Math Wars and the Epistemic Divide in Mathematics

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Abstract

(a) Why do school (K-12) students find mathematics especially difficult? (b) What is a good way to ameliorate these difficulties? (c) Would the new technology of computation *fundamentally* change the *content* of mathematics? Learning difficulties peculiar to mathematics are here traced to an epistemic schism in mathematics. Using "phylogeny is ontogeny" these difficulties are seen as reflections of actual historical difficulties. Much mathematics taught at the K-12 level is of Indo-Arabic origin: (1) arithmetic, (2) algebra, (3) trigonometry, (4) calculus. This mathematics arose in a different epistemic context, and Europe experienced difficulties in assimilating it because it recognized only a single "universal" European mathematics. This led to the real math wars, lasting for a thousand years, first over algorismus and zero and then over calculus and infinitesimals. Computers have precipitated a third math war by again greatly enhancing the ability to calculate in a way regarded as epistemically insecure. The suggested correction is to recognize the distinct epistemic setting of mathematics-as-calculation and teach it accordingly.

1 Introduction

1.1 The Math Wars in the US

In recent times, mathematics education in the US has been ravaged by the so called "Math Wars". Worry over the poor performance of US students in mathematics tests[1] again focused attention on mathematics education in the 1980's. This led to the formulation of a set of standards by NCTM[2] in 1989 (contested by

[1] National Commission on Excellence in Education, 1983, *A Nation at Risk: the Imperative for Educational Reform*, US Department of Education, Washington DC.

[2] National Council of Teachers of Mathematics, 1989, *Curriculum and Evaluation Standards for School Mathematics*, NCTM, Reston, VA.

e.g. California,[3] and updated[4] in 2000). The US Education Department brought out a White Paper on mathematics education,[5] and, in October 1999, endorsed as "promising" certain texts promoting "constructivist"[6] or discovery-learning methods of teaching mathematics. This "constructivist" curriculum has since been labeled "new new math", "fuzzy math",[7] and "no correct-answer math" by opponents, who include Field Medalists and Nobel Prize winners.[8] Worries about poor performance in mathematics persist,[9] and the TIMSS-R[10] sought to relate this poor performance to a variety of factors, (apart from the curriculum), such as university degrees of maths teachers, home education resources etc.

1.2 The epistemic divide

None of this addresses the root cause of learning difficulties specific to mathematics. The controversy surrounding the "new new math" of the 90's, like that surrounding the "new math" of the 1960's, is situated by this paper as only a symptom of a deeper and more persistent malaise, an epistemic schism within mathematics. The quarrel about *what* and *how* mathematics should be taught simply reflects fundamentally divergent perceptions of what mathematics *is*.

[3] California State Board of Education, 1997, *Mathematics Content Standards for California Public Schools, Kindergarten through Grade 12*, California Department of Education, Sacramento, CA, 1999.

[4] National Council of Teachers of Mathematics, 2000, *Principles and Standards for School Mathematics*, NCTM, Reston, VA.

[5] US Department of Education, 1997, *Mathematics Equals Opportunity: White Paper Prepared for US Secretary of Education, Richard Riley,* October 20, 1997.

[6] The constructivist approach, based on the theories of Jean Piaget, holds that all knowledge is constructed by individuals by assimilating new experiences into an existing base, or constructing new schemas. Radical constructivists believe that each person discovers truth anew. Hence, constructivists believe that instead of being handed down the rules of mathematics authoritatively by the teacher, students should be exposed to mathematical situations, and should discover the rules inductively from experience. For an empirical study in this direction, see e.g., Geoffrey B. Saxe, "The Mathematics of Child Street Vendors", *Child Development* **59** 1988 1415–25. Opponents of constructivism hold the view that students may fail to construct the right rules, and the wrong rules they construct may be difficult to deconstruct later on. R. Davis, C. Maher, and N. Noddings, *Constructivist Views on Teaching and Learning of Mathematics*, NCTM, Reston, VA, 1991. Constructivism must be distinguished from social constructivism, which raises even more fundamental questions of whether in mathematics, as in music, there is at all any "right" rule, independent of culture; see, e.g. Ubi D'Ambrosio, *Ethnomathematics*, and *Socio-Cultural Bases for Mathematics Education*, Unicamp, Campinas, 1985, Paul Ernest, *Social Constructivism as a Philosophy of Mathematics*, SUNY Series, Reform in Mathematics Education, 1998, and C. K. Raju, *Journal of Indian Council of Philosophical Research* **18** (2001) 267–270.

[7] Congressional Record of the US Senate, "A Failure to Produce Better Students", Senate, June 09, 1997, Congressional Record, p S5393. Robert Byrd, D-West Virginia.

[8] *Washington Post*, 18 Nov 1999.

[9] *New York Times Metro*, 24 October 2001, "Most Eighth Graders Fail Statewide Math and Reading Tests". D5.

[10] I.V.S. Mullis et al, *TIMSS 1999 International Benchmarking Report: Findings from IEA's Repeat of the Third International Mathematics and Science Study at the Eighth Grade*, Boston College, Chestnut Hill, MA, 2000. Also I V S Mullis et al, *Mathematics Benchmarking Report TIMSS 1999-Eighth Grade Achievement for US States and Districts in an International Context*, Boston College, Chestnut Hill, 2001.

This divide in mathematics is rooted in history. Much of what is today taught in K-12 mathematics—arithmetic, algebra, trigonometry, calculus—is a product of a complex historical process of cultural assimilation as some of the very names "algebra", "sine", "surd" and "algorithm" indicate.[11] Elementary arithmetic algorithms, for example, competed with abaci for over six hundred years in Europe because of the difficulties encountered in this process of assimilation.

1.3 Phylogeny is Ontogeny

This paper proposes that we learn from these historical difficulties by applying in a novel way the principle that "phylogeny is ontogeny"—that the learning process reflects the historical evolution of the subject, telescoped into a much shorter period of time. Thus, the attempt is to understand the difficulties that students today have in assimilating elementary mathematics by studying the difficulties that arose historically in the process of culturally assimilating that mathematics. Correction naturally follows a better understanding.

2 Detoxifying the history of mathematics

This, of course, requires a fresh approach to history not as an instrument of glorification, but as a means of understanding. This new approach makes epistemology the key to understanding the history of mathematics.

2.1 The two streams of mathematics

Briefly, Europe inherited not one but two mathematical traditions: (i) from Greece and Egypt[12] a mathematics that was spiritual, anti-empirical, proof-oriented, and explicitly religious, (ii) from India via Arabs a mathematics that was pro-empirical, and calculation-oriented, with practical objectives[13] Much mathematics taught at the K-12 level is of Indo-Arabic origin: (1) arithmetic, (2) algebra, (3) trigonometry, and (4) calculus.

[11]Elementary arithmetic algorithms from "Algorismus" being a Latinization of Al Khwarizmi, and his translation of the texts of Indian mathematicians like Brahmagupta, Bhaskara, and Mahavira; "algebra" from the Arabic *al jabr*;"sine" from the Latin *sinus*, being a translation of the Arabic *jaib*, being a misreading of *jībā* (from the Sanskrit *jīvā*, both being written without vowels as *jb*); "surd" from the Latin *surdus* = "deaf" from "bad ear", with "ear" = *karṇa* being a misreading of the Sanskrit *"karaṇī"* = diagonal (used for square root extraction). Some details of transmission may be found in Suzan Rose Benedict, *A Comparative Study of the Early Treatises Introducing into Europe the Hindu Art of Reckoning,* Ph.D. Thesis, University of Michigan, April 29, 1914, Rumford Press, Concord (no date given).

[12]Martin Bernal, *Black Athena: The Afroasiatic roots of Classical Civilization*, Vol. I: *The Fabrication of Ancient Greece 1785–1985.* Vintage, 1991. C. K. Raju, "How Should Euclidean Geometry be Taught", in: G. Nagarjuna (ed) *History and Philosophy of Science: Implications for Science Education*, Homi Bhabha Centre, Bombay, 2001, 241–260.

[13]C. K. Raju, "Computers, Mathematics Education, and the Alternative Epistemology of the Calculus in the Yuktibhāsā", *Philosophy East and West:* **51**(3) 2001, 325–62.

Despite the obviously different philosophical orientations of these two streams of mathematics Europe recognized only a single possible philosophy of a "universal" European mathematics, into which it forcibly sought to fit both mathematical streams. One can understand how this happened under the influence of religious politics as follows.

2.2 The role of religious politics

In Europe ever since state and church came together some 1700 years ago, history became a malleable instrument of religious politics. Through Constantine, Charlemagne, crusades, and colonisation, the church thrived on the most extreme agenda of hate and violence ever known to humanity. Papal *fatwas*, like the Bull Romanus Pontifex, promulgated a doctrine known as the "Doctrine of Christian Discovery",[14] which required, inter alia, that no "theologically incorrect" part of the world, could, in principle, make any significant contribution to knowledge or discovery. Though these Bulls have been widely regarded[15] as setting the agenda for religiously motivated genocide in the Americas,[16], they also set the agenda for intellectual genocide, by seeking to eliminate the contributions of the Persians, the Egyptians, and the Arabs, by the crude device of attributing all of it to the "Greeks". Furthermore, the extreme violence of the church was also directed inwards: in the days of the Inquisition, the slightest acknowledgment of "pagan" influence could easily have led to one being denounced by some rival, with grave and excessively painful consequences. Even in England, a Newton kept his theological deviance secret throughout his life, and the final version of his 8-volume *History of the Church* still remains a secret.[17] All this resulted in the amusing historical fantasy that mathematics originated in "Greece" (located in Africa!)

This distorted history inevitably impacted also the philosophy of mathematics, so that mathematics came to be defined in Europe as something that imitated the "Greek" method of proof—as sanitized by Christian rational the-

[14]Pope Nicholas V, bull Romanus Pontifex, 1453: "[W]e bestow suitable favors and special graces on those Catholic kings and princes... intrepid champions of the Christian faith... to invade, search out, capture, vanquish, and subdue all Saracens and pagans whatsoever, and other enemies of Christ wheresoever placed, and... to reduce their persons to perpetual slavery, and to apply and appropriate... possessions, and goods, and to convert them to... their use and profit." This was later followed by the bull Inter Cetera of Pope Alexander of 3 May 1493, giving the rights to conquest and subjugation of one part of the globe to Spain, and the other part to Portugal. F. G. Davenport, *European Treatises bearing on the History of the United States and its Dependencies to 1648*, vol. 1, Carnegie Institute of Washington, Washington, DC, 1917, pp. 20–26, and pp. 61–68. *Hence,* according to US law, Indians lost their right to their ancestral land upon being "discovered" by the Christian Columbus. *Johnson and Graham's Lessee V. McIntosh* 21 U.S. (8 Wheat) 543, 5 L.Ed. 681 (1823)

[15]e.g. Steven T. Newcomb, *Pagans in the Promised Land: Religion, Law, and the American Indian*, Indigenous Law Institute, 1995.

[16]Bartolomé de las Casas, *A Short Account of the Destruction of the Indies*, 1540/42, trans. (e.g.) Nigel Griffin, Penguin, 1992.

[17]C. K. Raju, "Newton's Secret", chp. 4 in *The Eleven Pictures of Time*, Sage, New Delhi, 2003.

ology.[18] A key element of this sanitization was the complete elimination of the empirical from mathematics, as in the current notion of mathematical proof due to Hilbert and Russell. The complete elimination of the empirical conveniently reduced mathematics to a branch of metaphysics.

3 The real math wars

Because of this agenda of forcing all knowledge to fit a convenient theological mould, Europe attempted to force the imported practical mathematics into a metaphysical mould of mathematics-as-certitude. This led to a protracted struggle lasting a thousand years: the resulting tensions were reflected not only in Clavius' advocacy of practical mathematics and his influential reform of the mathematics syllabus,[19] but also in popular satire[20] on Platonic mathematics. The difficulties with the infinitesimal calculus, and, more recently, computational mathematics, are some of the other high points of this struggle.

More systematically, in this thousand-year old and continuing clash of mathematical epistemologies, one can identify three phases, concerning algorismus, calculus, and computers, respectively.

(1) **(Algorithms and the First Math War)** Today's elementary arithmetic algorithms were accepted in Europe after some six hundred years of battle (from the 10th to the 16th c. CE) between earlier abacus methods and algorismus methods. Gerbert (Pope Sylvester II, d. 1003 CE) first used Indo-Arabic symbols on counters (*apices*) without understanding that method of computation.[21] Algorismus texts were based on (al Khwarizmi's) translations of Indian mathematical texts of the 7th c. CE, and these methods of arithmetical com-

[18]It is generally overlooked that Proclus justified the appeal to the empirical at the beginning of mathematics, as in the first few theorems of the *Elements*, since this was quite acceptable for his understanding of mathematics, as also for the understanding of mathematics in Islamic rational theology. However, in Christian rational theology, because mathematics was regarded as certain, or as incorporating necessary truth, and the empirical *had* to be regarded as contingent to permit God to create a world of his choice, therefore, the empirical had no place in mathematics.

[19]Christoph Clavius, c. 1575? "A Method of Promoting Mathematical Studies in the Schools of the Society", Document No. 34 in: E. C. Phillips, "The proposals of Father Christopher Clavius, SJ, for Improving the Teaching of Mathematics" *Bull. Amer. Assoc. Jesuit Scientists* (Eastern Section), Vol. XVIII, May 1941, No. 4, 203–206. Also, Christophori Clavii Bambergensis e Societate Iesv, *Epitome Arithmeticae Practacae*, Rome, Dominici Basae, 1583, Tr. into Chinese by Matteo Ricci.

[20]Jonathan Swift, *Gulliver's Travels, Part III, A Voyage to Laputa...*, Wordsworth Editions, 1992, p 125, "His Majesty discovered not the least curiosity to enquire into the laws, government, history, religion, or manners of the countries where I had been; but confined his questions to the state of mathematicks, and received the account I gave him, with great contempt and indifference...".

[21]Karl Menninger, *Number Words and Number Symbols: A Cultural History of Numbers*, trans. Paul Broneer, MIT Press, Cambridge, Mass., p 325, "Yet it would be wrong to see in the *apices* nothing more than a trivial innovation introduced by Gerbert. The truth is that he did adumbrate the use of the new numerals; he had heard marvelous things about the new computation which they made possible but which he, and perhaps also his informants, did not essentially understand."

putation, studied for their practical value by Florentine merchants, were viewed with great epistemological suspicion in Europe. The turning point of this war is usually placed in the 16th c. CE,[22] but the war truly ended only in 1834 with the burning of tally sticks which also burnt down the British Parliament.[23] The difficulties have usually been regarded as relating to the symbolic representation of numbers versus the concrete representation of numbers in the abacus. (The usual algorithms for addition, subtraction, multiplication, and division, are impossible with the Roman numerals used in Europe, and explicitly require a place-value system.) Thus, zero was problematic since it had "no value in itself, but added any amount of value on being placed after a number". But there were various other differences. The "Greek" notion attached a mystical significance to numbers, so that a typical challenge problem to a mathematician in 16th c. Europe was this, "Is unity a number?" (The expected answer being that unity is not a number.) The Indian notion, on the other hand, did not have such hang-ups. A more subtle problem related to the question of non-representable (*śūnya*, both infinitely large and infinitesimally small, later zero[24]). Thus a key problem was that, unlike Buddhist philosophy (particularly *Śūnyavāda*), idealist philosophy failed to seriously address the problem of non-representables. These difficulties, by the way, are not entirely over: look at the peculiar conventions relating to zero in the Java computing language: zero as integer behaves differently from zero as a floating point number!

[22]to coincide with the publication of Gregor Reisch's *Margarita Philosophica* (Basel, 1517) which shows a smiling Boethius and a glum Pythagoras, the former representing the algorismus, and the latter the abacus.

[23]Charles Dickens, *Speech to the Administrative Reform Association*, 18 June 1855, in K. F. Fielding (ed), *Speeches of Charles Dickens*, Clarendon, Oxford, 1960, p 206: "Ages ago a savage mode of keeping accounts on notched sticks was introduced into the Court of the Exchequer and the accounts were kept much as Robinson Crusoe kept his calendar on the desert island... it took until 1826 to get these sticks abolished. In 1834 it was found that there was a considerable accumulation of them; and the question arose, what was to be done with such worn-out, worm-eaten, rotten old bits of wood? The sticks were housed in Westminster, and it would naturally occur to any intelligent person that nothing could be easier than to allow them to be carried away for firewood by the miserable people who lived in that neighbourhood. However ...the order went out that they were to be privately and confidentially burned. It came to pass that they were burned in the stove in the House of Lords. The stove, over gorged with these preposterous sticks, set fire to the panelling; the panelling set fire to the House of Commons; the two houses were reduced to ashes; the architects were called in to build others; and we are now in the second million of the cost thereof." More details in Katherine Solender, *Dreadful Fire! The Burning of the Houses of Parliament*, Indiana University Press, 1984. This "Robinson Crusoe technology" of accounting was first introduced in Britain by Normans in the twelfth century, over six hundred years after Āryabhata; see, e.g., J. M. Pullan, *The History of the Abacus*, Praeger Publishers, New York, 1968, p 51.

[24]Zero played a key role in the transition from abacus to algorismus: "One might say, in a nutshell, that zero overcame the abacus. But its victory, which started in the Middle Ages, took a long time", Menninger, cited above, p 331. However, the role of śūnya in Brahmagupta etc. is far more sophisticated than a mere symbol used in algorithms to replace counters. Specifically, śūnya was used in a way similar to non-representables in modern-day floating-point computation. C. K. Raju, "The Mathematical Epistemology of Śūnya", summary of interventions in the *Seminar on the Concept of Śūnya*, IGNCA, and INSA, New Delhi, 1997, in: *The Concept of Śūnya*, ed. A. K. Bag and S. R. Sarma, IGNCA, INSA, and Aryan Books International, New Delhi 2002, pp. 168–181.

(2) **(Calculus and the Second Math War)** The infinitesimal calculus is another key aspect of mathematics-as-calculation, and the struggle to assimilate the calculus may be seen as exactly analogous to the case of the algorismus. As my earlier papers have sought to show, from the 16th c. onwards, Indian mathematics/astronomy texts of Āryabhata, Bhaskara, Nīlkaṇṭha, Śankara Vāriyar, and Jyeṣṭadeva, containing key results of the calculus were transmitted from Cochin[25] to Europe by Jesuits like Matteo Ricci[26] in connection with the European navigational problem (of determining latitude and longitude at sea), the related problem of computing precise trigonometric values,[27] and the related[28] calendar reform of 1582. Despite the obvious practical merits of the calculus, its inherently foreign epistemology was mathematically unacceptable to many in Europe, so that there followed another three centuries of warfare about the exact mathematical status and worth of "infinitesimals". Basically, the Indian infinitesimal techniques involved two features that were unacceptable in Europe. The first was that the Indian notion of *pramāṇa*, since it permitted the use of the empirical, was different from the European notion of mathematical proof. The second was that Indian techniques of calculation routinely used rounding, while the European notion of mathematics as certitude required that the smallest quantity should not be neglected. (This difference can still be seen in everyday commercial transactions today; in India, a vegetable vendor will routinely try to round off Rs 18 to Rs 20, by adding a small purchase, while Rs 20.50 will equally be rounded down to Rs 20. This is not the case in the West, and this cultural difference is not really to do with the non-availability of small change.) Thus, while valid *pramāṇa* was available for the infinite and indefinite series in Indian tradition, Cavalieri, Wallis, Gregory, Newton, Leibniz etc. struggled in vain to convert it into mathematical proof that was acceptable to Europeans. Despite the historical glorification with which we have been inundated, it is clear from Berkeley's objections[29] an actual epistemic advance had to await Dedekind's semi-formalisation of real numbers in the late 19th c., and the formalisation in the 20th c. of the set theory that it used. Thus it took a long time to assimilate the calculus within formalistic mathematics.[30]

[25]These books were readily available in Cochin where the first Catholic mission was established in 1500 CE and worked together with the local Syrian Christians.

[26]Matteo Ricci, letter to Petri Maffei, 1 Dec 1581, *Goa* **38** (I) ff 129r–130v; corrected and reproduced in *Documenta Indica*, XII, 472–477 (p 474). Also reproduced in Tacchi Venturi, *Matteo Ricci SI, Le Lettre Dalla Cina 1580–1610*, vol. 2, Macareta, 1613.

[27]Many navigational theorists were concerned with precise trigonometric values. See C. K. Raju, *Philosophy East and West* cited above, for references to the works of Pedro Nunes, Simon Stevin, and Christoph Clavius on sine (secant) tables used to calculate Mercator's loxodromes. For the use of these trigonometric values in traditional navigation, see C. K. Raju, "Kamāl or Rāpalagai", Paper presented at the *Xth Indo-Portuguese Conference on History*, INSA, New Delhi, 1998, to appear in Proc.

[28]C. K. Raju, "How and Why the Calculus was Imported into Europe", paper presented at the *International Seminar on East-West Transitions*, Bangalore, Dec 2000.http://www.IndianCalculus.info/Bangalore.pdf.

[29]George Berkeley, *The Analyst or a Discourse Addressed to an Infidel Mathematician*, London, 1734, ed. D. R. Wilkins, available online at http://www.maths.tcd.ie/˜dwilkins/Berkeley/

[30]The question remains partially open, for even the Schwartz theory of distributions is

(3) **(Computers and the Third Math War)** Computers, today, are rapidly widening this divide in mathematics. Numbers represented on a computer necessarily disobey key theoretical "laws", such as the associative law, required of numbers in formal number systems, and taught to K–12 students. However, using this floating point representation of numbers,[31] computers enable numerical calculations that stretch far beyond what can be mathematically proved; such calculations may have great practical value, as in solutions of stochastic differential equations driven by Lévy motion, used to estimate financial risk, or study perturbation related to controlled fusion, or in solutions of functional differential equations used in my proposal for a new physics.[32] Nevertheless, such numerical solutions continue to be regarded as mathematically valueless in the absence of a proof that the solution exists.

4 Resolving the math wars

The root cause of this thousand-year old math war may now be identified: each case of algorismus, calculus, and computers, enhanced the ability to calculate, but with techniques regarded as epistemologically insecure from the Platonic viewpoint. Being not indifferent to the practical value of the mathematics, Europeans sought to force this mathematics to be "theologically correct" by reinterpreting it. The difficulty of this task is what made the assimilation of mathematics in Europe so difficult that it took nearly a thousand years. Using "phylogeny is ontogeny", it is this superimposition of theology that makes mathematics difficult to learn today. To resolve the quarrel about the teaching of mathematics, we must first address this epistemic schism in mathematics.

inadequate to settle the way the calculus is practically used in quantum field theory (renormalization problem), and Non-Standard analysis showed incidentally that infinities and infinitesimals have a formal existence in non-Archimedian fields larger than the reals. C. K. Raju, "Products and Compositions with the Dirac Delta Function." *J. Phys. A: Math. Gen.* **15** (1982) 381–96; "On the Square of x^{-n}" *J. Phys. A: Math. Gen.* **16** (1983) 3739–53; "Renormalisation, Extended Particles and Non-Locality." *Hadronic J. Suppl.* **1** (1985) 352–70; "Distributional Matter Tensors in Relativity," in: *Proc. MG5*, D. Blair and M. J. Buckingham (ed), R. Ruffini (series ed.), World Scientific, Singapore, 1989, 421–23.

[31] Pat H. Sterbenz, *Floating-Point Computation*, Prentice Hall, Englewood Cliffs, NJ, 1974. The floating-point representation, however is not clear about what to do with non-representables. The IEEE standard 754 of 1985 specifies several categories of non-representables: NaN, overflow, underflow, INF, -INF. Confusion about non-representables and zero persists: thus the Java language treats these non-representables differently, depending upon whether the computation involved is an integer computation or a floating-point computation: thus, in Java, 2/0=NaN, while 2.0/0.0 = INFINITY.

[32] C. K. Raju, *Time: Towards a Consistent Theory*, Kluwer Academic, Dordrecht, 1994; Fundamental Theories in Physics, Vol. 65. While the book covers the relevant aspects of the existence theory for retarded functional differential equations, it advocates the use of mixed-type functional differential equations, for which no such mathematical results of any consequence are available. For the difficulties with assessing the reliability of risk estimation, using numerical solutions of stochastic differential equations driven Lévy motion, see C. K. Raju, "Supercomputing in Finance", *Pranjana*, **3** (2000) 11–36. Since large sums of money (USD 30 billion in the above case) are involved, the precise epistemological value of a numerical solution sans proof becomes critical.

We must first decide in a culturally neutral way: does mathematics relate to calculation or to proof? And, what are valid methods of proof?

On the one hand, from a formalist perspective, proof[33] has a higher epistemological value than calculation: it is today mathematically acceptable for a mathematical theorem to prove the existence of something without providing any accompanying method of calculation (or even construction), but no Field's medal was ever given for making a complex calculation, unsupported by a proof; for something that lacks proof would not today be regarded as mathematics, and would not, therefore, qualify for a Field's medal.

On the other hand, there is the undeniable fact that for all practical applications of mathematics, such as sending a man to the moon, it is not the existence theorem *per se* but the calculation that is important; and that calculation usually involves many layers of approximation, and potential sources of error, in obtaining a numerical approximation to an approximate solution of a physical model which is itself "approximate". Thus, the result of a typical calculation, though useful like the physical model, cannot but be "approximate", empirically based, and fallible—quite unlike the result of a mathematical proof, which is believed to be an exact, formal, perfect, and certain theorem.

That belief is questionable.[34] Briefly, Plato regarded mathematics as universal for he believed it concerned necessary truths. Formalists, while maintaining the Platonic divorce from the empirical, have shifted the locus of this necessary truth from theorem to proof, which is believed to connect arbitrary axioms to their necessary consequences. However, this belief too is incorrect, for proof uses logic, which is neither culturally universal (e.g. Buddhist or Jain logic[35]) nor empirically certain (e.g. quantum logic[36]). Furthermore, the notion of valid proof has varied across cultures: so formal mathematics contains no necessary or universal truths, but is purely a system of aesthetics like music.

This aesthetic does not suit practical mathematics-as-calculation which needs an alternative epistemological basis, a basis which acknowledges inexactitude, fallibility, differences from formal notions of "number", and accepts a role for the empirical ("contingent") within mathematics. Practical and useful mathematics, as decried by Plato, but as used in algorithms, calculus and numerical computation, needs a separate, non-Platonic, non-Neoplatonic ("non-Euclidean") epistemology, and it needs to be taught in a different way.

[33] as defined in the currently dominant notion of mathematics

[34] C. K. Raju, "Computers, Mathematics Education, and the Alternative Epistemology of the Calculus in the Yuktibhāsā", *Philosophy East and West* **51**(3), cited above.

[35] Buddhist or Jain logic is not two-valued or even truth-functional. C. K. Raju, "Mathematics and Culture", in: Daya Krishna and K. S. Murthy eds, *History, Culture and Truth*, Kalki Prakash, New Delhi 1999, 179–193. Reprinted in *Philosophy of Mathematics Education* **11** (1999), available at http://www.ex.ac.uk/~PErnest/pome/art18.htm. C. K. Raju, "Some Remarks on Ontology and Logic in Buddhism, Jainism and Quantum Mechanics." Invited talk at the conference on *Science et engagement ontologique*, Barbizon, October, 1999. Abstract attached.

[36] C. K. Raju, "Quantum Mechanical Time", chp. 6b in *Time: Towards a Consistent Theory*, Kluwer Academic, Dordrecht, 1994.

5 Correcting math teaching

So what does the revised history and "non-Euclidean" epistemology of mathematics mean for classroom teaching?

Briefly, since formal mathematics is no more than a culturally-dependent system of aesthetics, while it may continue to be taught like Western music, there is no need to impose its consequences on K-12 children. What we need to teach children is practical mathematics. And this can be taught much more easily in the epistemic setting in which it originated.

As a concrete example, consider the case of "Euclidean" geometry, which has been part of the traditional European mathematics curriculum almost since the inception of Oxford University, and part of the Arabic and Neoplatonic mathematical syllabus for centuries before that. Allowing unrestricted recourse to the empirical in mathematical proof trivialises the book.[37] On the other hand, Hilbert's[38] synthetic reinterpretation of the *Elements*, leading to the 1956 recommendations of the US School Mathematics Study Group,[39] still used in Indian schools, has serious problems that have already been discussed.[40] However, the fact that a certain book would get de-valued is hardly a valid reason for imposing a non-intuitive, non-metric geometry on K-12 students. Synthetic geometry should be set aside as an unsuccessful attempt to make "Euclid" theologically correct. Though teaching geometry in the traditional Indian way with a rope would involve a serious epistemic shift away from present-day formal mathematics, it is practical, free from artificial theological encumbrances, and is very easy for children to understand. Thus, the "Pythagorean" "theorem" can be established in one step instead of 47 steps. Any philosophical or theological problems with this could well be discussed at the appropriate advanced level, instead of forcing children to grapple with the consequences of obscure theological concerns.

There would be similar radical changes also in the way one teaches numbers, algorithms, calculus. For example, although the computer is ubiquitous, the way students are taught about calculations on a computers is roughly as follows. First, students are taught that numbers obey certain "laws" (note the theological overtones). Then, at an advanced level (provided they specialise

[37] C. K. Raju, "Euclid" cited earlier; for this reason, although Euclid's book was long known in India, it remained confined to religious instruction, and was not translated into Sanskrit (from Persian) until the 18th c. See, C. K. Raju, "Interaction between India, China, and Central and West Asia in Mathematics and Astronomy", A. Rahman ed, *Interaction between India, China, and Central and West Asia*, PHISPC, New Delhi, 2002.

[38] D. Hilbert, *The Foundations of Geometry*, Open Court, La Salle, 1902.

[39] School Mathematics Study Group, *Geometry*, Yale University Press, 1961.

[40] "Equality" in the *Elements* was related to political equity by Neoplatonists and Arab rationalists. While Hilbert reinterpreted this equality as congruence, this reinterpretation does not hold good for "equality", from *Elements* Prop. 1.35 onwards, which refers to equal areas. (Synthetic geometry does not define length, hence it is bit pointless to define area synthetically.) Consequently, the *Elements*, though known in India, remained part of sectarian education for centuries, until the book was finally translated from Persian to Sanskrit in the 18th c. See C. K. Raju, "Euclid", and "India, China and Central and West Asia" cited in 28 above, and C. K. Raju, *Philosophy East and West*, cited earlier.

in mathematics), they are taught the basis of those laws along with number systems like the real number system. Only then are they positioned to understand the rounding conventions used in floating point arithmetic, and the resulting "errors" as studied in numerical analysis. (Thus, most students, including many who specialise in mathematics, never learn about the actual way in which calculations are performed on a computer.[41]) Instead, of this long-drawn route, one could simply explain the technique of calculation with rounding, as done by Brahmagupta, for example, so that it would be very easy for even a K-12 student to grasp the process. The point here is not that one should copy what Brahmagupta did, but that one should proceed on practical rather than theological concerns.

In particular, it may be worth re-examining whether one might want to teach as entirely separate subjects, from the outset, the two mathematical streams: practical mathematics and formal mathematics, with their distinct notions of number and proof. At the same time, one may want to re-examine the feasibility of teaching the consequences of formal mathematics at an elementary level where formalist philosophy itself cannot be taught. Such a re-examination would be particularly timely since the sudden growth of computer technology has again upset the earlier balance (in the West) between mathematics as proof and mathematics as calculation, and this calls for a fundamental review of what mathematics should be taught and how.

It is not being proposed that one should rush into the classroom right away with the suggestions that arise from this work. These suggestions are to be seen as constituting a future research program, which is a clear consequence of the revised historical understanding. A more precise set of recommendations would need to be evolved and documented in consultation with a variety of people including students, historians and philosophers of science, math educators, computer scientists etc. The classroom trials of these new teaching recommendations should be taken up only after allowing a reasonable gap of at least a few years, to allow the documentation to circulate, to elicit reactions and suggestions from a wider circle of educators.

[41] and even a well-established formal mathematician slipped in stating that there was only one possible "accurate" way to do rounding.