so 4 full units, making the total as 92(=88+4). Then two sets of 4 quarters making the total as 94 and at last left with 3 quarters and 3 smaller units(squares), making 94.78 as the answer. She showed the addition (75+3=78) but when the teacher asked her to show that part on the graph, she felt that the last remaining part to be 7 and not 3. Then the teacher let her change and modify the calculation. Thus Group-3 used the strategy similar to Group-1, though this group used more of multiplication and approximation.

Raziya from Group-2 subtracted the value obtained by Group-3 from those obtained by Group-1 and Group-3. She came to the board and showed the following subtraction:

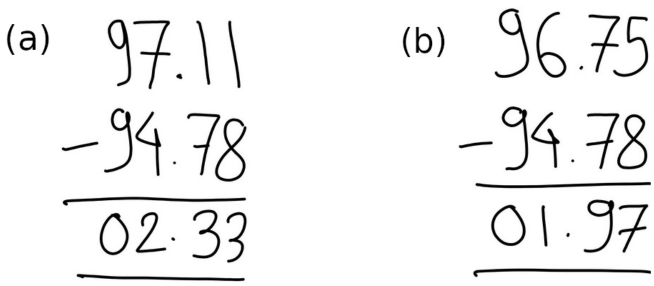


Figure 4.19: (a) Difference in A4's area between Group-3 and Group-2,
(b) Difference in A4's area between Group-3 and Group-2

Raziya's aim was to bring to the notice of the other groups how the strategy used by her group is more efficient and that there are errors in the area calculated by the other two groups.

4.4.4.4 Some observations on episode 4

Yackel & Cobb (1996) talked about the mathematical norms that students were exposed to in the classroom, which is irrespective of the mathematical content (e.g., algebra, fraction, geometry, etc.) being discussed in the classroom but more about the practice and the way it is being dealt with inside a classroom context. Again, the argumentation theory of Toulmin is applicable for any logical practice of reasoning irrespective of which discipline we are working with. However, the mathematical practice of establishing a mathematical claim is different from the general social discourse context, where the participants (or actors) try to convince each other through other forms of warrant, which can be subjective, verbal, have supporting examples, or analogy. The students were becoming aware of the classroom mathematical norms of justifying their reasoning with proper or acceptable mathematical explanations different from the general warrants. Group-2 was so convinced with their strategy of using the $l \times b$ formula, that their lead members (Raziya & Aliza) used different kinds of warrants to convince other students that it is correct. They used examples to show that the formula gives results that are consistent with unit-structuring, but also used the warrant or the authority of the mathematical formula and mathematical expression to support their claims for the size (or area) of A4 and A3 sheets. What Group-2 may have abstracted about the classroom mathematical norm of giving the warrant could be producing mere mathematical expressions. Students draw out the general practices happening inside the classroom as classroom norms. Thus, what they draw from mathematical argumentation as a part of the classroom mathematical norm is that something that involves a mathematical expression or numerical manipulation, can provide a warrant. That is why they may have used some mathematical forms or expressions to convince others that they are correct. Here it appears that the backing for the argumentation structure of students of Group-2, was coming from the mathematical norms or practices of using mathematical expressions and formulas without getting into their correctness and thus not engaging with the discursive argumentative practice (of reasoning and logic) when challenged by other groups. Thus, this form of backing or norm comes from the conventional classroom practices where mathematical formulas hold an independent authority without question.

Group-1 on the other hand presented a completely different strategy, where they extensively used the understanding of box/ square units. They raised a doubt about whether the formula can give the total number of squares units for an A4 or an A3 sheet, which led to the start of a series of argumentation between Group-1 and Group-2. This also shows that the jump (from concrete drawing of units to abstract formula) is not so easy to make. Group-2 used the warrant of mathematical formula and expression to convince Group-1, and they extended the warrant even to explain their result for the size of an A3 sheet. Raziya's warrant likely stemmed from her previous experience of using mathematical formulas, despite Merazuddin's tough challenge that A3 cannot be four times the A4 sheet by visually or spatially showing that it is only twice.

At one point Raziya from Group-2 admitted that their area for A3-sheet was coming out to be four times that of A4-sheet, and she said that they had doubled both the length and breadth while finding the area of A3. Although the teacher thought that the situation was resolved in the classroom, it was only when we watched the video again did we realize that Raziya doubled the length (longer side of A4) rather than the breadth (smaller side) of A4. Although both the calculations give the same value, spatially/ geometrically Raziya's explanation does not work for the present task. Raziya's numerical solution was not anchored in the appropriate spatial reasoning.

4.5 Discussion

I summarize some of the broad insights that can be drawn from these episodes below under the following themes: (i) Disconnect between spatial and numerical understanding (ii) Insights from the analysis of argumentation in the four episodes (iii) Reflections on socio-mathematical norms in a classroom and (iv) Coherence and contribution toward the network model.

4.5.1 Disconnect between the spatial and numerical understanding

In the first episode, students' argumentation was based on unit-structuring as they were more comfortable relying on counting the full units rather than accepting that the numerical value of the area was halved since the halves were congruent as shown by the teacher. Even though the two triangular halves were spatially or geometrically equivalent, students' assurance came from the numerical value of 15 units. This indicates a disconnect between students' spatial and numerical understanding of area-measurement. This also supports Sarama & Clements' (2009) claim that the problems in the learning of area-measurement could be due to difficulty in connecting the spatial and numerical aspects. The basis of students' reasoning was more aligned to the additive thinking of counting units rather than the multiplicative thinking reflected in the fact that halving an object will correspond to obtaining half the number of units compared to the original object. In other words, multiplicative thinking here involves the coordination of equipartitioning and the numerical operation of division.

Battista (2007) has emphasised that students must be able to extend their reasoning to different forms of units. The task used in the second episode is based on a fractional unit. In the second episode it was found that different students can identify and consider different fractional parts as their fractional unit, but while representing that fractional unit they had difficulty in assigning a numerical value to the fractional unit in terms of the full unit.

In the third episode, students were considering two different numerical values as the same, as they were derived through the same spatial structuring. This again points to the difficulty of moving between spatial and numerical understanding.

In the fourth episode, while Group-1 extensively relied on the spatial structuring and counting of the units, Group-2 used the numerical formula without fully abstracting it from the spatial structuring. Again for the A3 sheet, rather than just doubling the area of A4 to get the area of A3, they doubled both the linear dimensions of A4 to end up getting four times the area of A4 for A3. Thus, Group-2's reasoning was completely devoid of spatial reasoning.

4.5.2 Insights from the analysis of argumentation in the four episodes

The focus on argumentation structures that emerged in the various episodes gives insights into the students' struggles to construct the concept of AM. It highlights the key components that underlie this construction. It also shows the robustness of some of the students' prior conceptions, especially with regard to unit-structuring, the decomposition and recomposition of halves and quarters and their interpretation in terms of numerical representations. It indicates the gaps in what students know and the potential interventions that are pedagogically important.

In Episode-1, there was a difference between the argumentation structure of the students and the teacher. Despite realizing that the two triangular parts found by diagonally dividing a rectangle are congruent, there was a resistance on the part of the students in accepting that each triangular half has half the number of units as in the rectangle. From the argumentation point of view, we see that the basis of warrant for the students and the teacher were different. The students' warrant came from the unit structuring, specifically with the number of full units that can be made from one triangular part. However, the teacher gave a different warrant, which is that the two triangular halves are congruent, which did not convince the students since they were looking for a different warrant.

The core of this issue is the correspondence between actions done on the rectangle and the operations with the numerical value of area. For the teacher, the decomposition of the rectangle into two congruent halves had a correspondence in dividing the numerical value of the area by 2. For the students however, the correspondence needed to be established by means of unit structuring. By focusing on the argument structure and the differences in the warrants, we see the gaps in the warrants that are available to the teacher and to the students. This can guide instruction in designing tasks and in steering the discussion towards bridging these gaps. For example, the students already use their knowledge that the joining of shapes to give a larger shape corresponds to the addition of the numerical values for area. This is shown in the process by which they accumulate units, or fractional parts of units in all the episodes. However, this knowledge needs to be coordinated and re-applied in instances where the shapes being recomposed consist both of triangles and rectangles, as well as in other cases (Chambris, 2022). This we expect will lead to more robust correspondences being established between spatial and numerical aspects of AM.

Further, the argumentation perspective also reveals the complexity of the geometric intuition that the students are drawing upon. Pressing for further warrants and backing in the students' argument could lead to understanding the geometrical basis for Raziya's demonstration that the two triangular parts have equal number of units. Establishing the congruence of the small triangular pieces that are moved into empty slots, which Raziya showed by drawing arrows, requires detailed and, for the students, sophisticated geometric reasoning. This might take the path of symmetry and geometric transformations such as rotation, translation, etc., or might draw on theorems related to triangle congruence. (A possibility of such argument structures emerging also exists in the manner in which the congruence of the diagonal halves of the rectangle was established by overlapping. However, this did not emerge in the classroom although Sajjad and other students struggled to show that the two halves were congruent.)

In Episode-2, there are two task demands that the students are struggling with. The first is to find a suitable way of denoting the size in terms of the given unit of the fractional part that is appended to the rectangular shape. The second demand is to add this to the remaining units to arrive at a numerical value of the area of the entire shape. In Part-1, the response given by Suhana (3.3 units) is accurate. However, when pressed to provide a warrant, we see that her reasoning is whole number based, where she counts 3 whole units and 3 smaller units (which are rectangular strips). She completely ignores the relation between the smaller strips and the given unit. Hence, she is using the decimal point as a separator, where the units to the left and to the right are different, with an unspecified relation between these units. This is a well recognized difficulty that students face in learning decimal fractions (Takker & Subramaniam, 2019).

In Part-2, we see that Aliza does a careful recomposition of the smaller strips and tries to establish a relation to the given unit. However, this attempt is not successful, because she designates three units and a quarter (which is seen to be a small square) as 3.5, to which she adds the size of a smaller strip to obtain 3.6. Her reasoning is ambiguous in terms of whether she is using whole number thinking or thinking of the relation between the smaller units and the given unit. Raziya's intervention clearly pulls the focus back to the relation between the smaller part and the given unit, when she says that 3.5 means three and a half, but the smaller part is a quarter and not a half. This only shows that 3.6 is incorrect, but does not lead to the correct numerical representation of the total area.

In Part-3, Raziya takes a different approach to representing the size of the appended part of the rectangle. She uses a fractional notation, which she is confident about and denotes the size of the appended part as 3/10. She is also then able to easily represent the addition of this part to the rectangular part to write the final area as 3 3/10 units. By looking at the structure of the arguments that the students put forth, we see both what the students are comfortable with and what they find difficult. They are quite comfortable with decomposing and recomposing rectangular parts out of a larger rectangle and in seeing the relation of the smaller parts to the larger part in terms of such actions. However, not all the students have the resources to coordinate such robust spatial understanding with the numerical representation of fractional parts. Moreover, we also see the difficulty in making the transition from fractional representation to decimal representation. Again, this points to the close and deep connections between AM and the topic of fractions and decimals.

The argument used by Sajaad in Part-4 of Episode-2 is interesting because he draws on a different knowledge resource to represent fractional parts in terms of decimals. He sees the appended portion of the shape in terms of the small squares, which he counts as 30. He then identifies a quarter of the shape in relation to the given units and sees that this contains 25 small squares. He is able to draw on the analogy of money where he knows that a quarter rupee means 25 paise and is able to represent the final numerical value of the area as 3.30. This allows him to arrive at the correct representation, not through a place value based understanding of decimal fractions, but through an analogy with the representation of money. This suggests both the difficulty of the place value notation for decimal fractions as well as possible routes by which students may arrive at an understanding of this difficult concept.

The task in Episode-3 illustrates the principle of variation that was adopted in designing the tasks. Here, the given unit is a small strip instead of a larger unit as in the previous episodes. Thus, it gave rise to a process that was the inverse of the process seen in Episode-2 – here students chunked the given unit to form a larger unit, which was more convenient to measure the given shape. We see the students moving back and forth between using the chunked units to count and representing the size in terms of the given unit. Both the groups, whose arguments are presented, could successfully represent the size of the given shape in terms of the chunked unit. However, differences arose in moving back to represent the size in terms of the given unit. Aliza and Afia's group represented this as 60.5, while Sajaad's group correctly represented this as 65 units. The argument structure showed that the warrant offered by Aliza was that both the representations were of the same value. This was eventually refuted by Merajuddin comparing 60.5 and 61 and interpreting the former as 60 and a half, and hence less than 61. This showed that 60.5 was indeed less than 65 and hence could not be equal to 65. In this episode again, we see the struggles in representing numerical values for area, when the area is an addition of different kinds of units. We also see the importance of the representations for half, which are very familiar to the students, and can anchor warrants for claims about correct decimal representations.

Episode-4 captures extended discussion and complex argumentation among the students. We analyse this by identifying three different arguments. The first, which emerges as the teacher visits different groups of students and talks to them about the strategies, is concerned with the use of unit structuring to arrive at the size of the A4 sheet. Here, as in the previous episodes, students need to combine measures of whole units and fractional units. We see the detailed warrants that students offer to justify their procedures for identifying and counting different sized units, as well as their justifications for representing the sums of whole units and fractional units. It is also evident that students are drawing on their familiarity with fractions such as half, quarter, three-quarters, one-and-a-half, etc. (Urdu and many other Indian languages have a separate word for one-and-a-half: "*deedh*".) Thus, warrants based on out-of-school knowledge can play a crucial role in the social construction of concepts in the classroom.

The second argument centres around finding the size of A3, from the known size of A4. Establishing that the size of A3 is double the size of A4 was easy by folding and overlapping. Two groups had found the size of A4 by unit structuring and these groups did not have a difficulty in finding the size of A3 by doubling or adding. Raziya's group had used the $l \times b$ formula to find the size of A4, and they arrived at an incorrect result for the size of A3, which was challenged by Merajuddin's group. It was clear from the responses from Raziya's group that they preferred using the formula and ignored its connection to unit structuring. It was also evident that they had difficulty in coordinating the use of this formula with the spatial understanding of the task, leading to them doubling both the length and the breadth in moving from A4 to A3. It was striking that when Razia was asked to show where the length and breadth had doubled, she showed the doubling of length along one edge, but showed a parallel image of the breadth to show the doubling of breadth. (Her explanation suggests that she interpreted the doubling of breadth as the emergence of another breadth through the action of folding the A3 sheet.) The demand for warrants by Merajuddin's group created the conditions for further exploration of Raziya's AM conception, indicating the difficulty of coordinating the algebraic understanding (i.e., the use of the $l \times b$ formula), with spatial understanding. It also shows that unit structuring is closely tied to spatial understanding and hence is essential to making a robust connection with finding the numerical value for area. Even after Raziya's group accepted their error, and arrived at the correct value for the size of A3, the underlying spatial understanding was flawed. This was evident in the way their geometric representations of the A4 and A3 shapes. Clearly, the coordination between algebraic and spatial understanding is difficult even for capable students.

The third argument that emerges in Episode-4 is a branching of the discussion into the validity of the $l \times b$ formula. It is interesting that Merajuddin's group identified the use of the multiplication as one of the steps in argument by Razia's group that was open to challenge. Multiplication was used in two ways by Razia's group – in the $l \times b$ formula, as well as to obtain the length, breadth and area of A3 from those respectively of A4. However, the argument that emerged focused mainly on the use of the formula, likely because the teacher steered it in that direction. We find from the analysis of the discussion, that three different kinds of warrants were used to establish the validity of the $l \times b$ formula.

The first kind of warrant is based on authority, and is mentioned by Raziya when she is first challenged about the use of the formula. She responds by saying the teacher (in the regular school classroom) had given this as the formula for area. She also explicitly interprets "size" of the rectangle as the "area" of the rectangle, showing that a part of the warrant stems from drawing on formal mathematical vocabulary. This is backed by Raziya and Aliza bringing a poster, which had the formula, to show to the class. It appeared, however, that this warrant was not acceptable to the other groups. This is indicated by the additional warrants that the teacher offered, which showed that the argument was not brought to a closure by drawing on the authority of the poster.

The second kind of warrant for the validity of the $l \times b$ formula was to show that it gives a result which is consistent with finding the size using unit structuring. This was the approach taken by the teacher, where she produced several examples of rectangles with varying length and breadth. We note that although this may be convincing for the cases where the length and breadth are whole units, a more detailed argument is needed for the cases where the sides are not whole units of length. The teacher did extend the warrant by using a couple of examples with half units of length and breadth. This discussion is important and may need to be taken up again to establish a more robust connection between unit structuring and the $l \times b$ formula. We also note that the teacher appears to have assumed that students are comfortable with measuring length and breadth in fractional units, which requires the process of unit structuring to be applied in one dimension, along the length or along the breadth. Further discussion and potential challenges may reveal that the unit structuring involved in measuring length and representing the value of fractional lengths is also an important element in the network model of the learning of AM.

The third warrant was produced by Raziya and was something that we did not anticipate prior to the analysis of this episode. This warrant has to do with the accuracy of measuring the size of a rectangle. Raziya intervenes at two places in the discussion to show that the use of the $l \times b$ formula produces a more accurate value for size as compared to unit structuring. She does this in two ways – by pointing out that the process of unit structuring may be missing some parts, which are too small. She also finds the difference between the values for A4 size produced by the use of the formula and by unit structuring by the operation of subtraction, which gives the value of the "error" in using unit structuring. The warrant here that Raziya is offering is that the $l \times b$ formula produces a more accurate value for size. This brings in the aspect of approximation introduced by unit structuring, which is an important mathematical construct that is part of the learning of AM, and plays a key role in making the connection between the algebraic, numerical and spatial understanding of AM.

4.5.3 Reflections on socio-mathematical norms in a classroom

As elaborated in Section 4.2, "backing" is considered to be an important element of an argument structure and is defined to be a set of established norms, rules, or logic followed in a specific discipline. In the current study done in the classroom context, guided by the theory of social construction, the "backing" for the warrants in the arguments discussed stem mainly from the socio-mathematical norms established in the classroom. Such socio-mathematical norms established in the classroom act as the key agent in further supporting the warrant provided by the students. This can be seen across the episodes where students adopt the practices of social negotiation in a classroom by representing one's thought/argument through dialogue, using physical materials, drawing on the blackboard, and showing symbolic and numerical representations. These practices were collectively established as socio-mathematical norms in the classroom as a way of substantiating the warrant.

In Episode-4, we saw that Raziya's group associated area with $l \times b$, where only the linear dimensions are used; her explanations were mostly based on linear dimensions. Also, Raziya's group could not associate it with the 2-dimensional unit of area. However, the other students who did not come across the area formula frequently, mostly reasoned in terms of unit-structuring. What we further notice in this episode is that Raziya's group was repeatedly using the mathematical norms drawn from their regular conventional classroom as backing for their claim and warrant. However, as educators, we need to be aware of the unquestioned practices we are transacting through mathematical forms and expressions to our students, which students are expected to accept as mathematical truth and logic just because it can be expressed in some particular mathematical form. This in turn results in mathematical logic and reasoning becoming limited to symbolic representation and manipulation. It is only when Mirazuddin's group challenged Raziya's group for their procedure of measuring A4 and A3 (the use of $l \times b$ formula for A4, and the extension to $2l \times 2b$ for A3) that the gaps in the group's understanding of AM were revealed. This also raises the question, what are the factors that lead to the emergence of argumentation in the classroom as reflected in these episodes. First, it is important that the teaching design experiment incorporates some flexibility in the pedagogy to allow for the instruction to be guided by students' responses. Second, if we try to see what motivated the students to challenge the claims of their peers, leading thereby to the emergence of argumentation in the classroom, we can find that it is often a moment of doubt or curiosity that comes with not being able to comprehend the abstractions that are linked to the concrete modelling of the situations. A third factor may be the need for consistency between different ways of solving a problem, which we discuss in the next section.

Thus, this chapter of the thesis shows how actions with concrete representations get paired up with mathematical discourses/narratives, thus bringing in reasoning as an essential part of concept formation. And overall it contributes to understanding why reasoning is essential to concept formation and suggests that coherence among representations is essential for the stability of concept.

4.5.4 Coherence and contribution toward the network model

In numerous instances, the students challenged claims made by their peers or raised a doubt while striving to achieve consistency between different ways of arriving at a result. In Episode-1 for example, after Sajaad was convinced that each of the triangular pieces obtained from the rectangle was half the rectangle, he was struggling to find a reason for why the size of each piece would be 15 units. Raziya's response could be seen to be driven by the same motivation as she found a way of reconciling the size of the triangle arrived at by halving 30 units and by unit structuring. Raziya's response stands in contrast to the response by the teacher, whose argument structure is essentially different from the ones the students' and hence failed to make a connection with the students' argument. Raziya, on the other hand, placed her own warrant within the students' argument structure. I mentioned another instance where Raziya adjusted her own warrant in order to align with her discussant in section 4.4.2.3, where Aliza was disputing her claim that the size of the appended part of the rectangle was 3/10. Aliza argued that this must be 6/10. In response, Raziya pointed out that the fractional unit that Aliza was focusing on was 1/20 and not 1/10 and hence the size would be 6/20. This could be reduced to 3/10, thereby establishing the consistency of different ways of counting the fractional units. Recognizing and responding, even implicitly, to the argument structures in play is important in building coherence between perspectives and approaches.

The repeated emergence of struggles to achieve coherence points to the importance of the network model. The concept of area has rich interconnections with fractions, geometry and algebra, and these need to be repeatedly revisited to allow for a mutual strengthening of the networked concepts. From the discussion of the episodes, we see that students often display a robust but fragmented understanding of parts of this network. For instance, they show ready facility with decomposing and recomposing parts, and with identifying and describing halves, quarters or even one-tenths and one-fifths. However, they are unsure about how to represent these numerically and how these are incorporated into the numerical operation of addition. Thus these parts of the network need to be revisited repeatedly during instruction to strengthen the interconnections.

The discussion in this chapter shows that the measurement procedure of unit structuring is learned robustly and must be repeatedly invoked to support reasoning and learning of other aspects. In the network model for the learning of AM, unit structuring is a key node that is learned robustly and can potentially support other kinds of reasoning and learning. The network model suggests that learning a complex concept like area measurement, proceeds in an inter-connected rather than a linear manner. Critical learning outcomes like coordination between different ways of measurement area, learning the correspondence between actions on a model and numerical operations, etc., must be visited repeatedly through multiple kinds of tasks. More importantly, they must be visited through discussion/ argumentation in the classroom. This is captured well by a network model, where students need to traverse back and forth making connections between elements, rather than establish the relevant mathematical facts progressively in a linear fashion.

Figure 4.20: The classroom in chapter 4

5

Analysing the effect of material interaction on students' area-concept

This study emerged as a result of some of the unresolved questions of the previous studies reported in the thesis. In the study discussed in the previous chapter, the analysis focuses mainly on the aspects of social interaction while analysing the conceptual formation or construction happening within the classroom. Thus, even though a lot of effort went into integrating the conceptual underpinnings of area (as elaborated in Chapter 3) into the task-design, the significance of the task-design or the material environment could not be analysed further in the previous study. One reason for not being able to do so, is because the scope of the previous study was within the social interaction/ social construction model. The complex classroom setup poses several other challenges that limit any focused investigation, where one can zoom into studying the processes where a learner engages in manipulation of physical materials. It has been argued that manipulations are significant in bringing new discoveries or pathways for solving new problems (Chandrasekharan & Nersessian, 2015). Manipulations are "material interactions" in contrast to the social interaction discussed in the previous chapter. Thus, in this study the goal is to investigate the significance of material interaction in student's conception of area-measurement. The present study falls under the strand of enactivist theories, which is recognized by mathematics educators as an emerging promising strand both theoretically and methodologically (Reid, & Mgombelo, 2015; Abrahamson, Dutton, & Bakker, 2021). Although several studies have provided results showing the significance of manipulations (or material interaction) in math education, most studies end up relying on the (i) outcome/test based results or (ii) discourse (classroom social interaction or social form of explanation) provided by the learners. Both these ways of studying the significance of material interaction have limitations. The outcome based studies are open to the general criticism of assessment being separate from meaningful education or a safe learning environment (Sjøberg, 2018). Throughout this thesis, there is an attempt to focus on the processes (changes) of learning rather than the outcomes or post tests per se. In this particular study, our main focus is to understand the processes involved in the material-interaction rather than the outcome of such manipulation as evidenced in the students' assessment. Further, studies based on discourse or social interaction are impacted by the social hierarchy that exists in the society outside the classroom and gets reflected within the classroom, i.e., the same hierarchical dynamics of knowledge construction that exists in the society (Restivo, 2017). Although while designing a teaching experiment and establishing the socio-mathematical norms, one intends to design the classroom as a safe and democratic space for every child to express themselves, it is likely that the psycho-social reality (or hierarchy) experienced by the child comes in the way of their verbal expression. Thus, even though social interaction plays a primary role in the social construction of a concept, on most occasions, they tend to be limited to only a subgroup of students who are more vocal in the classroom, bringing in the need for a more accessible material interaction to students. Material interaction through physical manipulation brings newer pathways, concepts, or ideas (Chandrasekharan & Nersessian, 2015). In this particular thesis, material interaction leads to new arguments, definitions, and strategies that emerge not just through discourse or social interaction but also require the presence or experiences of matter, or "material interaction."

The advent of new media tools has opened up a new horizon in studying the importance of manipulations on cognition and learning under the umbrella of 4E-cognition[14] (Abrahamson, Nathan, Williams-Pierce, Walkington, Ottmar, Soto, & Alibali, 2020). However, the field is dominated by new technologies and digital tools, which the majority of the population in developing countries is still struggling to get access to (Sacristán, Rahaman, Srinivas, & Rojano, 2021). Thus, while this new strand of research on cognition focuses mainly on the digital affordances provided by the new media, it tends to ignore the economic affordability of the new media in the context of developing countries. Thus, in this chapter we intend to study the effect of affordable physical tools drawing on the insights of the recent advancements in the domain of cognitive sciences.

In the previous chapters, we reported studies on students' conception of area in several settings that included a naturalistic (e.g., school) setting, a task-based interaction setting and a classroom setting (intervention study). The main focus of the first set of studies described in Chapter 3 was to understand students' conception of area in the naturalistic setting and then in a more focused task-based interaction setting. In the second study, presented in Chapter 4, we moved on to understand how students construct the concept of area in a classroom setting through social interaction. In the classroom setting, our focus was to look at the active role played by a particular form of social interaction in the co-construction of a concept.

As can be seen in the previous chapters, apart from the social input or interaction, students were also interacting with and manipulating the materials at hand while solving area problems. In this chapter, we seek to understand whether such material interaction and manipulation influences students'

14 4E-cognition stands for a combination of embodied, embedded, enacted, and extended cognition

conception of area, and if it does, how material-interaction shapes the learner's understanding (or engagement) with the area concept. For this, we needed to design a study where students, before solving an area problem, interact purely with the material, with minimal to no interference of social interaction or inputs. In accordance with this need, we have used novel eye-tracking methodologies to capture the cognitive processes difficult to capture through communication alone. The potential of such methodologies for mathematics education research (MER) is being increasingly recognized (Strohmaier, MacKay, Obersteiner, & Reiss, 2020).

Thus, unlike the previous study, where the interaction-based data was mainly discursive, the present study focuses on interaction that is non-discursive and action oriented. Students interacted with material in our previous studies as well, but we could observe and analyse the interaction only at an overt macro level. To understand how material-interaction affects students' understanding or engagement with area, a more microscopic process analysis was required. For this, in addition to the video recorder, we used an eye tracker to have more microscopic observations and record of actions with hands and eyes. However, most of the studies using eye-tracker methodologies do not integrate different forms of data recorded along with the eye-trackers data while making sense of the observations (Strohmaier, et. al, 2020). Thus the present study also tries to integrate different forms of data to arrive at a coherent understanding.

The studies mentioned in this chapter are a collaborative work with the LSR[15] (Learning Science Research) group at HBCSE[16], and is thus markedly different[17] in scope, style and presentation from the previous studies (and chapters).

5.1 Literature Review

Manipulable instructional aids play a key role in learning-by-doing and constructivist educational approaches in general, particularly in the learning of mathematics at the primary and middle school levels (Martin, Lukong, & Reaves, 2007; Tchoshanov, 2011; Boggan, Harper, & Whitmire, 2010). The popularity of manipulatives as teaching tools is supported by studies showing that they scaffold the learning of both arithmetic and geometry (Uttal, Scudder, & DeLoache, 1997; Olkun, 2003;

15 More about the LSR group can be found here https://lsr.hbcse.tifr.res.in/

16 HBCSE stands for Homi Bhabha Centre for Science Education, it's the same institute from where I am pursuing my PhD.

17 This work is a collaborative work with the LSR group. Specifically, Sanjay Chandrasekharan provided overall guidance, Harshit Agrawal helped in data collection and Nisheeth Srivastava carried out the quantitative analysis of the eye-coordinate data. My contributions were: the design of the task and the study, data collection, qualitative data analysis, integration of this analysis with the quantitative data analysis, and interpretation of results.

Tchoshanov, 2011, Martin & Schwartz, 2005). However, the interactive process by which manipulatives change the cognitive system is not well understood. As manipulation-based learning of mathematics is popular and successful, this practice provides a well-structured paradigm instance to develop a study to look at the way knowledge emerges through interaction with external structures. A central objective of this chapter is to understand this interaction process by focussing on it at a micro level.

Apart from the interest in understanding the way material-interaction or manipulation affects the cognitive system, there is also significant application interest in this problem, as the design of new computational media for mathematics and science learning are aiming towards multi-touch and embodied manipulation of formal entities (Sarama & Clements, 2009a). Recent applications include systems to learn numbers (Sinclair & De Freitas, 2014), algebra (Ottmar, Weitnauer, Landy, & Goldstone, 2015; Weitnauer, Landy & Ottmar, 2016), vectors (Karnam, Agrawal, Mishra, & Chandrasekharan, 2016), proportions (Shayan, Abrahamson, Bakker, Duijzer, & Van der Schaaf, 2015), volume (Lakshmi et al., 2016), and equations and graphs (Majumdar et al., 2014). The design of such embodied interaction systems for learning are of interest from a theoretical perspective as well (Hutto, Kirchhoff & Abrahamson, 2015; Abrahamson, & Sánchez-García, 2016), because such designs work in dual mode -- as educational interventions as well as probes into the cognitive system -- thus providing insights into the way embodied interactions lead to the change/generation of internal cognitive structures. A related approach examines the role of gestures (Goldin-Meadow, Cook, & Mitchell, 2009; Alibali & Nathan, 2012) in mathematics learning. Actions on new computational media are also considered similar to the process of gesturing and drawing during the mathematical discovery process (de Frietas and Sinclair, 2014). These overt movements are hypothesised to be part of the mechanism that helps move body-based intuitions (about possible mathematical results) into externalised symbolic proofs, which are built using known and accepted mathematical structures (also see Sfard, 1994; Rotman, 2008; Marghetis, & Núnez, 2013).

Many cognitive models/explanations have been proposed to account for the way manipulatives contribute to the learning of early mathematical concepts such as fractions and area. One approach considers manipulatives as a specific instance of multiple representations, which help provide different perspectives (visual, symbolic, etc.) of the same concept, and thus improve students' understanding of the underlying mathematical principles (Moreno & Mayer, 1999). A second approach is the cognitive off-loading hypothesis, which suggests that physical manipulatives help distribute working memory load to the external environment, and this allows students to perform more sophisticated mental calculations than what their internal memory resources alone would support (Cary & Carlson, 1999). Finally, an action priming account suggests that manipulatives work by analogy, specifically analogous actions: “the best manipulatives are the ones that require physical manipulations that are analogous to the abstract mental manipulations required by the problem” (Hall, 1998). This analogous action account does not rule out the first two accounts (multiple perspectives, working memory offloading), but focuses more on the connection between procedures, which are embedded in the manipulative and the target concept to be learned.

However, the domain of MER is still reluctant to fully adapt or integrate an enactive cognition paradigm in MER (Schindler & Lilienthal, 2019). Although MER has adopted eye-tracking methodologies to some extent, it mostly interprets eye-tracking data as markers of attention. This revolves around the dilemma of the eye-mind hypothesis that assumes eye-movement as a marker of mind, which further assumes the mind as existing independent of the body (Strohmaier et al., 2020). Thus, there is a need for MER to revisit manipulation or material interaction by drawing upon the advancement in the theories of enactive cognition, and adopting the same in eye-tracking methodologies.

5.1.1 Research Questions and Study Design

The above section presents various studies beyond MER to highlight the significance of material interaction both from the educational and cognitive perspective. It also highlights the importance of drawing from the advancements in enactive cognition paradigm. As elaborated in the previous section the three major approaches (or cognitive accounts) to describe how manipulatives contribute to the learning of early mathematical concepts drawn from the enactive cognition paradigm are: (i) by providing multiple representations, (ii) by allowing cognitive off-loading, and (iii) by action priming (where, concrete manipulation primes the learner to do abstract mental manipulation). In contrast to the descriptive account mentioned in these three approaches, we are also interested in understanding the mechanism by which manipulatives support the learning of mathematical concepts, and seek to develop an account of this interaction process. Thus, the study attempts to move beyond the existing realm of MER from capturing student’s strategies to also capture the change in cognitive processes through manipulation. In this chapter, we seek to address the following questions:

1. What process change happens as a result of manipulation?
2. How does manipulation transform the process of solving an area-problem?

To address these two questions, we develop a new process-oriented study method, combining qualitative approaches (from education and problem-solving research) with an eye-tracking analysis based on transition matrices (an analysis method from neuroscience). The main components of this approach are not new, as both qualitative studies and eye tracking have been used for more than forty years to study children's problem-solving, starting with studies of “centered” and “decentered” eye fixations during the volume conservation task (O'Bryan & Boersma, 1971). We extend these approaches, to develop a novel analysis method that seeks to characterise task-oriented eye movements, where the eye is systematically moved to different locations in the task space during problem-solving. In this approach, the eye is treated as an actuator, and its movements are analysed, similar to the way the task-oriented movements of the hand are tracked during problem solving tasks such as Tower of Hanoi. Eye fixations during such problem-solving are considered to mark shifts in executive attention (Smith & Kosslyn, 2007), which allow the problem solver to track and integrate micro-level moves, sequences, and task switching during the solving of the problem.

In the present chapter, we have collected data from two studies based on this method. Analysis of this data provides insights into the way manipulatives change the problem-solving process. The design or outline of the studies (as shown in Figure 5.1) contributes towards addressing the problem of complex process analysis based on student actions and eye movements. To address Question 1 above (what process change happens as a result of manipulation), we outline a way to study how the actions done on external manipulatives transform the internal processes as in Study 5.1 (a cognitive account for the same is elaborated in section 5.4.1). For addressing Question 2 (how manipulation transforms the process of solving an area problem), we developed Study 5.2, to understand and uncover the essential factors involved in the design of the manipulatives or the nature of the material interaction.

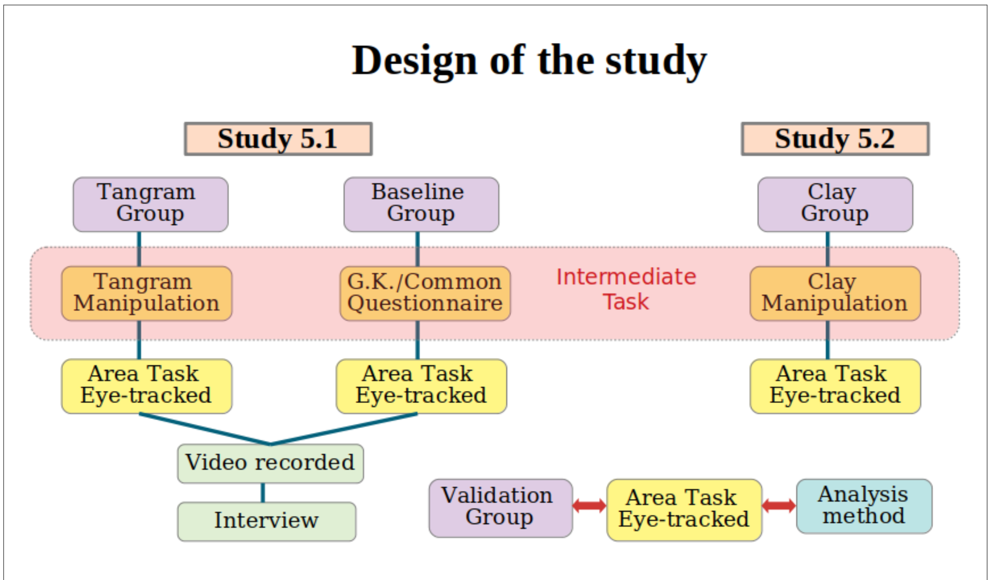


Figure 5.1: Design of the study ("G.K." refers to General Knowledge. The questionnaire can be found in Appendix V.)

5.1.2 Area problem and material interaction

The use of manipulatives in learning, while popular with teachers and supported by results from individual research studies, is not fully supported by converging evidence. Meta-analyses and reviews of studies comparing groups and classrooms that use and do not use manipulatives show no overall advantage for manipulative use (Sowell, 1989; McNeil, & Jarvin, 2007; also see Bosse et al., 2016). One way to start the process of reconciling this divergence between classroom practice and overall research findings is to characterise in detail:

1. The specific cognitive changes, if any, induced by manipulatives in the cognitive process of solving particular mathematics problems.
2. The ways in which these specific shifts in the cognitive process contribute to the understanding of mathematics concepts.

Based on such studies of the cognitive process, the specific contributions of manipulatives, in helping the student understand or generate mathematics procedures and concepts, could be characterised. Educational interventions could then be designed to exploit these specific cognitive elements. This type of cognitive analysis, seeking to characterise in detail the micro-level interactions involved in the problem-solving process, is the broad research approach followed here. The type of studies focused in MER so far have still not captured, inferred or integrated the various actions and interactions happening while solving problems (Schindler & Lilienthal, 2019; Strohmaier et al., 2020). Thus the current study attempts to integrate various interaction data, to develop a cognitive process account. The specific education domain examined in the analysis here is the problem of calculating area (or measuring area). We characterise how the cognitive process involved in solving area problems changes after working with manipulatives. This process account is developed through the tracking of hand and eye movements.

In the studies reported in the present chapter, the eye tracker is used as a micro-level observation device (similar to a microscope), to generate a highly detailed qualitative picture of the task-oriented eye movements during the problem-solving process. To develop an understanding of how the task-oriented eye movements change with the problem-solving context, we used a two condition (baseline, study) intervention approach. This study approach is similar to qualitative field studies in ethology and anthropology, as well as classroom studies in education, where controlled interventions are often used in combination with qualitative observation, to generate and characterise problem-solving behavior. The characterisation involved is similar to qualitative studies of problem-solving, which tracks the task-oriented actions and moves in the problem-space. Extending this method to eye tracking, we focus on task-oriented eye movements, and thus the actuator role of the eye, rather than the perceptor role. In the analysis approach we use, the fixation data is treated as an indicator of shifts in executive attention, which helps control and track the task-oriented movements of the eye (visits, returns, sequences, switching, etc.) while solving the problem.

Manipulation of physical dissection models, such as assembling of unit figures, tiling and covering using units, etc., are standard teaching approaches to make the area concept easier to learn (Outhred and Mitchelmore, 2000). Intuitively, this appears to be an effective (and required) approach to teach area, as area is a property of physical entities. However, it is not clear how these physical manipulations help in understanding the formal notion of area, particularly in bringing together the different mathematical constructs involved in the area concept. To understand this process, the studies we report here use an intervention similar to the use of manipulatives in classrooms, to examine how working with manipulatives just before doing two area tasks changes the process of solving the area problems. The results suggest that students who worked with the manipulative intervention chunk the test figure differently from the baseline group, and calculate area using a real-time approach to change the composition using smaller figures. However, this strategy does not lead to significant improvement in accuracy. This nuanced set of results demonstrate how learning by doing could be seen as failing (as indicated by the meta-analysis result discussed above), even when the manipulative intervention changes the problem-solving process in the right direction -- a case of partial transfer (Bransford & Schwartz, 1999). This dissociation between process and final solution could partially account for the conflict between teaching intuitions/practice and the meta-analysis result. A contributing factor might also be seeing only the final test result, which might even change due to very minor errors, rather than looking at the process of solution.

To systematically develop research designs that address this dissociation, particularly new media designs, a general cognitive account of how interaction with manipulatives changes the process of solving problems is needed. Here we develop such a general cognitive account, outlining possible mechanisms that underlie the changes in problem-solving processes generated by manipulatives. Thus, in this chapter we try to develop and design an enactive cognition approach to understand the way manipulatives change the problem-solving process.

Although, in this chapter we draw inspiration and guidance from the pedagogical use of manipulatives in solving mathematical problems, particularly related to area, our broader and general objective here is to understand the cognitive mechanisms underlying manipulative-based learning. So the focus is on how manipulatives change the problem-solving process, and which cognitive mechanisms support this change. We do not seek to provide an account of the way manipulatives (over a period of time) help solve the complex integration problem involved in learning area as a network of concepts. Understanding this integration problem requires a wider set of studies, which could use the implications we propose here as a starting point.

5.2 Study: Geometrical Manipulation

The first study (which we designate as “Study 5.1” in Figure 5.1) examined the following question: what change in the cognitive process, if any, is induced by a physical manipulation task when students are trying to solve area problems? To answer this question, two area calculation tasks were given to participants. Before starting the two area tasks, participants completed one of two pre-tasks – either manipulating and solving a single tangram-like puzzle (study group), or answering general knowledge questions (baseline group). Participants were randomly assigned to this pre-task condition.

Tangram is an old Chinese puzzle, where seven geometrical pieces are manipulated to generate various figures. Our manipulation task only had four pieces, as participants in the pilot testing phase found the seven-piece tangram too difficult to solve. Olkun (2003) reports that experience with solving tangrams, both concrete and computer-based, has a positive effect on students' two-dimensional geometric reasoning. Tangrams can also play an important role in the development of spatial ability, competency of rotation and space, geometrical knowledge, reasoning, geometrical imagination and conservation of area (Brincková, Haviar, & Dzúriková, 2007; Baran, Dogusoy, & Cagiltay, 2007; Lin, Shao, Wong, Li, & Niramitranon, 2011). Tangrams can also be used to bring together different domains of mathematics, such as number sense, algebra, geometry, and measurement (Tchoshanov, 2011).

5.2.1 Participants

Twenty two Grade 6 students (11-13 years; 12 female, 10 male) from two Mumbai schools (11 from an English medium school, 11 from a Marathi medium school) participated in the study. Participants were assigned to baseline and study groups randomly, giving us 11 participants in each group (See Table 5.1 for students' profile). The students and their parents provided consent before participation.

Table 5.1: Student's profile

School	Students' Profile			Total
School 1 (English medium)	Tangram Group	Boys	1	5
		Girls	4	
	Baseline Group	Boys	2	6
		Girls	4	
School 2 (Semi-English[18], Marathi medium)	Tangram Group	Boys	3	6
		Girls	3	
	Baseline Group	Boys	3	5
		Girls	2	

18 Semi-english schools stands for schools which have the local vernacular language as the medium of instruction (marathi in this case), but it will have some subjects offered in english

Area is introduced in Grade 5 in India (though some informal contexts are included in earlier grades). The students thus had one year of formal exposure to area as a mathematical concept. The teaching focus in schools is on learning the $l \times b$ formula, based on example pictures in the textbook, which are also drawn on the blackboard by the teacher. Manipulatives are not used in the classrooms, particularly in the schools we studied, which are in a middle to low income neighbourhood (refer Chapter 3 for other details).

5.2.2 Task

The primary task in our study was to calculate the area of two non-standard figures (see Figure 5.2) drawn on graph paper. The second figure (B) was given after the first one (A) was solved. A unit was shown in one corner of the graph paper, and students were asked the following area-problem question: *A full cake is shown in the figure. A piece of this cake is shown at the right corner of the graph paper. This piece costs Rupees 1/-*. What will be the cost of the entire cake? The question was rephrased or translated if needed. No time limit was set for completing the task. Most students completed the task and the following interview in 45 minutes.

Each student performed the task individually on a table, sitting in a height-adjustable chair. Before starting the area task, the study group was required to make a square out of four cardboard pieces of different shapes (see bottom left part of Figure 5.2 for the actual pieces). The baseline group was asked some general knowledge questions before they started the area task. The general knowledge questions were given so that both groups got a pre-task, and both completed them successfully. The students were settled in by a friendly researcher, who emphasised that all the tasks were exploratory and did not involve any kind of assessment.

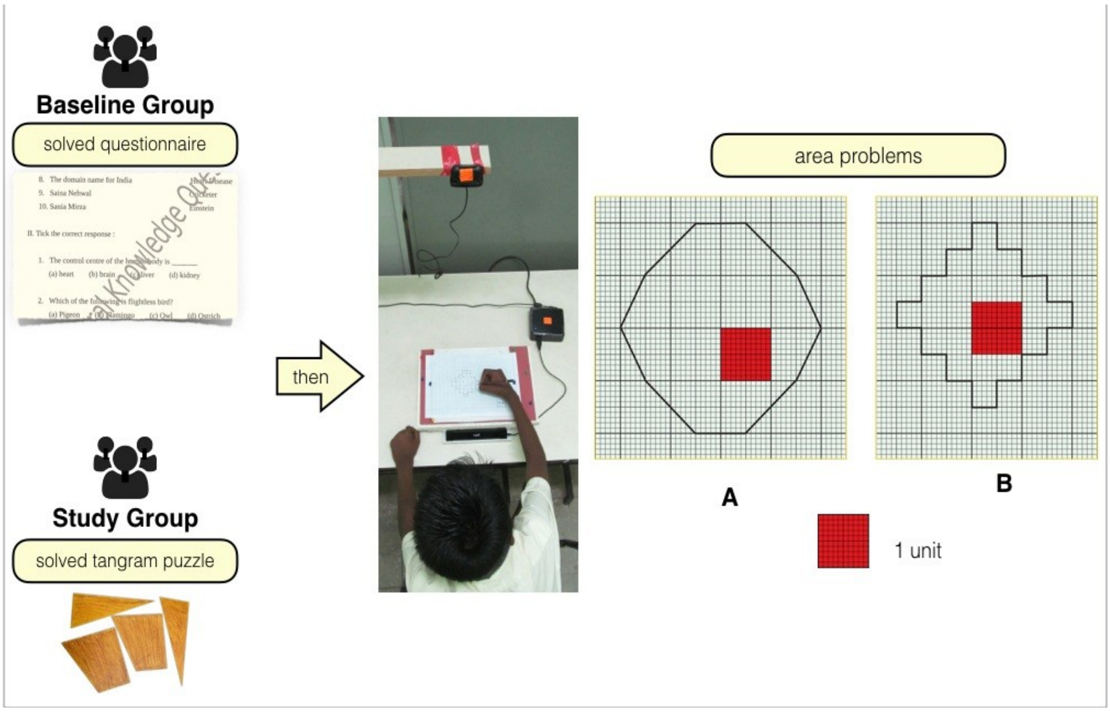


Figure 5.2: The experimental setup

5.2.3 Data sources

Multiple data sources were used that included eye-coordinates using eye-tracker and a webcam, then video data of the overall actions using video recorder, also the video data of the students' interviews conducted at the end.

5.2.3.1 Eye movements

A video camera (Logitech C 525, HD 720p Autofocus) was aligned vertically above the work surface. The video from this camera was synchronized with a Tobii static eye-tracker (Tobii Technology, Stockholm, Sweden), which was mounted on the work surface (see picture in Figure 1 for positioning). This non-standard configuration of the static eye tracker (which is usually used to track eye movements on a laptop screen) was developed in collaboration with Tobii technical personnel, who provided onsite help to calibrate the system. This setup allowed tracking of participants' task-oriented eye movements on the graph paper as they worked on the area calculation tasks. This setup was needed because we could not do the task on a computer, for three reasons. One, we needed to track pencil movements and marks on the graph paper. Second, many of the students we were working with came from a low-income neighbourhood, and were not familiar with computers. Finally, the task is more intuitive on paper than on a computer.

5.2.3.2 Video data

A separate camera was set up on a tripod in a corner of the room, and it captured video data for the pre-task, the area tasks and the post-task interviews.

5.2.3.3 Pencil marks and movements

Participants were given a pencil, and told that they could make markings on the graph paper. They were instructed to write the final answer for each problem on the graph paper sheet.

5.2.3.4 Interviews

Each participant was interviewed post-test, and was asked how they approached the problem and how they solved it. Some students changed their answer to the cake problem during the interview. They were asked about the strategy they used during the task, and why they changed their answer. There is also a possibility that during the interview the presence of the interviewer can act as providing the conditions of social interaction (Chapter 4) and hence students might have reconstructed their problem solving process or understanding rather than retracing or going back to what they did earlier.

These data sources gave us multiple windows into the problem-solving process. Apart from these process data, final answer values and time taken for the solutions were also collected for each participant.

The primary focus of our analysis was eye movements. The accuracy data and hand movement process data were analysed first, to develop a qualitative approach to characterise the task-oriented movements of the eye during problem solving.

5.2.4 Data analysis

Here we outline the methods used for analysing the video data and eye tracking data.

5.2.4.1 Video data

To tease out possible differences in the strategies used by students, we did a qualitative analysis of the video recordings and participant interviews. The different calculation strategies used by participants were coded systematically, in two phases: one, using their pencil movements and paper-based markings and calculations recorded on video, and second, the strategies they reported in the post-task interviews. The coding scheme for videos was developed using the videos and interviews of two students. This scheme, after discussions in the research group, was then fixed for all the other videos. These codes were also validated against self-reported strategies in post-task interviews. Table 5.2 shows the coding scheme that was used for the qualitative analysis of the video data.

Table 5.2: Strategies Used by Students

Strategy name	**Description**
1. Pointing	*Just pointing at a unit or part, without marks*
2. Marking	*Making marks on unit or part*
3. Numbering	*Putting numbers (1, 2, 3) to units or parts, or adding them using numbers*
4. Closer shifting	*Shifting parts to immediate neighbours or adjacent positions*
5. Distant shifting	*Shifting parts to places other than immediate neighbourhood, and also moving parts from within units.*
6. Making outlines (Explicit partitioning)	*Making marks or outline for parts/ partitions*
7. Assigning numerical values to parts	*Explicit mention of the values for the parts, giving fractional or decimal values to parts.*
8. Estimating	*Estimating values of different units. Could be 4 quarters making 1 unit or any two halves making 1 unit.*
9. Approximating (not numerical or geometrical)	*Values where it is not clear how a particular value is assigned by the student. Also values reported by students without justification for why that value was assigned.*
10. Totaling	*Final adding together of units or parts*

5.2.4.2 Eye movement data

Based on the above qualitative analysis, we moved to a characterization of the patterns of task-oriented eye movements corresponding to the spatial chunking and counting strategies (indicated by the qualitative analysis of videos and interviews). The qualitative analysis roughly indicates that the tangram (or study) group made large shifts within the task figures, and some participants used a style of partitioning that combined elements in different ways (see results of the qualitative analysis in the next section). This implied that they approached the area task in a global, whole diagram fashion, dividing the whole diagram up into manageable, possibly non-contiguous components, and adding each of these components separately.

In terms of task-oriented eye movements, this approach would be indicated by:

1. the eye movement pattern staying stable within the subtasks (sub-components) of the given problem space, and
2. when the eye movement pattern changes, it changes to a greater extent in physical space within the given problem space.

On the other hand, for participants who use a primarily numerical strategy, just counting squares locally, the task-oriented eye movement patterns will change more often, but also more incrementally.

It was these two specific patterns that we sought to identify in the eye tracking data.

Note that the area problems here cannot be solved by depending on some direct numerical strategy but it requires one to do visual (or physical) operation of the partial elements with respect to the square unit. That is either one needs to see and find the value of the partial element with respect to the whole unit or combine the partial elements to make the whole. So the two task-oriented eye movement strategies discussed above would be used by everyone, but possibly at different levels. Our analysis sought to identify whether systematic patterns existed in the use of these strategies, relative to the intervention conditions.

One standard approach towards characterizing eye movement data is based on an assumed bijective relation between fixations and visual attention. The focus there is on the role of the eye as a perceptor, where the information distribution in the given figures guides the eye to fixate on specific areas. In such analysis, fixations and saccades[19] track visual attention and the way it shifts, and patterns in this data provide an indication of the way the eye gathers information. Following this approach, the pattern of behaviour we are looking for could, intuitively, correspond to the expectation that the Tangram group would show:

(P) Longer fixations on average, since stable gaze within subtasks is likely to be construed as long fixations by velocity-sensitive fixation classification algorithms, and

(Q) Greater kurtosis in the distribution of saccade lengths, since the saccades within and across subtask components would be heterogeneous in size, reflecting the hierarchical nature of the eye's engagement with the task.

Figure 5.3 outlines the results of this analysis: while participants in the Tangram/experiment group did seem to have longer fixations on average, the difference is not statistically significant ($0.2<p <0.25$ in both tasks). Whereas we expected the kurtosis of the Tangram/experiment group's saccade length distribution to be larger, these were approximately equal in both samples for both tasks (Task A: {5.26, 5.07}, Task B: {6.06, 5.03} for the control and tangram groups respectively).

The inconclusive nature of this analysis stems considerably from its generality. By treating fixations and saccades as the basic unit of analysis, we ignored the task-relevant spatial locations of the actual area subunits that our participants are manipulating internally to solve the problems. Remaining agnostic about the task-relevant spatial contents of the scene is a useful strategy for a general analysis of eye movement patterns, particularly to understand the way information is clustered in figures and texts, and how this clustering directs visual attention and perception. But this approach is suboptimal in tracking the differences in task-oriented actions and moves in a problem-solving space. The task-

19 Fixations refers to the duration the eye gets fixated or focused in the given space, while Saccades refers to the distance covered during rapid eye transitions.

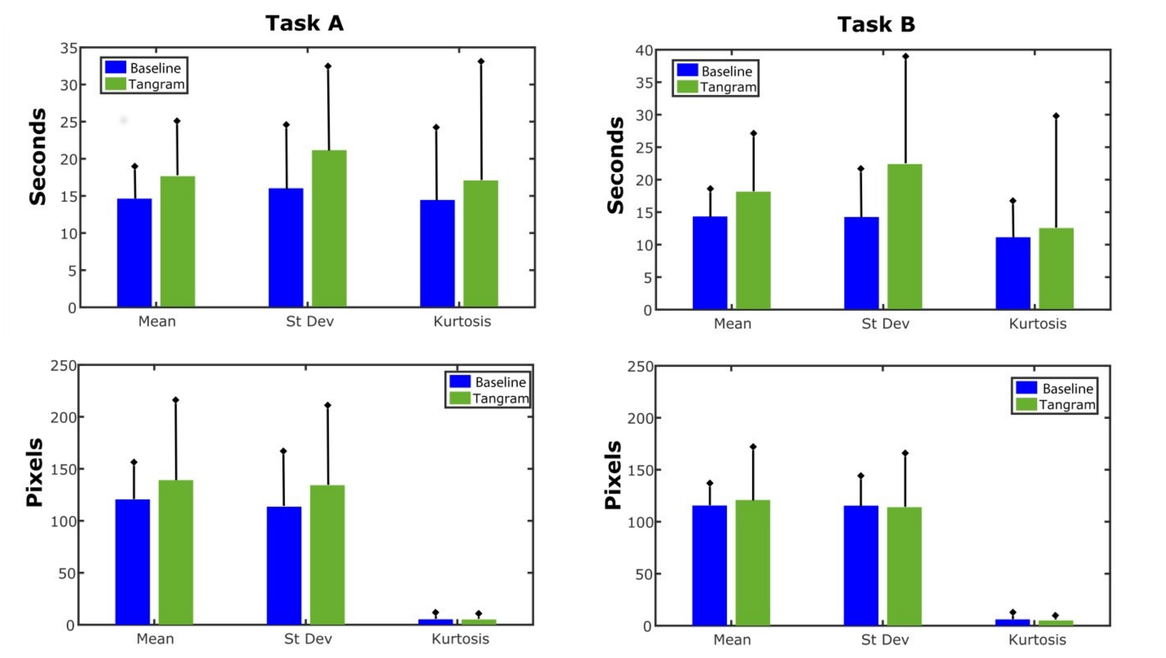


Figure 5.3: Fixation duration and saccade length statistics for eye movement for the baseline and tangram group

oriented movements of the eye engages executive attention, which is involved in monitoring, sequencing, and task switching during problem solving (Smith & Kosslyn, 2007; Botvinick, Braver, Barch, Carter & Cohen, 2001; Fernandez-Duque, Baird & Posner, 2000). Our analysis sought to identify fixations that embed these executive attention processes, which are closely tied to task-relevant transitions. An analysis based on all fixations and saccades would not focus on these executive attention elements, and thus not allow the characterization of the differences in the task-oriented actions and moves related to problem-solving.

To characterise the task-oriented eye movements in our tasks, and to analyse and identify possible patterns across the two conditions, the following steps were done:

a. each area image was divided into sets of states

b. transition probabilities between these states were calculated

c. task-relevant transitions by frequency were identified within fixed time windows

d. large changes in the set of relevant transitions between contiguous time windows were identified (as a marker of shift in gaze patterns).

First, we divided each of the area test diagrams into a set of states S, and identified their coordinate locations for each eye-tracking study setting. Fixation data for each participant then became a finite string of state occurrences s ε S. Then, we calculated pair-wise transition probabilities for each state

s_a with respect to all other states, such that

$$p(s_a \rightarrow s_b) = \frac{n(a \rightarrow b)}{n(a \rightarrow\ )},$$

to obtain transition matrices T, s.t. $T_{ab} = p(a \rightarrow b)$ for each participant.

We further obtained a matrix enumerating salient transitions, by selecting transition probabilities that were not too low $(p < 1/(|S|-1))$, ignoring self-transitions, and transitions between states outside the diagram. Thus, we obtained a binary matrix containing salient transitions per participant M, s.t. $M_{ab} = 1 \Leftrightarrow T_{ab} \geq 1/(|S|-1)$, and 0 otherwise. While binarizing the transition matrix throws away information about the absolute value of the transition frequencies, these depend heavily on the size of the time window chosen, which is a free parameter in our account. Binarizing the transition matrix provides us information more resilient to the value of this parameter. As a quantitative measure of this resilience, comparing the entries of this binarized matrix obtained using all time window sizes between 1 and 10 seconds, we obtain a median mismatch rate = 0.053 ± 0.031 averaged across all participants, and all pairwise state comparisons, which was acceptably low.

We divided every student's overall gaze sequence up into equal-sized time segments (t=5 secs for all our results below). In this way, we obtained an incremental view of which patterns emerge and which fall away as the participant progresses through the task. Videos showing the evolution of new gaze patterns for each student while completing the first task were made. In each video, faint blue lines marked salient existing gaze patterns and dark blue segments indicated the emergence of a new gaze pattern.

The changes in these videos were too dense and rapid, so it was not possible to make qualitative judgements of the micro-level changes captured by these videos. To gain insight into the micro-level changes, we summarized this information quantitatively, by measuring the extent of change in pattern as the quantitative difference between two transition matrices via the Frobenius norm,

$$\delta_t = \sqrt{\sum_a \sum_b |M_{ab}^{(t+1)} - M_{ab}^{(t)}|^2}.$$

We tested both our postulates (mentioned in the beginning of this section) using this metric (of the extent of change of the gaze pattern) defined above. Specifically, postulate (1) was operationalized as follows: the average change for the baseline group (no tangram) will be larger than that for the study group (tangram). Postulate (2) was operationalized as – the maximal change in the study group will be larger than the maximal change in the baseline group.

5.2.5 Study to validate the analysis method

Since the measurement of transition probabilities in this analysis is contingent on the time window size we have used, we also conducted a post hoc validation study to verify that, in its current calibration, this analysis is indeed sensitive to differences in transition patterns when one group uses a counting strategy, and another uses a spatial recombination strategy.

To do this, we randomly assigned a follow-up group of 10 adults (Age: 17-37, 6 Male, 4 Female) to one of two cohorts, both solving the same area tasks as in our main study design. One cohort was instructed to solve the problems using the counting strategy; the other was asked to solve them using the "chunking" strategy. The instructions were: "Use counting of the units as your dominant strategy to calculate the given area. Trace your actions using the pencil" (counting strategy), and " *Use shifting of parts to make whole units as your dominant strategy to calculate the given area. Trace your actions using the pencil*" (chunking strategy). Both cohorts were asked to make pencil movements while solving the area problem, so that we could make sure that they were following the instructions.

If our $\boldsymbol{\delta}$ measurements are indeed sensitive to difference in strategy, then an observer blind to the identity of the cohort assignments should be able to identify the strategies using the values themselves, using the criteria that participants using a chunking strategy should have a lower mean ($\boldsymbol{\delta}$) and a higher max ($\boldsymbol{\delta}$). To see whether this was possible, the data from the 10 participants were first anonymized and named using numbers in a random fashion. This dataset was then sent to the research group member then based in the United States. He was blind to the cohort identities, but was aware that both cohorts had five participants. He processed the eye-tracking data for all 10 participants, and operationalized the criteria above to assign cohort labels to them. For both criteria, we simply summed the scores obtained across both tasks, and then ranked them to split our sample into 5/5 cohorts, with the mean criterion assigning the lowest ranks to the chunking cohort and the max criterion doing so for the highest ranks.

The mean criterion correctly predicted 8 out of 10 labels; the max criterion correctly predicted 10 out of 10 labels (Table 5.3). This performance is clearly better than chance for the max criterion (95% confidence interval for best fit binomial distribution excludes p = 0.5) and almost certainly so for the mean criterion (95% confidence interval for best fit binomial distribution p = {0.02, 0.55}). Thus, while our measurement of changes in the transition pattern does depend on parameters contingent to the time-scales of our particular study, the results from the validation study suggest that it is well-calibrated to pick out the changes we set out to identify.

5.2.6 Results

Different results of the analysis are presented here:

5.2.6.1 Area estimation accuracy was not different across the two groups

The first element of our analysis was designing a measure of competence in area calculation. Simply counting how many students got the problem right showed that more students got the right answer in the tangram group for both diagrams, but there is very little difference between the tangram and baseline group (see Figure 5.4 A). Since counting the number of correct responses penalizes answers that are quite close to the correct value and wild guesses equally, an alternative measure, the percentage error off the true value, was calculated. Even by this measure though, the error percentages of the two groups were not significantly different (Figure 5.4 B). These findings are in line with earlier observations in related work. In a similar study, Olkun (2003) found that the post-test scores for a mathematical problem set were statistically indistinguishable between the test group primed with physical manipulatives and a control group.

Table 5.3: Validation Study Results

	Mean		**Prediction**	**Max**		**Prediction**	***True label***
Subject	**Task 1**	**Task 2**		**Task 1**	**Task 2**		
1	17.7	13.3	Chunk	27	17	Chunk	***Chunk***
2	18	17.3	Count	25	13	Count	***Count***
3	24.3	19.7	Count	23	9	Count	***Count***
4	14.7	6.3	Chunk	31	20	Chunk	***Chunk***
5	28.7	10.0	Count	18	18	Count	***Count***
6	15	15.0	Chunk	39	12	Chunk	***Chunk***
7	17.3	5.0	Chunk	21	10	Count	***Count***
8	25	8.0	Count	41	16	Chunk	***Chunk***
9	25	10.3	Count	25	17	Count	***Count***
10	15	12.3	Chunk	44	16	Chunk	***Chunk***

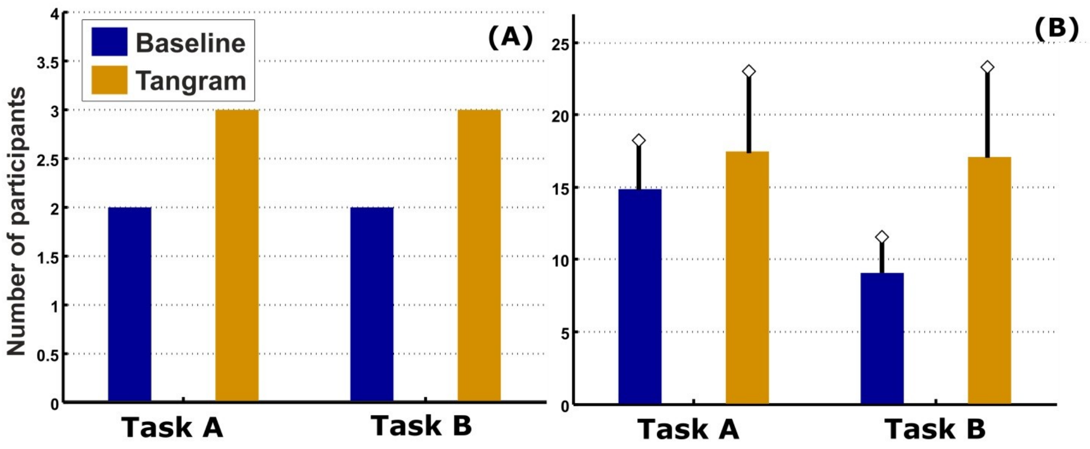


Figure 5.4: The Tangram intervention does not improve area calculation outcomes. (A) Number of students who solved either area calculation and got the correct answer precisely. (B) Average error percentage in area calculations. Error bar represents 1 SEM.

5.2.6.2 Video data and interviews suggest differential strategy use in the tangram group

Both baseline and study populations showed a relatively balanced proportion of strategy use across both tasks, with significant positive correlation in inter-task strategy use within participants (average $\rho = 0.53$). This correlation was computed by binarizing instances of strategy use in both tasks per participant, computing correlations between these two binary vectors, and then averaging correlations across participants. There were, though, potentially interesting deviations in strategy use, particularly in instances of large shifts within the diagram (as measured by finger or pencil movements), explicit partitioning of portions of the diagram with pencil marks, and approximations (coded based only on self-reports) (see Figure 5.5).

Intuitively, the former two strategies would be expected to be more prevalent in participants who were chunking the space within diagrams and adding the chunks; the latter would be more prevalent in participants who were using simpler counting-based strategies. The general trend of the deviations we observed supports this intuition - the tangram group showed more occurrences of the first two strategies, and considerably fewer of the third. While these deviations were not statistically significant

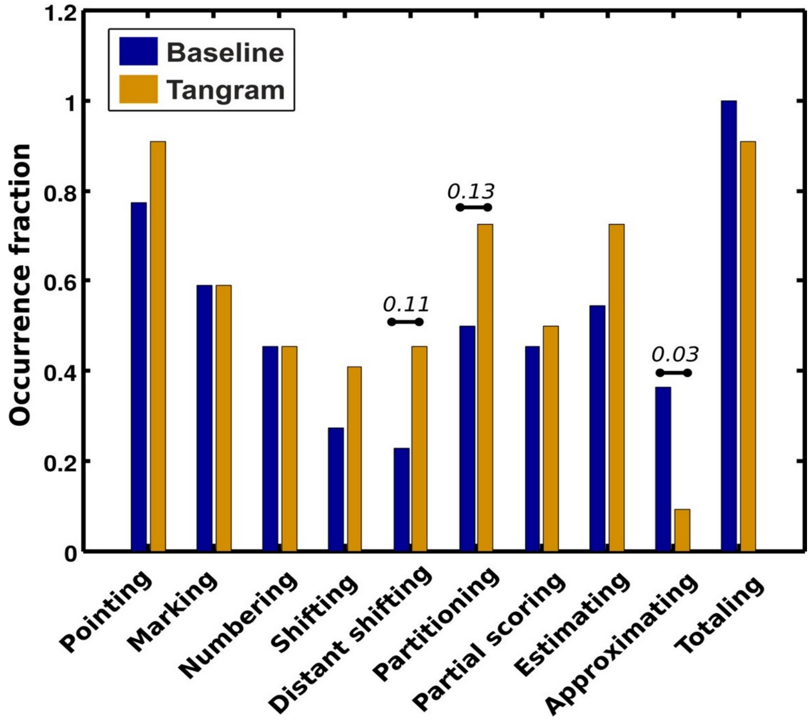


Figure 5.5: Qualitative coding (out of video data) of various strategies used by students in the experiment. The behavioural strategies were plotted on the x-axis. Occurrence counted only the use or non-use of a particular strategy rather than the number used by students. Number above bar pairs represent p-values for corresponding two sample t tests.

on an individual basis, as is evident from the p-values shown in Figure 5.5, they provided insight into what patterns to look for in the eye movement analysis.

5.2.6.3 Analysis of task-oriented eye movements revealed patterns consistent with spatial chunking

As discussed in the data analysis section, the qualitative analysis suggested that the tangram group made large shifts within the task figures, and some participants used a style of partitioning that combined elements in different ways. This implied that they approached the area task in a global, whole diagram fashion, dividing the whole diagram up into manageable, possibly non-contiguous components, and adding each of these components separately. In terms of task-oriented eye movements, this approach would be indicated by:

(1) the eye movement pattern staying stable within the subtasks (components), and

(2) when the eye movement pattern changes, it changes to a greater extent in physical space.

Postulate (1) was operationalized as – the average change for the baseline group (no tangram) will be larger than that for the study group (tangram). Postulate (2) was operationalized as follows: the maximal change in the study group will be larger than the maximal change in the baseline group.

Figure 5.6 shows the results of the task-specific analysis of eye movements. As can be seen, it is much less noisy than the conventional fixation-saccade analysis provided in Figure 5.3. Although postulate 2 is only marginally borne out in task A (t20 = 1.63, p = 0.14 for difference between

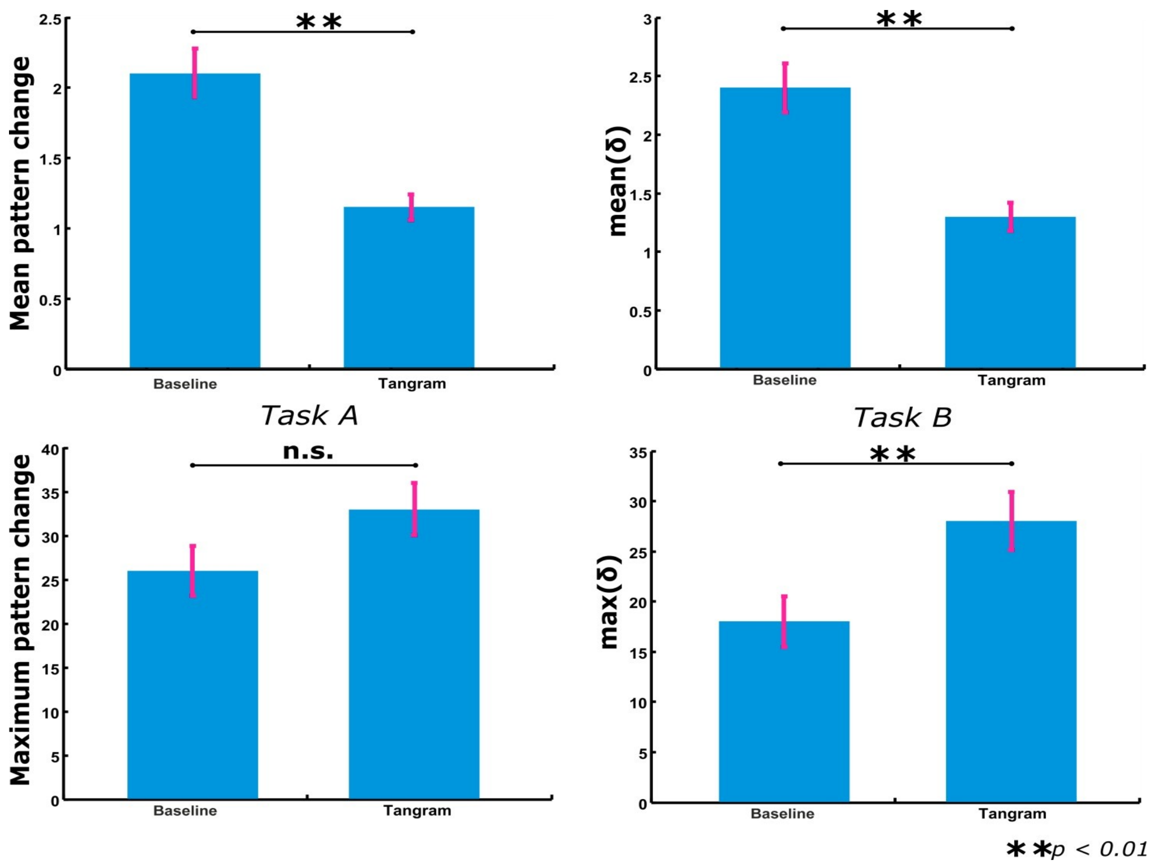


Figure 5.6: Matrices for measuring spatial chunking behaviour show statistically significant differences between the baseline and tangram group. Subjects in the tangram group showed smaller changes in gaze patterns (top panel) but the changes that did occur were more drastic in magnitude (bottom panels) across both area calculation tasks. Error bars represent +/- SEM within populations.

baseline and tangram outcomes), postulate 1 is supported strongly (t20 = 2.88, p = 0.0094) at the p<0.01 level. For task B, both postulate 1 (t20 = 3.33, p = 0.0033) and postulate 2 (t20 = 2.95, p = 0.0078) are strongly supported at the p<0.01 level. These results together indicate that participants in the tangram cohort made less frequent, but larger jumps in eye gaze patterns, which is the expected gaze signature of the use of the flexible (i.e. many different combinations) spatial partitioning strategies indicated by the qualitative analysis of video data.

5.2.7 Discussion

The results from the analysis of task-oriented eye movements indicate that there are significant process differences between the two groups. Particularly, the tangram group appears to follow a recombining approach, partitioning the figure in a highly changing fashion, starting with components bigger than the given unit. The baseline group, on the other hand, appears to follow a less flexible counting process, starting with the given standard unit and smaller components.

This result only indicates that the use of the manipulative leads to a strategy change. It does not show that manipulation can lead to a better understanding of area. However, if the recombining strategy supports learning the key concepts involved in area (Chapter 3), and also helps integrate these concepts, then manipulation could possibly improve the understanding of area.

As our focus here is characterizing the changes in the cognitive process induced by the manipulative, our next study examined further the nature of the process change induced by the manipulative.

5.3 Study: Clay Manipulation

This is the second study done under this chapter and is referred to as Study 5.2 (see figure 5.1). The first study illustrated two points. One, the general direction of the change in cognitive process induced by the manipulative task is indicated at the macro-level by the qualitative analysis of hand and pencil movements and self-reports of strategies used. Two, this process change indicated by the macro-level analysis can be more clearly captured by a micro-level analysis of task-oriented eye movements, as the nature of the change (in actions and moves in the problem space) can be characterised in more detail using the eye movement analysis method we have developed. In this follow-up study, we sought to characterise the connection between the manipulation and the problem-solving process in more detail. Specifically, this study asked the question: Which aspect of the manipulation task, actions or structure, is leading to the shift in the problem-solving process?

In particular, we were interested in finding out whether manipulation alone could lead to the difference we observed in study 1, or whether the geometric structure of the tangram is also needed. If the geometric structure is needed, the manipulation task is working in a coagulative fashion, combining both structure and actions from the manipulation task, and mapping this coagulated structure to the area task. If just physical manipulation is enough, then the shift in process could come just from actions. Particularly, there is something about the process of executing actions that leads to a shift in the cognitive process. One possibility here is the integration/binding ability of actions (Kothiyal et al, 2014; Majumdar et al, 2014). Since actions require constant and real-time integration of both motor elements and sensory elements, actions have an in-built integration capability, which would be activated by the manipulation task. This activation could then transfer to the area task, leading to a shift in cognitive process, particularly to a strategy where integration is a key feature. Note that support for the former coagulative possibility, which is what we found, does not rule out the integrating role of actions.

5.3.1 Procedure

Ten new sixth-grade participants with similar age profiles as the previous study were recruited (all from a Semi-English, Marathi medium school). They did the area study with a new pre-task: manipulating clay dough into any figure they wanted. All other design elements were kept the same as the first study. The figures students chose to make were either animals, birds or flowers (see fig. 5.7). There was no baseline group in this case, as the results from the first baseline group could be compared with the clay group. The conversation happened in Hindi as most students were well versed with Hindi language.

Figure 5.7: Clay models made by students

5.3.2 Results

Here we report results comparing data from both studies. The clay group's accuracy performance was similar to that of the other two groups. Given the way the task-oriented eye movement analysis supported the qualitative analysis in the earlier study, only the eye data was analysed to understand the problem-solving process in this condition. Comparing the results from both studies, interesting and nuanced differences (see Figure 5.8) emerged across the groups. Building the clay model shifted the average change in gaze pattern in much the same way as the tangram manipulation, such that mean(δ) is statistically indistinguishable between the clay and tangram conditions (Task A: $t19 = 1.26$, $p = 0.23$, Task B: $t19 = 1.15$, $p = 0.26$).

However, in the tangram condition, this drop in average gaze transition frequency was accompanied by the occurrence of large transitions (across chunked sub-units of the diagrams). In the clay condition, the opposite pattern is seen -- the size of the largest transition is quite significantly lower in this condition than the baseline case (Task A: $t_{19} = 2.48$, $p = 0.02$, Task B: $t_{19} = 1.5$, $p = 0.15$), and clearly lower than in the tangram case ($p < 0.001$, for both problem tasks).

These results show that building the clay model reduces the average transition occurrence to the same

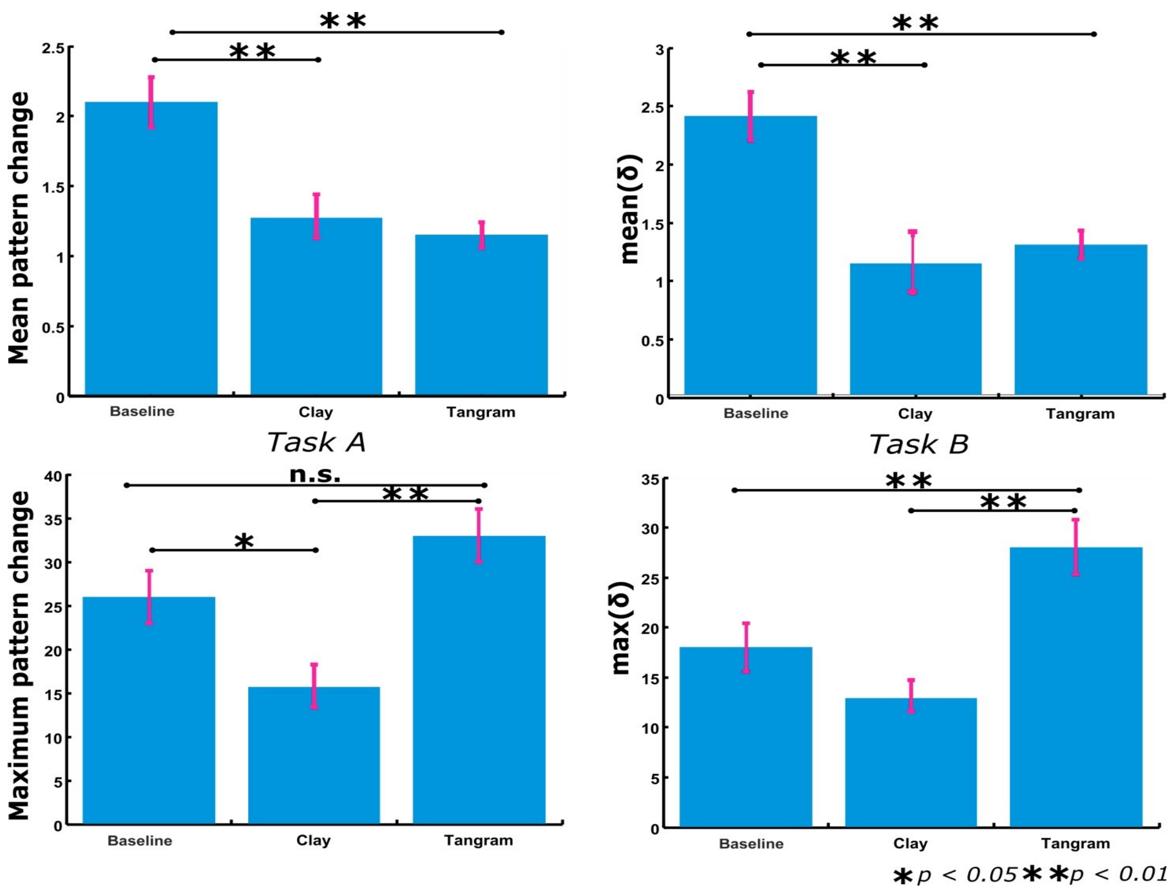


Figure 5.8: Combined result from both studies. The clay condition leads to less transitions, similar to the Tangram condition (top panel). However, the movements are not as spread in space.

degree as the tangram manipulation, but does not change the size of the largest transitions. The task-oriented eye movement patterns of these participants thus indicates lesser visual exploration of the diagrams. Overall, these results suggest that participants primed using clay-modelling did use chunking (as they solved the problem using less moves than the baseline group), but to a much lesser extent than the tangram group, and using a less global (whole figure) and changing process (as they did not chunk far away elements).

Figure 5.9 shows the results from the standard saccade analysis (results from both studies). The results are in the same direction as the task-oriented eye movement analysis above. As in the case of the standard analysis in the tangram case, there are no significant differences in this analysis as well.

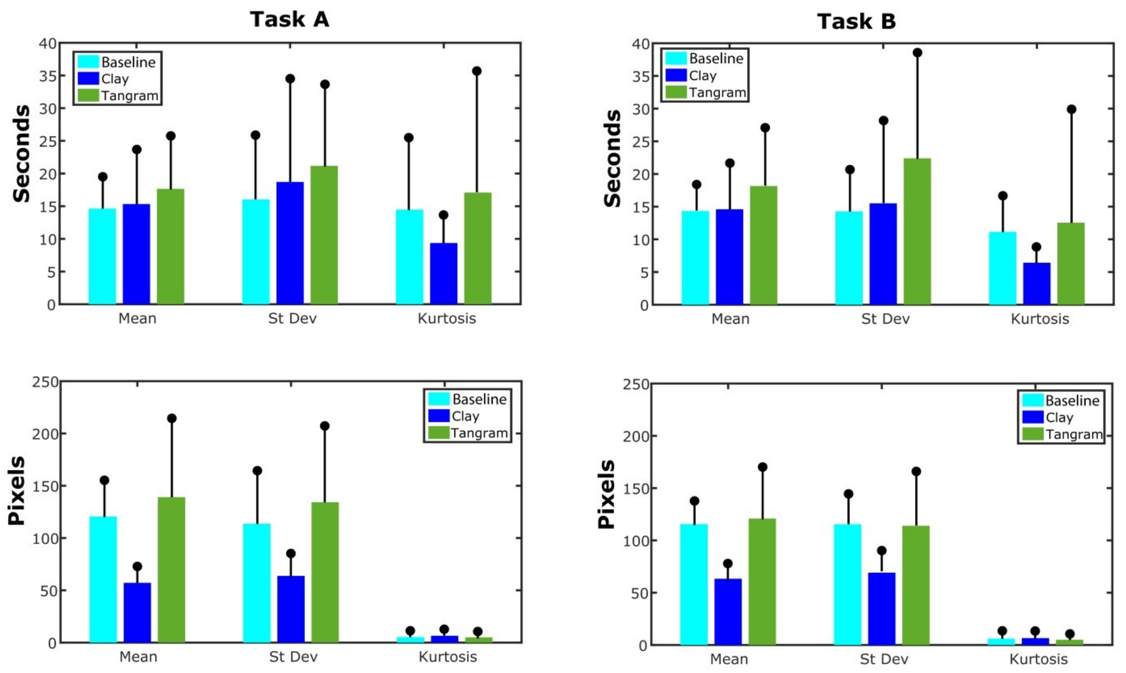


Figure 5.9: Combined results from the two studies. Fixation duration and saccade length statistics for the baseline, Tangram, and clay groups. Bars show standard deviations for the specific statistic. None of the differences reach statistical significance.

One possible concern with these results -- particularly when reflecting on these studies from the perspective of our finding that geometrical structure is needed for the Tangram effect -- is that the tangram task has geometry, while this element is missing in the clay and knowledge tasks. This difference may have worked as a possible confound, as the effect we are reporting could derive from the presence/absence of geometry.

However, this “only geometry” interpretation requires concluding that action is not required for learning mathematical concepts such as area and fraction, and the actions on manipulatives are superfluous. Since the consensus in the literature is that actions are central to the learning effects based on manipulatives, the geometry interpretation is not very persuasive.

Secondly, note that the “geometry-less” groups (clay and knowledge test) in study 1 and 2 show different eye movement patterns. This suggests manipulation does have an effect, as the clay study has manipulation, while the knowledge test does not. Since manipulation has an effect in the non-geometry clay case, it is likely that the effect in the Tangram case (the geometry case) also comes partially from the manipulation. Since this partial effect is all we are claiming, the possible confound based on geometry does not undermine our results.

5.3.2.1 Study discussion

The eye tracking analysis method we use is a simpler variation of the one used by Anderson (2012) to track the second-by-second thinking while students solved algebra problems. This type of process analysis is mostly done with neural response datasets (King & Dehaene, 2014). The method we present here extends this analysis to eye tracking data, particularly task-oriented eye movements during problem solving, where the focus is on the role of the eye as an actuator.

A significant chunk of eye tracking studies, particularly in user research, focus on where someone is looking in a given figure or text during task performance, and this data is usually used to understand the way information is distributed in the given figure or area, and how this distribution grabs visual attention. In such studies, the strong relationship between eye position and visual attention is used as a bijective map between eye movement and cognition. But this is not the only possible way to link eye movements to cognition. Eye tracking studies examining problem solving, as well as studies in education, also consider eye movements as driven by the requirements of the task (Schneider, Maruyama, Dehaene & Sigman, 2012; Inglis & Alcock, 2012; Smith, Mestre, & Ross, 2010.; Epelboim & Suppes, 2001; Susac, Bubic, Kaponja, Planinic, & Palmovic, 2014). The analysis we report here follows this approach, considering eye movements as task-oriented actions, changing in relation to the moves in the given task (similar to hand movements). The saccadic analysis we report for both studies, on the other hand, treats eye movements as context free, and as indicating shifts in just visual attention, which is considered as directed by information in the given figure. Our task-oriented analysis does not deny this role of attention, which focuses on the role of the eye as a perceptor. The task-oriented approach we take just shifts the focus to the role of the eye as an actuator, and the role played by executive attention in controlling the eye in this role. The task-oriented movements of the eye provide detailed process information about the way the task is performed (such as which point was visited, the sequence of visits, integration of visited points, etc.), thus capturing the most significant actions while solving the area problem.

In summary, using eye movement data to understand changes in the process of problem solving requires associating task-oriented eye movements with cognitive markers (such as strategies), which can be isolated using qualitative studies of problem solving. The characterization approach we report here combines such qualitative studies with eye tracking, and this could be a very productive way to gain insight into the micro-level changes involved in problem-solving.

5.3.3 Summary

The above two studies guide us in addressing the two questions raised above, by showing that:

1) Manipulating or interacting with material effects/influences the process of solving an area problem.

(Both tangram & clay groups have reduced mean pattern change as illustrated in Figure 5.8 indicating more strategic and focused eye movements.)

2) The geometry-specific manipulations have an advantage over generic manipulations, in terms of generating a chunking strategy in the problem solving process. (Maximum pattern change is better mainly in the tangram group as illustrated in Figure 5.8)

5.4 Discussion: Cognitive account Vs Math Education account

The discussion of the results reported in the present study is divided into two parts. In the first part, we have developed a cognitive explanation to ground the results of the present study while the second part adopts a math education perspective, by taking into account the new developments in technology and media. Thus, the first part provides a cognitive explanation for the results of the study, while the second part is about revisiting math education from a more interdisciplinary perspective, where, apart from elaborating what the present study means for math education, I will also explain how such study or the results contribute to broaden the field of math education.

5.4.1 Cognitive account

From a theoretical account the two studies revealed nuanced task-oriented eye movement patterns linked to strategy use across the three intervention conditions (baseline, tangram, clay), and the results suggest the following:

1) Systematic manipulation of any material before the area task can prime an action level shift in the problem-solving process, just through the actions involved in the manipulation (as seen in the lower transitions in both the clay and tangram case).

2) However, a more strategy-level shift is primed when there is structure embedded in the manipulation task. The embedded structure leads to a systematic pattern of manipulation actions, which, in turn, lead to systematic shifts in the problem-solving process (larger transitions across the figure seen in the tangram task).

In the following section, we develop a two-step model that accounts for both these results. Note that this model is presented as a general model of how manipulatives change problem-solving, even though the data used to derive it comes from a single representative study. The last section of the chapter covers a discussion on the rationale for proposing such a general model, and the limitations of this proposal.

5.4.1.1 Recombinant enaction: how manipulatives generate strategy shifts

In the general cognitive model we propose below, the mechanism underlying manipulative-based learning, and learning-by-doing in general, is an augmentation of the "mutability" (Kahneman and Miller, 1986) or the "slippability" (Hoftstadter, 1982) of the problem-solving trajectory. This change in mutability is brought about by latent actions from the manipulation, which are carried over to imagination, particularly to executive attention operations in working memory. In this model, the actions during manipulation first primes the action system, and this process then expands the "action space", i.e., the space of actions available while doing the problem task.

Next, when the problem task is encountered (in our case, the solving of the area problem), task-oriented eye movements (exploring the task space) and imagined actions (to restructure the task space) are generated. Note that this shift of actions to imagination occurs because the task elements in the given figure are not manipulable physically. During this exploration-and-imagination process, the primed actions (latent from the manipulation task) introduce branch points into the stream of imagined actions. This recombination process (recombining stored actions and imagined actions) creates new trajectories in the problem-solving space, which eventually leads to a problem solution.

This model suggests that the shift in the problem solving process does not come from a direct one-to-one mapping between the actions in the manipulative and the actions in the problem-solving process, as proposed by Hall (1998). Instead, the shift comes from a recombination process, where actions executed on the manipulative are combined with imagined actions. This recombination process extends the "action space" of imagined actions (see Chandrasekharan, 2009; 2014 for related discussions).

In this proposal, all latent actions carried over from the manipulation task will generate branch points in the imagination process. This accounts for the first result from the clay condition, which indicates that actions have an effect by themselves. However, because the long-distance movements seen in the tangram condition were not seen in the clay condition, all actions in the manipulation task (which translates to many generated branch points in the imagined action process) does not lead to the actual shift in the problem-solving process seen in the tangram case. This suggests the shift in the problem-solving process seen in the tangram case is dependent on an action-structure coagulation, combining the primed actions and the structure embedded in the actions. That is, only actions with certain embedded features (geometric features in the tangram case), particularly features that fit the action-possibilities provided by the task space (area in our case), lead to the problem-solving process shifting to a new pattern. The next section discusses neural mechanisms that could support this effect.

In sum, all manipulatives prime the action system, and these latent actions generate branch points in the imagined task space. Manipulatives thus work, in general, as systems that generate actions that could be recombined to generate new imagined procedures. They thus expand the action space, the space of actions possible in the imagined task environment (Step 1 in figure 5.10). However, which branch points in the action space are actually chosen for problem-solving depends on the saliency of the branch points. This saliency is driven by a matching between the action possibilities provided by structures in the task space (affordances, Gibson, 1977; 1979) and the latent actions carried over from the manipulation task (Step 2 in figure 5.10). Thus manipulatives that extend the action-space productively, i.e., in such a way that the generated imagined actions match the action-possibilities provided by the task space, are more likely to generate interesting shifts in strategy. Given this mapping (between the task space structure and the structure embedded in action traces) the shifts in strategy generated by the manipulative may turn out to be moves in the task environment that could lead to possible solutions.

The recombinant action model consolidates all the results from this study, and thus offers a *general* process account of how manipulatives generate changes in thinking. We have elaborated this process model of learning-by-doing further in the *Cognitive Science* paper reporting this study[20]. This account extends three mechanisms proposed by recent work in cognitive neuroscience. We also discuss a range of theoretical implications of this general model of how actions change thinking. Please see the paper for details.

20 Rahaman, J., Agrawal, H., Srivastava, N., Chandrasekharan, S. (2018). Recombinant enaction: manipulatives generate new procedures in the imagination, by extending and recombining action spaces. *Cognitive Science*, 42(2), 370–415.

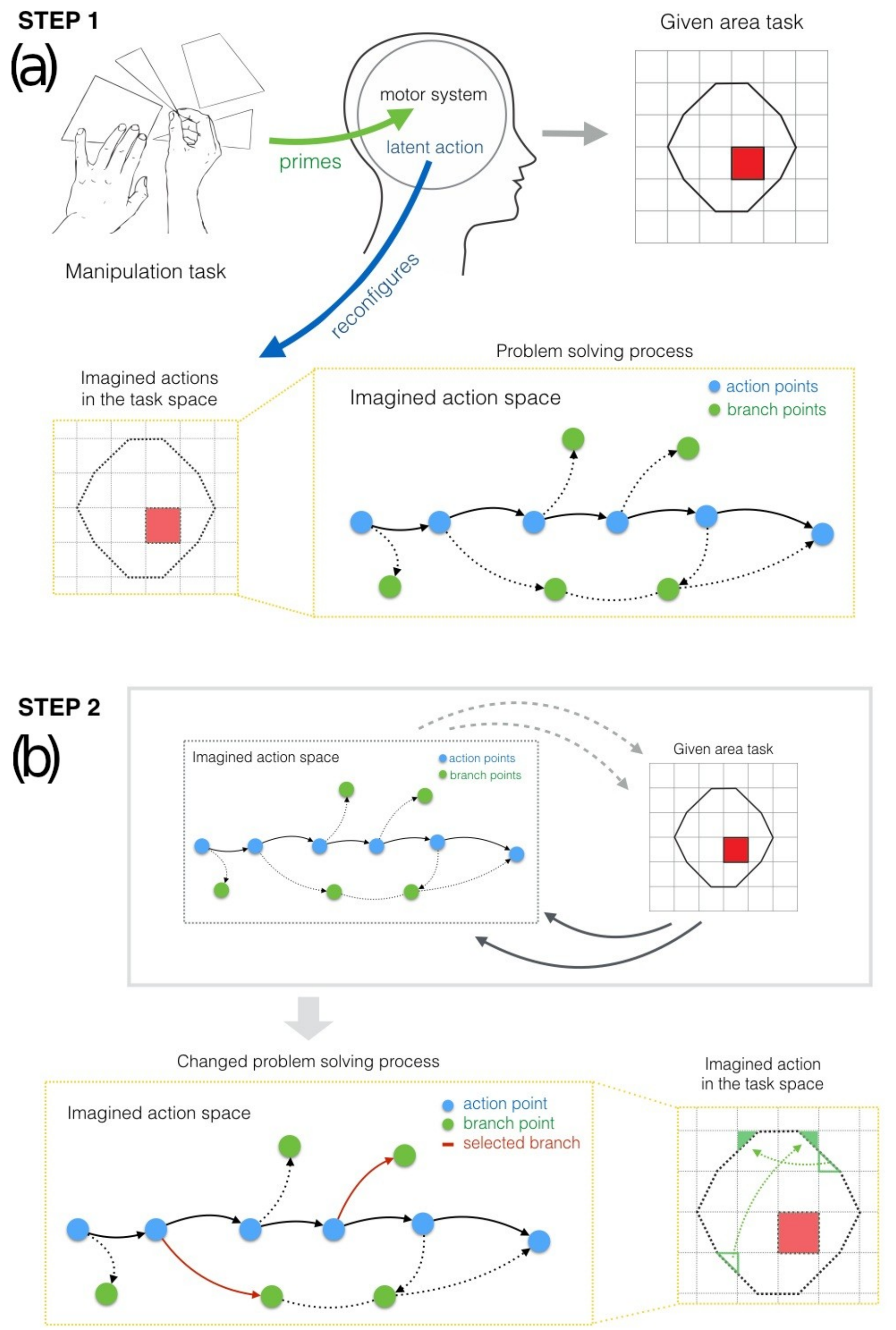


Figure 5.10: (a) The cognitive model of manipulatives priming the action system, (b) The recombinant action model

5.4.2 Mathematics education account

To summarize the discussion so far, the thesis starts with understanding students' conceptions of and students' struggles to make sense of AM. The studies in Chapters 3 and 4 of the thesis use constructivism and social-constructivism respectively to unravel the roots of these struggles or difficulties in order to find ways in which students can be supported to grapple with AM abstraction better. The present chapter, in contrast, uses the embodied cognition perspective, to understand the role of material and manipulation in providing cognitive advantage in understanding AM.

For the current study students were randomly assigned to three groups: Baseline, Tangram and Clay. The baseline group received a general questionnaire, the Tangram group received a simplified Tangram to manipulate and the clay group received a clay to mold it into different objects before receiving the area problem. The Tangram group's engagement with the Tangram type task was video recorded. The video data indicates students' broad actions while engaging with the Tangram type manipulation. All of them moved the pieces, tried various combinations, checking the sides of the pieces to join and finally all of them could successfully solve the task (i.e. arrange all the given pieces into a square). To reduce the complexity and the time taken, students were given a reduced version of Tangram type task (having four pieces) as the actual Tangram task (having seven pieces) took students (in a pilot study) more than 30-40 minutes to get to the solution. However, the simpler four piece Tangram took 2 to 16 minutes for students to get to the solution.

The area problems given to students were not conventional regular shapes but non-conventional shapes; one was a convex polygon with 10 sides of different lengths while the other one was a 28 sided concave polygon having unit side length, with 16 right angles and 12 reflex angles (see Figure 5.11). Thus, using the formal abstraction or direct application of the $l \times b$ formula will not lead to the solution. As we saw, students face difficulty in abstracting or applying their formal understanding of area to different shapes other than the conventional shapes, such as L-shaped figures (Cavanagh, 2007; Zacharos, 2006). Moreover, presentation of the unit and contextualizing the question to find the value of units in the given shapes or figure will not mislead students to mindlessly use formulas, but identify the units in the given figure. One way to solve the task is by directly identifying and counting each of the units or part of units in the given shape and sum them up at the end. The other strategy could be to chunk the whole problem space into different sections efficiently, whereby the units can be easily located and counted on and then some remaining parts at the edges that can be shifted to other corners to chunk them into units. The chunking strategy is a relatively more efficient strategy to solve this area problem.

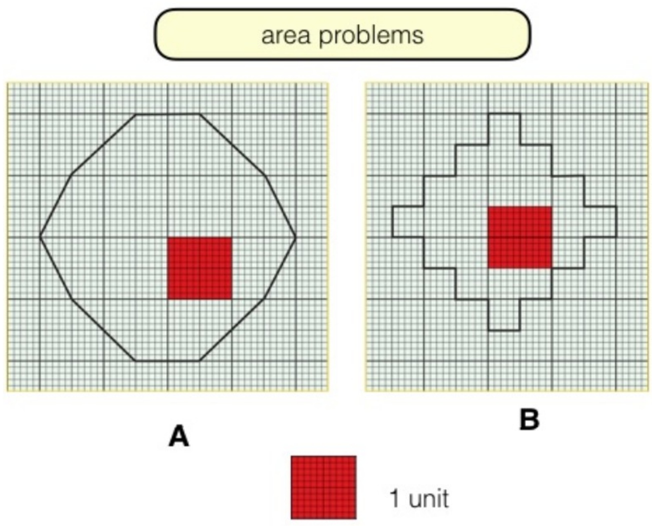


Figure 5.11: The area problems given to the students

The studies mentioned in this chapter show significant difference in the eye movement pattern between the baseline and the Tangram group with the Tangram group showing significantly less mean pattern change and significantly higher maximum pattern change (Figures 5.6 and 5.8). We interpreted the large mean pattern change to be aligned more with the counting strategy, where one focuses on parts or smaller sections while the large maximum pattern change was interpreted to be aligned with looking at the task space more holistically and at the corners, also joining and shifting parts from different edges or sections, and thus more aligned with the chunking strategy. This result is further validated by the validation group, where participants, who were asked to follow the chunking strategy, showed their eye movement pattern aligned with the Tangram group in terms of having large maximum pattern change and lower mean pattern change, while the group that was instructed to follow counting strategy had eye movement pattern similar to the baseline group. The eye movement pattern of the clay group was different from both the Tangram and the baseline groups. The maximum pattern change was less than the Tangram group but the mean pattern change was less than the baseline group but comparable to the Tangram group. This indicates that the clay group was following a different eye-pattern compared to both Tangram and the baseline group. Thus the clay group was following a strategy different from the baseline and Tangram groups. The reduced minimum pattern movement indicated they were not as much aligned to focusing on smaller parts as the baseline group, but the reduced maximum pattern movement indicates that they were also not aligned to chunking or longer shifting that might characterise the Tangram group (Figure 5.8). A possible interpretation is that their strategies were different from that of the baseline group in not getting stuck to counting but not as efficient as the Tangram group, which appeared to use the longer shifting or chunking strategies. However, it is challenging to infer anything conclusive about the strategies used by the clay group, as we could gather and analyse the eye movement data of the clay group and not their video or interview data. Thus, the present study, while strengthening our assumptions related to the importance of material interaction in engaging with AM, provides concrete evidence to infer what kind of material manipulation gives more cognitive advantage to the learner while dealing with AM (e.g., Tangram manipulation is better than Clay manipulation). In the literature review chapter we argued for the role of tool use, that involves material manipulation, from the perspective of how mathematically rich they are. The current study also reveals the significance of material manipulation from the cognitive perspective in providing one with efficient or flexible strategies. While material interaction was prominent across the previous studies, the current study adds another dimension to our understanding of area conception and learning in terms of its significance from the embodied cognition perspective.

Although the number of students in the Tangram group attaining the correct answer was more compared to the baseline group, the difference is not significant indicating the failure of the outcome based studies in capturing the role of interventions in general. Outcome based studies generally focus on the end result rather than capturing the processes involved in solving a problem. Thus the present study also highlights the importance of focusing on the correct processes while solving math problems rather than just the correct answer. As the study design focuses on the *cognitive processes,* both in the intervention (in Tangram group) and also in the given area-task, here we highlight the role of cognitive processes in education (unlike most outcome based studies), and the need to focus on correct processes and not correct answers, in pedagogy, and also in testing and assessment. The study also highlights the need to move away from just emphasizing symbolic manipulation as the main mathematical process. We need to be cognizant of other non-symbolic (visual, spatial or material) processes involved in AM. In particular, geometry in general, and perhaps other mathematical concepts such as multiplication, fraction, decimal, probability, calculus, etc. We have discussed earlier how the area-model is an important pedagogical tool to learn these other concepts.

As argued in the beginning of the chapter, the field of math education needs to consider the advancements in other disciplines and technologies in order to further grow as a discipline. Thus, even though manipulations are found to be effective, their role is still considered in some limited form to only help the transition to the abstract (Sarama, & Clements, 2016). Perhaps this is because the field is still heavily influenced by Piagetian theories of constructivism and also the idea of stages which indicates that a child moves from concrete manipulation to abstraction (Uttal, Scudder, & DeLoache, 1997). Thus, concrete manipulation is identified as a lower level thinking which has to be eventually given up after reaching abstraction and higher stages. This, in turn, has created a divide or hierarchy between the concrete and abstract, perhaps making it further difficult for students to access or engage with abstractions. Uttal, Scudder, & DeLoache, (1997) have argued for concrete manipulatives to be treated equivalent to symbols, implying manipulations to be treated as symbolic operations, which presently occupy most part of the school syllabus. Again, the Vygotskian and social constructivism strands have highlighted the primacy of material or concrete manipulatives as a tool in learning. Though most studies under the social-constructivist strand brought in a rich resource of studies of tools and artifacts developed and used in various cultural historical contexts and practices, they tend to be limited to specific contexts and communities, and tend to overlook the new tools and technologies produced in the modern age. Thus, there is a need to include studies which can investigate the emerging, dominant and specially the efficient, accessible and affordable practices of the advancing modern techno-social world. In other words, with the advent of new tools and technologies, we need to redefine the older methodologies and designs to rethink and accommodate and extrapolate our understanding of learning beyond the conventional classroom to a more accessible environment. However, rather than becoming a blind advocate of new media, we need to revisit the potential of our existing resources and tweak them to maximise their mental affordances.

Further, there is a need to integrate and update the social theories by incorporating the advancements in psychological theories (Restivo, 2017). This study presents a case, where multiple data sources were used to address some of the complex realities of the learning context. Again there are several other studies that use manipulatives but tend to focus on other important aspects of the learning rather than the manipulation itself or looking at the nature of the interaction itself. The studies focusing purely on manipulation mostly end up having a conventional way of judging the performance, or rely on the verbal or written (symbolic) explanation as a marker for effect or change, i.e., the transfer from manipulation to abstraction and then to reproducing it into conventional math problems (mostly symbolic in form) is considered to be the norm in such evaluations of manipulation in learning (Olkun, 2003; Tchoshanov, 2011). However, the present study brings a case where the process data was analysed i.e., the learning setup itself rather than studying the effect in an evaluation setup. The present study provides empirical support for the positive effect of manipulative-based teaching and learning using newer units of analysis and novel methods. Thus, from this perspective as well, the The study provides a model for math education research to design and construct new meanings and ways to study the role of manipulation and also to revisit and rethink what we mean by mathematical ability. Also, such studies can lead to newer discoveries about the nature of learning, and eventually an account of knowledge as dynamically changing through actions.

5.5 Limitations and future work

In the cognitive perspective, we present a new account of how doing becomes thinking, bringing together: 1) education and cognition questions, 2) a novel task-oriented combination of eye tracking, neural data analysis methodology and qualitative methods from education and problem solving, and

3) a cognitive theoretical explanation. The empirical and theoretical approaches we illustrate in this chapter offer a starting point to understand the very complex process by which formal knowledge emerges from procedural knowledge, and the way these two knowledge systems interact in learning and discovery situations, particularly through the generation of epistemic actions and structures (Chandrasekharan & Nersessian, 2015). The study offers a very preliminary account of the way manipulatives change the problem-solving process, and offers a way to further develop cognitive and motor neuroscience approaches to problems in education (Howard-Jones, 2014; Varma, McCandliss, & Schwartz, 2008). In the following discussion, we outline some of the major limitations of the model, and indicate some ongoing and future work to address these issues.

5.5.1 Grounding

Our narrative is constrained and grounded by data from a single illustrative study. However, given the role of manipulatives in connecting different areas of cognition such as distributed and embodied cognition and transfer, and their current application significance, we have proposed not just an account of our results, but a general account of how manipulatives change the problem-solving process. Since this model generalises from the results of a specific study, it is very likely that the model would not account for other cases, such as situations where there are no separate manipulation and problem-solving phases, as in the case of Abrahamson, Shayan, Bakker, & van der Schaaf (2016). However, since the study follows a study design that serves a general purpose of connecting action to imagination, it could account for data from other studies as well. Given this possibility, our general study design is only illustrative, seeking to meet two pragmatic objectives:

1) Function as a seed design, working either as a convergence case (when other study designs are in line with the one presented here), or as a contrast case (when other designs differ drastically from the one presented here).

2) Help develop theoretical guidelines for designing embodied media for learning mathematics and science (Abrahamson & Sánchez-García, 2016).

The study design is thus just a starting point, and it will be revised or improvised significantly, based on future studies.

5.5.2 Data

The findings emerged from a focus on examining process data, which is very difficult to gather and analyse. This has limited the scope of the study we report, which is too small to make definitive claims about the validity of the findings. Going forward, what kind of data could provide ways to validate or reject the model? The critical component of the model is the way the manipulative extends the action space of the learner. One way to validate or reject the model would be to examine whether working with the manipulative actually leads to the extension of the body schema, as in the tool use case (Maravita & Iriki, 2004). Recent work in cognitive neuroscience provides some experimental paradigms to examine this hypothesis (see Chandrasekharan 2014 for a review), and we are currently developing studies based on these empirical approaches.

5.5.3 Domain

Manipulatives are used in many contexts other than mathematics, such as learning chemical structures, anatomy, engineering design, etc. The role they play in such contexts would be quite different from the mathematics case, particularly because the operations in imagination are more concrete in these areas. Similarly, the role played by manipulatives in laboratories, and the way they change imagination in this context, would be very different from the above cases. The study design we propose is thus limited to cases where manipulatives embed abstract procedures, and the way they change the process of solving procedural problems. However, we hope that the current version provides a starting point to better understand the two-way interaction between actions and formal conceptual knowledge, and the way manipulatives mediate this interaction.

6

Conclusion

In this chapter, I provide a brief recap of the whole thesis, from where it all started, to why particular routes were chosen, to what emerged from the pathways, to finally ponder upon the learning and to contemplate the journey ahead. Although the thesis mainly focuses on the conception of area-measurement, in a broad sense, it also highlights some emerging trends within mathematics education research.

6.1 Introduction

The thesis starts with how measurement was a prevalent practice of human society historically, and could be the primary source for the emergence of geometry as we know it today. The significance of measurement as a root topic in mathematics education, in contrast to beginning with numbers, has been argued in the work of Davydov (1975), who argues for learning mathematics meaningfully as comparison of quantities, which comes naturally to children. However, contemporary mathematics curricula give less importance to measurement, and separate it from the topic of geometry, despite their deep connections. School mathematics gives more importance to geometry, which occupies a significantly large portion in the curriculum compared to measurement. School geometry is mainly dominated by Euclidean geometry, which is based on an abstract metaphysical viewpoint, away from real objects, making it difficult for a learner to meaningfully understand geometry. And as geometry carries a large weightage in mathematics curricula, not being able to deal with geometry has an impact on students' overall mathematical performance. This eventually can be a reason for many students to give up or hate mathematics. On the contrary, measurement, specifically geometric-measurement, has a more practical and realistic aspect to it, which can assist students' transition to geometry and other abstract forms of mathematics (Smith, Males, & Gonulates, 2016). Thus, geometric-measurement can bridge the gap between real-life roots of geometry and its abstractions, thus making geometry and eventually mathematics more meaningful for students to engage with.

In the school curriculum, the topics covered under geometric-measurement start with length measurement, and then move on to area and volume measurement. The poor performance of young students on length measurement tasks, which is widely reported in the literature, gets further reduced with area-measurement (AM) tasks (Battista, 2007). Length measurement tasks are resourced with physical tools like scales or rulers, which were developed as cultural tools but also support learners to build their own mental tool (or mental ruler) for length measurement (Clements, 1999). No such tool is readily available for measuring area. The lack of any tool or instrument for AM partly explains the greater challenges in accessing AM, besides the abstraction involved in the $l \times b$ formula, resulting in students' poorer performance in AM compared to length measurement. Apart from the non-availability of any measuring tool for AM, it is also the first quantity a student encounters that introduces multiplication of two extensive quantities (Smith, Males, & Gonulates, 2016). Thus, AM turns out to be the first concept where students learn to arrive at a new quantity by multiplying two other quantities. This has further application, for understanding higher advanced topics, such as force, volume, weight, momentum, etc. The area model also has applications in several math topics, for e.g., multiplication, fractions, algebra, functions, probability, calculus, measure theory (Dreyfus & Hershkowitz, 2017; de Freitas & Sinclair, 2020; Sisman & Aksu, 2016; Sarama & Clements, 2009; Outhred & Mitchelmore, 2000). Thus, AM can act as a foundation to broaden students' learning, to move to an integrated and interdisciplinary understanding of mathematics. Considering the significance of the topic of AM, the thesis tries to uncover students' conception of AM, primarily through the three main studies covered in Chapters 3, 4 and 5 respectively.

6.2 Research Objectives

The thesis broadly tries to investigate why, what, and how questions around students' AM conception. The thesis starts with the 'why' question of why this topic is worth pursuing. Secondly, what are students' conceptions of area measurement (AM) or more broadly what leads to their conceptions and how we can transform them into richer conceptions. While the introduction chapter mainly covers the why question (the need to focus on this topic), the literature review chapter presents studies around this topic, which are broadly categorised into four themes: conceptual, curricular, tool use, and multiplicative thinking.

Chapter 3 follows the literature review chapter and presents the first study of the thesis, which explores students' conception of AM by adapting Piagetian constructivist theories. Since students' conceptions are a product of their environment, we adopted the naturalistic paradigm to understand students' conceptions in their existing set up through observational studies. Thus, Chapter 3 consists of a set of studies, broadly grouped under three categories: first, those that were done in the students' existing school setting, since schooling plays a significant role in students' conception; second, analysis of the textbooks, and third, studies conducted in a research set-up with individual students, through task based interviews. Table 6.1 lists the research questions that were addressed by these studies.

Table 6.1: Initial research questions explored in Chapter 3

	Guiding Questions
1.1	What are students' conceptions of "area"?
1.2	What is the conception of "area" reflected in classroom practices?
1.3	How does the curriculum (or textbook) handle the "area" conception?
1.4	(a) What are students' conceptions of "conservation" of area and perimeter ?
	(b) What are students' representations for area and perimeter?
	(c) How do students interpret area and perimeter for unfamiliar figures?
	(d) What are students' conceptions of unit structuring in area-measurement ?
1.5	What is the connection between area-measurement and multiplicative thinking?
1.6	(a) What is a good model of learning area-measurement?
	(b) Why should the proposed network model of learning the concept of area be adopted?

The first part of the study was conducted in the regular school setting, starting with classroom observations to explore question 1.2 (in table 6.1) followed by students' interviews (within the school premises) to explore question 1.1. For question 1.3, I analysed mathematics textbooks used in two different school board systems for classes 5, 6, and 7. The findings and observations done under these broad questions led to some new insights. Informed by the literature review, the new findings led to the formulation of some new guiding questions, 1.4 (a) - (d) in Table 6.1). This time they were investigated in a research setup, inspired mainly by Piagetian methodology. This new set of studies were structured (or planned) cognitive studies, done through task-based semi-structured interviews with students, adopting constructivist paradigms. The fourth theme of the literature review chapter led to the emergence of another study guided by question 1.5. The results of the above studies were consolidated into a coherent whole, by arguing for area as a "network" concept, requiring integration of the spatial, numerical and algebraic aspects. This argument is also investigated through the two didactical questions 1.6 (a) and (b).

Chapter 4 covers the second major study of the thesis. It is an intervention study, to investigate "how do students construct the concept of AM in a classroom?" using teaching design experiment as the methodology. Unlike the previous chapter, which was about investigating students' individual constructions of AM, this chapter moves beyond the individual construction, to social construction happening within a classroom context. As part of this study, different lessons and activities were designed, to investigate the use of different tools and tasks in engaging with AM. I also wanted to investigate whether certain tools can act as culturally developed tools within the classroom micro culture, and how such tools might aid in bridging the gap between geometrical & numerical understandings of AM. Since classroom culture or context is a much more complex setting, with the component of social interaction, the framework of argumentation and socio-mathematical norm was adapted, to conceptualise and analyse different argumentation structures in students' constructions (Toulmin, 2003; Yackel, & Cobb, 1996). Thus, the methods of analysis of the second study of the thesis are social interaction and social constructivism.

Chapter 5 outlines the third study of the thesis. It extends the use of tools and materials to develop a focused study on the role of material interaction and manipulation in students' approaches to solving area problems, particularly the strategies and process changes during this problem-solving process. Two main research questions investigated here are:

5.1. What process change happens as a result of manipulation?

5.2. How does manipulation transform the process of solving an area-problem?

This study was done in an experimental setup, using a video recorder and eye tracking methodologies, to identify the cognitive processes and strategies involved in students' problem-solving actions.

In the next section, I discuss the different results and findings of the three studies mentioned above, followed by some broad implications of the thesis.

6.3 Results and findings

Since the thesis consists of three broad studies, investigating different sets of questions, some of the broad results and insights of the studies are summarized here, in the order followed in the thesis.

1. In study 1, some worries were raised by our observations during the initial interviews with students from 4th-6th grade. It was noticed that when asked about "what is an area?" Most of the students uttered some formulas involving dimension, for example, $l \times b$ or $2\times(l + b)$ etc. When some students were further asked to elaborate or explain the definition of area by writing or drawing, they mostly drew some conventional shapes like squares or rectangles. When they were shown irregular or curved closed figures and asked about their area, a few of them said they didn't have an area. These observations brought forth concerns about the gap in the formal and the physical or material sense of area and AM.
2. Further, when some students were asked to shade the area of given closed shapes, different

parts were shaded by different students. For example, they darkened the boundary, or made some squares inside the given shape, shading only one small square among them. This clearly indicates that the students were not understanding which attribute is measured through area. These observations were further validated from the observations of pedagogy and curriculum, which mostly focus on computational activities, and have hardly any tasks or activities that allow students to represent (or shade) the area of given shapes. Such an activity, apart from highlighting the attribute for area, also highlights the continuous character of area, instead of the discrete counting of the units (Kobiela, & Lehrer, 2019). Thus, there is a need for practical tasks like painting/ shading for area and highlighting boundaries for perimeter, to distinguish between the two measures. This was done in the tasks discussed in Chapter 3, thus clarifying the covering and continuous aspects of area better than physical tiling.

3. Students were asked to compute the area of a non-conventional shape like an L-shaped figure. Again, a few students computed the area of an L-shaped figure by multiplying all the given dimensions rather than by partitioning it into rectangular pieces to compute it accurately. This indicated the disconnect between area and its numerical operations or calculations.
4. The pedagogy and the curriculum was studied through classroom observations and textbook analysis. There were parallels between students' conceptions of AM emerging in the interviews, and the way the curriculum and the pedagogy handles AM, suggesting that students' conceptions were a direct product of the pedagogy and the curriculum. That is, students draw the meaning of area from what they come across in the context of AM in their school, and thus associate only the computational notion, and limited shapes like rectangles and squares, with the conception of area. This pointed to a network approach to understanding of area, where students draw the meaning of AM by linking different bits and pieces they come across in the context of AM. The curriculum misses several other important conceptual components as well, like recognising the attribute for area, unit, array, conservation, dimensions, multiplicative operation, etc.
5. We eschewed the much-used covering task in order to avoid the risk of pre-structuring the task. Covering tasks with physical tiles may tend to become a mechanical task (Outhred & Mitchelmore, 2000) performed without understanding. Thus, asking students to shade the area of a region instead of mechanically covering it with tiles can make spatial understanding of area more explicit. To explore or uncover the numerical understanding of area, instead of using covering or tiling task with unit blocks, we designed some new tasks using unit cards (made of card paper), as a reference for students to compare the area of different shaped sheets and to further explore and unpack the connection between AM and multiplicative thinking.
6. Textbook analysis also revealed the nature of curricular tasks on area and AM. The tasks were

mainly "solve" or "find" type questions, rather than tasks involving construction or creation of shapes. Most area problems or area calculation in textbooks require students to measure the linear dimensions (length, breadth) for a rectangle and (base, height) for a triangle. Thus the idea of area gets associated with linear dimensions, rather than the 2-dimensional surface. This can be an important factor causing students to associate only computational meaning for area, without any association with the space or the attribute for area.

a. The topic of length measurement in the textbook includes tasks that require students to measure a given line segment using a ruler and also the reverse task, of constructing or drawing line segments having a given length measurement (e.g. 5cm). Having both of these tasks not only exposes students to varied practical exercises but also ensures understanding the reverse or inverse operation. That is, it allows students to move from spatial to numerical representations and the reverse for length measurement. This appears to be missing for AM-specific tasks in the textbook or curriculum, where students are asked to find the area of a given shape, but they are not asked to draw a shape with a given area.
b. Most area tasks are of the form where the shape is fixed (for a given area measure). However there are hardly any occasions in the curricula that allows students to see that for the same area (magnitude or measure), different shapes are possible. Thus, such exercises are missing. Tasks around this could strengthen students' understanding of the conservation of area concept.
c. The idea of inverse operation or inverse relation is very important and it can be found in most mathematical operations, such as addition-subtraction, multiplication-division, unit-measure (unit size and measure value), differentiation-integration, etc. Thus the idea and exposure of inverse operation is central to mathematics in general, and any mathematical operation in particular. However the curriculum presents only uni-directional area tasks (as reported in our textbook analysis), where the task takes the form of finding the area (number) when given a shape.
d. In general, geometric measurement is the process of moving from space to number by quantifying the space into number. In the context of area or AM the shift is also from additive counting (in the context of length or discrete units) to multiplication. Moving to a multiplication operation on the other hand is built on the understanding of multiplicative thinking. Thus the connection of area measurement with the domain of multiplicative thinking – a very widely researched area in math education research – needs to be established.

7. The set of studies reported in Chapter 3 of the thesis were consolidated through the "network model" of area (see Figure 3.13 and 3.14, Chapter 3). The need for the network model was

also inferred through the task-based interviews and observations. These indicated that the understanding of area is intricately associated with a network of related concepts. This is contrasted with length measurement, where hidden conceptual elements such as attribute, conservation, transitivity, equal partitioning, iteration of unit, accumulation of distance, origin and the relation to number (Sarama & Clements, 2009) are packed within a ruler or scale. Thus, a ruler acts as a compact tool, holding together all the connected conceptual elements involved in length measurement, forming an ‘enactable’ network.

8. The network model supports a spiral curriculum instead of a linear curriculum. In the former, students experience the content initially in an intuitive authentic form before revisiting it in the formal sense (Bruner, 1996). This allows students to experience the concept in the original form before rediscovering it in the formal sense.

   The spiral curriculum is aligned with the network model, as such a curriculum can strengthen the connections between different concepts. It contrasts with the linear curriculum, which deals with different concepts separately in isolated forms. The spiral curriculum is also aligned with the way concepts are present in nature, where concepts don’t stand out separately but are present in an integrated fashion. Thus, the curriculum needs to integrate the network of concepts that are deemed to be essentially involved in the understanding of area.

9. As we argued in the introduction chapter, students mainly draw meanings of various concepts from their school exposure. In the context of area-measurement, students tend to associate what they come across in their school curriculum as the meaning for area. Thus, most students tend to associate specific limited shapes like rectangles and squares with area, and tend to assume only these shapes have area and not others. Understanding area as a network will help students understand the concept better. For example, associating “area” with things (words or objects) they have come across in other contexts, even out-of-school contexts is needed, rather than just associating area with some conventional shapes and its associated formula.
10. I highlighted the significance of tools. For length measurement, the ruler acts as a tool or instrument that has all the conceptual elements involved in the process of length measurement packed within it in a compact fashion. The hidden conceptual processes can be unpacked either by separating out each component and dealing with it separately or by handling all components together as an enacted network. The network model calls attention to the need for an instrument or tool to strengthen the connection between different concepts involved in area and AM. One way of doing it could be to have a diverse and wide use of this instrument for different purposes. This would allow learners to appreciate its wide application, similar to the way a scale or other instrument can be explored or unpacked through multiple forms of use, where the purpose served by the instrument can be made more explicit. The theoretical

motivation for the construction or usability of such an instrument comes from the "Network model" or the network understanding of a concept, which is elaborated in Chapter 3 of the thesis.

These results and findings from the first set of studies of the thesis, discussed in Chapter 3, concern the current status of AM conception, pedagogy and curriculum without intervention. The second study (chapter 4), in contrast, involves designing tasks based on the insights gained from the literature and the studies in Chapter 3.

The study reported in Chapter 4 focussed on social interaction as an important component in conceptualising AM. It draws from the social interaction and social constructivism paradigm, based on tasks and interactions. Thus, the unit of analysis is not just actions and gestures, but also the verbal reasoning in the characterization of stages. Piaget et al. (1960) did not write about how a child's intuitive understanding keeps changing not just through the material interactions but even when the child socially interacts with the researcher/teacher and other children. In our study, apart from the social interaction, careful attention was also paid to the tasks and the materiality of the interaction in students' construction of a particular concept. This builds on the idea of a culturally developed tool, for measuring area in a compact, accessible and handy form:

i. Through multiple use

ii. Including aspects physical access and graspability

This was attempted in Chapter 4, using the graph or grid paper as a culturally developed tool within the classroom microculture, and multiple forms of tasks based on shapes drawn on graph paper. The utility of graph paper as a local culturally developed tool evolved in the classroom context. Students reasoned with graphs and grids extensively, specially for engaging with both the spatial and numerical aspect of area, through several kinds of tasks based on shapes drawn on graph sheets.

Several insights can be drawn from chapter 4. Methodologically, paying attention to the argument structure and the nuanced differences in the warrants revealed the gaps in the warrants that were available to the students and the teacher. Students seemed quite fluent in their actions on decomposing and recomposing different parts of a rectangular piece and seeing the relation between the parts to some extent, however they still struggled to coordinate the spatial and numerical representation of the fractional parts. This indicates, on the one hand, the deep connection between AM and the topics of fraction and decimals, and on the other, the challenges in seeing the connection. Moreover, the struggle of students in tasks requiring seeing or representing the area as addition of different kinds of unit, is important in building their understanding of fraction and decimal numeral representation. The students are familiar with the various representations of half, which can support them in building such an understanding. The use of money analogy, as seen in a student's response further concretises the decimal representation and sets up an example of a warrant coming from a widely available out-of-school context supporting the social construction in a classroom. The repeated error by Raziya's group in finding the area of A3, despite providing strong warrants for using the algebraic generalisation $l \times b$ for A4, indicated the complexity in coordinating algebraic and spatial understanding among students. Further, analysis of the arguments advanced by this group turn highlights the significance of approximation emerging from unit structuring, which is an important mathematical construct that is part of the learning of AM, and plays a key role in making the connection between the algebraic, numerical and spatial understanding of AM. The episodes of argumentation discussed in Chapter 4 also raise the question, what are the factors that lead to the emergence of argumentation in the classroom as reflected in these episodes. The students' curiosity to understand their peers' methods or solutions, and the drive to achieve consistency and coherence between contrasting solutions and approaches are important among such factors. Chapter 4 also presents how actions with concrete representations get paired up with mathematical discourses/narratives in argumentation, and how this process contributes to concept formation. The back and forth arguments between students using multiple representations also affirmed the network understanding of area.

Another rich activity that emerged through these tasks was the Tangram activity, which has some core common pedagogical characteristics of a good activity: it is accessible, hands-on, low floor high ceiling, allows multiple arrangements and possibilities and hence solutions. Thus, it allows learners to explore different facets of AM, including the network of concepts connected and extended with AM. It also provides scope for an integrated approach to learning a particular concept, including the extended verbal support/ prompt to facilitate students' reasoning. Considering this potential of the Tangram activity, a version of it was used in Chapter 5 to study the cognitive processes involved in learning with a material interaction, such as a Tangram type activity. Chapter 5 thus explored the role of material interaction in a focused way, reducing the complexities of social interaction using a controlled lab set-up.

The study in Chapter 5 showed prevalence of the chunking strategy among students who did the tangram type manipulation, while the baseline group's eye movement indicated the counting strategy. Thus students having the same profile showed differences in their eye-movement based on the experiences they had immediately had before doing the area task. The group that did tangram type manipulation showed the presence of an efficient chunking strategy while the baseline group indicated the presence of an inefficient counting strategy. The clay manipulation group indicated less of the counting strategy but showed significantly less presence of large shiftings indicating the lack of chunking strategy among them. Thus the study helps us to recognise the specific kinds of activities or manipulation that can be provided to learners to lead them to efficient strategies.

Apart from providing a general process account of how doing leads to changes in thinking, this study showed that as math educators, we need to re-imagine new methods to capture all forms of dynamic processes and not just discourse, and there is a need to engage closely with multiple forms of interaction. This study contributed to developing new methodologies to address math education problems, by using newly developed tools to capture different forms of process data.

6.4 Implications: Contributions to theory or knowledge

One of the contributions of the first study is to add new components to Piagetian studies, by bringing in the importance of other major factors that play a significant role in students' conceptions. Thus, the first study shows how students' conceptions are closely related to the school curriculum and teaching. This highlights the significance of adopting a holistic and ongoing approach towards material design and teacher training, addressing the gaps or difficulties associated with students' conceptions. This, along with the network model proposed at the end of Chapter 3, challenges the deficit model of research that ascribes misconceptions or wrong conceptions to students, rather than the complex network of other factors affecting or moulding students' conceptions. The study highlights the parallels between the elements of students' conceptions and the corresponding limiting factors present in the curriculum. For example, the range of typical geometric objects or representations used in textbooks can act as a source for gaps in students' conceptions. The observations are in congruence with the interpretation that the existing school curriculum focuses only on a limited number of elements of the network of area, and does so in a disconnected manner. One of the general implications of the thesis is thus the possible use of the network of concepts involved in AM to develop better curricular material and pedagogy, which provide ways to strengthen the connections between various elements of the network.

The implications of the thesis are discussed below under four sections drawing from four kinds of contribution made by the thesis: (i) Curricular, (ii) Pedagogical, (iii) Research (including method) and (iv) Future implications.

6.4.1 Curricular Implications

A major curricular implication of the thesis is that it points out specific gaps in the curriculum, and thus provides specific suggestions/revisions to address these gaps in the AM curriculum, as elaborated below:

I. Students' interviews in Chapter 3 indicate that lack of association with experiences outside the school or connected with real life, leads to a very limited understanding of the term or the concept of area or AM. Thus, the thesis clearly highlights the importance of using local and non-formal terms for area to extend students' understandings with respect to area. Throughout the studies done in the thesis, the generic word “size” is used to refer to area, but the specific aspects of area are brought in through the context of the questions such as which field is larger?, which sheet can hold a greater number of the given square cards? So the focus moves from knowing the term to using it in practice, to construct or associate its meaning from the contexts experienced.

II. The thesis highlights the need to move beyond typical shapes, and connect with real life examples. One of the observations was that some of the students considered area or AM to be associated only with the typical shapes that they see in textbooks like squares, rectangles, and triangles. Since visual information impacts the concept image deeper than the definitions, it is important to also provide examples of non-typical shapes like irregular curved shapes, so that students' conceptions get extended and generalised to any closed shape (Srinivas, Rahaman, Kumar, & Bose, 2019).

III. The textbook review identifies the need to go beyond the “find” or “solve” type problems, to draw and design type tasks for learning AM. As elaborated in the 6th point of Section 6.3, the textbook exercise questions on AM are mainly of the form of find/ solve type questions, where some information is given in the form of dimensions such as length or breath and the expected solution is to numerically calculate the AM using the information provided in the question. Thus, the find/solve type questions generally have a single common solution and a fixed path. The thesis recommends new tasks for AM of the form where students are asked to make different shapes on a graph paper when they are given a particular numerical value for area. In particular, tasks that work in two directions – one where students are given a shape and are asked to calculate the area and the other where students are given a numerical value and are asked to draw a shape with such an area – could further strengthen the spatial and numerical connection for AM. Such kinds of tasks are relevant for learning AM because they allow students to create and explore multiple spatial possibilities. They also provide opportunities for learners to practically explore the principle of conservation of area.

IV. The thesis provides examples of tasks requiring a richer and more meaningful use of grids /graph paper. Graph paper can function as a culturally developed tool for studying area within the classroom microculture, similar to using rulers for length measurement. The textbook analysis, classroom observation and students’ interviews showed a very procedural use or understanding of grids. Grid/graph paper was extensively used in the classroom study in Chapter 4.

V. The thesis highlights the importance of physical tasks, especially Tangram type tasks, making the important aspects of such tasks explicit. As argued in Chapter 2, the literature has provided us with several important and relevant tasks for AM. Although many such tasks are used for assessing learning, some of them hold a lot of potential for learning itself. Instead of using tasks as an assessment for capturing students' errors and mistakes, those tasks or situations can be improvised to make them resourceful and challenging learning situations. Thus, rather than acknowledging the strength of students' reasoning processes, the focus turns into what (formal) aspects of measurement students know with less focus on understanding their intuitive (non-formal) reasoning or spatial reasoning. The end result of this use is just to label students' reasoning as inadequate, rather than using it as a learning tool to address students' reasoning. In the literature on AM, one of the important tasks used is the covering task with physical tiles. The challenge with physical tiles is that it may pre-structure the task (Outhred & Mitchelmore, 2000). However, the extension tasks to find the actual measurement without physical tiles turned out to be too abstract for students to make the precise marking without a tool. Physical materials or tools like Geoboard, provide a useful resource to introduce area or AM as a countable measurement, where different polygons can be measured in terms of a unit, by partitioning a given shape into different segments, thus making AM more apparent/visible/graspable and countable as they are unitized through the equi-spaced nails. However, given such a structure of nails in a geoboard can raise the risk of students tending to count the nails instead of noticing the space (Cavanagh, 2007). Further, confusing area with linear dimensions is also prevalent among students. Thus it is not enough to just ground the task with physically graspable materials. There is a need to focus on more meaningful and conceptual tasks that have the potential to highlight relevant aspects or attributes of AM over other features. The choice of Tangram type tasks over other such physical tasks is favoured because tangram tasks do not have this prestructured counting ingrained in them. They also potentially integrate the foundational components of mathematics: spatial, numerical and even algebraic aspects. Considering this rich potential of Tangram type tasks, they were also used in Chapter 5 to test for their impact on cognitive processes. The study in Chapter 5 provides evidence for the way Tangram type tasks prime students to tend to use chunking strategies for solving AM tasks.

VI. The strands of literature following a linear curriculum highlight significant gaps with respect to AM curriculum, and provide developmental models of learning, which in turn streamline our curriculum into a linear progressive model. However, my curricular review and textbook analysis revealed that the topics appear disconnected and isolated from each other in the context of AM. The thesis reveals that the concept or context of AM relies on several mathematical concepts and topics in a network form. Thus, the thesis argues for a spiral integrated curriculum in contrast to the existing disconnected linear curriculum. The thesis proposes and provides an exemplar for the spiral curriculum in the context of AM, which can be further applied and extended to other math concepts and topics.

VII. Finally, one of the significant implications of the thesis is that it highlights the foundational role of AM. Davydov's curriculum is based on measurement. He argued that it is more coherent, logical and hence psychologically meaningful for students. Drawing from Davydov's argument, the thesis argues for a coherent curriculum based on meaningful grounding of the foundational math topics. The thesis uncovers the potential of AM to act as a foundational topic to open up avenues for other advanced mathematical topics like fractions, calculus, multiplication, geometry, functions and statistical ideas.

As argued earlier, area-measurement acts as the first physical quantity where students come across multiplication in the form of a product of two other quantities. And this opens up the horizon to explore all such quantities defined as a product of two other quantities, as found in the discipline of Physics (Smith, Males, Gamlates, 2016). Thus the thesis provides an integrated and interdisciplinary model for mathematics education.

6.4.2 Pedagogical Contribution:

In addition to the curricular contribution, the thesis also has some pedagogical implications. The thesis reiterates the importance of social constructivism as a pedagogical approach in the classroom. It presents exemplars of the application of social constructivism in a classroom through collective construction using the framework of argumentation. It also adds to the significance of socio-mathematical norms to achieve social construction in the classroom. The thesis also cautions about exercising or practising the norms in a mechanical manner, as a tool for argumentation rather than engaging in depth to rationalise for collective construction. Overall, the thesis exemplifies classroom practices that encourage social constructivism through argumentation and collective construction.

6.4.3 Research Contribution:

Apart from the curricular and pedagogical contribution, the thesis provides some specific research and theory contributions to the discipline of mathematics education and cognition. The thesis exemplifies a case for adapting three broad paradigms: constructivist, social constructivist and enactivist theories in MER, and contributes in terms of the design and development of new tasks that would allow moving in this direction. Drawing from the literature, which provides many relevant tasks to support AM learning, the analysis of such tasks builds on their strengths and addresses the gaps. For instance, I highlight the significance of the multiplicative composition of units rather than additive counting, in addition to addressing the continuous nature of AM. Thus, the thesis provided new ways to imagine the creation of tasks for AM in particular, and math education research (MER) in general. The thesis also strengthens the network model. The use of Tangam as one of the key tasks in Chapter 4 (Study 2) builds the potential of using the Tangram to integrate several foundational mathematical topics, concepts and subconcepts through social interaction (Chapter 4). The potential/role of Tangram are further explored from the perspective of enactivist theories as key mathematical manipulatives in

Chapter 5 (Study 3), as an instance of learning concepts through material interaction. I draw on, and also extend, enactivist theories, recognized as an emerging promising research direction, both theoretically and methodologically (Reid, & Mgombelo, 2015; de Freitas and Sinclair, 2014). While MER is mostly dominated by the hierarchy between the concrete and abstract, researchers have argued for the relevance of concrete manipulatives as equivalent to symbolic ones (Uttal, Scudder, & DeLoache, 1997). Thus the thesis further highlights the importance of non-symbolic (visual, spatial or material) processes in MER.

The thesis adds to the research literature of enactivist theories by providing a theory of cognitive mechanisms occurring in manipulations. This adds to the literature on studying cognitive mechanisms and provides experimental evidence for the same. Similarly, Chapter 5 provided us with a new study design, and a novel experimental set-up was developed to capture the process data. Unlike the usual scenario of installing a static eye-tracker on a computer, here a static eye-tracker was used to capture students' hand and eye movement. A new analytical method was developed to capture, analyse and interpret the process data. The constructs of action strategy: chunking and counting, were defined to interpret the process data, which were further validated by the validation group. Using this analysis method, the role of tangram type manipulation was studied over a baseline and a clay group. Thus, the thesis adds to theories of enactivism, however also highlighting the significance of specific material interaction happening in Tangram type manipulation over clay manipulation. Chapter 5 also highlights the failure or limitation of outcome-based studies, and brings our attention back to the significance of processes over answers. Tasks were designed and developed to require students to engage with the process of AM, rather than direct application of formula or additive counting.

The branch of enactivism in recent times mainly focuses on digital interactives or digital manipulation over concrete or physical manipulation. However, considering the challenges of accessing new-media or digital interactive media in developing countries, the thesis highlights the importance of integrating and developing perspectives for physical manipulatives (Sacristán, Rahaman, Srinivas, & Rojano, 2021).

Overall, the thesis provides an imagination of learning beyond the conventional classroom to more accessible environments for learners.

6.5 Limitations and challenges

This section reports the limitations and challenges encountered in the thesis project. One of the common challenges faced across all of the three major studies in the thesis, was to avoid the formal word "area" when interacting with students, as it tends to limit students to think only about the formal or school based notion or practice associated with the term. That is, students tended to directly look for some numbers or dimensions to apply to some formula or do numerical operations without engaging more deeply and meaningfully with the given task. Thus I wanted to use an alternative term for area that would be closer to students' real life or practical context. However, one of the major limitations of the thesis in this regard is the challenges in accessing students' local languages or local contexts for AM. Since the study setup was limited to the school or institute context, the studies could not reveal or access other local terms or contexts of AM used at students' homes or native places.

The interactions with students in the studies reported in the thesis were mediated in three languages English, Hindi and Urdu. However, India is a multicultural and multilingual country, and as the setting of study is in an urban hub, it was a major challenge to explore or extract the understanding of AM from all such multiple contexts familiar to the students. As a researcher, I was not aware of many spoken languages used by students at their native places. Thus, after extended deliberation, I resolved to use the word "size" as an alternative to the term "area" considering it was being used in the contexts I was aware of. Though the term "size" carried ambiguity in the sense of not indicating the exact attribute of interest, it was enriched with the context of AM to make this explicit.

Another related limitation of the study is the location of the study, which is an urban city. Some of the native contexts of the students, like farming or grazing on a land or field, were thus missed. Thus, though the initial studies of the thesis tried to explore student's intuitive conceptions and reasoning, it is likely that a lack of exposure to students' life experiences and language sub-cultures came in the way of bringing out students' conception of AM.

6.6 Further implications and recommendations

Most of the implications are already reported in the previous sections, however, this section covers insights worth improvising or extending further to future work. The thesis exemplifies the network model of learning in the context of AM. However the same can be applied and explored in the context of other mathematical topics and concepts. Thus, one of the future extensions of this thesis is widening the scope of the network model to encompass other topics.

The thesis exemplifies the design and development of proper materials and tasks to address the gaps in AM learning, the same can be extended to other topics and concepts to further address the gaps in MER.

Building on the ideas of the use of cultural tools, the thesis proposes the use of a grid as a cultural tool within the classroom microculture. The thesis presents a very rich use of the grid by providing a wide range of tasks based on it (in Chapter 4), unlike the procedural use of a grid found in existing curricula (textbooks and pedagogy). As one of the concrete recommendations of the thesis, just like a transparent ruler, I would like to recommend the use of a transparent grid (see figure 6.1) as a tool to measure AM. This could be designed and made widely available to school students.

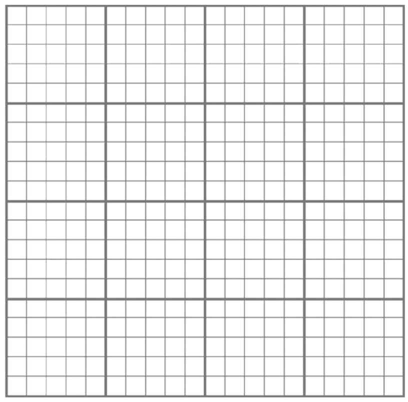


Figure 6.1: Transparent grid

Thus, just like all the conceptual properties of length measurement are held in a compact form within a ruler, similarly a transparent grid could hold all conceptual characteristics of AM within it. The thesis also recommends moving beyond typical shapes of geometry in general and AM in particular. Thus, apart from using a transparent grid, students should also be provided with tasks that encourage them to measure irregular and curved shapes and even country maps, to provide students with multiple tools to visualise the planar measure of any shape or surface through the lens of a grid. This will also integrate the spatial and numerical aspects of AM, which is found to be disconnected in the existing curriculum and challenging to students to connect them.

Appendix

Appendix I

Consent Form

HOMI BHABHA CENTRE FOR SCIENCE EDUCATION

TATA INSTITUTE OF FUNDAMENTAL RESEARCH

V. N. Purav Marg, Mankhurd , Mumbai 400088.

A study will be conducted at Homi Bhabha Centre for Science Education on 29 February, 2012. This study will consist of informal interaction with the student for about 20-30 minutes. The aim of the study is to explore students' strategies when they deal with some specific mathematical concepts.

Interested students may participate in the study by filling the details in the form below.

Student Name : ____

Std :____

Address :____

____

Contact Number of Parents : ______________________________________

Details of the Study

Date : 29th February, 2012

Duration: 20-30 minutes

Time : between 1.30 pm to 7.30 pm

Place : Homi Bhabha Centre for Science Education, Room no. 119

Participant's parents are kindly requested to bring their child on 29 February, 2012 and take them back from the Homi Bhabha Centre For Science Education. Kindly give your consent for video recording the interview session for further research and analysis. I assure you that we will abide by the research ethics and confidentiality. We appreciate your cooperation.

Parent Name :______________________________

Signature and Date : ___________________________________

Appendix II

MSB 5	MSB 6	MSB 7
1. Numbers 2. Operations on Numbers: Addition and Subtraction 3. Operation on Numbers: Multiplication and DIvision 4. Unitary Method 5. Divisibility 6. Profit and Loss 7. Measurement of Time 8. Equivalent Fractions 9. Addition and Subtraction of Fractions 10. Multiplication and Division of Fractions 11. Geometry: Basic Concepts 12. Angle and Triangle 13. Roman Numerals 14. Decimal Fractions: Introduction 15. Decimal Fractions: Addition, Subtraction, Multiplication 16. Measurement	1. Divisibility 2. Order of Operations and the Use of Brackets 3. The Use of Letters in Place of Numbers 4. Point, Line, Plane 5. Angle 6. Pair of Angles 7. Natural Numbers and Whole Numbers 8. Indices 9. Square and Square Roots 10. Decimal Fractions - Divisions 11. Ratio and Proportion 12. Profit and Loss 13. Perimeter PartII. 14. Integers 15. Algebraic Expressions 16. Addition and Subtraction of Algebraic Expressions 17. Equations with one Variable 18. Percentage 19. Simple Interest	PART ONE Properties of Triangles Squares and Square Roots Indices Averages Variation Theorem of Pythagoras Product of Algebraic Expressions Construction of Triangles Quadrilaterals Equations in One Variable Simple Interest Rational Numbers Miscellaneous Problems: Set 1 PART TWO Operations on Rational Numbers Profit and Loss Congruence Types of Quadrilaterals Area Identity Factors of Algebraic Expressions Joint Bar Graphs Volume and Surface Area Circle Construction of Quadrilaterals

17. Segment: Measurement and Construction 18. Properties and Rectangles and Squares 19. Circle 20. Perimeter 21. Area 22. Statistical Data	20. Triangles and Types of Triangles 21. Properties of Triangles 22. Geometric Constructions 23. Bar Graphs 24. Area 25. Volume 26. Circle.	Miscellaneous Problems: Set 2 Answers
NCERT 5	NCERT 6	NCERT 7
NCERT 5 MATH-MAGIC What is inside this book? The Fish Tale Shapes and Angles How Many Squares? Parts and Wholes Does it Look the Same? Be My Multiple, I'll be Your Factor Can You See the Pattern? Mapping Your Way Boxes and Sketches Tenths and Hundredths Area and its Boundary Smart Charts Ways to Multiply and Divide How Big? How Heavy?	NCERT 6 Knowing our Numbers Whole Numbers Playing with Numbers Basic Geometrical Ideas Understanding Elementary Shapes Integers Fractions Decimals Data Handling Mensuration Algebra Ratio and Proportion Symmetry Practical Geometry Answers Brain-Teasers	NCERT 7 Foreword Preface Integers Fractions and Decimals Data Handling Simple Equations Lines and Angles The Triangle and its Properties Congruence of Triangles Comparing Quantities Rational Numbers Practical Geometry Perimeter and Area Algebraic Expressions Exponents and Powers Symmetry Visualising Solid Shapes Answers Brain-Teasers

Appendix III

Questions to be discussed with student

Name: ....

Age: ....

Class: …..........

Paper sheets and pencil will be given to the student to write whenever required and a voice recorder will be used for recording.

Ask the student to draw a rectangle and a square of any dimension and in case the student is not able to draw those figures, explain to the student what are rectangles and squares by drawing and verbally defining.

1(i). A school playground is rectangular in shape with length 200m and breadth 100m.
Can you draw the playground on this sheet, is it fine? Are you sure?

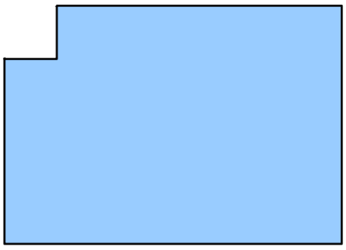
A square help desk of side 10m in which all first aids are kept is to be placed within the playground. The head mistress feels that the desk should be located in one corner of the playground.
(Ask them to draw the square help desk in the rectangular field, in case s/he is confused about which corner, tell her that any corner is fine).

The games teacher feels that the square desk should be along the edge of the playground because the games teacher thinks then there will be more space for the children to play. Is your games teacher right, what is your opinion ?
(Ask them to draw another rectangular field with the help desk in the new location).

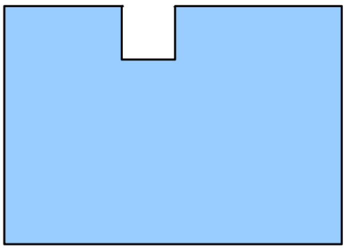
1(ii). A company that makes a lawn by cultivating grass in the field wishes to make lawn for the playground by growing grass. It charges according to how much space it covers by grass. Which lawn will cost more to the school?

Repeat the question if required and ask them to use the given sheets for justifying their response.
In case they are not able to draw the arrangements, show them the figures given below using a

different sheet in which these figures were drawn in advance.



Arrangement 1



Arrangement 2

Arrangements for playground

1(iii). A jogging path is made along the boundary of the playground for the first case in the previous situation and Kajal runs along that path. Can you trace her path?

Ask them to darken that path either in their own figure or in the figure provided to the student.

A jogging path is made along the boundary of the playground in the second situation and Kajal runs along that path. Can you trace her path?

Ask them to darken that path for the second situation.

Does she cover the same or different distances in the two situations?

Tell the student to justify their response.

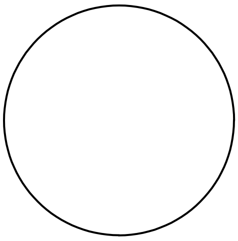
2(i). Show them the figures drawn in the sheet 2(i) [separate sheets provided].

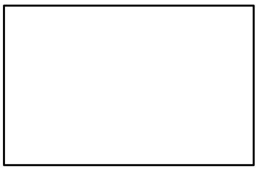
Can you show me the area here?

Can you colour the area of the first two figures with pencil?

(See whether they are able to show the spread for area)

2(i).





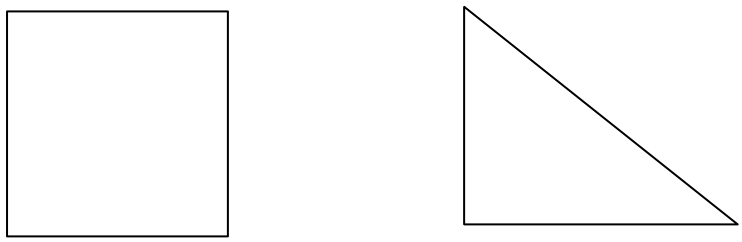
2(ii). Show them the figures drawn in the sheet 2(ii) [separate sheets provided].

Can you show the perimeter of the two figures with pencil?

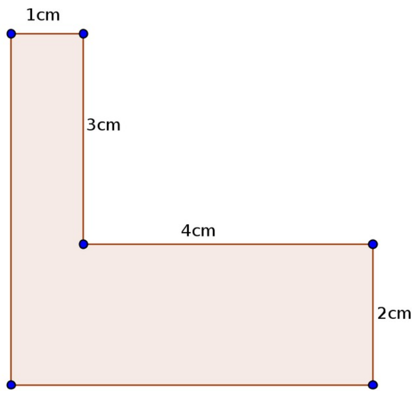
Can you darken the perimeter ?

(See whether they are able to trace the perimeter)

2(ii).



3. Student is given a sheet with the diagram as below and is asked to find the perimeter and area of the following figure without using scale.

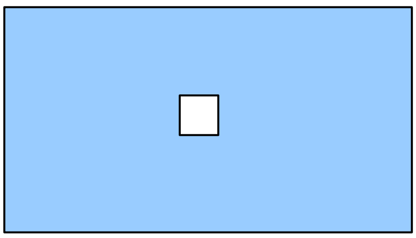


4. Draw a shape having an area 16 square cm. Can you draw some other shapes both rectangular and non-rectangular having the same area. (Give graph paper for drawing the shapes.)

5. A room has a rectangular floor of length 20m and breadth 10m. A pillar is made in the middle of the room with a square base of side 2m.

What is the area and perimeter(optional) of the remaining floor area ?

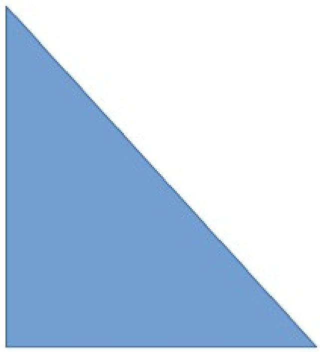
In case they are not able to draw the arrangements, show them the figures given below using a different sheet in which these figures were drawn in advance.



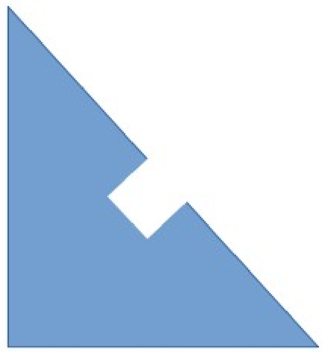
Ask them to write in the sheet provided to them.

6. A triangular piece of paper of perimeter 80cm from which a square of perimeter of 20cm is cut off as shown.

Show the student the figure shown below using a different sheet of paper.



Triangular piece



After a square part is cut off

What is the perimeter of the remaining piece?
Ask them to write in the sheet provided to them.

7. The student is given a rectangular sheet of sides 21cm and 12cm, and three different paper tiles of dimensions 2×2, 3×4, and 6×2, given one at a time.
Can this tile, if pasted repeatedly, cover the sheet? If Yes, ask the student: If the same tile is pasted repeatedly, how many of such tiles will be required to cover the sheet.
Ask the student to write it on the sheet provided.

8. You want to cover a rectangular floor with a length of 19m and a breadth of 6m using tiles. Can you cover the floor with a rectangular tile having sides of 3m and 2m? What other tiles can you use to cover this floor?

9. You want to cover a rectangular floor with a length of 15m and a breadth of 8m using tiles. Can you cover the floor with tiles that are right triangles of height 2m and base 5m?
Optional question: What other tiles can you use to cover this floor?

Appendix IV

Consent form used in Chapter 4 in Hindi and English

Consent form used in ** School

होमी भाभा विज्ञान शिक्षा केन्द्र

टाटा मूलभूत अनुसन्धान संस्थान

वि. न. पुरव मार्ग, मुम्बई 400088

<u>सहमति-पत्र इक़रार नामा</u>

होमी भाभा विज्ञान शिक्षा केन्द्र (एच. बी. सी. एस.ई), 'टाटा मूलभूत अनुसन्धान संस्थान' (टि. आय. एफ. आर) मुम्बई का एक राष्ट्रीय केन्द्र है। इसका खास मक्सद प्राथमिक स्कूलों से लेकर ग्रेजुएशन तक विज्ञान और गणित की पढ़ाई में काबलियत को बढ़ावा देना, देश में विज्ञान शिक्षा की तरक्की और समाजी तरक्की को बढ़ावा देना, और खोज (तहकीकी) और सामग्री विकास करना शामिल है। पिछले ढाई सालों से हम ** उर्दू और अंग्रेजी स्कूल में बच्चों के स्कूल के बाहरी जिन्दगी' से गणित के अलग-अलग पहलू सीखने के तरीकों को समझने की कोशिश कर रहे हैं। इसी सिलसिले में उर्दू स्कूल के छठी और सातवीं क्लास के बच्चों के लिए दो हफ़्तों का रियाजी पर स्पेशल क्लास (छुट्टी-कैम्प) रखा गया है जो 12 अप्रैल से 27 अप्रैल 2013 तक चला। इन स्पेशल क्लासों का विडियो रेकॉर्डिंग किया गया जिनका इस्तेमाल सिर्फ़ अनुसन्धान कार्य (तहकीकी) और भविष्य के कैम्पों का खाका तैयार करने में किया जाएगा। साथ ही साथ इससे बच्चों की हुनर और जरूरतों का भी पता चलेगा।

इस कार्यक्रम में आपकी सहमति (रजामन्दी) जरुरी है। आपकी दी गई जानकारी गुप्त (खुफ़िया) रखी जाएगी।

तालीब इल्म का नाम:

क्लास :

पता:

वालिदेन का दस्तख़त :

फोन नम्बर:

तारीख :

Homi Bhabha Centre for Science Education

Tata Institute of Fundamental Research

V. N Purav Marg, Mumbai 400088

Letter of Consent

Homi Bhabha Centre for Science Education (HBCSE), a national center in 'Tata Institute of Fundamental Research' (TIFR), Mumbai. Its specific objectives include promoting competence in science and mathematics studies from primary schools to graduation, promoting the advancement of science education and social progress in the country, and research and content development. For the last two and a half years, we have been trying to understand the different aspects of mathematics learning in ** Urdu and English School. In this regard, a special class (holiday-camp) has been organized for the students of class VI and VII of Urdu school for two weeks which ran from 12th April to 27th April 2013. These special classes were video-recorded and will be used only for research purposes and to prepare blueprints for future camps. At the same time, it will reveal the skills and needs of the children.

Your consent is required in this program. The information you provide will be kept confidential.

Name of the Student:

Class :

Address:

Parent's signature:

Phone Number:

Date:

Appendix V

Worksheet

Name: ____________________________

School: ___________________________

1. Match the following :

1.	MS Dhoni	Badminton
2.	Black Pepper Charles	Darwin
3.	Alladin Isaac	Newton
4.	The laws of gravitation	Magic Lamp
5.	Cardiologist	Tennis
6.	The theory of evolution	.in
7.	The theory of relativity	A spice
8.	The domain name for India	Heart Disease
9.	Saina Nehwal	Cricketer
10.	Sania Mirza	Einstein

2. Tick the correct response :

1. The control centre of the human body is ___________

   (a) heart (b) brain (c) liver (d) kidney

2. Which of the following is a flightless bird?

(a) Pigeon (b) Flamingo (c) Owl (d) Ostrich

3. Which pair is different from the rest

(a) 2, 8 (b) 4, 16 (c) 5, 25 (d) 8, 64

4. Which of the following countries have the largest population?

(a) India (b) China (c) Japan (d) USA

5. Who is Pranab Mukherjee?

(a) Prime Minister (b) Cabinet Minister (c) President (d) Army Chief

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